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Universally Kuratowski–Ulam spaces

by

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Abstract. We introduce the notions of Kuratowski-Ulam pairs of topological spaces and universally Kuratowski-Ulam space. A pair (X, Y) of topological spaces is called a Kuratowski–Ulam pair if the Kuratowski–Ulam Theorem holds in $X \times Y$. A space Y is called a universally Kuratowski–Ulam (uK–U) space if (X, Y) is a Kuratowski–Ulam pair for every space X. Obviously, every meager in itself space is uK–U. Moreover, it is known that every space with a countable π-basis is uK–U. We prove the following:

• every dyadic space (in fact, any continuous image of any product of separable metrizable spaces) is uK–U (so there are uK–U Baire spaces which do not have countable π-bases);

• every Baire uK–U space is ccc.

1. Kuratowski–Ulam pairs. We use standard set-theoretical notions. In particular, ordinal numbers will be identified with the set of their predecessors and cardinal numbers with the initial ordinals. For a set A and a cardinal κ, $[A]^{<\kappa}$ is the family of all subsets of A with cardinality less than κ. Similarly we define the families $[A]^{\kappa}$ and $[A]^{\leq\kappa}$.

The symbols X, Y, Z denote topological spaces, $\mathcal{M}(X)$ denotes the family of all meager subsets in X. For $E \subset X \times Y$ and $x \in X$, E_x denotes the x-section of E, etc.

A family $\mathcal{U}$ of non-empty open subsets of X is called a *pseudo-basis* (*π-basis* for short) of X if every non-empty open set W in X contains a $U \in \mathcal{U}$. A topological space X is *κ-cc* if there is no family of size κ of

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open, pairwise disjoint sets in X. Note that the ccc property is the same as ω_1-cc.

For a given space X we will use the following two cardinals:

$$\operatorname{add}(\mathcal{M}(X)) = \min\Big\{|\mathcal{D}| : \mathcal{D} \subset \mathcal{M}(X) \ \& \ \bigcup \mathcal{D} \notin \mathcal{M}(X)\Big\},$$
$$\pi(X) = \min\{|\mathcal{U}| : \mathcal{U} \text{ is a } \pi\text{-basis for } X\}.$$

A pair of topological spaces (X, Y) is called a *Kuratowski–Ulam pair* (briefly, *K–U pair*) if the Kuratowski–Ulam theorem holds in $X \times Y$:

K–U: If $E \in \mathcal{M}(X \times Y)$, then $\{x \in X : E_x \notin \mathcal{M}(Y)\} \in \mathcal{M}(X)$.

Kuratowski and Ulam proved that (X, Y) is a K–U pair whenever $\pi(Y) < \operatorname{add}(\mathcal{M}(X))$. (See, e.g., [KK] or [JO, Theorem 15.1, p. 56]. For applications of this method for Ellentuck topologies generated by filters see [IR].) Thus any pair (X, Y), where Y is a topological space with a countable π-basis, is a K–U pair. This fact suggests the consideration of the following property of topological spaces.

Definition 1. A topological space Y is called a *universally Kuratowski–Ulam* space (*uK–U space* for short) if (X, Y) is a K–U pair for any topological space X.

Thus, by the Kuratowski–Ulam theorem, every space Y with a countable π-basis is a uK–U space. Note also that every space Y meager in itself is uK–U.

The scheme of this paper is the following. First we show that there are uK–U Baire spaces without countable π-basis. Next we prove that every Baire uK–U space satisfies the ccc condition and give some examples of Baire ccc spaces which are not uK–U. We finish with descriptions of properties of the class of uK–U spaces.

