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Weakly α-favourable measure spaces

by

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Abstract. I discuss the properties of α-favourable and weakly α-favourable measure spaces, with remarks on their relations with other classes.

1. Introduction. Seeking to understand the special properties of Lebesgue measure, regarded as an abstract measure without looking directly at its relation with the topology of the real line or Euclidean space, we are led more or less naturally to six classes of measure space: compact, countably compact and monocompact measure spaces, α-favourable and weakly α-favourable measure spaces, and perfect measure spaces. (See 1A below for the definitions.) These form a straightforward hierarchy, each class, as I have listed them, being included in the next. In this paper I look at the class of weakly α-favourable spaces from the point of view of pure measure theory, examining in particular its permanence properties under standard operations.

We find that an image of a weakly α-favourable measure is weakly α-favourable (2B), and that the product of any number of weakly α-favourable probability spaces is weakly α-favourable (2D); in fact, there is a general result on measures on product spaces with weakly α-favourable marginals (2C), in the spirit of Kolmogorov's theorem on measures with perfect marginals. I do not know whether there are corresponding results for α-favourable spaces, but I include a partial result in this direction: the product of an α-favourable space with a countably compact space is α-favourable (2E).

In §3, I discuss some conditions under which we can be sure that a weakly α-favourable space is in fact α-favourable, or regularly monocompact, or even countably compact. In §4, I give examples (i) of a regularly monocompact probability measure which is not countably compact, and (ii) of a perfect probability measure which is not weakly α-favourable (adapting

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an example of K. Musiał). I conclude with a brief discussion of some open problems.

1A. *Basic definitions.* (a) Following [19], I will say that a *compact class* is a family $\mathcal{K}$ of sets such that $\bigcap\mathcal{K}' \neq \emptyset$ whenever $\mathcal{K}' \subseteq \mathcal{K}$ has the finite intersection property, and that a *countably compact class* is a family $\mathcal{K}$ of sets such that $\bigcap\mathcal{K}'$ is non-empty for every countable set $\mathcal{K}' \subseteq \mathcal{K}$ with the finite intersection property. (Note that most authors up to 1980 or so, following [14], used the phrases "compact class", "compact measure" to mean what I am calling "countably compact class" and "countably compact measure".) Following [24] and [5], I say that a *monocompact class* is a family $\mathcal{K}$ of sets such that $\bigcap_{n\in\mathbb{N}} K_n \neq \emptyset$ for any non-increasing sequence $\langle K_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{K}$.

(b) Now let $\mathcal{K}$ be a family of non-empty sets. Consider the infinite game $\Gamma(\mathcal{K})$ for two players in which the players choose alternately sets K_0, K'_0, K_1, $K'_1, \ldots$ in $\mathcal{K}$ with $K_0 \supseteq K'_0 \supseteq K_1 \supseteq K'_1 \supseteq \ldots$ and the first player wins if $\bigcap_{i\in\mathbb{N}} K_i = \emptyset$, while the second player wins if $\bigcap_{i\in\mathbb{N}} K_i \neq \emptyset$. I will say that a *strategy* for the second player in $\Gamma(\mathcal{K})$ is a function $\sigma : \bigcup_{n\ge1}\mathcal{K}^n \to \mathcal{K}$ such that $\sigma(K_0,\ldots,K_n) \subseteq K_n$ for every $K_0,\ldots,K_n \in \mathcal{K}$, and that such a strategy σ is a *winning strategy* if $\bigcap_{n\in\mathbb{N}} K_n \neq \emptyset$ whenever $\langle K_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{K}$ such that $K_{n+1} \subseteq \sigma(K_0,\ldots,K_n)$ for every $n \in \mathbb{N}$. Similarly, a *tactic* (or "stationary strategy") for the second player in $\Gamma(\mathcal{K})$ is a function $\tau : \mathcal{K} \to \mathcal{K}$ such that $\tau(K) \subseteq K$ for every $K \in \mathcal{K}$, and τ is a *winning tactic* if $\bigcap_{n\in\mathbb{N}} K_n \neq \emptyset$ whenever $\langle K_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{K}$ such that $K_{n+1} \subseteq \tau(K_n)$ for every $n \in \mathbb{N}$. If $\tau : \mathcal{K} \to \mathcal{K}$ is any tactic for the second player, we have an associated strategy σ defined by setting $\sigma(K_0,\ldots,K_n) = \tau(K_n)$ for every $K_0,\ldots,K_n \in \mathcal{K}$, and σ will be a winning strategy iff τ is a winning tactic.

I will say that a non-empty family $\mathcal{K}$ of non-empty sets is a *weakly α-favourable class* if the second player has a winning strategy in $\Gamma(\mathcal{K})$, and an *α-favourable class* if the second player has a winning tactic. In this context, it is convenient to say that if $\mathcal{K}$ is empty (so that there are no plays in the game $\Gamma(\mathcal{K})$) then the empty tactic is a winning tactic for the second player, so that $\mathcal{K}$ is α-favourable.

For a variety of measure-theoretic and topological questions to which the idea of "weakly α-favourable class'" is relevant, see [8].

(c) Recall that a measure space (X,Σ,μ) is (*countably*) *compact* if there is a (countably) compact class $\mathcal{K} \subseteq \Sigma$ such that μ is inner regular with respect to $\mathcal{K}$, in the sense that $\mu E = \sup\{\mu K : K \in \mathcal{K},\ K \subseteq E\}$ for every $E \in \Sigma$ ([14] and [19], or [10], §342 and §451). (X,Σ,μ), or μ, is (*weakly*) *α-favourable* if $\Sigma \setminus \mathcal{N}_\mu$ is a (weakly) α-favourable class, where I write $\mathcal{N}_\mu$ for the ideal of μ-negligible sets ([8]).

Next, if (X, Σ, μ) is a measure space and $\mathcal{K}$ a family of sets, we say that $\mathcal{K}$ *μ-approximates Σ* if whenever $E \in \Sigma$ and $0 \le \gamma < \mu E$ there are $F \in \Sigma$ and $K \in \mathcal{K}$ such that $F \subseteq K \subseteq E$ and $\mu F \ge \gamma$. (Note that, on the definitions above, μ is inner regular with respect to $\mathcal{K}$ iff $\mathcal{K} \cap \Sigma$ μ-approximates Σ.) (X, Σ, μ), or μ, is *monocompact* if there is a monocompact class which μ-approximates Σ.

Three natural variations on this idea are perhaps worth signalling, even though it is at present quite unclear which, if any, of them correspond to different measure spaces.

(α) Let us say that a measure space (X, Σ, μ) is *regularly monocompact* if there is a monocompact class $\mathcal{K}$ such that μ is inner regular with respect to $\mathcal{K}$ (equivalently, if there is a monocompact class $\mathcal{K} \subseteq \Sigma$ which μ-approximates Σ).

(β) Let us say that a measure space (X, Σ, μ) is *weakly monocompact* if there is a monocompact class $\mathcal{K}$ such that for every $E \in \Sigma \setminus \mathcal{N}_\mu$ there are $F \in \Sigma \setminus \mathcal{N}_\mu$, $K \in \mathcal{K}$ such that $F \subseteq K \subseteq E$.

(γ) Let us say that a measure space (X, Σ, μ) is *weakly regularly monocompact* if there is a monocompact class $\mathcal{K}$ which is coinitial with $\Sigma \setminus \mathcal{N}_\mu$.

To complete this list, recall that a measure space (X, Σ, μ) is *perfect* if whenever $f : X \to \mathbb{R}$ is a Σ-measurable function, $E \in \Sigma$ and $\mu E > 0$, then there is a compact set $K \subseteq f[E]$ such that $\mu f^{-1}[K] > 0$ ([20], or [10], §342 and §451).

1B. *Elementary results.* (a) A compact class is countably compact. A countably compact class of non-empty sets is monocompact. A monocompact class is α-favourable (since $K \mapsto K$ is a winning tactic). An α-favourable class is weakly α-favourable. Any subset of a compact, or countably compact, or monocompact class is a class of the same kind.

(b) It will be useful to have the following facts available.

(i) A family $\mathcal{K}$ of subsets of a set X is a compact class iff there is a compact topology (not necessarily Hausdorff) on X such that every member of $\mathcal{K}$ is closed ([19], 3.2, or [10], 342D). It follows that if $\mathcal{K}$ is any compact class of sets, there is a compact class $\mathcal{K}^* \supseteq \mathcal{K}$ such that $K \cup K' \in \mathcal{K}^*$ for all K, $K' \in \mathcal{K}^*$ and $\bigcap \mathcal{K}' \in \mathcal{K}^*$ for every non-empty $\mathcal{K}' \subseteq \mathcal{K}^*$.

(ii) If $\mathcal{K}$ is a countably compact class, there is a countably compact class $K^* \supseteq \mathcal{K}$ such that $K \cup K' \in \mathcal{K}^*$ for all K, $K' \in \mathcal{K}^*$ and $\bigcap_{n\in\mathbb{N}} K_n \in \mathcal{K}^*$ for every sequence $\langle K_n \rangle_{n\in\mathbb{N}}$ in $\mathcal{K}^*$ ([14] or [10], 413R).

1C. Lemma. *If $\mathcal{K}$ is a family of non-empty sets, and $\mathcal{L}$ is a coinitial subset of $\mathcal{K}$, then $\mathcal{L}$ is (weakly) α-favourable iff $\mathcal{K}$ is.*

Proof. For "weakly α-favourable"', this is [8], 2P. For "α-favourable", let $\theta : \mathcal{K} \to \mathcal{L}$ be any function such that $\theta(K) \subseteq K$ for every $K \in \mathcal{K}$. If $\tau : \mathcal{K} \to \mathcal{K}$ is a winning tactic for the second player in $\Gamma(\mathcal{K})$, then $L \mapsto \theta\tau(L) : \mathcal{L} \to \mathcal{L}$ is a winning tactic in $\Gamma(\mathcal{L})$. If $\tau : \mathcal{L} \to \mathcal{L}$ is a winning tactic in $\Gamma(\mathcal{L})$, then $K \mapsto \tau\theta(K) : \mathcal{K} \to \mathcal{K}$ is a winning tactic in $\Gamma(\mathcal{K})$.

1D. In the definition of "strategy" above, the function σ determining the moves of the second player is defined solely in terms of the moves of the first player, it being understood that a play of the game will proceed in the form

$$K_0, \sigma(K_0), K_1, \sigma(K_0, K_1), \ldots,$$

so that there is no need to name the moves of the second player in the definition of σ. A different approach describes a play of the game by the sequence of the *second* player's moves, and it is useful to have a condition for weak α-favourability in terms of such sequences.

LEMMA. *Let $\mathcal{K}$ be a family of non-empty sets. Then $\mathcal{K}$ is weakly α-favourable iff there is a family $Q \subseteq \bigcup_{n\ge1} \mathcal{K}^n$ such that*

(i) *whenever $(K'_0, \ldots, K'_n) \in Q$ then $K'_0 \supseteq K'_1 \supseteq \ldots \supseteq K'_n$;*

(ii) *for every $K \in \mathcal{K}$ there is a $K' \subseteq K$ such that the one-term sequence (K') belongs to Q;*

(iii) *whenever $(K'_0, \ldots, K'_n) \in Q$ and $K \in \mathcal{K}$ and $K \subseteq K'_n$, then there is a $K' \subseteq K$ such that $(K'_0, \ldots, K'_n, K') \in Q$;*

(iv) *whenever $\langle K'_n \rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{K}$ such that $(K'_0, \ldots, K'_n) \in Q$ for every n, then $\bigcap_{n\in\mathbb{N}} K'_n \ne \emptyset$.*

Proof. (a) If $\mathcal{K}$ is weakly α-favourable, let $\sigma : \bigcup_{n\ge1} \mathcal{K}^n \to \mathcal{K}$ be a winning strategy for the second player in $\Gamma(\mathcal{K})$. Define $Q_n \subseteq \mathcal{K}^{n+1}$, $\phi_n : Q_n \to \mathcal{K}^{n+1}$ inductively, for $n \in \mathbb{N}$, as follows. Start by setting $Q_0 = \{\sigma(K) : K \in \mathcal{K}\}$, and let $\phi_0 : Q_0 \to \mathcal{K}$ be any function such that $K' = \sigma(\phi_0(K'))$ for every $K' \in Q_0$. Given Q_n and ϕ_n, let Q_{n+1} be the set of finite sequences $\boldsymbol{s}^\frown K'$ in $\mathcal{K}$ for which $\boldsymbol{s} = (K'_0, \ldots, K'_n) \in Q_n$ and there is some $K \in \mathcal{K}$ such that $K \subseteq K'_n$ and $K' = \sigma(\phi_n(\boldsymbol{s})^\frown K)$; now take $\phi_{n+1} : Q_{n+1} \to \mathcal{K}^{n+1}$ to be any function such that if $\boldsymbol{s} \in Q_n$ and $\boldsymbol{s}^\frown K' \in Q_{n+1}$, then $\phi_{n+1}(\boldsymbol{s}^\frown K')$ is of the form $\phi_n(\boldsymbol{s})^\frown K$ where $K \subseteq K'_n$ and $K' = \sigma(\boldsymbol{s}^\frown K)$. At the end of the induction, set $Q = \bigcup_{n\in\mathbb{N}} Q_n \subseteq \bigcup_{n\ge1} \mathcal{K}^n$.

