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Homotopy and homology groups of the n-dimensional Hawaiian earring

by

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Abstract. For the n-dimensional Hawaiian earring $\mathbb{H}_n$, $n \geq 2$, $\pi_n(\mathbb{H}_n, o) \simeq \mathbb{Z}^\omega$ and $\pi_i(\mathbb{H}_n, o)$ is trivial for each $1 \leq i \leq n-1$. Let CX be the cone over a space X and $CX \vee CY$ be the one-point union with two points of the base spaces X and Y being identified to a point. Then $\mathrm{H}_n(X \vee Y) \simeq \mathrm{H}_n(X) \oplus \mathrm{H}_n(Y) \oplus \mathrm{H}_n(CX \vee CY)$ for $n \geq 1$.

1. Introduction. The n-dimensional *Hawaiian earring* $\mathbb{H}_n$ is the subspace of the $(n+1)$-dimensional Euclidean space defined by

$$\mathbb{H}_n = \bigcup_{m=0}^{\infty} \{(x_0, \ldots, x_n) : (x_0 - 1/m)^2 + x_1^2 + \ldots + x_n^2 = 1/m^2\}$$

and we let $o = (0, \ldots, 0)$. The fundamental group of the 1-dimensional Hawaiian earring was first studied by H. B. Griffiths [8] and is known to be somewhat complicated. See also [9], [2] and [5]. The abelianization of $\pi_1(\mathbb{H}_1, o)$, that is, the first integral singular homology group, is explicitly presented in [6]: $\mathrm{H}_1(\mathbb{H}_1) \simeq \mathbb{Z}^\omega \oplus \bigoplus_c \mathbb{Q} \oplus \widehat{\bigoplus_c \mathbb{Z}}$, where $\mathbb{Z}^\omega$ is the direct product of countable many copies of the integer group, c is the cardinality of the real line, and $\widehat{\bigoplus_c \mathbb{Z}}$ is the $\mathbb{Z}$-adic completion of the free abelian group of rank c. It seems difficult to give a topological interpretation of the direct summand $\bigoplus_c \mathbb{Q} \oplus \widehat{\bigoplus_c \mathbb{Z}}$, but one way to explain the complexity would be as follows: The first homology group of a path-connected space is the quotient group of the fundamental group factored by its commutator subgroup. An element of the commutator subgroup can reverse the order of group multiplications of elements only finitely many times, while an element of $\pi_1(\mathbb{H}_1, o)$ that corresponds to a loop

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in $\mathbb{H}_1$, the projection of which to each circle is null-homotopic, need not be "canceled" by finitely many commutativity relations. This has an effect on that complexity.

On the other hand, it is known that $\pi_n(X,x)$ is an abelian group for each $n \geq 2$ and for each pointed space (X,x). Moreover, as is implicitly stated in [1, p. 295], one can disregard the order of group multiplications on $\pi_n(X,x)$ in an infinitary sense. In the present paper, we shall make this situation clearer and prove that $\pi_n(\mathbb{H}_n,o)$ is isomorphic to $\mathbb{Z}^\omega$ for each $n \geq 2$. This result follows from the main theorem which is stated after some preliminary definitions.

Let (X_i,x_i) be pointed spaces such that $X_i \cap X_j = \emptyset$ for $i \neq j$. The underlying set of a pointed space $(\widetilde{\bigvee}_{i\in I}(X_i,x_i),x^*)$ is the union of all X_i's obtained by identifying all x_i to a point x^* and the topology is defined by specifying the neighborhood bases as follows:

(1) If $x \in X_i \setminus \{x_i\}$, then the neighborhood base of x in $\widetilde{\bigvee}_{i\in I}(X_i,x_i)$ is the one of X_i;

(2) The point x^* has a neighborhood base, each element of which is of the form $\widetilde{\bigvee}_{i\in I\setminus F}(X_i,x_i)\vee\bigvee_{j\in F}U_j$, where F is a finite subset of I and each U_j is an open neighborhood of x_j in X_j for $j \in F$.

A space X is said to be *semi-locally strongly contractible* at $x \in X$ if there exist a neighborhood U of x and a continuous map $r : U \times \mathbb{I} \to X$ such that $r(u,0) = u, r(u,1) = x$ for $u \in U$ and $r(x,t) = x$ for $0 \leq t \leq 1$. For abelian groups A_i, $\widetilde{\prod}_{i\in I}A_i$ denotes the subgroup of the direct product $\prod_{i\in I} A_i$ consisting of all f's such that $\{i \in I : f(i) \neq 0\}$ is at most countable. In the next section, we prove:

Theorem 1.1. *Let $n\geq 2$ and X_i be an $(n-1)$-connected space which is semi-locally strongly contractible at x_i for each $i\in I$. Then $\pi_n(\widetilde{\bigvee}_{i\in I}(X_i,x_i),x^*)$ is isomorphic to the group $\widetilde{\prod}_{i\in I}\pi_n(X_i,x_i)$.*

Since $\mathbb{H}_n$ is homeomorphic to $\widetilde{\bigvee}_{i\in\mathbb{N}}(S_i,x_i)$, where S_i is the n-sphere, we have

Corollary 1.2. *For the n-dimensional Hawaiian earring $\mathbb{H}_n$, $n \geq 2$, $\pi_n(\mathbb{H}_n,o) \simeq \mathbb{Z}^\omega$ and $\pi_i(\mathbb{H}_n,o)$ is trivial for each $1 \leq i \leq n-1$.*

In the last section, we study the singular homology groups of the one-point unions of spaces. The next theorem generalizes [4, Theorem 1.2].