Theorem 1. *If S is a dense subspace of 2^κ, Z a regular topological space and $f : S \to Z$ a continuous surjection, then Z is a uK–U space.*

Fix the following notation. By Φ we denote the family of all functions $\varphi : A \to 2$ where $A \in [\kappa]^{<\omega}$. There is a canonical isomorphism U between the family Φ and the family $\mathcal{U}$ of all *basic open sets in* 2^κ: $U(\varphi) = \{y \in 2^\kappa : \varphi \subset y\}$. Note that if $\varphi \subset \psi$, $\varphi, \psi \in \Phi$, then $U(\psi) \subset U(\varphi)$. We say that a set $U \subset S$ is *basic open in* S if $U = \widetilde{U} \cap S$ for some basic open set $\widetilde{U} \subset 2^\kappa$. We say that a set $A \subset 2^\kappa$ is *determined by a set of coordinates* $\tau \subset \kappa$ if $A = \{y \in 2^\kappa : y|\tau \in A^*\}$ for some $A^* \subset 2^\tau$. By $\operatorname{cl}(A)$, $\operatorname{int}(A)$ we denote the closure and interior of A in the space 2^κ, and by $\operatorname{cl}_S(A)$, $\operatorname{int}_S(A)$ the closure and interior of A in S.

We will use the following lemma:

LEMMA 1. *If $W \subset S$ is a regular open set then it is a countable union of basic open sets in S.*

Proof. Note that S, as a dense subspace of 2^κ, has the ccc property. Thus we can find a sequence $\langle B_n \rangle_{n<\omega}$ of basic open sets in 2^κ such that $S \cap \bigcup_{n<\omega} B_n$ is a dense subset of W. Each B_n is determined by $\tau_n \in [\kappa]^{<\omega}$. Consider $\tau = \bigcup_{n<\omega} \tau_n \in [\kappa]^{\le\omega}$. Then $\mathrm{cl}(W) = \mathrm{cl}(\bigcup_{n<\omega} B_n)$ is determined by τ. Therefore $\mathrm{int}(\mathrm{cl}(W))$ is determined by τ, so it is expressible as $\bigcup_{n<\omega} U_n$, where U_n are basic open sets in 2^κ.

Now consider $U = S \cap \bigcup_{n\in\omega} U_n$. Observe that $W = \mathrm{int}_S(\mathrm{cl}_S(W)) = S \cap \mathrm{int}(\mathrm{cl}(W))$, so $W = U$. ∎

COROLLARY 1. *If V is a non-meager open set in Z, then there exists a basic open set $W \subset S$ such that $f[W]$ is non-meager and $f[W] \subset V$.*

Proof. Because Z is regular, there exists a non-meager open V' such that $\mathrm{cl}_Z(V') \subset V$. In fact, for each $z \in V$ there is an open set V_x such that $x \in V_x \in \mathrm{cl}_Z(V_x) \subset V$. If all V_x are meager then by the Banach Category Theorem [JO, Theorem 16.1, p. 62], $V = \bigcup_x V_x$ is meager, a contradiction.

Put $W_0 = f^{-1}[V']$ and $W_1 = \mathrm{int}_S(\mathrm{cl}_S(W_0))$. Note that W_1 is regular open in S and $V' \subset f[W_1] \subset V$. By Lemma 1, W_1 is a countable union of basic open sets in S, so the image of one of them is non-meager. ∎

Proof of Theorem 1. Let X be any topological space and $E \subset X \times Z$ be a nowhere dense closed set.

Let $\mathcal{P}$ be the set of all pairs (G, I) where G is an open set in X and $I \in [\kappa]^{<\omega}$. Define a relation $\prec$ on $\mathcal{P}$ by $(H, J) \prec (G, I)$ if

- $H \subset G$ and $J \supset I$, and
- if $W \subset S$ is a basic open set determined by I then either
  - — $H \times f[W] \subset E$, or
  - — there exists $W' \subset W$, a basic open set determined by J, and an open set $U \subset Z$ such that $f[W'] \subset U$ and $(H \times U) \cap E = \emptyset$.