The construction ensures that Q satisfies (i). Setting $K' = \sigma(K)$, we see that Q satisfies condition (ii). If $\boldsymbol{s} = (K'_0, \ldots, K'_n) \in Q$, $K \in \mathcal{K}$ and $K \subseteq K'_n$, set $K' = \sigma(\phi(\boldsymbol{s})^\frown K)$; then $K' \subseteq K$ and $\boldsymbol{s}^\frown K' \in Q$, so Q satisfies (iii). If $\langle K'_n \rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{K}$ such that $(K'_0, \ldots, K'_n) \in Q$ for each n, then observe that $\phi_{n+1}(K'_0, \ldots, K'_{n+1})$ extends $\phi_n(K'_0, \ldots, K'_n)$ for every n, so we have a sequence $\langle K_n \rangle_{n\in\mathbb{N}}$ in $\mathcal{K}$ such that $(K_0, \ldots, K_n) = \phi_n(K'_0, \ldots, K'_n)$ for every n. In this case, by the choice of Q_{n+1} and ϕ_{n+1}, we have $K_{n+1} \subseteq K'_n =$

$\sigma(K_0,\dots,K_n)$ for every n. Because σ is a winning strategy, $\bigcap_{n\in\mathbb{N}} K'_n = \bigcap_{n\in\mathbb{N}} K_n$ is non-empty, as required by condition (iv).

(b) If there is such a family Q, then choose $\sigma_n : \mathcal{K}^{n+1} \to \mathcal{K}$ inductively, for $n \in \mathbb{N}$, as follows. For $K \in \mathcal{K}$, $\sigma_0(K)$ is to be any member of $\mathcal{K}$, included in K, such that the single-element sequence $(\sigma_0(K))$ belongs to Q; there is such an element because Q satisfies condition (i). Given $\sigma_n : \mathcal{K}^{n+1} \to \mathcal{K}$ and $(K_0,\dots,K_{n+1}) \in \mathcal{K}^{n+2}$, set $K'_r = \sigma_r(K_0,\dots,K_r)$ for $r \le n$. If $(K'_0,\dots,K'_n) \in Q$ and $K_{n+1} \subseteq K'_n$, then, because Q satisfies condition (ii), there is some $K \subseteq K_{n+1}$ such that $(K'_0,\dots,K'_n,K') \in Q$; take $\sigma_{n+1}(K_0,\dots,K_{n+1})$ to be any such K'. In any other case, set $\sigma_{n+1}(K_0,\dots,K_{n+1}) = K_{n+1}$. At the end of the induction, set $\sigma(\boldsymbol{s}) = \sigma_n(\boldsymbol{s})$ for $n \in \mathbb{N}$, $\boldsymbol{s} \in \mathcal{K}^{n+1}$.

The construction of the σ_n certainly ensures that σ is a strategy for the second player in $\Gamma(\mathcal{K})$. If now $\langle K_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{K}$ such that $K_{n+1} \subseteq \sigma(K_0,\dots,K_n)$ for every $n \in \mathbb{N}$, then, inducing on n, we see that $(K'_0,\dots,K'_n) \in Q$ for every $n \in \mathbb{N}$, where $K'_n = \sigma(K_0,\dots,K_n)$; so $\bigcap_{n\in\mathbb{N}} K_n = \bigcap_{n\in\mathbb{N}} K'_n$ is non-empty, by condition (iii) on Q. As $\langle K_n\rangle_{n\in\mathbb{N}}$ is arbitrary, σ is a winning strategy and $\mathcal{K}$ is weakly α-favourable.

1E. In §§3–4 below, we shall be looking at measures on $\{0,1\}^A$, for various sets A, and it will be helpful to have the following facts set out clearly.

LEMMA. *Let $\langle X_i\rangle_{i\in I}$ be any family of topological spaces, with product X.*

(a) *For any closed set $F \subseteq X$, there is a smallest set $A_F \subseteq I$ such that F is determined by coordinates in A_F, in the sense that $x \in F$ whenever $x \in X$, $y \in F$ and $x{\restriction}A_F = y{\restriction}A_F$.*

(b) *If $\langle F_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence of closed sets in X with intersection F, then $A_F \subseteq \bigcup_{n\in\mathbb{N}}\bigcap_{m\ge n} A_{F_m}$.*

(c) *Suppose that μ is a strictly positive topological measure on X which is a product measure in the sense that $\mu(E\cap F) = \mu E\cdot\mu F$ whenever $E, F \in \operatorname{dom}\mu$ and there is some set $J \subseteq I$ such that E is determined by coordinates in J and F is determined by coordinates in $I\setminus J$. Suppose that E and F are measurable sets such that $E \mathbin{\triangle} F$ is negligible, F is closed, and F is self-supporting in the sense that $\mu(F\cap U) > 0$ whenever $U \subseteq X$ is an open set meeting F. If $J \subseteq I$ is such that E is determined by coordinates in J, so is F.*

Proof. (a) Let $\mathcal{J}$ be the family of those subsets J of I such that F is determined by coordinates in J. Then $I \in \mathcal{J}$ and $J \cap J' \in \mathcal{J}$ for any J, $J' \in \mathcal{J}$ ([10], 254T). Set $A_F = \bigcap\mathcal{J}$.

Suppose, if possible, that F is not determined by coordinates in A_F. Then we have $y \in F$, $x \in X\setminus F$ such that $x{\restriction}A_F = y{\restriction}A_F$. Because F is closed, there is a finite set $K \subseteq I$ such that $\{z : z \in X, z\restriction K = x\restriction I\}$ is disjoint from F. Because $\mathcal{J}$ is closed under finite intersections, there is a $J \in \mathcal{J}$ such that $K \cap J = A_F \cap J$. But now consider $z \in X$ defined by saying that $z(i) = x(i)$ if $i \in K$, $z(i) = y(i)$ if $i \in I \setminus K$. Then $z\restriction J = y\restriction J$, so $z \in F$, because F is determined by coordinates in J; but also $z\restriction K = x\restriction K$, so $z \notin F$, which is impossible.

Thus F is determined by coordinates in A_F, and A_F is the unique smallest member of $\mathcal{J}$, as claimed.

(b) Set $A = \bigcup_{n\in\mathbb{N}} \bigcap_{m\geq n} A_{F_m}$. If $i \in I \setminus A$, then $L = \{n : i \notin A_{F_n}\}$ is infinite, so $F = \bigcap_{n\in L} F_n$. Since every F_n, for $n \in L$, is determined by coordinates in $I \setminus \{i\}$, so is F ([10], 254M), and $A_F \subseteq I \setminus \{i\}$. As i is arbitrary, $A_F \subseteq A$, as claimed.

(c) Suppose, if possible, otherwise; then there are $x \in F$, $y \in X \setminus F$ such that $x\restriction J = y\restriction J$. Because F is closed, there is a basic open set containing y and disjoint from F; we can express this in the form $U \cap V$ where U and V are open, U is determined by coordinates in J and V is determined by coordinates in $I \setminus J$. Now $x \in U$, so $U \cap F \neq \emptyset$ and $\mu(U \cap F) > 0$. But now

$$0 = \mu(F \cap U \cap V) = \mu(E \cap U \cap V)$$

because $E \triangle F$ is negligible

$$= \mu(E \cap U) \cdot \mu V$$

because $E \cap U$ is determined by coordinates in J, V is determined by coordinates in $I \setminus J$, and μ is a product measure

$$= \mu(F \cap U) \cdot \mu V > 0$$

(because μ is strictly positive). But this is absurd.

2. Measure spaces. I come now to the main work of this paper, the study of (weakly) α-favourable measure spaces.

2A. *Elementary facts.* Compact measure spaces are countably compact, countably compact measure spaces are α-favourable, and α-favourable measure spaces are weakly α-favourable, just because we have corresponding results for the abstract classes of §1. It is also the case that semi-finite weakly α-favourable measure spaces are perfect; see [8], 6Bg. (The result there is stated only for totally finite measure spaces, because these are the only case considered in [20]. If we extend the definition as in 1Ac above, then a semi-finite measure μ is perfect iff the subspace measure μ_E on E is perfect whenever $\mu E < \infty$; and it is also easy to see that μ is weakly α-favourable iff μ_E is weakly α-favourable for every set E of finite measure, because $\{E : 0 < \mu E < \infty\}$ is coinitial with $\operatorname{dom}\mu \setminus \mathcal{N}_\mu$.) If (X, Σ, μ) is "countably separated", in the sense that there is a sequence of measurable sets separating the points of X, then μ is compact iff it is perfect ([10], 343K), so all the classes considered here coincide for such spaces. All Radon measures are of course compact, since by definition they are inner regular with respect to the compact sets in some Hausdorff space, which form a compact class.

Since any countably compact class is monocompact, we see at once that, for arbitrary measure spaces,

countably compact ⇒ regularly monocompact

⇒ monocompact and weakly regularly monocompact

and

monocompact or weakly regularly monocompact ⇒ weakly monocompact.

It is also easy to see that a weakly monocompact space (X, Σ, μ) is α-favourable; if $\mathcal{K}$ is a monocompact class witnessing the definition in 1Ac(γ) above, choose any function $\tau : \Sigma \setminus \mathcal{N}_\mu \to \Sigma \setminus \mathcal{N}_\mu$ such that for every $E \in \Sigma \setminus \mathcal{N}_\mu$ there is a $K \in \mathcal{K}$ such that $\tau(E) \subseteq K \subseteq E$; then τ will be a winning tactic for the second player in the game $\Gamma(\Sigma \setminus \mathcal{N}_\mu)$.

It is worth noting that a semi-finite measure space (X, Σ, μ) is perfect iff $(X, \Sigma_0, \mu\restriction\Sigma_0)$ is compact for every countably generated σ-algebra $\Sigma_0 \subseteq \Sigma$ iff $(X, \Sigma_0, \mu\restriction\Sigma_0)$ is countably compact for every countably generated σ-algebra $\Sigma_0 \subseteq \Sigma$ ([18], or [10], 451F). In particular, for measures defined on countably generated σ-algebras, all the classes considered in this paper coincide.

It is easy to see that if (X, Σ, μ) is a measure space and $E \in \Sigma$, then E, with the induced measure, is compact, or countably compact, or monocompact (of any variety), or α-favourable, or weakly α-favourable, or perfect, if X is. Non-measurable subsets, on the other hand, relatively rarely inherit these properties; for instance, no non-measurable subset of $\mathbb{R}$ can be perfect in its subspace measure. Concerning measurable images and products there is something more interesting to say, as follows.

2B. THEOREM. *If (X, Σ, μ) is a weakly α-favourable measure space, (Y, T, ν) a semi-finite measure space and $f : X \to Y$ an inverse-measure-preserving function (that is, $\mu f^{-1}[F]$ is defined and equal to νF for every $F \in T$), then (Y, T, ν) is weakly α-favourable.*

Proof. (a) Write $\mathcal{L}$ for $\{F : F \in T, 0 < \nu F < \infty\}$. Write $\mathcal{K}$ for the set of those $K \in \Sigma \setminus \mathcal{N}_\mu$ such that *either* $K \cap f^{-1}[F]$ is μ-negligible for every $F \in \mathcal{L}$ *or* $f[K] \cap F$ is ν-negligible whenever $F \in T$ and $K \cap f^{-1}[F]$ is μ-negligible. We shall have to remember that while $K \cap f^{-1}[F]$ will always be measurable, there is no guarantee that $f[K] \cap F$ is measurable. Note however that if $E \in \Sigma$ and $f[E]$ is negligible, then there is some negligible $F \in T$ such that $f[E] \subseteq F$, so that $E \subseteq f^{-1}[F]$ is negligible, because f is inverse-measure-preserving.

(b) $\mathcal{K}$ is coinitial with $\Sigma \setminus \mathcal{N}_\mu$. To see this, take any $E \in \Sigma \setminus \mathcal{N}_\mu$. If $E \cap f^{-1}[F]$ is negligible whenever $F \in \mathcal{L}$, then $E \in \mathcal{K}$ and we can stop. Otherwise, take $F_0 \in \mathcal{L}$ such that $E_0 = E \cap f^{-1}[F_0]$ is not negligible. Set $\mathcal{F} = \{F : F \in T, F \subseteq F_0, E_0 \cap f^{-1}[F] \text{ is negligible}\}$. By the principle of exhaustion ([10], 215A–215B), there is a countable set $\mathcal{F}_0 \subseteq \mathcal{F}$ such that $F \setminus F_1$ is negligible for every $F \in \mathcal{F}$, where $F_1 = \bigcup \mathcal{F}_0$. Set $K = E_0 \setminus f^{-1}[F_1]$. Now $\mathcal{F}$ is certainly closed under countable unions, so $F_1 \in \mathcal{F}$, and $E_0 \setminus K = E_0 \cap f^{-1}[F_1]$ is negligible; but this means that K is not negligible.