Theorem 1.3. *Let CX be the cone over a space X and $CX \vee CY$ be the one-point union with two points of the base spaces X and Y being identified to a point. Then*

$$\mathrm{H}_n(X\vee Y) \simeq \mathrm{H}_n(X) \oplus \mathrm{H}_n(Y) \oplus \mathrm{H}_n(CX\vee CY) \quad \textit{for } n \geq 1.$$

As an application, we shall show

COROLLARY 1.4. *The homotopy group $\pi_n(C\mathbb{H}_n \vee C\mathbb{H}_n, o)$ is trivial for $n \geq 2$. On the other hand, $\pi_{m+n-1}(C\mathbb{H}_m \vee C\mathbb{H}_n, o)$ is non-trivial for $m, n \geq 2$.*

We remark that $\pi_1(C\mathbb{H}_1 \vee C\mathbb{H}_1, o)$ is non-trivial [7]. Corollary 1.2 and the first half of Corollary 1.4 seem to show a difference between the cases $n = 1$ and $n \geq 2$. However, the second half of Corollary 1.4 indicates that the difference just reflects the effect of the Whitehead products which reduce to the commutators in the case $m = n = 1$. (We refer the reader to [4] for the case $n = 1$.) The last corollary also shows that we cannot drop the assumption of semi-local strong contractibility in Theorem 1.1. Also we cannot drop the $(n-1)$-connectedness of the space X_i either. An example which indicates this is given by means of the Whitehead product (see [10, Chapter 4, Section 7]). Throughout this paper, all spaces are Tikhonov spaces and all maps are assumed to be continuous unless otherwise stated.

2. Lemmas and proof of Theorem 1.1

DEFINITION 2.1. A continuous map $f : \mathbb{I}^n \to \widetilde{\bigvee}_{i \in I}(X_i, x_i)$ with $f(\partial \mathbb{I}^n) = \{x^*\}$ is said to be *standard* if there exists a sequence $(i_m : m < \omega)$ such that $i_l \neq i_m$ for distinct l and m and $f(\mathbb{I}^{n-1} \times [1/2^{m+1}, 1/2^m]) \subset X_{i_m}$ for each $m < \omega$.

Observe that $f(\partial(\mathbb{I}^{n-1} \times [1/2^{m+1}, 1/2^m])) = x^*$ in the definition above. The following is the key step for the proof of Theorem 1.1.

LEMMA 2.2. *Let $n \geq 2$ and let X_i be an $(n-1)$-connected space which is semi-locally strongly contractible for each $i \in I$. Then, for any continuous map $f : \mathbb{I}^n \to \widetilde{\bigvee}_{i \in I}(X_i, x_i)$ with $f(\partial \mathbb{I}^n) = x^*$, there exists a standard map which is homotopic to f relative to the boundary.*

Since the proof of the above lemma is long and somewhat technical, we outline it below, focusing on the main difficulty. Our goal is to construct a homotopy relative to ∂I^n between the given map f and a standard map. Throughout the remainder of this section, unless otherwise stated, a "homotopy" means a "homotopy relative to $\partial \mathbb{I}^n$". Two major obstacles prevent us from proceeding straightforwardly.

1) We have no information, at the beginning, what the preimage $f^{-1}(X_i)$ looks like. All we could say is that it is a compact subset of $\mathbb{I}^n$ which may be very complicated. For example, it might be homeomorphic to, say, "Wada Lake". Nor do we know the relationship between two preimages $f^{-1}(X_i)$ and $f^{-1}(X_j)$. They may be linked in a rather complicated way. So our first task is to change f by a homotopy in such a way that the preimage $f^{-1}(X_i)$ has a

"good shape" (such a subset will be called a canonical neighborhood below) for each i. This process is described in Lemmas 2.4 and 2.5. These lemmas produce a map $g : \mathbb{I}^n \to (\widetilde{\bigvee}_{i\in I}(X_i, x_i), x^*)$ which is homotopic to f such that $g^{-1}(X_i)$ is the disjoint union of the closures of some canonical neighborhoods.

2) Next we need to "rearrange" these preimages by a homotopy to obtain a standard map. Thus $g^{-1}(X_i)$ should be contained in $\mathbb{I}^{n-1} \times [1/2^{i+1}, 1/2^i]$. Our method here is modeled on the proof of the well known fact: $\pi_n(X)$ is abelian for each $n \geq 2$. It is somewhat technical but not so difficult to move each of the canonical neighborhoods to the "right place." The difficulty here is that we need to rearrange *infinitely many* canonical neighborhoods by induction and the continuity of the resulting maps on each inductive step should be retained during the whole procedure. This requires a rather careful construction and the point of this construction is that, once we rearrange a standard neighborhood, all the later stages must keep it fixed. This process is described as the iterated application of the "basic construction" below.

Let us start with auxiliary arguments.

DEFINITION 2.3. A *canonical neighborhood* in $\mathbb{I}^n$ is an open set of the form $\prod_{i=1}^n (a_n, b_n)$.