CLAIM. *For any $(G, I) \in \mathcal{P}$ and any non-empty open set $G_0 \subset G$ there exists $(H, J) \in \mathcal{P}$ such that $(H, J) \prec (G, I)$ and $H \subset G_0$.*

In fact, let $|I| = n$ and $\{W_i : 0 < i \le 2^n\}$ be the finite sequence of all basic open sets determined by I. For each $i \le 2^n$ consider two cases. If $G_{i-1} \times f[W_i] \subset E$, set $G_i = G_{i-1}$ and $J_i = J_{i-1}$. (Here $J_0 = I$.) Otherwise find $W_i' \subset W_i$, a basic open set in S determined by J_i, and open sets $U_i \subset Z$, $G_i \subset G_{i-1}$ with $f[W_i'] \subset U_i$ and $(G_i \times U_i) \cap E = \emptyset$. Finally, set $H = G_{2^n}$ and $J = \bigcup_{0<i\le 2^n} J_i$.

Now choose inductively a sequence $\mathcal{P}_n \subset \mathcal{P}$ such that

- $\mathcal{P}_0 = \{(X, \emptyset)\}$.
- If $(H, J), (H', J')$ are distinct members of $\mathcal{P}_n$ then $H \cap H' = \emptyset$.

- For $(H,J) \in \mathcal{P}_{n+1}$ there exists $(G,I) \in \mathcal{P}_n$ such that $(H,J) \prec (G,I)$.
- $\mathcal{P}_{n+1}$ is a maximal family which satisfies the conditions above.

Then all the $G_n^* = \bigcup\{H : (H,J) \in \mathcal{P}_n\}$ are open and dense, so $\bigcap_{n<\omega} G_n^*$ is comeager in X.

Take any $x \in \bigcap_{n<\omega} G_n^*$. We have to prove that E_x is meager. It is sufficient to prove that for any non-meager open set $V \subset Z$ there exists a non-empty open set $V_0 \subset V$ with $E_x \cap V_0 = \emptyset$.

Fix a non-meager open set $V \subset Z$. By Corollary 1 there exists a basic open set $W_0 \subset S$ such that $f[W_0] \subset V$ is non-meager. Assume that W_0 is determined by $J \in [\kappa]^{<\omega}$. For x there is a sequence $\langle (H_n, J_n)\rangle_n$ such that for each n,

- $(H_n, J_n) \in \mathcal{P}_n$;
- $x \in H_n$;
- $(H_{n+1}, J_{n+1}) \prec (H_n, J_n)$, so $J_{n+1} \supset J_n$.

Since J is finite, there exists n with $J_{n+1} \cap J = J_n \cap J$. Note that J_n determines a finite partition of S. Since $f[W_0]$ is non-meager, there exists an open basic set W determined by J_n such that $f[W \cap W_0]$ is not meager. Since E is nowhere dense, $H_{n+1} \times f[W] \not\subset E$. Therefore there exists $W' \subset W$, a basic open set of S determined by J_{n+1}, and an open set $U \subset Z$ such that $(H_{n+1} \times U) \cap E = \emptyset$ and $f[W'] \subset U$. Now $W' \cap W_0 \neq \emptyset$, so $f[W' \cap W_0] \neq \emptyset$ and $U \cap V \neq \emptyset$. We have $x \in H_{n+1}$ and $(H_{n+1} \times (U \cap V)) \cap E = \emptyset$, so $(U \cap V) \cap E_x = \emptyset$. ∎

In particular, for every cardinal κ the space 2^κ is uK–U.

COROLLARY 2. *There exists a uK–U Baire space Y without a countable π-basis.*

Proof. Consider $Y = 2^{\omega_1}$. By Theorem 1, Y is a uK–U space. On the other hand, it is well known that $\pi(Y) = \omega_1$. In fact, let $\{U_n : n < \omega\}$ be a sequence of basic open sets in Y. For each n there exists $A_n \in [\omega_1]^{<\omega}$ and $\varphi_n : A_n \to 2$ such that $U_n = U(\varphi_n)$. Then $A = \bigcup A_n$ is countable. Choose $\alpha \in \omega_1 \setminus A$ and take $V = \{y \in Y : y(\alpha) = 1\}$. Then V is open in Y and no U_n is contained in V. Thus $\{U_n : n \in \omega\}$ is not a π-basis for Y. ∎

A compact space X is said to be *dyadic* if it is a continuous image of the space 2^κ for some cardinal κ (cf. [RE, p. 285]). Thus Theorem 1 implies the following.