If $F \in T$ and $K \cap f^{-1}[F]$ is negligible, then $F \cap F_0 \in \mathcal{F}$ so $F \cap F_0 \setminus F_1$ is negligible. But $f[K] \subseteq F_0 \setminus F_1$, so $f[K] \cap F$ is negligible. As F is arbitrary, $K \in \mathcal{K}$. Of course, $K \subseteq E$; as E is arbitrary, $\mathcal{K}$ is coinitial with $\Sigma \setminus \mathcal{N}_\mu$, as claimed.

(c) By 1C, $\mathcal{K}$ is a weakly α-favourable class, and there is a winning strategy σ for the second player in $\Gamma(\mathcal{K})$. Define $\phi_n : \mathcal{L}^{n+1} \to \mathcal{K}^{n+1}$, $\sigma'_n : \mathcal{L}^{n+1} \to \mathcal{L}$ inductively, for $n \in \mathbb{N}$, as follows.

If $L \in \mathcal{L}$ take $K \in \mathcal{K}$ such that $K \subseteq f^{-1}[L]$; let $\phi_0(L)$ be the one-term sequence (K), and let $\sigma'_0(L)$ be a measurable envelope of $f[\sigma(K)]$ included in L. (Such an envelope exists because νL is finite and $f[\sigma(K)] \subseteq f[K] \subseteq L$; see [10], 132E.) Note that as $\sigma(K)$ is not negligible, nor is $f[\sigma(K)]$, and $\sigma'_0(L) \in \mathcal{L}$. Now suppose that ϕ_n, σ'_n have been defined and $\boldsymbol{s} \in \mathcal{L}^{n+1}$, $L \in \mathcal{L}$. Choose $K \in \mathcal{K}$ such that $K \subseteq f^{-1}[L]$ and, if possible, $K \subseteq \sigma(\phi_n(\boldsymbol{s}))$; set $\phi_{n+1}(\boldsymbol{s}^\frown L) = \phi_n(\boldsymbol{s})^\frown K$, and let $\sigma'_{n+1}(\boldsymbol{s}^\frown L)$ be a measurable envelope of $f[\sigma(\boldsymbol{s}^\frown K)]$ which is included in L. Continue.

(d) At the end of the induction, set $\sigma'(\boldsymbol{s}) = \sigma'_n(\boldsymbol{s})$ whenever $n \in \mathbb{N}$ and $\boldsymbol{s} \in \mathcal{L}^{n+1}$. The construction ensures that σ' is a strategy for the second player in $\Gamma(\mathcal{L})$.

Now suppose that $\langle L_n \rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{L}$ such that L_{n+1} is included in $\sigma'(L_0, \ldots, L_n)$ for every $n \in \mathbb{N}$. Then we have a sequence $\langle K_n \rangle_{n\in\mathbb{N}}$ in $\mathcal{K}$ defined by saying that $(K_0, \ldots, K_n) = \phi_n(L_0, \ldots, L_n)$ for every $n \in \mathbb{N}$. Set $K'_n = \sigma(K_0, \ldots, K_n)$ for each n. Then K'_n belongs to $\mathcal{K}$, is non-negligible and included in $K_n \subseteq f^{-1}[L_n]$, so, by the definition of $\mathcal{K}$, $f[K'_n] \cap F$ is negligible whenever $F \in T$ and $K_n \cap f^{-1}[F]$ is negligible. Examining the construction, we see that for each $n \in \mathbb{N}$ we have $L_{n+1} \subseteq \sigma'(L_0, \ldots, L_n)$, which is a measurable envelope of $f[K'_n]$. But this means that $L_{n+1} \cap f[K'_n]$ is not negligible, so $K'_n \cap f^{-1}[L_{n+1}]$ is not negligible. When we came to choose K_{n+1}, therefore, we had the option of taking it to be a subset of K'_n, so we did, and $K_{n+1} \subseteq \sigma(K_0, \ldots, K_{n+1})$.

Since this is true for every $n \in \mathbb{N}$, $\bigcap_{n\in\mathbb{N}} K_n$ is not empty. But $K_n \subseteq f^{-1}[L_n]$ for every n, so $\bigcap_{n\in\mathbb{N}} L_n \supseteq f[\bigcap_{n\in\mathbb{N}} K_n]$ is non-empty. As $\langle L_n \rangle_{n\in\mathbb{N}}$ is arbitrary, σ' is a winning strategy, and $\mathcal{L}$ is a weakly α-favourable class.

(e) Because ν is semi-finite, $\mathcal{L}$ is coinitial with $T\setminus\mathcal{N}_\nu$, so (by 1C in the other direction) ν is weakly α-favourable.

REMARK. There is a corresponding result for countably compact measures ([17], or [10], 452I). The same is true of perfect measures ([20], or [10], 451E), but not of compact measures (e.g., take (X,μ) to be $[0,1]$ with Lebesgue measure, (Y,ν) to be $[0,1]$ with the countable-cocountable measure, and f the identity map). I do not know whether the image, in this sense, of an α-favourable measure must again be α-favourable.

2C. I turn now to products of weakly α-favourable spaces. I approach these through a more general result on projective limit measures on product spaces.

THEOREM. *Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a non-empty family of weakly α-favourable measure spaces. Let μ be a probability measure on X, inner regular with respect to the σ-algebra $\widehat{\bigotimes}_{i\in I}\Sigma_i$ generated by $\{\pi_i^{-1}[E]:i\in I,\ E\in\Sigma_i\}$, where $\pi_i:X\to X_i$ is the canonical map for each $i\in I$. If every π_i is inverse-measure-preserving, then μ is weakly α-favourable.*

Proof. (a) If any X_i is empty, the result is trivial; let us suppose that every X_i is non-empty. For each $i\in I$ set $\mathcal{C}_i=\{\pi_i^{-1}[E]:E\in\Sigma_i\}$, and write $\mathcal{C}=\bigcup_{i\in I}\mathcal{C}_i$. Let $\mathcal{K}$ be the set of non-negligible subsets of X expressible in the form

$$K=\bigcap_{n\in\mathbb{N}}\bigcup_{r\le k_n}\bigcap_{i\in J_{nr}}C_{nri},$$

where $k_n\in\mathbb{N}$ for $n\in\mathbb{N}$, J_{nr} is a finite subset of I for $n\in\mathbb{N}$, $r\le k_n$, and $C_{nri}\in\mathcal{C}_i$ for every $n\in\mathbb{N}$, $r\le k_n$, $i\in J_{nr}$. For each $K\in\mathcal{K}$, fix such a representation; say

$$K=\bigcap_{n\in\mathbb{N}}\bigcup_{r\le k_n(K)}\bigcap_{i\in J_{nr}(K)}C_{nri}(K).$$

Note that μ is inner regular with respect to $\mathcal{K}$ ([10], 454A).

(b) For each $i\in I$, let $Q_i\subseteq\bigcup_{n\in\mathbb{N}}(\Sigma_i\setminus\mathcal{N}_{\mu_i})^{n+1}$ satisfy the conditions of Lemma 1D. For $n\in\mathbb{N}$, let $S_n\subseteq\mathcal{C}^{n+1}$ be the set of sequences $(C_0,\dots,C_n)$ in $\mathcal{C}$ such that if $i\in I$ and $\{r:r\le n,\ C_r\in\mathcal{C}_i\setminus\{X\}\}$ is a non-empty set enumerated in ascending order as $r_0,\dots,r_s$, then $(E_0,\dots,E_s)\in Q_i$, where $C_{r_j}=\pi_i^{-1}[E_j]$ for $j\le s$. Observe that if $\langle C_n\rangle_{n\in\mathbb{N}}$ is any sequence in $\mathcal{C}$ such that $\langle C_r\rangle_{r\le n}\in S_n$ for every n, then $\bigcap_{n\in\mathbb{N}}C_n\ne\emptyset$, because $\bigcap_{n\in L_i}C_n\ne\emptyset$ for every $i\in I$, where $L_i=\{n:C_n\in\mathcal{C}_i\setminus\{X\}\}$.

Fix on any well-ordering $\preccurlyeq_1$ of $\mathbb{N}^3$ with order type ω, and any total ordering $\preccurlyeq_2$ of I; let $\preccurlyeq$ be the lexicographic ordering of $\mathbb{N}^3\times I$.

(c) We can define $\phi_n:\mathcal{K}^{n+1}\to S_n$, for $n\in\mathbb{N}$, in such a way that

(i) if $\phi_n(K_0,\ldots,K_n) = (C_0,\ldots,C_n)$ then also $\phi_m(K_0,\ldots,K_m) = (C_0,\ldots,C_m)$ for every $m \le n$,

(ii) if $\langle K_n\rangle_{n\in\mathbb{N}}$, $\langle C_n\rangle_{n\in\mathbb{N}}$ are sequences in $\mathcal{K}$, $\mathcal{C}$ respectively such that $\phi_n(K_0,\ldots,K_n) = (C_0,\ldots,C_n)$ and $K_{n+1} \subseteq K_n \cap \bigcap_{r\le n} C_r$ for every n, then $\bigcap_{n\in\mathbb{N}} K_n = \bigcap_{n\in\mathbb{N}} C_n$.

The construction proceeds as follows. For simplicity at the start, just take $\phi_0(K) = X$ for every $K \in \mathcal{K}$. Given that $K_0,\ldots,K_{n+1} \in \mathcal{K}$ and $\phi_n(K_0,\ldots,K_n) = (C_0,\ldots,C_n) \in S_n$, then, if $K_{n+1} \subseteq K_n \cap \bigcap_{r\le n} C_r$, look at the quadruples (l,m,r,i) such that $l \le n$, $m \in \mathbb{N}$, $r \le k_m(K_l)$ and $i \in J_{mr}(K_l)$. Take the $\preccurlyeq$-first quadruple (l,m,r,i), if there is one, such that no C_j, for $j \le n$, is either included in or disjoint from $C_{mri}(K_l)$. Look at those $j \le n$ such that $C_j \in \mathcal{C}_i \setminus \{X\}$. These, if any, are of the form $(\pi_i^{-1}[E_0],\ldots,\pi_i^{-1}[E_s])$ where $(E_0,\ldots,E_s) \in Q_i$. There must therefore be an $E \in \Sigma_i$ such that $(E_0,\ldots,E_s,E) \in Q_i$ and $\pi_i^{-1}[E]$ is either included in $C_{mri}(K_l)$ or disjoint from it and $K_{n+1} \cap \pi_i^{-1}[E]$ is not negligible (since there is a sequence of such E's with conegligible union in E_s, and K_{n+1} is a non-negligible subset of $\pi_i^{-1}[E_s]$). Set $\phi_{n+1}(K_0,\ldots,K_{n+1}) = (C_0,\ldots,C_n,\pi_i^{-1}[E_i]) \in S_{n+1}$.

If $K_{n+1} \not\subseteq K_n \cap \bigcap_{r\le n} C_r$ or there is no appropriate quadruple (l,m,r,i), set $\phi_{n+1}(K_0,\ldots,K_{n+1}) = (C_0,\ldots,C_n,X)$.

The construction ensures (i). To see that (ii) is true, take $\langle K_n\rangle_{n\in\mathbb{N}}$, $\langle C_n\rangle_{n\in\mathbb{N}}$ such that $\phi_n(K_0,\ldots,K_n) = (C_0,\ldots,C_n)$ and $K_{n+1} \subseteq K_n \cap \bigcap_{r\le n} C_r$ for every n. Consider the set L of all quadruples (l,m,r,i) such that l, $m \in \mathbb{N}$, $r \le k_m(K_l)$ and $i \in J_{mr}(K_l)$. Observe that because every $J_{mr}(K_l)$ is finite, these quadruples form a set of order type at most ω for $\preccurlyeq$. Suppose, if possible, that $(l,m,r,i) \in L$ is such that no C_n is either disjoint from or included in $C_{mri}(K_l)$. In this case, every C_n, at least for $n > l$, was chosen to be disjoint from or included in some different $C_{m'r'i'}(K_{l'})$ where $(l',m',r',i') \preccurlyeq (l,m,r,i)$; and there are only finitely many of these.

Thus, for every $(l,m,r,i) \in L$, there is some $n \in \mathbb{N}$ such that C_n is either included in or disjoint from $C_{mri}(K_l)$. Take any $l,m \in \mathbb{N}$; set $H_{lm} = \bigcup_{r\le k_m(K_l)} \bigcap_{i\in J_{mr}(K_l)} C_{mri}(K_l)$. Since $k_m(K_l)$ and every $J_{mr}(K_l)$ are finite, there is some $n \ge l$ such that $\bigcap_{j\le n} C_j$ is either disjoint from H_{lm} or included in H_{lm}; since

$$\emptyset \ne K_{n+1} \subseteq K_l \cap \bigcap_{j\le n} C_j \subseteq H_{lm},$$

we have $\bigcap_{j\le n} C_j \subseteq H_{lm}$. But this means that

$$\bigcap_{j\in\mathbb{N}} C_j \subseteq \bigcap_{l,m\in\mathbb{N}} H_{lm} = \bigcap_{l\in\mathbb{N}} K_l,$$

as claimed.

(d) Now define $\sigma : \bigcup_{n\in\mathbb{N}} \mathcal{K}^n \to \mathcal{K}$ by writing

$$\sigma(K_0,\ldots,K_n) = K_n \cap \bigcap_{r\le n} C_r$$

whenever $\phi_n(K_0,\ldots,K_n) = (C_0,\ldots,C_n)$ and the intersection is not negligible, K_n otherwise; this is a winning strategy for the second player in $\Gamma(\mathcal{K})$. Since μ is inner regular with respect to $\mathcal{K}$, it is weakly α-favourable.