LEMMA 2.4. *Let X be an $(n-1)$-connected space and P be an n-dimensional polyhedron in $\mathbb{I}^n$ which is the union of finitely many n-simplexes. Let $\{D_i \mid 0 \leq i \leq m\}$ be the decomposition of P into those n-simplexes. Suppose that $f : \mathbb{I}^n \to X$ is a continuous map such that $f(\partial P \cup \partial \mathbb{I}^n) = \{x^*\}$. Then there exists a homotopy $H : \mathbb{I}^n \times \mathbb{I} \to X$ such that*

$$\begin{cases} H(u,0) = f(u) & \text{for } u \in \mathbb{I}^n, \\ H(u,t) = x^* & \text{for } u \in \partial\mathbb{I}^n,\ 0 \leq t \leq 1, \\ H(u,t) = f(u) & \text{for } u \notin \operatorname{int} P,\ 0 \leq t \leq 1, \\ H(u,1) = x^* & \text{for } u \in \bigcup_{i=0}^m \partial D_i. \end{cases}$$

Proof. Let $E = \bigcup_{i=0}^m \partial D_i \supset \partial P$. Since X is $(n-1)$-connected, $f|E$ is homotopic to a constant map relative to ∂P. Let $H' : E \times \mathbb{I} \to X$ be the relevant homotopy, i.e.

$$\begin{cases} H'(u,0) = f(u) & \text{for } u \in E, \\ H'(u,1) = x^* & \text{for } u \in E, \\ H'(u,t) = x^* & \text{for } u \in \partial P,\ 0 \leq t \leq 1. \end{cases}$$

Since $D_j \times \{0\} \cup \partial D_j \times \mathbb{I}$ is a retract of $D_j \times \mathbb{I}$ for each j, we can extend H' to a map $H : P \times \mathbb{I} \to X$ such that $H(u,t) = x^*$ for $u \in \partial P$ and $0 \le t \le 1$. Then H naturally extends to the required homotopy $H : \mathbb{I}^n \times \mathbb{I} \to X$ by defining $H(u,t) = f(u)$ for each $u \in \mathrm{cl}(\mathbb{I}^n \setminus P)$. ∎

LEMMA 2.5. *Let $n \ge 2$, let X_i be an $(n-1)$-connected space which is semi-locally strongly contractible at x_i for each $i \in I$, and let $X = \widetilde{\bigvee}_{i\in I}(X_i, x_i)$. Then each continuous map $f : \mathbb{I}^n \to X$ with $f(\partial \mathbb{I}^n) = x^*$ is homotopic relative to $\partial \mathbb{I}^n$ to a map $g : \mathbb{I}^n \to X$ such that*:

- *For each i there exist finitely many pairwise disjoint canonical neighborhoods $\{O_j \mid 0 \le j \le m\}$ such that $f(O_j) \subset X_i$, $f(\partial O_j) = \{x^*\}$ and $f^{-1}(X_i \setminus \{x_i\}) \subset \bigcup_{j=0}^m O_j$.*

Proof. We shall obtain the map g as the limit $\lim_{m\to\infty} f_m$ of maps $f_m : \mathbb{I}^n \to X$, to be constructed inductively. We start with a preliminary construction.

Preliminary construction. Fix $i \in I$ and let $O = f^{-1}(X_i \setminus \{x_i\})$. There exists a neighborhood U of x_i which semi-locally strongly contracts to x_i in X_i. Since X_i is a Tikhonov space, there exists a continuous function $F : \widetilde{\bigvee}_{i\in I}(X_i, x_i) \to [0,1]$ such that

$$F(x) = \begin{cases} 0 & \text{if } x \in X_i \setminus U, \\ 1 & \text{if } x = x^* \text{ or } x \notin X_i. \end{cases}$$

(The existence of such a function on X_i directly follows from the definition of the Tikhonov spaces. The function naturally extends to a function on $\widetilde{\bigvee}_{i\in I}(X_i, x_i)$ by the above formula, and the continuity follows from the definition of the topology of $\widetilde{\bigvee}_{i\in I}(X_i, x_i)$.) Let $r : U \times \mathbb{I} \to X_i$ be a contraction such that $r(x_0, t) = x_0$, $r(u,1) = x_0$, and $r(u,0) = u$ ($u \in U$), and let $K : X \times \mathbb{I} \to X$ be defined by

$$K(x,t) = \begin{cases} x & \text{if } x \notin X_i \text{ or } F(x) = 0, \\ r(x, t \cdot \min\{1, 2F(x)\}) & \text{otherwise.} \end{cases}$$

Define $H : \mathbb{I}^n \times \mathbb{I} \to X$ by $H(u,t) = K(f(u),t)$. Let $h(u) = H(u,1)$ and $O' = h^{-1}(X_i \setminus \{x_i\})$. Then H is a homotopy between f and h relative to $\partial \mathbb{I}^n$, and we have $\mathrm{cl}\, O' \subset O$. There exists an n-dimensional polyhedron P which is the union of the closures of finitely many canonical neighborhoods such that $\mathrm{cl}\, O' \subset \mathrm{int}\, P \subset P \subset O$. Now, applying Lemma 2.4 to h and P, we obtain a map h' which is homotopic to h relative to $\partial \mathbb{I}^n$ such that $h'^{-1}(X_i \setminus \{x_i\})$ is covered by a pairwise disjoint collection of finitely many canonical neighborhoods whose boundaries are all mapped to $\{x^*\}$ by h'.

Here we remark that the homotopy between f and h' above fixes points in $f^{-1}(\widetilde{\bigvee}_{j\in I\setminus\{i\}}(X_j, x_j))$.