COROLLARY 3. *Every dyadic space is uK–U.* ∎

A topological space X is said to be *quasi-dyadic* if it is a continuous image of the Tikhonov product $\prod_\alpha X_\alpha$ of a family $\{X_\alpha : \alpha < \kappa\}$ of metric separable spaces (see [FG]).

THEOREM 2. *Every regular quasi-dyadic space is uK–U.*

Proof. We start with the following lemma.

Lemma 2. *Every metric separable space is a continuous image of a dense subset of the space* 2^ω.

Proof. This is a consequence of the fact that every metric separable space is homeomorphic to a subspace of the Hilbert cube I^ω (see e.g. [AK, Theorem 4.14, p. 22]) and that I^ω is a continuous image of 2^ω. Thus every metric separable space is a continuous image of some subspace of 2^ω. On the other hand, it is easy to prove that every subset of a Cantor set is a continuous image of a dense subset of 2^ω. ∎

To complete the proof of Theorem 2, assume that Y is a regular space, X_α, $\alpha < \kappa$, are metric separable spaces, and $f : \prod_{\alpha<\kappa} X_\alpha \to Y$ is a continuous surjection. For every $\alpha < \kappa$ there exists a continuous surjection $f_\alpha : A_\alpha \to X_\alpha$, where A_α is a dense subspace of 2^ω. Then the set $\prod_{\alpha<\kappa} A_\alpha$ is dense in $2^{\omega\kappa}$ and $f \circ \prod_{\alpha<\kappa} f_\alpha$ is a continuous surjection from $\prod_{\alpha<\kappa} A_\alpha$ onto Y. By Theorem 1, Y is a uK–U space. ∎

Theorem 3. *Assume that X is a non-meager space, Y is a Baire space and (X, Y) is a K–U pair. Then Y is* $\mathrm{add}(\mathcal{M}(X))$*-cc.*

Proof. Suppose that $\kappa = \mathrm{add}(\mathcal{M}(X))$ and $\mathcal{B} = \{B_\alpha : \alpha < \kappa\}$ is a family of open, non-empty, pairwise disjoint sets in Y. Let $\mathcal{A} = \{A_\alpha : \alpha < \kappa\}$ be a family of nowhere dense sets in X with $\bigcup \mathcal{A} \notin \mathcal{M}(X)$. Define $W \subset X \times Y$, $W = \bigcup_{\alpha<\kappa} A_\alpha \times B_\alpha$. Note that W is nowhere dense in $X \times Y$. In fact, fix a basic open set $U \times V$ and consider two cases. If $V_0 = V \setminus \mathrm{cl}_Y(\bigcup_{\alpha<\kappa} B_\alpha) \neq \emptyset$ then $U \times V_0$ is open and disjoint from W. Otherwise $V_0 = \emptyset$. Then $V \cap B_\alpha \neq \emptyset$ for some $\alpha < \kappa$, so for an open, non-empty set $U' \subset U \setminus A_\alpha$ we find that $U' \times (V \cap B_\alpha)$ is a non-empty open set disjoint from W.

On the other hand,

$$\{x : W_x \notin \mathcal{M}(Y)\} = \bigcup \mathcal{A} \notin \mathcal{M}(X)$$

thus (X, Y) is not a K–U pair. ∎

Remark. There exist completely regular spaces X non-meager in themselves with $\mathrm{add}(\mathcal{M}(X)) = \omega_1$. In fact, it is well known that $X = 2^{\omega_1}$ has this property. (All sets $E_\alpha = \{x \in X : x(\xi) = 0 \text{ for } \xi \geq \alpha\}$ are closed and nowhere dense in X, but $\bigcup_{\alpha<\omega_1} E_\alpha \notin \mathcal{M}(X)$. Another example: the space (ω^ω, τ_d) from Example 1 below; see [R].)

Thus we have the following.

Corollary 4. *Every Baire uK–U space satisfies the ccc condition.* ∎

Now we will show that the assumption of ccc for a Baire space Y is not sufficient to make it uK–U.