2D. Corollary. *If $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ is any family of weakly α-favourable probability spaces, the product measure on $\prod_{i\in I} X_i$ is also weakly α-favourable.*

Remarks. Once again, this corresponds to known results about the other classes. The product of compact probability spaces is compact ([10], 342G), the product of countably compact probability spaces is countably compact ([14]), and the product of perfect probability spaces is perfect ([20]); all these may be found in [10], 451J, and the extensions corresponding to Theorem 2C are also true ([10], 454A). If we use the "c.l.d. product measure" of [10], §251, we find that the product of finitely many compact, or countably compact, or perfect measure spaces of any magnitude is again compact or countably compact or perfect ([10], 451I). However, I do not know whether the product of two α-favourable probability spaces is always α-favourable. I can give the following partial result.

2E. Proposition. *If (X,Σ,μ) is an α-favourable measure space and (Y,T,ν) is a countably compact measure space, then the c.l.d. product measure on $X\times Y$ is α-favourable.*

Proof. Write λ for the c.l.d. product measure on $X\times Y$, and Λ for its domain. (For a full description of the "c.l.d. product measure", see [10], §251. The only special feature we need here is the fact that

if $\lambda W > 0$ then there are $E\in\Sigma$, $F\in T$, both of finite measure, such that $\lambda(W\cap(E\times F)) > 0$,

as in [10], 251Id. Of course, this is only relevant when one of μ, ν is not σ-finite.)

Let $\tau : \Sigma\setminus\mathcal{N}_\mu \to \Sigma\setminus\mathcal{N}_\mu$ be a winning tactic for the second player in $\Gamma(\Sigma\setminus\mathcal{N}_\mu)$. Let $\mathcal{K}\subseteq T$ be a countably compact class of sets such that ν is inner regular with respect to $\mathcal{K}$; by 1Bb(ii), we may suppose that $\mathcal{K}$ is closed under finite unions and countable intersections. Now let $\mathcal{W}$ be the family of sets $W\in\Lambda$ such that $\lambda W>0$, $\pi_1[W] = \{x : (x,y)\in W\}\in\Sigma$, and all the vertical sections of W belong to $\mathcal{K}$. Then $\mathcal{W}$ is coinitial with $\Lambda\setminus\mathcal{N}_\lambda$. To see this, take $W\in\Lambda$ such that $\lambda W>0$, and let $E\in\Sigma$, $F\in T$ be such that $\mu E<\infty$, $\nu F<\infty$ and $\lambda(W\cap(E\times F))>0$. Let $F'\in\mathcal{K}$ be such that $F'\subseteq F$ and $\lambda(W\cap(E\times F'))>0$. Let $\langle E_n\rangle_{n\in\mathbb{N}}$, $\langle F_n\rangle_{n\in\mathbb{N}}$

be sequences in Σ, T respectively such that $\sum_{n=0}^{\infty}\mu E_n\cdot\nu F_n<\mu E\cdot\nu F'$ and $\bigcup_{n\in\mathbb{N}}E_n\times F_n\supseteq(E\times F')\setminus W$. Choose $F'_n\in\mathcal{K}$, for $n\in\mathbb{N}$, such that $F'_n\subseteq F'\setminus F_n$ for each n and

$$\mu E\cdot\nu F'>\sum_{n=0}^{\infty}(\mu E_n\cdot\nu F_n+\mu(E_n\cap E)\cdot\nu(F'\setminus(F_n\cup F'_n)));$$

then $W'=\bigcap_{n\in\mathbb{N}}((E\setminus E_n)\times F')\cup(E_n\times F'_n)$ is a non-negligible subset of W and every vertical section of W' is in $\mathcal{K}$. By Fubini's theorem ([10], 252F), there is a non-negligible measurable set $E'\subseteq\{x:\nu W[\{x\}]>0\}$; setting $W''=W'\cap(E'\times Y)$, we have a member of $\mathcal{W}$ included in W, as required.

Now define $\tau':\mathcal{W}\to\mathcal{W}$ by setting $\tau'(W)=W\cap(\tau(\pi_1[W])\times Y)$ for every $W\in\mathcal{W}$. This is a tactic for the second player in the game $\Gamma(\mathcal{W})$. If $\langle W_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{W}$ such that $W_{n+1}\subseteq\tau'(W_n)$ for every n, then $\pi_1[W_{n+1}]\subseteq\tau(\pi_1[W_n])$ for every n, so there is an $x\in\bigcap_{n\in\mathbb{N}}\pi_1[W_{n+1}]$. For each $n\in\mathbb{N}$, the vertical section $W_n[\{x\}]$ is a non-empty member of $\mathcal{K}$, so $\bigcap_{n\in\mathbb{N}}W_n[\{x\}]$ is non-empty and $\bigcap_{n\in\mathbb{N}}W_n$ is non-empty. As $\langle W_n\rangle_{n\in\mathbb{N}}$ is arbitrary, τ' is a winning strategy and $\mathcal{W}$ is an α-favourable class. As $\mathcal{W}$ is coinitial with $\Lambda\setminus\mathcal{N}_\lambda$, λ is α-favourable, as claimed.

3. When weakly α-favourable spaces must have stronger properties. I have already noted that a semi-finite countably separated measure space is compact iff it is perfect. In this section I give further results showing that for measure spaces which are "simple" in one sense or another some of the classes above coincide.

3A. It will be convenient to be able to call on the following straightforward idea.

LEMMA. *Suppose that (X,Σ,μ) is a totally finite measure space, and that $Q\subseteq\bigcup_{n\ge1}(\Sigma\setminus\mathcal{N}_\mu)^n$ satisfies the conditions* (i)–(iii) *of Lemma* 1D. *Let $\langle V_i\rangle_{i\in A}$ be any family in Σ. Set $S=\bigcup_{n\ge1}A^n$. Then we can find a family $\langle Q_s\rangle_{s\in S}$ of subsets of Q such that*

(α) *whenever $n\ge1$ and $s\in A^n$ then Q_s is a countable subset of $(\Sigma\setminus\mathcal{N}_\mu)^n$*;

(β) *if s, t belong to S and s extends t, then every member of Q_s extends a member of Q_t*;

(γ) *if we set $\mathcal{H}_s=\{K_n:(K_0,\dots,K_n)\in Q_s\}$ for $s\in A^{n+1}$, then $\mathcal{H}_s$ is a disjoint family with conegligible union*;

(δ) *if $s\in A^{n+1}$ and $K\in\mathcal{H}_s$, then K is either included in $V_{s(n)}$ or disjoint from it.*

Proof. Choose Q_s for short sequences s first, as follows. Let us say, conventionally, that if $s\in A^0$ is the empty sequence then Q_s consists of the empty sequence alone. Given $q \in Q_s$, where $s \in A^n$, and $i \in A$, set

$$\mathcal{E}_q = \{K : q^\frown K \in Q,\ K \subseteq V_i \text{ or } K \cap V_i = \emptyset\},$$

and let $\mathcal{E}'_q \subseteq \mathcal{E}_q$ be a maximal disjoint set; define $Q_{s^\frown i} = \{q^\frown K : q \in Q_s, K \in \mathcal{E}'_q\}$, and continue. An easy induction on n shows that (γ) is true for every s, and the other requirements are obviously satisfied.

3B. The first main result of this section extends a theorem of [5], where it is shown that a monocompact space in which the algebra of measurable sets is ω_1-generated is countably compact. In order to express the new result in its full strength we need a couple of definitions.

Definitions. (a) Let P be any partially ordered set. Then the *additivity* of P is

$$\operatorname{add} P = \min\{\#(Q) : Q \subseteq P \text{ has no upper bound in } P\},$$

writing $\min \emptyset = \infty$, and the *cofinality* and *coinitiality* of P are

$$\operatorname{cf}(P) = \min\{\#(Q) : Q \subseteq P \text{ is cofinal with } P\},$$
$$\operatorname{ci}(P) = \min\{\#(Q) : Q \subseteq P \text{ is coinitial with } P\}.$$

(b) If $\mathfrak{A}$ is a Boolean algebra, write $\operatorname{wdistr}(\mathfrak{A})$ for the least cardinal κ such that $\mathfrak{A}$ is not weakly (κ, ∞)-distributive ([13], §14.6), that is, the least cardinal of any set $\mathcal{E} \subseteq \mathcal{P}\mathfrak{A}$ such that

(i) for some $a_0 \in \mathfrak{A} \setminus \{0\}$, $E \subseteq \mathfrak{A}$ and $\sup E = a_0$ for every $E \in \mathcal{E}$,

(ii) if $a \in \mathfrak{A} \setminus \{0\}$ and $a \subseteq a_0$ then there is an $E \in \mathcal{E}$ such that $a \not\subseteq \sup E'$ for any finite $E' \subseteq E$.

(If there is no such family $\mathcal{E}$, that is, if $\mathfrak{A}$ is purely atomic, say that $\operatorname{wdistr}(\mathfrak{A}) = \infty$.)

(c) For a general discussion of these cardinals, see [8], §6. We need to know that if (X, Σ, μ) is a probability space with measure algebra $\mathfrak{A}$, then both $\operatorname{add}(\mathcal{N}_\mu)$ and $\operatorname{wdistr}(\mathfrak{A})$ have uncountable cofinality. In fact, $\operatorname{add} P$ is either 2 or ∞ or a regular infinite cardinal for any partially ordered set P, and $\operatorname{add}(\mathcal{N}_\mu)$ is certainly uncountable, so has uncountable cofinality. As for $\operatorname{wdistr}(\mathfrak{A})$, this is either ∞ (if μ is purely atomic), or $\operatorname{add}(\mathcal{N}_L)$, where $\mathcal{N}_L$ is the ideal of Lebesgue negligible sets (if μ has countable Maharam type and is not purely atomic), or ω_1 (if μ has uncountable Maharam type); see [9], 6.3d and 6.13a. So this too always has uncountable cofinality.

3C. We need a little calculation with these cardinals.

Lemma. *Let* (X, Σ, μ) *be a measure space, with* $\mu X > 0$, *and suppose that* $\mathbb{E}$ *is a family of countable subsets of* Σ *such that* $\bigcup \mathcal{E}$ *is conegligible for every* $\mathcal{E} \in \mathbb{E}$ *and* $\#(\mathbb{E}) < \min(\operatorname{add}(\mathcal{N}_\mu), \operatorname{wdistr}(\mathfrak{A}))$, *where* $\mathfrak{A} = \Sigma / \Sigma \cap \mathcal{N}_\mu$ *is the measure algebra of* (X, Σ, μ). *Then* μ *is inner regular with respect to* *the family $\mathcal{K}$ of sets $K \in \Sigma$ such that K is covered by a finite subfamily of $\mathcal{E}$ for every $\mathcal{E} \in \mathbb{E}$.*

Proof. Since $\mathcal{K}$ is closed under finite unions, it is enough to show that $\mathcal{K} \setminus \mathcal{N}_\mu$ is coinitial with $\Sigma \setminus \mathcal{N}_\mu$ (see [10], 412Aa). So take any $F \in \Sigma \setminus \mathcal{N}_\mu$. Consider $a = F^\bullet$ in $\mathfrak{A}$, and for $\mathcal{E} \subseteq \Sigma$ consider

$$A_{\mathcal{E}} = \{(E \cap F)^\bullet : E \in \mathcal{E}\}.$$

If $\mathcal{E} \in \mathbb{E}$, then, because $\mathcal{E}$ is countable and the map $E \mapsto E^\bullet : \Sigma \to \mathbb{R}$ is a sequentially order-continuous Boolean homomorphism,

$$\sup A_{\mathcal{E}} = \Big(F \cap \bigcup \mathcal{E}\Big)^\bullet,$$

which is equal to $F^\bullet = a$ because $\bigcup \mathcal{E}$ is conegligible. Because $\#(\mathbb{E}) < \mathrm{wdistr}(\mathfrak{A})$, there is a non-zero $c \subseteq a$ such that, for every $\mathcal{E} \in \mathbb{E}$, there is a finite set $\mathcal{E}' \subseteq \mathcal{E}$ such that $c \subseteq \sup A_{\mathcal{E}'}$. Let $K \in \Sigma$ be such that $K^\bullet = c$. Then $K \setminus \bigcup \mathcal{E}'$ is negligible for every $\mathcal{E} \in \mathbb{E}$; and also, of course, $K \setminus F$ is negligible. Because $\#(\mathbb{E}) < \mathrm{add}(\mathcal{N}_\mu)$, there is an $H \in \Sigma \cap \mathcal{N}_\mu$ including $(K \setminus F) \cup \bigcup_{\mathcal{E} \in \mathbb{E}} (K \setminus \bigcup \mathcal{E}')$. Setting $K' = K \setminus H$ we have a non-negligible member of $\mathcal{K}$ included in F.