Inductive construction of f_m. Let i_m ($m < \omega$) be an enumeration of the indices i such that $(X_i \setminus \{x_i\}) \cap \mathrm{Im}(f) \neq \emptyset$. We inductively define continuous maps $f_m : \mathbb{I}^n \to X$ and homotopies $H_m : \mathbb{I}^n \times [1 - 1/2^m, 1 - 1/2^{m+1}] \to X$ from f_m to f_{m+1} for each m. Let $f_0 = f$. Suppose that we have defined a map f_m. We perform the preliminary construction above for $i = i_m$ and obtain a map f_{m+1} via a homotopy $H_m : \mathbb{I}^n \times [1 - 1/2^m, 1 - 1/2^{m+1}] \to X$ relative to $\partial\mathbb{I}^n \cup f_m^{-1}(\widetilde{\bigvee}_{j\in I\setminus\{i_m\}}(X_j, x_j))$. This completes the inductive construction.

Since $f_m = f_l$ on $f_l^{-1}(X_{i_l})$ for each $m > l$, it follows that the limit map $g = \lim_{m\to\infty} f_m$ exists and is continuous. Now, it is easy to see that g has the desired properties. This completes the proof of Lemma 2.5. ∎

Next we introduce a "Basic Construction" as the composition of some auxiliary homotopies (a) and (b) below. Given a map f as in the conclusion of Lemma 2.5, the Basic Construction describes the process of "moving up" a canonical neighborhood P with $f(\partial P) = x^*$. As previously mentioned, the construction is inspired by the proof of the fact that the higher homotopy groups are abelian.

Auxiliary homotopies. (a) Let $\prod_{i=1}^{n}(c_i, d_i)$ be a canonical neighborhood and let $f : \mathbb{I}^n \to X = \widetilde{\bigvee}_{i\in I}(X_i, x_i)$ satisfy $f(\prod_{i=1}^{n-1}[c_i, d_i] \times \{c_n, d_n\}) = \{x^*\}$ and

(a.1) $f(\prod_{i=1}^{n-2}[c_i, d_i] \times \{d_{n-1}\} \times [c_n, d_n]\}) = \{x^*\}$ or

(a.2) $f(\prod_{i=1}^{n-2}[c_i, d_i] \times \{c_{n-1}\} \times [c_n, d_n]\}) = \{x^*\}$.

We shall homotope f to a map f_0 such that

$$f_0\Big(\prod_{i=1}^{n-2}[c_i, d_i] \times [(c_{n-1} + d_{n-1})/2, d_{n-1}] \times [c_n, d_n]\}\Big) = \{x^*\}$$

(if (a.1) holds) or

$$f_0\Big(\prod_{i=1}^{n-2}[c_i, d_i] \times [c_{n-1}, (c_{n-1} + d_{n-1})/2] \times [c_n, d_n]\}\Big) = \{x^*\}$$

(if (a.2) holds) respectively.

When $f(\prod_{i=1}^{n-2}[c_i, d_i] \times \{d_{n-1}\} \times [c_n, d_n]) = \{x^*\}$, we define a homotopy relative to $\partial\mathbb{I}^n$ from f to a map f_0 which is constantly x^* on $\prod_{i=1}^{n-2}[c_i, d_i] \times [(c_{n-1} + d_{n-1})/2, d_{n-1}] \times [c_n, d_n]$. The homotopy is given as follows. Define $K_0 : (\mathbb{I}^{n-2} \times \mathbb{I} \times \mathbb{I}) \times \mathbb{I} \to X$ by

$$K_0((s,u,v),t) = \begin{cases} f\Big(s, c_{n-1} + \dfrac{2(d_{n-1}-c_{n-1})}{(d_{n-1}-c_{n-1})(1+t)}(u-c_{n-1}), v\Big) & \\ \quad \text{for } s \in \prod_{i=1}^{n-2}[c_i,d_i], & \\ \quad c_{n-1} \le u \le (c_{n-1}(1-t)+d_{n-1}(1+t))/2, & \\ \quad c_n \le v \le d_n; & \\ x^* \quad \text{for } s \in \prod_{i=1}^{n-2}[c_i,d_i], & \\ \quad (c_{n-1}(1-t)+d_{n-1}(1+t))/2 \le u \le d_{n-1}, & \\ \quad c_n \le v \le d_n; & \\ f(s,u,v) \quad \text{otherwise.} & \end{cases}$$

Then $K_0((s,u,v),1) = f(s,u,v)$, and $f_0(s,u,v) = K_0((s,u,v),0)$ has the desired property. When $f(\prod_{i=1}^{n-2}[c_i,d_i] \times \{c_{n-1}\} \times [c_n,d_n]) = \{x^*\}$, we similarly define a homotopy K_1 from f to a map f_1 which is constantly x^* on $\prod_{i=1}^{n-2}[c_i,d_i] \times [c_{n-1},(c_{n-1}+d_{n-1})/2] \times [c_n,d_n]$.