For $s \in \omega^{<\omega}$ and $f \in \omega^\omega$ with $s \subset f$ define

$$(s, f) = \{g \in \omega^\omega : s \subset g \text{ and } f \leq g\}.$$

Note that the family of such pairs forms a basis for a ccc topology τ_d on ω^ω. It is known that (ω^ω, τ_d) is a completely regular, Baire space (see [R]). Moreover, let τ denote the standard topology on ω^ω. For $f, g \in \omega^\omega$ the symbol $f \leq^* g$ means that the set $\{n \in \omega : f(n) > g(n)\}$ is finite.

Example 1. *$((\omega^\omega, \tau), (\omega^\omega, \tau_d))$ and $((\omega^\omega, \tau_d), (\omega^\omega, \tau_d))$ are not K–U pairs.*

Proof. Define $W = \{(f, g) \in (\omega^\omega)^2 : f \leq^* g\}$.

Claim 1. *W is meager in the topologies $\tau_d \times \tau_d$ and $\tau \times \tau_d$.*

Put $W_n = \{(f, g) \in \omega^\omega \times \omega^\omega : \forall_{k>n}\ f(k) \leq g(k)\}$. We will verify that all W_n are nowhere dense in the topology $\tau_d \times \tau_d$. Let $(s, f) \times (r, h)$ be a basic set. Fix $\overline{k} > n$ such that $\overline{k} \notin \operatorname{dom}(s) \cup \operatorname{dom}(r)$. Choose $s_1, r_1 \in \omega^{<\omega}$ such that $s \subset s_1$, $r \subset r_1$, $s_1(\overline{k}) > r_1(\overline{k})$, $s_1 \geq f|\operatorname{dom}(s_1)$, and $r_1 \geq h|\operatorname{dom}(r_1)$. Let f_1 be any extension of s_1 with $f_1 \geq f$ and h_1 be any extension of r_1 with $h_1 \geq h$. Then $(s_1, f_1) \times (r_1, h_1) \subset (s, f) \times (r, h)$. Observe that $e(\overline{k}) > g(\overline{k})$ for each $(e, g) \in (s_1, f_1) \times (r_1, h_1)$. Thus $(s_1, f_1) \times (r_1, h_1) \cap W_n = \emptyset$, so W_n is nowhere dense, and consequently W is meager in the topology $\tau_d \times \tau_d$.

Similarly we can prove that W is meager in the topology $\tau \times \tau_d$.

Claim 2. *$W_f \notin \mathcal{M}(\tau_d)$ for each $f \in \omega^\omega$.*

Note that $W_f = \{h : f \leq^* h\}$. Fix a basic set (s, g) and define $g_1 \in \omega^\omega$ such that $g_1(i) = h(i)$ if $i \in \operatorname{dom}(s)$ and $g_1(i) = \max(h(i), f(i))$ otherwise. Then $(s, g_1) \subset (s, g) \cap W_f$. Therefore W_f is comeager in the topology τ_d. ∎

Corollary 5. *The space (ω^ω, τ_d) is not a uK–U space.* ∎

We also have another better known example of a ccc space which is not uK–U. Let d denote the density topology on the real line. Recall that $(\mathbb{R}, d)$ is a Baire space with the ccc property, and $A \subset \mathbb{R}$ is d-nowhere dense iff it is d-meager iff $\mathrm{m}(A) = 0$. Here m denotes the Lebesgue measure. (The basic properties of this topology are described in [JO]. See also [FT] for more details.)

Example 2. *For $X = (\mathbb{R}, d)$ the pair (X, X) is not a K–U pair.*

Proof. Consider

$$A = \{(x, y) : x - y \notin \mathbb{Q}\}.$$

As is easily seen, both A and its complement are $d \times d$-dense (this is a consequence of Steinhaus' Theorem [HS], see also [AL]). Moreover, A is a G_δ subset of the plane with full Lebesgue measure, so it contains a closed set E (in Euclidean topology so also in $d \times d$ topology) with positive measure. The set E is nowhere dense in $(\mathbb{R}^2, d \times d)$ and, by Fubini's Theorem,

$$\{x : E_x \notin \mathcal{M}(d)\} = \{x : \mathrm{m}(E_x) > 0\} \notin \mathcal{M}(d). \ \blacksquare$$

2. Properties of the class of uK–U spaces. In this section we present more results and problems about uK–U spaces. We omit some proofs because they are standard.