3D. Theorem. *Let (X, Σ, μ) be a weakly α-favourable probability space, and suppose that there is a set $\mathcal{V} \subseteq \Sigma$ such that $\#(\mathcal{V}) \le \min(\mathrm{add}(\mathcal{N}_\mu), \mathrm{wdistr}(\mathfrak{A}))$, where $\mathfrak{A}$ is the measure algebra of μ, and μ is inner regular with respect to the Σ-algebra T generated by $\mathcal{V}$. Then (X, Σ, μ) is countably compact.*

Proof. (a) Set $\kappa = \min(\mathrm{add}(\mathcal{N}_\mu), \mathrm{wdistr}(\mathfrak{A}))$, and let $\langle V_\xi \rangle_{\xi<\kappa}$ run over $\mathcal{V} \cup \{X\}$. For each $\xi < \kappa$ let T_ξ be the σ-algebra generated by $\{V_\eta : \eta < \xi\}$. Because $\mathrm{cf}(\kappa) > \omega$, $T = \bigcup_{\xi<\kappa} T_\xi$. Let $Q \subseteq \bigcup_{n\ge1} (\Sigma \setminus \mathcal{N}_\mu)^n$ be a family witnessing that $\Sigma \setminus \mathcal{N}_\mu$ is a weakly α-favourable class, as in 1D; and choose $Q_s \subseteq Q$, $\mathcal{H}_s \subseteq \Sigma \setminus \mathcal{N}_\mu$, for $s \in \bigcup_{n\ge1} \kappa^n$, by the construction of Lemma 3A.

(b) Let $\mathcal{W}$ be the subalgebra of T generated by $\{V_\xi : \xi < \kappa\}$, and $\mathcal{W}_\delta$ the family of sets expressible as intersections of sequences in $\mathcal{W}$. Then μ is inner regular with respect to $\mathcal{W}_\delta$. To see this, observe that because $\mathcal{W}$ is closed under finite unions, $\mathcal{W}_\delta$ is closed under finite unions and countable intersections, while $\mathcal{W}$ is closed under complementation. By [10], 412C, or otherwise, $\mu \restriction T$ is inner regular with respect to $\mathcal{W}_\delta$, so μ also is, by [10], 412Ab.

For each $\xi < \kappa$, write $\mathcal{W}^{(\xi)}$ for the algebra of subsets of X generated by $\{V_\eta : \eta < \xi\}$, and $\mathcal{W}_\delta^{(\xi)}$ for the set of intersections of sequences in $\mathcal{W}^{(\xi)}$; then $\mathcal{W}_\delta = \bigcup_{\xi<\kappa} \mathcal{W}_\delta^{(\xi)}$, again because κ has uncountable cofinality.

(c) For $\xi < \kappa$ let $\mathcal{K}_\xi$ be the family of those sets $K \in \mathcal{W}_\delta^{(\xi)}$ such that whenever $s \in \bigcup_{m\ge1} \xi^m$ there is a finite subset of $\mathcal{H}_s$ covering K. Set

$\mathcal{K} = \bigcup_{\xi<\kappa} \mathcal{K}_\xi$. Then μ is inner regular with respect to $\mathcal{K}$. To see this, suppose that $E \in \Sigma$ and that $0 \le \gamma < \mu E$. Choose sequences $\langle K_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{W}_\delta$ and $\langle \xi_n\rangle_{n\in\mathbb{N}}$ in κ inductively, as follows. Start with any $K_0 \in \mathcal{W}_\delta$ such that $K_0 \subseteq E$ and $\mu K_0 > \gamma$. Given that $K_n \in \mathcal{W}_\delta$ and that $\mu K_n > \gamma$, take $\xi_n < \kappa$ such that $K_n \in \mathcal{W}_\delta^{(\xi_n)}$ and $\xi_n \ge \xi_i$ for $i < n$. By Lemma 3C, there is a $K_{n+1} \in \Sigma$ such that $K_{n+1} \subseteq K_n$, $\mu K_{n+1} > \gamma$ and for every $s \in \bigcup_{m\ge1} \xi_n^m$ there is a finite subset of $\mathcal{H}_s$ covering K_{n+1}; shrinking K_{n+1} a trifle if need be, we may arrange that $K_{n+1} \in \mathcal{W}_\delta$. Continue.

At the end of the induction, set $K = \bigcap_{n\in\mathbb{N}} K_n$, $\xi = \sup_{n\in\mathbb{N}} \xi_n$. Then $K \in \mathcal{W}_\delta \cap T_\xi$, $K \subseteq E$ and $\mu K \ge \gamma$. If $s \in \bigcup_{m\ge1} \xi^m$, there is some $n \in \mathbb{N}$ such that $s \in \bigcup_{m\ge1} \xi_n^m$, because $\langle \xi_n\rangle_{n\in\mathbb{N}}$ is non-decreasing; and now $K \subseteq K_{n+1}$ is covered by a finite subset of $\mathcal{H}_s$. Thus $K \in \mathcal{K}_\xi$ and $K \in \mathcal{K}$. As E and γ are arbitrary, μ is inner regular with respect to $\mathcal{K}$.

(d) $\mathcal{K}$ is a countably compact class. For suppose that $\langle K_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{K}$ with the finite intersection property. Let $\mathcal{F}$ be an ultrafilter on X containing every K_n. For each n take $\xi_n < \kappa$ such that $K_n \in \mathcal{K}_{\xi_n}$, and a countable subset J_n of ξ_n such that K_n is expressible as $\bigcap_{i\in\mathbb{N}} W_{ni}$, where each W_{ni} belongs to the algebra of subsets of X generated by $\{V_\eta : \eta \in J_n\}$.

Let $\langle s(i)\rangle_{i\in\mathbb{N}}$ be a sequence running over $\bigcup_{n\in\mathbb{N}} J_n$. (I pass over the trivial case in which every J_n is empty, since in this case every K_n must be X.) For every $n \in \mathbb{N}$, $s_n = \langle s(i)\rangle_{i\le n}$ belongs to ξ_r^{n+1} for some r, so that K_r is covered by a finite subfamily of $\mathcal{H}_{s_n}$, and there is a $K'_n \in \mathcal{H}_{s_n}$ belonging to $\mathcal{F}$. Now $(K'_0, \ldots, K'_n) \in Q$ for every n. For we know that $K'_n \in \mathcal{H}_{s_n}$, so there is a $q \in Q_{s_{n-1}}$ such that $q^\frown K'_n \in Q_{s_n}$. (If $n = 0$ take q to be the empty sequence.) But this means that, for every $i < n$, $q(i) \supseteq K'_n$, so that $q(i) \in \mathcal{F}$, while $q(i) \in \mathcal{H}_{s_i}$; and since $\mathcal{H}_{s_i}$ is disjoint, we must have $q(i) = K'_i$ for every $i < n$, and $(K'_0, \ldots, K'_n) = q^\frown K'_n \in Q$.

Accordingly there is an $x \in \bigcap_{n\in\mathbb{N}} K'_n$. Now suppose, if possible, that $x \notin K_r$ for some $r \in \mathbb{N}$. Then there is an $i \in \mathbb{N}$ such that $x \notin W_{ri}$. There are therefore finite sets I, I' of J_r such that $x \in W$ and $W \cap W_{ri} = \emptyset$, where $W = \bigcap_{\eta\in I} V_\eta \setminus \bigcup_{\eta\in I'} V_\eta$.

If $\eta \in I$, there is some $n \in \mathbb{N}$ such that $s(n) = \eta$; by the choice of Q_{s_n}, K'_n is either included in V_η or disjoint from it. Since $x \in V_\eta$, we have $K'_n \subseteq V_\eta$, and $V_\eta \in \mathcal{F}$. Similarly, if $\zeta \in I'$, there is an m such that $s(m) = \zeta$, and in this case we must have $K'_m \subseteq X \setminus V_\zeta$ and $X \setminus V_\zeta \in \mathcal{F}$. So we see that $W \in \mathcal{F}$; but we started by taking $\mathcal{F}$ to contain K_r, so it also contains W_{ri}, which is disjoint from W. This contradiction shows that $x \in \bigcap_{r\in\mathbb{N}} K_r$. As $\langle K_r\rangle_{r\in\mathbb{N}}$ is arbitrary, $\mathcal{K}$ is a countably compact class. Putting this together with (c), we see that μ is countably compact, as claimed.

3E. Remarks. We can identify the following cases in which Theorem 3D can be applied.

(a) Since $\min(\mathrm{add}(\mathcal{N}_\mu), \mathrm{wdistr}(\mathfrak{A}))$ is always at least ω_1, we see that if (X, Σ, μ) is any weakly α-favourable probability space in which Σ is generated by ω_1 sets, then it is countably compact. This extends Corollary 1.3 of [5], and complements the result from [18] that if (X, Σ, μ) is perfect and Σ is countably generated, then μ is compact.

(b) Searching for adequate generating sets to use in Proposition 3D, we have the following simple fact. If $\mathcal{V}$ is any coinitial subset of $\Sigma\setminus\mathcal{N}_\mu$, then μ is inner regular with respect to the algebra generated by $\mathcal{V}$; so if $\mathrm{ci}(\Sigma\setminus\mathcal{N}_\mu) \leq \min(\mathrm{add}(\mathcal{N}_\mu), \mathrm{wdistr}(\mathfrak{A}))$, the result will be applicable. Now suppose that Martin's Axiom is true and that μ is a quasi-Radon probability measure of countable Maharam type, with measure algebra $\mathfrak{A}$. In this case, $\mathrm{add}(\mathcal{N}_\mu) \geq \mathfrak{c}$ ([7], 32F–G). In particular, $\mathrm{add}(\mathcal{N}_L) = \mathfrak{c}$, so $\mathrm{wdistr}(\mathfrak{A}) \geq \mathfrak{c}$. On the other hand, the family $\mathcal{K}$ of non-empty self-supporting closed subsets of X is coinitial with $\Sigma\setminus\mathcal{N}_\mu$, and the map $K \mapsto K^\bullet : \mathcal{K} \to \mathfrak{A}$ is injective; so $\mathrm{ci}(\Sigma\setminus\mathcal{N}_\mu) \leq \#(\mathfrak{A}) \leq \mathfrak{c}$ (because $\mathfrak{A}$ is a separable metrizable space). Thus we have

[MA] if (X, Σ, μ) is a weakly α-favourable quasi-Radon probability space with countable Maharam type, then it is countably compact.

Subject to the continuum hypothesis we can lift this one step, since a probability algebra of Maharam type $\mathfrak{c}$ also has cardinal $\mathfrak{c}$; so we get

[CH] if (X, Σ, μ) is a weakly α-favourable quasi-Radon probability space with Maharam type at most ω_1, then it is countably compact.

3F. For the next proposition, and also for the examples in §4, we shall need some further well known facts. Recall that a topological measure is "completion regular" if it is inner regular with respect to the zero sets, that is, sets of the form $f^{-1}[\{0\}]$ for some continuous real-valued function ([10], 411I); and that a Radon measure ν on $\{0,1\}^A$ is completion regular iff the self-supporting closed sets are all determined by coordinates in countable sets ([2], Lemma 2). The usual measure on $\{0,1\}^A$ ([10], 254J) is completion regular, by a theorem of Kakutani ([10], 415E and 417E).

Suppose that ν is a completion regular Radon measure on $\{0,1\}^A$, that X is any subset of $\{0,1\}^A$, and that μ is the subspace measure on X. (For the general theory of subspace measures, see [10], §214.) Let W be a measurable envelope of X. Write $\mathcal{K}$ for the family of non-empty compact self-supporting sets included in W. Then every member of $\mathcal{K}$ is determined by a countable set of coordinates, and μ is inner regular with respect to $\mathcal{K}_X = \{K\cap X : K \in \mathcal{K}\}$ (cf. [10], 412O). If $K, K' \in \mathcal{K}$ and $K\cap X \subseteq K'\cap X$, then $\nu(K\cap K') = \nu K$. Since K is self-supporting, it must be included in K'. Thus $K \mapsto K\cap X : \mathcal{K} \to \mathcal{K}_X$ is an order-preserving bijection.

3G. Theorem. *Suppose A is a set and that ν is a completion regular Radon probability measure on $\{0,1\}^A$. Let $X \subseteq \{0,1\}^A$ be a set such* *that the induced subspace measure μ on X is weakly α-favourable. Then μ is α-favourable. If ν is the usual measure of $\{0,1\}^A$, then μ is regularly monocompact.*

Proof. (a) Let W be a measurable envelope of X. Write $\mathcal{K}$ for the family of non-empty self-supporting compact subsets K of W, and $\mathcal{K}_X = \{K \cap X : K \in \mathcal{K}\}$, as in 3F, so that the subspace measure ν_W on W is inner regular with respect to $\mathcal{K}$, and μ is inner regular with respect to $\mathcal{K}_X$. By Lemma 1C, $\mathcal{K}_X$ is a weakly α-favourable class. Let $Q_0 \subseteq \bigcup_{n\geq 1} \mathcal{K}_X^n$ be a family satisfying the conditions of Lemma 1D. Set

$$Q = \{(K_0,\ldots,K_n) : n \in \mathbb{N},\ K_r \in \mathcal{K} \text{ for } r \leq n,\ (K_0 \cap X,\ldots,K_n \cap X) \in Q_0\}.$$

Then $Q \subseteq \bigcup_{n\geq 1} \mathcal{K}^n$ and

(i) $K_0 \supseteq \ldots \supseteq K_n$ for every $(K_0,\ldots,K_n) \in Q$;

(ii) for every $K \in \mathcal{K}$ there is a $K' \subseteq K$ such that the one-term sequence (K') belongs to Q;

(iii) whenever $(K_0,\ldots,K_n) \in Q$ and $K \in \mathcal{K}$ and $K \subseteq K_n$, then there is a $K' \subseteq K$ such that $(K_0,\ldots,K_n,K') \in Q$;

(iv) whenever $\langle K_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{K}$ such that $(K_0,\ldots,K_n) \in Q$ for every n, then $X \cap \bigcap_{n\in\mathbb{N}} K_n \neq \emptyset$.