(b) Next we consider a map $f : \mathbb{I}^n \to X$ such that $f(\prod_{i=1}^{n-1}[c_i,d_i] \times [\alpha,d_n]) = f(\partial(\prod_{i=1}^{n}[c_i,d_i])) = \{x^*\}$, where $c_n < \alpha \le \beta < d_n$, and construct a homotopy $K_2 : (\mathbb{I}^{n-1} \times \mathbb{I}) \times \mathbb{I} \to X$ that "moves continuously the subset $\prod_{i=1}^{n-1}[c_i,d_i] \times [c_n,\alpha]$ up to the level $\prod_{i=1}^{n-1}[c_i,d_i] \times [\beta,d_n]$." Precisely, let

$$K_2((s,u),t) = \begin{cases} f\Big(s, c_n + \dfrac{u - tc_n - (1-t)\beta}{t(\alpha-c_n)+(1-t)(d_n-\beta)}(\alpha - c_n)\Big) & \\ \quad \text{for } s \in \prod_{i=1}^{n-1}[c_i,d_i], u \in [tc_n+(1-t)\beta, t\alpha+(1-t)d_n]; & \\ x^* \quad \text{for } s \in \prod_{i=1}^{n-1}[c_i,d_i], & \\ \quad u \in [c_n, tc_n+(1-t)\beta] \cup [t\alpha+(1-t)d_n, d_n]; & \\ f(s,u) \quad \text{otherwise.} & \end{cases}$$

Basic Construction. Let $f : \mathbb{I}^n \to X$ be a map satisfying the conclusion of Lemma 2.5 such that $f(s,p) = f(s,q) = x^*$ for $s \in \mathbb{I}^{n-1}$, where $0 \le p < q \le 1$. Take a pairwise disjoint family $\mathcal{P}$ of canonical neighborhoods so that $f^{-1}(X \setminus \{x^*\}) \cap (\mathbb{I}^{n-1} \times [0,q]) \subset \bigcup \mathcal{P} \subset \mathbb{I}^{n-1} \times [0,q]$ and $f(\partial P) = \{x^*\}$ for each $P \in \mathcal{P}$. Fix $P = \prod_{i=1}^{n}(a_i,b_i) \in \mathcal{P}$ with $b_n \le p$. We shall construct a map h such that:

(B1) h is homotopic to f relative to $\partial\mathbb{I}^n \cup (\mathbb{I}^{n-1} \times [q,1])$.

(B2) Define $f' : \mathbb{I}^{n-1} \times [0,p] \to X$ by: $f'(s,u) = x^*$ if $(s,u) \in P$ and $f'(s,u) = f(s,u)$ otherwise. Then f' and $h|\mathbb{I}^{n-1} \times [0,p]$ are homotopic relative to $\partial(\mathbb{I}^{n-1} \times [0,p])$.

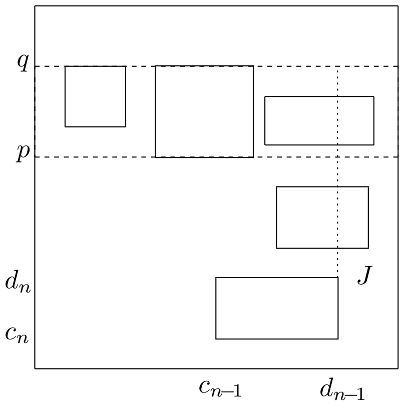

Fig. 1

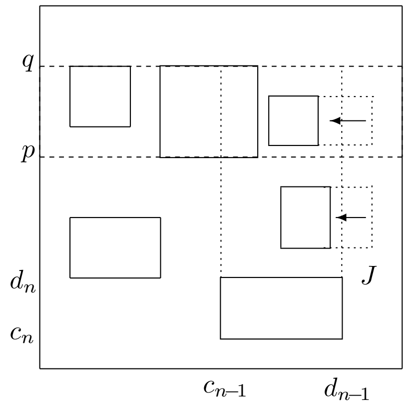

Fig. 2

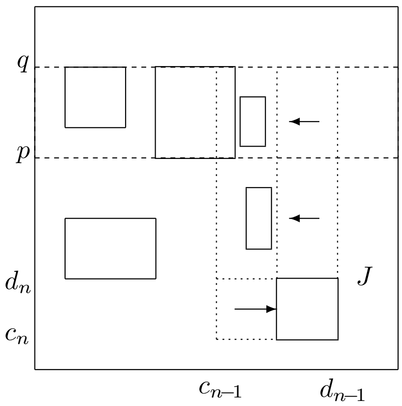

Fig. 3

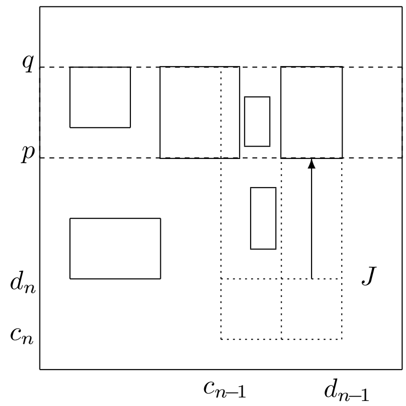

Fig. 4

STEP 1: Let $J = \mathbb{I}^{n-2} \times \{b_{n-1}\} \times [b_n, q]$ (see Figure 1). For each $Q \in \mathcal{P}$, take a standard neighborhood $Q' \subset Q$ so that $Q' \cap J = \emptyset$ and let $\mathcal{P}' = \{Q' : Q \in \mathcal{P}\}$. Applying (a) above to construct homotopies in each Q separately and putting these together, we have a map g_0, the continuity of which is guaranteed by the topology of X, such that

- $g_0(Q \setminus Q') = \{x^*\}$ for $Q \in \mathcal{P}$,
- g_0 is homotopic to f relative to $\mathbb{I}^n \setminus \bigcup \mathcal{P}$,
- $g_0^{-1}(X \setminus \{x^*\}) \subset \bigcup \mathcal{P}'$, and
- $g_0(J) = \{x^*\}$

(see Figure 2).