PROPERTY 1. *The product of finitely many uK–U spaces is also a uK–U space.*

Proof. Assume that Y and Z are uK–U spaces, X is a topological space and E is a closed nowhere dense subset of $X \times Y \times Z$. Let $E' = \{(x, y) \in X \times Y : E_{(x,y)} \notin \mathcal{M}(Z)\}$. Then $E' \in \mathcal{M}(X \times Y)$. Since (X, Y) is a K–U pair, we have $\{x \in X : (E')_x \notin \mathcal{M}(Y)\} \in \mathcal{M}(X)$.

Now observe that if $E_x \notin \mathcal{M}(Y \times Z)$ then $(E')_x \notin \mathcal{M}(Y)$. In fact,

$$E_x = \{(y, z) : (x, y, z) \in E\} = \{(y, z) : z \in E_{(x,y)}\}$$

and this set is closed. Then $\mathrm{int}(E_x) \notin \mathcal{M}(Y \times Z)$, and by the Banach Category Theorem, there exists an open set $U \times V \subset E_x$ with $U \times V \notin \mathcal{M}(Y \times Z)$. Therefore $U \notin \mathcal{M}(Y)$, $V \notin \mathcal{M}(Z)$, and

$$U \subset \{y : (E_x)_y \notin \mathcal{M}(Z)\} \notin \mathcal{M}(Y),$$

so

$$(E')_x = \{y : E_{(x,y)} \notin \mathcal{M}(Z)\} \notin \mathcal{M}(Y).$$

Thus

$$\{x \in X : E_x \notin \mathcal{M}(Y \times Z)\} \subset \{x \in X : (E')_x \notin \mathcal{M}(Y)\} \in \mathcal{M}(X). \ \blacksquare$$

PROPERTY 2. *The product of countably many uK–U spaces is a uK–U space.*

Proof. Suppose that $\{Y_n\}_{n<\omega}$ are uK–U spaces, X is any topological space and $W \subset X \times \prod_{n<\omega} Y_n$ is a dense open set. Put $\pi_n(x, y) = (x, y|n)$ for $n < \omega$ (that is, π_n is the natural projection from $X \times \prod_n Y_n$ onto $X \times \prod_{i<n} Y_i$). Let $W_n = \pi_n[W]$; then W_n is a dense open set in $X \times \prod_{i<n} Y_i$. Because finite products of uK–U spaces are uK–U (cf. Property 1), $\{x \in X : (W_n)_x \text{ is dense}\}$ is comeager for every n, so $H = \{x \in X : (W_n)_x \text{ is dense for every } n\}$ is comeager in X.

Now, if there is an $x \in H$ such that W_x is not dense in $\prod_{n<\omega} Y_n$, there are $n < \omega$ and non-empty open sets $G_i \subset Y_i$ for $i < n$ such that W_x does not meet $\prod_{i<n} G_i \times \prod_{i\ge n} Y_i$. But then $(W_n)_x$ does meet $\prod_{i<n} G_i$, which is impossible. $\blacksquare$

Applications. Recall that the product $X \times Y$ of Baire spaces may be non-Baire. (Some conditions for X and Y which imply that $X \times Y$ is a Baire space are described in [HMC].) Note that if X and Y are Baire spaces and (X, Y) is a K–U pair, then $X \times Y$ is a Baire space. Similarly, the product $X \times Y$ of a Baire space X and a uK–U Baire space Y is a Baire space.

REMARK. Property 2 leads to the natural problem whether the product of *any* family of uK–U spaces is always uK–U. This problem has been solved recently by D. Fremlin [DF] in the affirmative.