For $i \in A$, set $V_i = \{x : x \in W,\ x(i) = 0\}$. Set $S = \bigcup_{n\geq 1} A^n$, and choose $Q_s \subseteq Q$, $\mathcal{H}_s \subseteq \mathcal{K}$ from the family $\langle V_i\rangle_{i\in A}$ as in 3A above.

(b) For closed sets $F \subseteq \{0,1\}^A$, let A_F be the smallest set such that F is determined by coordinates in A_F (see Lemma 1E). Now define $\tau : \mathcal{K} \to \mathcal{K}$ in such a way that

whenever $K \in \mathcal{K}$, then $\tau(K) \in \mathcal{K}$, $\tau(K) \subseteq K$, and $\tau(K) \subseteq \bigcup \mathcal{H}_s$ for every $s \in \bigcup_{n\in\mathbb{N}} A_K^{n+1}$;

this is possible because there are only countably many sequences s to deal with, and $\bigcup \mathcal{H}_s$ is always conegligible in W.

(c) If $\langle K_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{K}$ such that $K_{n+1} \subseteq \tau(K_n)$ for every $n \in \mathbb{N}$, then $X \cap \bigcap_{n\in\mathbb{N}} K_n$ is non-empty. For set $K = \bigcap_{n\in\mathbb{N}} K_n$. Since every K_n is compact and $K_{n+1} \subseteq K_n$ for every n, K cannot be empty. If $K = \{0,1\}^A$, the result is trivial. Otherwise, take any $z \in K$, and let $\langle i_n\rangle_{n\in\mathbb{N}}$ be a sequence running over A_K. (A_K must be countable because it is included in $\bigcup_{n\in\mathbb{N}} A_{K_n}$, or otherwise, and it is not empty because we are supposing that $K \neq \{0,1\}^A$.) If $n \in \mathbb{N}$, then for any m large enough we have $i_r \in A_{K_m}$ for every $r \leq n$, by Lemma 1Eb above; in this case, $K \subseteq \tau(K_m)$ is covered by $\mathcal{H}_{s_n}$, where $s_n = (i_0,\ldots,i_n)$, so that $z \in E_n$ for some $E_n \in \mathcal{H}_{s_n}$.

At this point, note that, because $\mathcal{H}_{s_n}$ is disjoint, $E_{n+1} \in \mathcal{H}_{s_{n+1}}$ must always be included in E_n, and $(E_0, \ldots, E_n)$ must belong to Q_{s_n} for each n. We also know, by the construction of Q_{s_n}, that E_n is either included in H_{i_n} or disjoint from it.

Since $(E_0, \ldots, E_n) \in Q$ for every n, there is a point $x \in X \cap \bigcap_{n\in\mathbb{N}} E_n$. Now $x(i_n) = z(i_n)$ for every n, so $x\restriction A_K = z\restriction A_K$, and $x \in K$. So $X \cap K \neq \emptyset$, as claimed.

(d) If we now set $\tau'(K\cap X) = \tau(K)\cap X$ for every $K \in \mathcal{K}$, $\tau' : \mathcal{K}_X \to \mathcal{K}_X$ witnesses that $\mathcal{K}_X$ is an α-favourable class, so that μ is α-favourable.

(e) Now suppose that ν is the usual measure on $\{0,1\}^A$. Write $\mathcal{L}$ for the set of those $K \in \mathcal{K}$ such that $K \subseteq \bigcup \mathcal{H}_s$ for every $s \in \bigcup_{n\in\mathbb{N}} A_K^{n+1}$. The point is that μ is inner regular with respect to $\mathcal{L}_X = \{K \cap X : K \in \mathcal{L}\}$. To see this, argue as follows. Given $E \in \operatorname{dom}\mu$ and $0 \le \gamma < \mu E$, express E as $F \cap X$ where $F \subseteq W$ is measured by ν, and choose inductively a sequence $\langle L_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{K}$ such that

$$L_0 \subseteq F, \quad \nu L_0 > \gamma,$$

and, for each $n \in \mathbb{N}$, given that $\nu L_n > \gamma$,

$$L_{n+1} \subseteq L_n,\ L_{n+1} \subseteq \bigcup \mathcal{H}_s \text{ for every } s \in \textstyle\bigcup_{n\in\mathbb{N}} A_{L_n}^{n+1} \text{ and } \nu L_{n+1} > \gamma.$$

At the end of the induction, set $L = \bigcap_{n\in\mathbb{N}} L_n$; then $A_L^n \subseteq \bigcup_{m\in\mathbb{N}} A_{L_m}^n$ for every $n \in \mathbb{N}$, by Lemma 1Eb, so $L \subseteq \bigcup \mathcal{H}_s$ for every $s \in \bigcup_{n\in\mathbb{N}} A_L^{n+1}$. If we now take $K \in \mathcal{K}$ such that $K \subseteq L$ and $\mu(L \setminus K) = 0$, then $A_K \subseteq A_L$, by Lemma 1Ec, so $K \in \mathcal{L}$ and $K \cap X \in \mathcal{L}_X$; and of course $K \cap X \subseteq E$ and $\mu(K \cap X) \ge \gamma$. As E and γ are arbitrary, μ is inner regular with respect to $\mathcal{L}_X$.

(f) The definition of τ in (b) above allows us, if we choose, to set $\tau(K) = K$ for every $K \in \mathcal{L}$. Then $\tau' : \mathcal{K}_X \to \mathcal{K}_X$ will still be a winning tactic, as before. But $\tau'(E) = E$ for every $E \in \mathcal{L}_X$. So if $\langle E_n\rangle_{n\in\mathbb{N}}$ is any non-decreasing sequence in $\mathcal{L}_X$, we shall have $E_{n+1} \subseteq \tau'(E_n)$ for every n, and $\bigcap_{n\in\mathbb{N}} E_n \neq \emptyset$. This shows that $\mathcal{L}_X$ is a monocompact class, so that μ is regularly monocompact.

4. Examples. It is easy to find a countably compact measure which is not compact (for instance, the countable-cocountable measure on any uncountable set). Separating the other classes here seems to be harder. [16] provides an example of a perfect probability space which is not countably compact; since it is a subspace of $\{0,1\}^{\omega_1}$ with its usual measure, 3Ea tells us that it is also not weakly α-favourable. The principal example here (4B) is a regularly monocompact space which is not countably compact, answering a question suggested by [24] (p. 380). I show how Musiał's example can, subject to the continuum hypothesis, be adapted to have countable

Maharam type (4C), and conclude by describing an interesting example of a weakly α-favourable class which is not α-favourable (4E).

4A. *A combinatorial result.* The example relies on an important construction due to S. Todorčević ([23]). Since it is not stated in exactly the required form in that paper, and since it is of great independent interest, I give the proof.

LEMMA. *Let κ be the successor of a regular infinite cardinal. Then there is a function $\phi : [\kappa]^2 \to [\kappa]^{<\omega}$ such that whenever $D \in [\kappa]^\kappa$ then $\bigcup_{I\in[D]^2} \phi(I) = \kappa$.*

Proof. (a) Let λ be the cardinal predecessor of κ. Write $S = \{\delta : \delta < \kappa, \operatorname{cf}(\delta) = \lambda\}$, so that S is a stationary subset of κ ([12], p. 58). By Solovay's theorem ([12], Theorem 85) there is a partition $\langle S_\xi \rangle_{\xi<\kappa}$ into stationary sets. For each $\alpha < \kappa$, let C_α be a cofinal subset of α of order type $\operatorname{cf}(\alpha)$. (If $\alpha = \beta + 1$ is a successor ordinal, $C_\alpha = \{\beta\}$.) Now define $J_{\alpha\beta} \in [\kappa]^{<\omega}$ inductively, for $\alpha \le \beta$, as follows. $J_{\alpha\alpha} = \{\alpha\}$ for every $\alpha < \kappa$. If $J_{\alpha\gamma}$ has been defined for $\alpha \le \gamma < \beta$, where $\alpha < \beta < \kappa$, set $\gamma = \min(C_\beta \setminus \alpha)$, so that $\alpha \le \gamma < \beta$, and set $J_{\alpha\beta} = J_{\alpha\gamma} \cup \{\beta\}$.

(b) If $\delta \in S$ and $\delta \le \beta < \kappa$, then there is an $\alpha_0 < \delta$ such that $\delta \in J_{\alpha\beta}$ whenever $\alpha_0 \le \alpha < \delta$. We can prove this by induction on β. If $\beta = \delta$ we can take $\alpha_0 = 0$, since $\{\alpha, \beta\} \subseteq J_{\alpha\beta}$ for all $\alpha \le \beta$. For the inductive step to $\beta > \delta$, set $\gamma = \min(C_\beta \setminus \delta)$, so that $\delta \le \gamma < \beta$. We have

$$\operatorname{otp}(C_\beta \cap \delta) = \operatorname{otp}(C_\beta \cap \gamma) < \operatorname{otp}(C_\beta) = \operatorname{cf}(\beta) \le \lambda = \operatorname{cf}(\delta),$$

so $\alpha_1 = \sup C_\beta \cap \delta$ is less than δ. If $\alpha_1 < \alpha < \delta$, then $\min(C_\beta \setminus \alpha) = \gamma$, so that $J_{\alpha\beta} \supseteq J_{\alpha\gamma}$. Now the inductive hypothesis tells us that there is an α_0 such that $\alpha_1 < \alpha_0 < \delta$ and $\delta \in J_{\alpha\gamma}$ whenever $\alpha_0 \le \alpha < \delta$, so the induction proceeds.

(c) Set $\phi(\{\alpha, \beta\}) = \{\xi : \xi < \kappa, S_\xi \cap J_{\alpha\beta} \ne \emptyset\}$ whenever $\alpha < \beta < \kappa$. This is always finite because the S_ξ are disjoint. Now suppose that $D \in [\kappa]^\kappa$ and that $\xi < \kappa$. Then

$$D' = \{\delta : \delta < \kappa, \delta = \sup(D \cap \delta)\}$$

is a closed unbounded set in κ, so meets the stationary set S_ξ; take any $\delta \in S_\xi \cap D'$. Because $\#(D) = \kappa$, there is a $\beta \in D \setminus \delta$. By (b) there is an $\alpha_0 < \delta$ such that $\delta \in J_{\alpha\beta}$ whenever $\alpha_0 \le \alpha < \delta$. Now $\delta \in D'$, so there is an $\alpha \in D \cap \delta \setminus \alpha_0$, and in this case $\delta \in J_{\alpha\beta}$ so $\xi \in \phi(\{\alpha, \beta\})$.

As D and ξ are arbitrary, ϕ has the required property.

4B. EXAMPLE. *Let κ be a cardinal such that*

(α) *κ is the successor of a regular infinite cardinal*,
(β) $\kappa^\omega = \kappa$,

(γ) $\kappa \geq \omega_2$.

Give $\{0,1\}^\kappa$ *its usual measure* ν. *Then there is a subset* X *of* $\{0,1\}^\kappa$ *such that the subspace measure on* X *is regularly monocompact but not countably compact.*

REMARK. The smallest cardinal for which (α)–(γ) can be proved without special axioms is $\mathfrak{c}^{++}$. I ought to remark that for $\kappa = \mathfrak{c}^{++}$ a very much stronger result than 4A can be proved ([21]).

Proof of the Example. (a) By 4A, there is a function $\phi : [\kappa]^2 \to [\kappa]^{<\omega}$ such that $\kappa = \bigcup_{I\in[D]^2} \phi(I)$ for every $D \in [\kappa]^\kappa$. Let $\psi : \kappa \to [\kappa]^\omega$ be a surjection. For $B \subseteq \kappa$ set $H_B = \{x : x \in \{0,1\}^\kappa,\ x(\xi) = 0 \text{ for every } \xi \in B\}$.

(b) Let $\mathcal{A}$ be the family of sets expressible in the form $I \cup \psi(\xi)$, where $I \in [\kappa]^2$ and $\xi \in \phi(I)$. Then

(i) every member of $\mathcal{A}$ is infinite;

(ii) $\mathcal{A} \cap \mathcal{P}J$ is countable for any countable $J \subseteq \kappa$;

(iii) for any family $\langle J_\xi\rangle_{\xi<\kappa}$ of countable sets, there is an $A \in \mathcal{A}$ such that $\eta \notin J_\xi$ for any distinct $\xi, \eta \in A$.