STEP 2. Applying (a) again to g_0 for $c_i = 0$, $d_i = 1$ ($i \leq n-2$), $c_{n-1} = a_{n-1}$, $d_{n-1} = b_{n-1}$, $c_n = b_n$ and $d_n = q$, we obtain a map g_1

which is homotopic to g_0 relative to $\partial\mathbb{I}^{n-1} \times [q,1]$ and such that $g_1(\mathbb{I}^{n-2} \times [(a_{n-1}+b_{n-1})/2, b_n] \times [b_n, q]) = x^*$. Apply (a) once more to g_1 for $c_i = a_i$, $d_i = b_i$ $(i \le n)$ to obtain a map g_2 such that

- $g_2(\prod_{i=1}^{n-2}[a_i,b_i] \times [a_{n-1},(a_{n-1}+b_{n-1})/2] \times [a_n,b_n])) = \{x^*\}$,
- $g_2|\prod_{i=1}^{n}[a_i,b_i]$ is homotopic to $g_0|\prod_{i=1}^{n}[a_i,b_i]$ relative to $\partial\prod_{i=1}^{n}[a_i,b_i]$;
- $g_2(\prod_{i=1}^{n-2}[a_i,b_i] \times [(a_{n-1}+b_{n-1})/2g, b_{n-1}] \times [b_n,q]) = \{x^*\}$;
- g_2 is homotopic to g_0 relative to $\mathbb{I}^{n-1} \times [q,1]$

(see Figure 3).

Step 3. Finally, applying (b) to g_2 for $c_i = a_i$, $d_i = b_i$ $(i \le n-2)$, $c_{n-1} = (a_{n-1}+b_{n-1})/2$, $d_{n-1} = b_{n-1}$, $c_n = a_n$, $d_n = q$, $\alpha = b_n$, and $\beta = p$, we obtain a map h which is homotopic to f relative to $\partial\mathbb{I}^n \cup \mathbb{I}^{n-1} \times [q,1]$ (see Figure 4).

Now, we can see that the composition of the above four homotopies (between f and h) is a homotopy relative to $\partial\mathbb{I}^n \cup (\mathbb{I}^{n-1} \times [q,1])$, and h satisfies the required condition. This construction of h and a homotopy between f and h are called the *Basic Construction* below.

Proof of Lemma 2.2. We may assume that f satisfies the conclusion of Lemma 2.5 and also $f(s,u) = x^*$ for $1/2 \le u \le 1$. Let $(i_m : m < \omega)$ be an enumeration of $\{i \in I : \mathrm{Im}(f) \cap (X_i \setminus \{x_i\}) \neq \emptyset\}$ without repetition. (If the last set is finite, the conclusion is well known and so we assume, at the outset, that infinitely many X_i's intersect $\mathrm{Im}(f)$.) We define maps $f_m : \mathbb{I}^n \to X$ and homotopies $H_m : \mathbb{I}^n \times [1/2^{m+1}, 1/2^m] \to X$ by induction. First (= the 0th step), let $f_0 = f$ and take a collection $\mathcal{P}_0$ of finitely many pairwise disjoint canonical neighborhoods which covers $f_0^{-1}(X_{i_0})$. Performing the Basic Construction for each member of the collection $\mathcal{P}_0$ and for $p = 1/2 + 1/4$, $q = 1$, we have a homotopy $H_0 : \mathbb{I}^n \times [1/2, 1] \to X$ relative to $\partial\mathbb{I}^n$ such that $H_0(a,1) = f(a)$ and $f_1(a) = H_0(a, 1/2)$ satisfies $f_1^{-1}(X_{i_0}) \subset \mathbb{I}^{n-1} \times [3/4, 1]$.

In the mth step, we consider the subspace X_{i_m} and take a collection $\mathcal{P}_m$ of finitely many pairwise disjoint canonical neighborhoods which covers $f_m^{-1}(X_{i_m})$. Performing the basic constructions finitely many times for the map f_m and $p = 1/2 + 1/2^{m+2}$, $q = 1/2 + 1/2^{m+1}$, we obtain a homotopy $H_m : \mathbb{I}^n \times [1/2^{m+1}, 1/2^m] \to X$ such that

- $H_m(s,u,t) = f_m(s,u)$ for $(s,u) \in \mathbb{I}^{n-1} \times [1/2 + 1/2^{m+1}, 1] \cup \partial\mathbb{I}^n$, $1/2^{m+1} \le t \le 1/2^m$;
- if $H_m(s,u,1/2^{m+1}) \in X_{i_k} \setminus \{x_{i_k}\}$, then $1/2 + 1/2^{k+2} \le u \le 1/2 + 1/2^{k+1}$ for each $k \le m$.