PROPERTY 3. *Any open subspace of a uK–U space is itself uK–U.* ∎

PROPERTY 4. *If Y_0 is a dense subspace of a uK–U space Y, then it is also a uK–U space.* ∎

PROPERTY 5. *Assume that Y_0 is a subspace of a uK–U space Y such that $Y_0 \subset \mathrm{int}_Y(\mathrm{cl}_Y(Y_0))$. Then Y_0 is also a uK–U space.* ∎

EXAMPLE 3. *There exists a subspace Y_0 of a uK–U space Y which fails to be a uK–U space.*

Proof. Take Y_0 to be the discrete space of size ω_1. As Y_0 has weight ω_1, it embeds into $Y = [0,1]^{\omega_1}$ (see e.g. [RE, Theorem 2.3.11, p. 113]). By Theorem 2, Y is uK–U, but Y_0 is not ccc, so it is not uK–U, by Corollary 4. ∎

We say that a set $A \subset X$ is *nowhere meager* in a space X if $U \cap A \notin \mathcal{M}(X)$ for every open, non-meager set $U \subset X$.

PROPERTY 6. *Suppose that Y_0 is a uK–U dense subspace of a space Y. If Y_0 is nowhere meager in Y then Y is a uK–U space.* ∎

The assumption about Y_0 cannot be omitted.

EXAMPLE 4. *There exists a non-uK–U space Y with a dense uK–U subspace Y_0.*

Proof. Let Y be any complete dense-in-itself metric space which is non-ccc. By Corollary 4, Y is not uK–U space. For every $n > 0$ choose a discrete set $Y_n \subset Y$ which forms a $1/n$-net in Y. Then $Y_0 = \bigcup_{n>0} Y_n$ is dense in Y, dense in itself and meager in itself. Thus Y_0 is a uK–U space. ∎

PROPERTY 7. *Suppose that $\{Y_i : i < \omega\}$ is a sequence of uK–U subspaces of a topological space Y. Then $\bigcup_i Y_i$ is also a uK–U space.* ∎

COROLLARY 6. *The topological sum of countably many uK–U spaces is a uK–U space.* ∎

EXAMPLE 5. *The topological sum of uncountably many uK–U spaces may fail to be a uK–U space.*

Proof. Let Y be a discrete space of size ω_1. Then Y is not ccc, so it is not a uK–U space. On the other hand, every singleton is a uK–U space. ∎

PROPERTY 8. *The homeomorphic image of a uK–U space is also a uK–U space.* ∎

Property 9. *The image of a uK–U Baire space under a continuous open function is a uK–U space.* ∎

Note that any space Y is a continuous image of the space $Y \times \mathbb{Q}$ meager in itself. Thus any Y is a continuous image of a uK–U space.

Remark. The results above lead to the problem whether any continuous image of a uK–U Baire space is also uK–U. This problem has recently been solved by D. Fremlin [DF] in the negative.

References

[AL] R. Anantharaman and J. P. Lee, *Planar sets whose complements do not contain a dense set of lines*, Real Anal. Exchange 11 (1985–86), 168–179.

[RE] R. Engelking, *General Topology*, PWN, Warszawa, 1976.

[FG] D. H. Fremlin and S. Grekas, *Products of completion regular measures*, Fund. Math. 147 (1995), 27–37.

[DF] D. H. Fremlin, *Universally Kuratowski–Ulam spaces*, preprint.

[HMC] R. C. Haworth and R. C. McCoy, *Baire spaces*, Dissertationes Math. 141 (1977).

[AK] A. S. Kechris, *Classical Descriptive Set Theory*, Springer, Berlin, 1995.

[KK] K. Kuratowski, *Topologie I*, PWN, Warszawa, 1958.

[ĄR] G. Łabędzki and M. Repický, *Hechler reals*, J. Symbolic Logic 60 (1995), 444–458.

[JO] J. Oxtoby, *Measure and Category*, Springer, 1980.

[IR] I. Recław, *Fubini properties for σ-centered σ-ideals*, Topology Appl., to appear.

[FT] F. D. Tall, *The density topology*, Pacific Math. J. 62 (1976), 275–284.

[HS] H. Steinhaus, *Sur les distances des points des ensembles de mesure positive*, Fund. Math. 1 (1920), 99–104.

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