Of these, (i) is trivial, because $\psi(\xi)$ is infinite for every $\xi < \kappa$. (ii) is true because $[J]^2$ is countable whenever J is, and if $A \in \mathcal{A}$, $A \subseteq J$ then $A = I \cup \psi(\xi)$ for some $I \in [J]^2$, $\xi \in \phi(I)$. As for (iii), given such a family $\langle J_\xi\rangle_{\xi<\kappa}$, there is a $D \in [\kappa]^\kappa$ such that $\eta \notin J_\xi$ for any distinct $\xi, \eta \in D$ (because κ can be expressed as the union of ω_1 such free sets, by [6], Theorem 44.1, so at least one of them has cardinal κ). Now there are a $\xi < \kappa$ such that $\psi(\xi) \subseteq D$, and an $I \in [D]^2$ such that $\xi \in \phi(I)$, so $A = I \cup \phi(\xi)$ has the required properties.

(c) Now let $X \subseteq \{0,1\}^\kappa$ be $\{0,1\}^\kappa \setminus \bigcup_{A\in\mathcal{A}} H_A$. Let $\mu = \nu_X$ be the subspace measure on X, and Σ its domain.

(d) Write $\mathcal{K}$ for the family of non-empty closed subsets of $\{0,1\}^\kappa$ which are self-supporting for ν; then every $K \in \mathcal{K}$ meets X. To see this, recall that K is determined by coordinates in a countable set $J \subseteq I$. Now $\mathcal{A} \cap \mathcal{P}J$ is countable (by b(ii) above), so there is a $y \in K \setminus \bigcup\{H_A : A \in \mathcal{A},\ A \subseteq J\}$; if we set $x(\xi) = y(\xi)$ for $\xi \in J$ and 1 otherwise, then $x \in K \cap X$. Thus X is of full outer measure for ν, μ is a probability measure inner regular with respect to $\mathcal{K}_X = \{K \cap X : K \in \mathcal{K}\}$, and $K \mapsto K \cap X : \mathcal{K} \to \mathcal{K}_X$ is an order-isomorphism, as in 3F.

(e) μ is α-favourable. To see this, consider a tactic τ constructed as follows. For every non-negligible $E \in \Sigma$, choose $K_E \in \mathcal{K}$ such that $K_E \cap X \subseteq E$. K_E is determined by a countable set A_E of coordinates. Now let $\tau(E) \subseteq K_E \cap X$ be such that $\mu(\tau(E)) > 0$ and for every $I \in [A_E]^2$, $\xi \in \phi(I)$ there is a finite $J \subseteq \psi(\xi)$ such that H_J does not meet $\tau(E)$;

such a set exists because

$$\inf_{J\subseteq\psi(\xi)\text{ is finite}} \nu H_J = \nu H_{\psi(\xi)} = 0,$$

and there are only countably many pairs (I,ξ) to be looked at. Clearly, τ is a tactic for the second player in the game $\Gamma(\Sigma \setminus \mathcal{N}_\mu)$.

Let $\langle E_n\rangle_{n\in\mathbb{N}}$ be a sequence of non-negligible sets in Σ such that $E_{n+1} \subseteq \tau(E_n)$ for every n. Then

$$K_{E_{n+1}} \cap X \subseteq E_{n+1} \subseteq \tau(E_n) \subseteq K_{E_n} \cap X$$

for every n, so $\langle K_{E_n}\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence of compact sets in $\{0,1\}^\kappa$, and $K = \bigcap_{n\in\mathbb{N}} K_{E_n}$ is non-empty. By Lemma 1Eb, K is determined by coordinates in $A^* = \bigcup_{n\in\mathbb{N}}\bigcap_{m\ge n} A_{E_m}$.

Take any $z' \in K$, and consider $z \in \{0,1\}^\kappa$, defined by setting $z(i) = z'(i)$ for $i \in A^*$ and 1 otherwise. Then $z \in K$. If $z \notin X$, then there is an $A \in \mathcal{A}$ such that $z(i) = 0$ for every $i \in A$. Of course, $A \subseteq A^*$. Express A as $I\cup\psi(\xi)$, where $I \in [A]^2$ and $\xi \in \phi(I)$; then there is some n such that $I \in [A_{E_n}]^2$. But in this case there is a finite set $J \subseteq \psi(\xi)$ such that $K_{E_n} \cap H_J = \emptyset$, and $z \notin K_{E_n}$, which is impossible.

Thus $z \in K \cap X \subseteq \bigcap_{n\in\mathbb{N}} E_n$. As $\langle E_n\rangle_{n\in\mathbb{N}}$ is arbitrary, τ is a winning tactic and μ is α-favourable.

By 3G, μ is regularly monocompact.

(f) For the last step, I argue by contradiction. Suppose, if possible, that μ is countably compact.

(i) There is a countably compact class $\mathcal{R} \subseteq \Sigma$ such that μ is inner regular with respect to $\mathcal{R}$. By 1Bb(ii), we may suppose that $\mathcal{R}$ is closed under countable intersections. Write $\mathcal{Z}$ for the family of zero sets in $\{0,1\}^\kappa$ and $\mathcal{Z}_X$ for $\{Z \cap X : Z \in \mathcal{Z}\}$. Then $\mathcal{Z}_X$ is also closed under countable intersections, and μ is inner regular with respect to $\mathcal{Z}_X$; so μ is inner regular with respect to $\mathcal{R} \cap \mathcal{Z}_X$ ([10], 412Ac).

(ii) For each $\xi \in \kappa$, let $\langle Z_{\xi n}\rangle_{n\in\mathbb{N}}$ be a sequence in $\mathcal{Z}$ such that $Z_{\xi n}\cap X \in \mathcal{R}$ and $Z_{\xi n} \cap X \subseteq H_{\{\xi\}}$ for every $n \in \mathbb{N}$, and $\lim_{n\to\infty} \mu(Z_{\xi n} \cap X) = 1/2$. We may of course replace $Z_{\xi n}$ by $Z_{\xi n}\cap H_{\{\xi\}}$, so that $Z_{\xi n} \subseteq H_{\{\xi\}}$ for every n. Let $J_\xi \subseteq \kappa$ be a countable set such that every $Z_{\xi n}$ is determined by coordinates in J_ξ. Note that if $F \in \operatorname{dom}\nu$ is such that $\nu F > 0$ and F is determined by coordinates in $\kappa \setminus \{\xi\}$, there is an $n \in \mathbb{N}$ such that $\nu(F \cap Z_{\xi n}) > 0$, because

$$\lim_{n\to\infty} \nu(H_{\{\xi\}} \triangle Z_{\xi n}) = \lim_{n\to\infty} \mu(X \cap H_\xi \setminus Z_{\xi n}) = 0,$$

so

$$\lim_{n\to\infty} \nu(F \cap Z_{\xi n}) = \nu(F \cap H_\xi) = \tfrac{1}{2}\nu F > 0.$$

(iii) By b(iii), there is a set $A \in \mathcal{A}$ such that $\xi \notin J_\eta$ whenever ξ, η are distinct members of A. Enumerate A as $\langle \xi_i \rangle_{i\in\mathbb{N}}$ and choose $\langle n_i \rangle_{i\in\mathbb{N}}$ inductively in such a way that $F_m = \bigcap_{i\le m} Z_{\xi_i n_i}$ is never negligible; this is possible by the last remark in (ii) just above. In this case, $\langle X \cap Z_{\xi_i n_i} \rangle_{i\in\mathbb{N}}$ is a sequence in $\mathcal{R}$ such that every finite subset has non-empty intersection, because $F_m \cap X$ is never empty. There ought therefore to be a point $x \in X \cap \bigcap_{i\in\mathbb{N}} Z_{\xi_i n_i}$. But for this x, we have $x(\xi_i) = 0$ for every $i \in \mathbb{N}$, that is, $x(\xi) = 0$ for every $\xi \in A$, and $x \notin X$.

This contradiction shows that μ is not countably compact, and has all the declared properties.

4C. In [16], there is an example of a perfect probability measure which is not countably compact; since it is a subspace of $\{0,1\}^{\omega_1}$, 3Ea tells us that it is also not weakly α-favourable. I offer a minor adaptation of the argument to show that, subject to the continuum hypothesis, we can achieve the same phenomenon with a measure of countable Maharam type.

Example. *Let ν be any strictly positive completion regular Radon probability measure on $\{0,1\}^{\omega_1}$. Then there is a set $X \subseteq \{0,1\}^{\omega_1}$, of full outer measure for ν, such that the subspace measure on X is perfect but not weakly α-favourable.*

Proof. (a) I start by giving the construction. Let Ω be the set of non-zero countable limit ordinals. For each $\delta \in \Omega$, let $\langle \theta_\delta(n) \rangle_{n\in\mathbb{N}}$ be a strictly increasing sequence in δ with supremum δ, and set $A_\delta = \{\theta_\delta(n) : n \in \mathbb{N}\}$. Choose $w_\delta \in \{0,1\}^{A_\delta}$ such that $\nu\{x : x{\restriction}A_\delta = w_\delta\} = 0$.

If we write

$$\Omega_0 = \{\delta : \delta \in \Omega,\ w_\delta(\xi) = 0 \text{ for infinitely many } \xi \in A_\delta\},$$
$$\Omega_1 = \{\delta : \delta \in \Omega,\ w_\delta(\xi) = 1 \text{ for infinitely many } \xi \in A_\delta\},$$

at least one of these is a stationary set in ω_1. Take $\varepsilon \in \{0,1\}$ such that Ω_ε is stationary.

Now define X by writing

$$X = \{x : x \in \{0,1\}^{\omega_1},\ x{\restriction}A_\delta \neq w_\delta \text{ for every } \delta \in \Omega_\varepsilon\}.$$

(b) The first step is to check that $\nu^* X = 1$. For suppose, if possible, otherwise. Then, because ν is completion regular, there is a non-negligible zero set $K \subseteq \{0,1\}^{\omega_1}$ which is disjoint from X. This K is determined by some countable set of coordinates, so there is a $\zeta < \omega_1$ such that K is determined by coordinates in ζ. Now there is a $z \in K$ such that $z{\restriction}A_\delta \neq w_\delta$ for any $\delta \in \Omega_\varepsilon$ such that $\delta \le \zeta$, because there are only countably many such δ, so we have only countably many negligible sets to avoid. Set $x(\xi) = z(\xi)$ for $\xi < \zeta$, $x(\xi) = 1 - \varepsilon$ for $\xi \in \omega_1 \setminus \zeta$. Then $x \in K$ because $x{\restriction}\zeta = z{\restriction}\zeta$. If $\delta \in \Omega_\varepsilon$ and $\delta \le \zeta$, then $x{\restriction}A_\delta = z{\restriction}A_\delta \neq w_\delta$. If $\delta \in \Omega_\varepsilon$ and $\delta > \zeta$, then

$\{\xi : \xi \in A_\delta,\ x(\xi) = \varepsilon\} \subseteq A_\delta \cap \zeta$ is finite, so again $x{\restriction}A_\delta \neq w_\delta$. Accordingly, $x \in K \cap X$, which is supposed to be impossible. Thus X must indeed have full outer measure.

(c) Next, I have to show that the subspace measure μ on X is perfect. To see this, let $f : X \to \mathbb{R}$ be a measurable function and $E \in \operatorname{dom}\mu$ a non-negligible set. Then there is a measurable function $g : \{0,1\}^{\omega_1} \to \mathbb{R}$ extending f, and an $F \in \operatorname{dom}\nu$ such that $F \cap X = E$. There is a function $h : \{0,1\}^{\omega_1} \to \mathbb{R}$, determined by coordinates in a countable set, equal almost everywhere to g; let $H \subseteq \{0,1\}^{\omega_1}$ be a conegligible set, a countable union of zero sets, such that $h{\restriction}H = g{\restriction}H$. Let F_0 be a non-negligible zero set included in F.

Let $\zeta < \omega_1$ be such that F_0, H and h are all determined by coordinates less than ζ. As in (b) just above, the set $D = \{\delta : \delta \in \Omega_\varepsilon,\ A_\delta \subseteq \zeta\}$ is countable, and $G = \{x : x \in \{0,1\}^{\omega_1},\ x{\restriction}A_\delta \neq w_\delta \text{ for every } \delta \in D\}$ is conegligible. Of course, ν is perfect, so there is a compact set $K \subseteq h[F_0 \cap G \cap H]$ such that $\nu h^{-1}[K] > 0$. But, just as in (b), if we have any $z \in G$ there is an $x \in X$ such that $x{\restriction}\zeta = z{\restriction}\zeta$, and now $h(x) = h(z)$. This means that $h[B \cap X] \supseteq h[B \cap G]$ for any set $B \subseteq \{0,1\}^{\omega_1}$ determined by coordinates less than ζ. In particular,

$$\begin{aligned} f[E] = g[E] = g[F \cap X] &\supseteq g[H \cap F_0 \cap X] \\ &= h[H \cap F_0 \cap X] \supseteq h[H \cap F_0 \cap G] \supseteq K. \end{aligned}$$

On the other hand, $f^{-1}[K] \supseteq H \cap X \cap h^{-1}[K]$, so

$$\mu f^{-1}[K] \geq \nu^*(H \cap X \cap h^{-1}[K]) = \nu h^{-1}[K] > 0.$$

As f is arbitrary, μ is perfect.