Finally, define

$$\begin{cases} H(s,u,t) = H_m(s,u,t) & \text{for } 1/2^{m+1} \le t \le 1/2^m, \\ H(s,u,0) = x^* & \text{if } 0 \le u \le 1/2, \\ H(s,u,0) = H_m(s,u,1/2^{m+1}) & \text{if } 1/2 + 1/2^{m+1} \le u \le 1/2 + 1/2^m, \end{cases}$$

and $h(s,u) = H(s,(1+u)/2,0)$. Then H is continuous and consequently h is homotopic to f relative to the boundary. It is easy to see that h is standard. ∎

Proof of Theorem 1.1. Let $r_i : \widetilde{\bigvee}_{i\in I}(X_i,x_i) \to (X_i,x_i)$ be the canonical retraction and $r_{i*} : \pi_n(\widetilde{\bigvee}_{i\in I}(X_i,x_i),x^*) \to \pi_n(X_i,x_i)$ the induced homomorphism. Define $h : \pi_n(\widetilde{\bigvee}_{i\in I}(X_i,x_i),x^*) \to \prod_{i\in I}\pi_n(X_i,x_i)$ by $p_i h(x) = r_{i*}(x)$, where $p_i : \prod_{i\in I}\pi_n(X_i,x_i) \to \pi_n(X_i,x_i)$ is the projection. Since $\mathbb{I}^n$ is separable, it follows that $\mathrm{Im}(h) \subset \widetilde{\prod}_{i\in I}\pi_n(X_i,x_i)$. Since, for each countable sequence $(i_m : m < \omega)$ with $i_m \ne i_n$ $(m \ne n)$, $(X_{i_m} : m < \omega)$ converges to x^*, h is surjective.

Now, it suffices to show that h is injective. Let $h([f]) = 0$, where $[f]$ denotes the homotopy class relative to the boundary. By Lemma 2.2, we may assume that f is standard; let $(i_m : m < \omega)$ be the corresponding sequence. Then $r_{i_m *}([f]) = 0$ for each $m < \omega$. Since $r_{i_m} f(s,u) = x^*$ for $u \le 1/2^{m+1}$ or $u \ge 1/2^m$, $f|\mathbb{I}^{n-1} \times [1/2^{m+1},1/2^m]$ is null-homotopic relative to the boundary; let $H_m : \mathbb{I}^{n-1} \times [1/2^{m+1},1/2^m] \times \mathbb{I} \to X$ be the relevant homotopy. Define $H : \mathbb{I}^n \times \mathbb{I} \to X$ by $H|\mathbb{I}^{n-1} \times [1/2^{m+1},1/2^m] \times \mathbb{I} = H_m$ for each $m < \omega$ and $H(s,0,t) = x^*$. Then H is a homotopy from f to the constant map relative to the boundary. ∎

3. Proofs of Theorem 1.3 and Corollary 1.4. The cone CX over a space X is the quotient space of the cylinder $X \times \mathbb{I}$ obtained by shrinking $X \times \{1\}$ to a point. Let $p : X \times \mathbb{I} \to CX$ be the canonical projection. For a subset A of $\mathbb{I}$, let $C_A X = p(X \times A) \subset CX$. We identify X with the subset $p(X \times \{0\})$ of CX. The one-point union of pointed spaces (X,x) and (Y,y) is the quotient space obtained from the disjoint union $X \cup Y$ by identifying x and y. It is denoted by $(X,x) \vee (Y,y)$ and frequently abbreviated to $X \vee Y$ when no confusion occurs. Throughout the remaining part of the paper, we assume that the base point of the cone CX of a pointed space (X,x) is $p(x,0)$ and is simply denoted by x under the above identification.

Proof of Theorem 1.3. Let $A = C_{(1/3,1]}X \cup C_{(1/3,1]}Y$ and $B = C_{[0,2/3)}X \vee C_{[0,2/3)}Y$. Then A and B are open subsets of $CX \vee CY$ with $A \cup B = CX \vee CY$, and $A \cap B$ has the same homotopy type as the disjoint union $X \cup Y$. Consider the following part of the Mayer–Vietoris sequence:

$$\begin{aligned} \mathrm{H}_n(A\cap B) \xrightarrow{\alpha} \mathrm{H}_n(A) \oplus \mathrm{H}_n(B) &\xrightarrow{\beta} \mathrm{H}_n(A\cup B) \xrightarrow{\partial} \mathrm{H}_{n-1}(A\cap B) \\ &\xrightarrow{\gamma} \mathrm{H}_{n-1}(A) \oplus \mathrm{H}_{n-1}(B). \end{aligned}$$

For simplicity, we assume that $n \geq 2$. The argument for the case $n = 1$ is an easy modification. First notice that $\mathrm{H}_i(A \cap B) \simeq \mathrm{H}_i(X) \oplus \mathrm{H}_i(Y)$. Since A has the homotopy type of the space of two points and $n \geq 2$, we see that $\mathrm{H}_n(A) = \mathrm{H}_{n-1}(A) = 0$. In addition, B has the homotopy type of the one-point union $X \vee Y$, and hence $\mathrm{H}_i(B) \simeq \mathrm{H}_i(X \vee Y)$. Let $i_X : X \hookrightarrow X \vee Y$ and $i_Y : Y \hookrightarrow X \vee Y$ be the inclusion maps, and let $r_X : X \vee Y \to X$ and $r_Y : X \vee Y \to Y$ be the canonical retractions. Clearly $r_X \circ i_X = \mathrm{id}$ and $r_Y \circ i_Y = \mathrm{id}$. Under the above isomorphisms, the homomorphisms α and γ correspond to the homomorphism induced by $(i_X)_* + (i_Y)_* = h$ and hence both have the left inverse $(r_X)_* + (r_Y)_*$. In particular, γ is injective. Also it is easy to see that the homomorphism β corresponds to the one induced by $i : X \vee Y \hookrightarrow CX \vee CY$. Therefore the above sequence reduces to the following split short exact sequence:

$$0 \longrightarrow \mathrm{H}_n(X) \oplus \mathrm{H}_n(Y) \xrightarrow{h} \mathrm{H}_n(X \vee Y) \xrightarrow{i_*} \mathrm{H}_n(CX \vee CY) \longrightarrow 0$$