(d) These arguments are minor modifications of the corresponding ones in [16]. Similarly, to see that μ is not weakly α-favourable, I adapt the argument used in [16] to show that it is not countably compact. Write $\mathcal{K}$ for the family of non-empty closed self-supporting sets in $\{0,1\}^{\omega_1}$; every member of $\mathcal{K}$ is determined by a countable set of coordinates, μ is inner regular with respect to $\mathcal{K}_X = \{K \cap X : K \in \mathcal{K}\}$ and $K \mapsto K \cap X : \mathcal{K} \to \mathcal{K}_X$ is an order-preserving bijection (see 3F above).

(e) Let $\sigma : \bigcup_{n \geq 1} \mathcal{K}_X \to \mathcal{K}_X$ be any strategy for the second player in the game $\Gamma(\mathcal{K}_X)$. (I seek to show that σ is not a winning strategy.) Define $\sigma' : \bigcup_{n \geq 1} \mathcal{K}^n \to \mathcal{K}$ by saying that

$$\sigma(K_0 \cap X, \ldots, K_n \cap X) = \sigma'(K_0, \ldots, K_n) \cap X$$

for all $K_0, \ldots, K_n \in \mathcal{K}$.

(f) The next step is to choose an increasing family $\langle M_\alpha \rangle_{\alpha < \omega_1}$ of countable sets. These can be described as elementary submodels for an appropriate fragment of set theory; but for readers unfamiliar or uncomfortable with model theory, I give the details of a straightforward construction. Let $\mathcal{M}$ be the family of countable subsets M of $\mathcal{K} \cup \omega_1$ such that

- whenever $K_0, \ldots, K_n \in M \cap \mathcal{K}$, then $\sigma'(K_0, \ldots, K_n) \in M$,
- whenever $K, K' \in M \cap \mathcal{K}$ and $K \cap K'$ is not negligible, there is an $L \in M \cap \mathcal{K}$ such that $L \subseteq K \cap K'$,
- whenever $K \in M \cap \mathcal{K}$ then K is determined by coordinates in $M \cap \omega_1$,
- whenever $\xi \in M \cap \omega_1$ and $\eta < \xi$, then $\eta \in M$,
- whenever I is a finite subset of $M \cap \omega_1$ and $w \in \{0,1\}^I$, then $\{x : x \in \{0,1\}^{\omega_1},\ x\restriction I \in w\}$ belongs to M.

Then it is easy to see that if $M \subseteq \mathcal{K} \cup \omega_1$ is any countable set, there is an $M' \in \mathcal{M}$ such that $M \subseteq M'$; if $M \in \mathcal{M}$, then $M \cap \omega_1 \in \omega_1$; and if $\langle M_n \rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathcal{M}$, then $\bigcup_{n\in\mathbb{N}} M_n \in \mathcal{M}$. Let $\langle M_\alpha \rangle_{\alpha<\omega_1}$ be a family in $\mathcal{M}$ such that $M_0 = \emptyset$, $M_\alpha \cup \{M_\alpha \cap \omega_1\} \subseteq M_{\alpha+1}$ for every α, and $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ for $\alpha \in \Omega$.

If we set $\gamma_\alpha = \sup(M_\alpha \cap \omega_1)$ for $\alpha < \omega_1$, $\langle \gamma_\alpha \rangle_{\alpha<\omega_1}$ is strictly increasing and $\gamma_\alpha = \sup_{\beta<\alpha} \gamma_\beta$ for $\alpha \in \Omega$; so $\{\gamma_\alpha : \alpha \in \Omega\}$ is a club in ω_1, and meets the stationary set Ω_ε. Let $\alpha \in \Omega$ be such that $\gamma_\alpha = \delta \in \Omega_\varepsilon$. Set $M'_0 = \emptyset$, $M'_n = M_{\theta_\alpha(n)}$ for $n \geq 1$, so that $\langle M'_n \rangle_{n\in\mathbb{N}}$ is an increasing sequence in $\mathcal{M}$ with union M_α.

(g) Consider the set A_δ. For each $n \in \mathbb{N}$, $I_n = A_\delta \cap M'_{n+1} \setminus M'_n$ is finite, because $M'_n \cap \omega_1 < \delta$. Set $H_n = \{x : x \in \{0,1\}^{\omega_1},\ x\restriction I_n = w_\delta\restriction I_n\}$; then $H_n \in M'_{n+1}$. Choose $\langle K_n \rangle_{n\in\mathbb{N}}$ in $\mathcal{K}$ inductively, as follows. $K_0 = H_0 \in M'_1$. Given that $K_i \in M'_{i+1}$ for $i \leq n$, then $K'_n = \sigma'(K_0, \ldots, K_n) \in M'_{n+1}$. Now K'_n is determined by coordinates in $M'_{n+1} \cap \omega_1$, while H_{n+1} is determined by coordinates in I_{n+1}, which is disjoint from M'_{n+1}; so $K'_n \cap H_{n+1} \neq \emptyset$ and (because K'_n is self-supporting) $\nu(K'_n \cap H_{n+1}) > 0$. Since both K'_n and H_{n+1} belong to M'_{n+2}, there is a $K_{n+1} \in M'_{n+2} \cap \mathcal{K}$ such that $K_{n+1} \subseteq K'_n \cap H_{n+1}$. Continue.

At the end of the induction, consider $K = \bigcap_{n\in\mathbb{N}} K_n$. Because $K_n \subseteq H_n$ for every n, $x(\xi) = w_\delta(\xi)$ whenever $x \in K$ and $\xi \in A_\delta$, so $K \cap X = \emptyset$. But this means that $\langle K_n \cap X \rangle_{n\in\mathbb{N}}$ has empty intersection, while

$$K_{n+1} \cap X \subseteq K'_n \cap X = \sigma(K_0 \cap X, \ldots, K_n \cap X)$$

for every $n \in \mathbb{N}$. So $\langle K_n \cap X \rangle_{n\in\mathbb{N}}$ witnesses that σ is not a winning strategy. As σ is arbitrary, μ is not weakly α-favourable.

4D. Remark. If the continuum hypothesis is true, then by Theorem 7 and Lemma 3 of [2] there is a strictly positive completion regular Radon measure on $\{0,1\}^{\omega_1}$ with countable Maharam type, and we can start from this in Example 4C to obtain a perfect probability space of countable Maharam type which is not weakly α-favourable.

4E. Since most of us find it surprising that any class of sets can be weakly α-favourable but not α-favourable, and since, as far as I know, the following example has not been published explicitly (though it is implicit in [4]), I set out the following fact.

PROPOSITION. *Let $\widehat{\mathcal{B}}$ be the algebra of subsets of $\mathbb{R}$ with the Baire property, and $\mathcal{M}$ the ideal of meager subsets of $\mathbb{R}$. Then $\widehat{\mathcal{B}} \setminus \mathcal{M}$ is a weakly α-favourable class which is not α-favourable.*

Proof. (a) Because $\mathbb{R}$ is a complete metric space, $\mathbb{R}$ is an α-favourable topological space in the sense of [3], that is, the family of non-empty open subsets of $\mathbb{R}$ is an α-favourable class. By [8], 7I, it follows that $\widehat{\mathcal{B}} \setminus \mathcal{M}$ is weakly α-favourable.

(b) Let τ be a tactic for the second player in $\Gamma(\widehat{\mathcal{B}} \setminus \mathcal{M})$. (I seek to show that τ is not a winning tactic.) Let $\mathcal{U}$ be a countable base for the topology of $\mathbb{R}$, not containing the empty set.

(i) The key to the argument is the following fact: for any $U \in \mathcal{U}$ there is a $V \in \mathcal{U}$ such that $\{M : M \in \mathcal{M}, V \setminus \tau(U \setminus M) \in \mathcal{M}\}$ is cofinal with $\mathcal{M}$. For suppose, if possible, otherwise. Then we can find for each $V \in \mathcal{U}$ a set $M_V \in \mathcal{M}$ such that $V \setminus \tau(U \setminus M) \notin \mathcal{M}$ whenever $M \in \mathcal{M}$ and $M \supseteq M_V$. Because $\mathcal{M}$ is a σ-ideal of sets, so that $\operatorname{add}(\mathcal{M}) > \omega = \#(\mathcal{U})$, $M^* = \bigcup_{V \in \mathcal{U}} M_V$ belongs to $\mathcal{M}$. But now $\tau(U \setminus M^*) \in \widehat{\mathcal{B}} \setminus \mathcal{M}$, so there must be some non-empty open set G such that $G \triangle \tau(U \setminus M^*) \in \mathcal{M}$. Taking $V \in \mathcal{U}$ such that $V \subseteq G$, we have $V \setminus \tau(U \setminus M^*) \in \mathcal{M}$ while $M^* \supseteq M_V$, which is supposed to be impossible.

(ii) We can therefore choose a sequence $\langle V_i \rangle_{i \in \mathbb{N}}$ in $\mathcal{U}$ such that, for each $i \in \mathbb{N}$, $\{M : V_{i+1} \setminus \tau(V_i \setminus M) \in \mathcal{M}\}$ is cofinal with $\mathcal{M}$; shrinking V_i if need be, we can suppose that $\operatorname{diam} V_i \le 2^{-i}$ for each i, so that $N = \bigcap_{i \in \mathbb{N}} V_i$ contains at most one point. Now choose $\langle N_i \rangle_{i \in \mathbb{N}}$ in $\mathcal{M}$ in such a way that $N_0 \supseteq N$ and, for each i,

$$V_{i+1} \setminus \tau(V_i \setminus N_i) \in \mathcal{M}, \qquad N_{i+1} \supseteq V_{i+1} \setminus \tau(V_i \setminus N_i).$$

Now, setting $E_i = V_i \setminus N_i$, we have a sequence $\langle E_i \rangle_{i \in \mathbb{N}}$ in $\widehat{\mathcal{B}} \setminus \mathcal{M}$ such that $E_{i+1} \subseteq \tau(E_i)$ for every $i \in \mathbb{N}$, while $\bigcap_{i \in \mathbb{N}} E_i \subseteq \bigcap_{i \in \mathbb{N}} V_i \setminus N$ is empty.

(iii) Thus τ is not a winning tactic; as τ is arbitrary, $\widehat{\mathcal{B}} \setminus \mathcal{M}$ is not α-favourable, as claimed.

REMARK. Note that if we set $\mu E = 0$ for $E \in \mathcal{M}$ and ∞ for $E \in \widehat{\mathcal{B}} \setminus \mathcal{M}$, then $(\mathbb{R}, \widehat{\mathcal{B}}, \mu)$ is a measure space in which $\mathcal{M} = \mathcal{N}_\mu$; so that, if we read the definitions in 1A literally, it is a weakly α-favourable measure space which is not α-favourable. Of course, it is not semi-finite. Observe also that $(\mathbb{R}, \widehat{\mathcal{B}}, \mathcal{M})$ is a complete ω_1-saturated measurable space with negligibles in the sense of

[8], and indeed is “Ka-regular” and therefore “semi-perfect” in the terminology of that paper. Thus except for the crucial fact that $\pi(\widehat{\mathcal{B}}/\mathcal{M}) = \omega$ it is similar to the best-behaved of probability spaces.

5. Problems. As remarked above, the results here leave open the following questions:

(a) Is there a weakly α-favourable probability space which is not regularly monocompact?

(b) If (X, Σ, μ) is an α-favourable probability space and T is a σ-subalgebra of Σ, is $(X, T, \mu\restriction T)$ necessarily α-favourable?

(This is a special case of the question raised after Proposition 2B.)

(c) (i) Is the product of two α-favourable probability spaces again α-favourable?

(ii) If $\langle(X_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ is a family of probability spaces such that the measure on $\prod_{i\in J} X_i$ is α-favourable for every finite $J \subseteq I$, is the product measure on $\prod_{i\in I} X_i$ necessarily α-favourable?

Of course, a negative answer to (a) would lead at once to positive answers to (b) and (c), and to corresponding results for monocompact spaces; but this would be surprising, and it is more natural to seek a negative answer to one of the questions (b) or (c)(i) to provide a negative answer to (a). A difficulty with the questions in (c) is that we cannot start from the α-favourable space in Example 4B; any power of this is (isomorphic to) a weakly α-favourable subspace of some $\{0,1\}^A$, and is therefore α-favourable, by Proposition 3G. And similarly, it will be no help if one of the factors is α-favourable and the other is countably compact, by Proposition 2E. So until we have another example of an α-favourable measure which is not countably compact, we cannot approach (c)(i) effectively. On the other hand, it might be useful to look at 4B in the context of question (b) here.

As noted in 4D above, the simplest form of the example in 4C has Maharam type ω_1, and we need a special axiom, such as the continuum hypothesis, to achieve an example of this kind with countable Maharam type. Similarly, the example in 4B has Maharam type between $\max(\omega_2, \mathfrak{c})$ and $\mathfrak{c}^{++}$, and while conceivably the techniques of [2] and [22] may allow some reduction in this (as in 4C), they cannot bring us to Maharam type less than ω_2, by 3Ea. So the questions arise:

(d) Is it consistent to suppose that every perfect probability space of countable Maharam type is countably compact?

G. Plebanek has found a construction using a relatively weak special axiom (valid, in particular, if either there is a Sierpiński set in $\mathbb{R}$ or Martin’s axiom is true) of a perfect probability space of countable Maharam type which is not weakly α-favourable.

(e) Is every (weakly) α-favourable probability space of countable Maharam type countably compact?

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