Therefore the conclusion of the theorem follows. ∎

Proof of Corollary 1.4. We show that $\pi_i(C\mathbb{H}_n \vee C\mathbb{H}_n) = \{0\}$ for $1 \leq i \leq n$ by induction on i. Since the space $\mathbb{H}_n$ is locally simply connected, $C\mathbb{H}_n$ is also locally simply connected. In addition, $C\mathbb{H}_n$ is first countable (as a metric space) and contractible, and $C\mathbb{H}_n \vee C\mathbb{H}_n$ is simply connected by [7, Theorem 1]. (See also [3].) Suppose that we have shown $\pi_j(C\mathbb{H}_n \vee C\mathbb{H}_n) = \{0\}$ for $j < i$ ($\leq n$). Then, by the Hurewicz isomorphism theorem, we have $\pi_i(C\mathbb{H}_n \vee C\mathbb{H}_n) \simeq \mathrm{H}_i(C\mathbb{H}_n \vee C\mathbb{H}_n)$. Since $\mathbb{H}_n \vee \mathbb{H}_n$ is homeomorphic to $\mathbb{H}_n$, the proof of Theorem 1.1 shows that $\mathrm{H}_i(\mathbb{H}_n \vee \mathbb{H}_n)$ is naturally isomorphic to $\mathrm{H}_i(\mathbb{H}_n) \oplus \mathrm{H}_i(\mathbb{H}_n)$ via the isomorphism $(i_{\mathbb{H}_n})_* + (i_{\mathbb{H}_n})_*$. By Theorem 1.3 we see that $\mathrm{H}_i(C\mathbb{H}_n \vee C\mathbb{H}_n) = \{0\}$. Therefore, $\pi_i(C\mathbb{H}_n \vee C\mathbb{H}_n) \simeq \mathrm{H}_i(C\mathbb{H}_n \vee C\mathbb{H}_n) = \{0\}$. This completes the proof of the first part of the corollary.

Let X and Y be copies of the pointed spaces $(\mathbb{H}_m, o)$ and $(\mathbb{H}_n, o)$ respectively and $X \vee Y$ their one-point union. Let $r_X : X \vee Y \to X$ and $r_Y : X \vee Y \to X$ be the canonical retractions. As pointed out in the proof of Theorem 1.3, the surjection $\mathrm{H}_{m+n-1}(X \vee Y) \to \mathrm{H}_{m+n-1}(CX \vee CY)$ is induced by the inclusion $i : X \vee Y \hookrightarrow CX \vee CY$. Let $X_j \subset X$ and $Y_j \subset Y$ be the corresponding copies of $\{(x_0, \ldots, x_m) : (x_0 - 1/j)^2 + x_1^2 + \ldots + x_m^2 = 1/j^2\}$ and $\{(x_0, \ldots, x_n) : (x_0 - 1/j)^2 + x_1^2 + \ldots + x_n^2 = 1/j^2\}$ respectively.

Here we recall the proof of [1, Theorem 2]. Let $\alpha_j \in \pi_m(X_j)$ and $\beta_j \in \pi_n(Y_j)$ be non-trivial elements and $[\alpha_j, \beta_j] \in \pi_{m+n-1}(X \vee Y)$ be their Whitehead product. Since $[\alpha_j, \beta_j]$ can be realized by a map into $X_j \vee Y_j$, an element $\gamma \in \pi_{m+n-1}(X \vee Y)$ is defined with representation $\gamma = \sum_{j=1}^{\infty} [\alpha_j, \beta_j]$. (This γ is defined as a homotopy class containing a standard map to $\widetilde{\bigvee}_{j<\omega} X_j \vee Y_j$. We refer the reader to [1, p. 295] for the precise definition of γ.) The Hurewicz homomorphism $\pi_* \to \mathrm{H}_*$ from the homotopy groups to the singular homology groups is denoted by φ. Since $r_{X*}([\alpha_j,\beta_j])$ is trivial for each j and the homotopy track of the contraction is contained in X_j, $r_{X*}(\gamma)$ is trivial and hence $r_{X*}\varphi(\gamma)$ is also trivial. By [1, Theorem 2], $\varphi(\gamma)$ is non-trivial and hence $i_*\varphi(\gamma) = \varphi i_*(\gamma)$ is non-trivial and so is $i_*(\gamma) \in \pi_{m+n-1}(CX \vee CY, o)$. ■

Remark 3.1. The proof of Theorem 1.3 shows that the homomorphism $i_* : \mathbb{H}_n(X \vee Y) \to \mathbb{H}_n(CX \vee CY)$ induced by the inclusion $i : X \vee Y \to CX \vee CY$ is surjective. Due to the simple structure of connected open subsets of $[0,1]$, we can see that $i_* : \pi_1(X \vee Y, x) \to \pi_1(CX \vee CY, x)$ is surjective. However, we do not know whether $i_* : \pi_n(X \vee Y, x) \to \pi_n(CX \vee CY, x)$ is surjective for $n \geq 2$.

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