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Knots in $S^2 \times S^1$ derived from $\mathrm{Sym}(2, \mathbb{R})$

by

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Abstract. We realize closed geodesics on the real conformal compactification of the space $V = \mathrm{Sym}(2, \mathbb{R})$ of all 2×2 real symmetric matrices as knots or 2-component links in $S^2 \times S^1$ and show that these knots or links have certain types of symmetry of period 2.

1. Introduction. In [6], to complete a semisimple Jordan algebra V of classical type to a symmetric space, B. Makarevich used the notion of geodesics in V that originate at zero. In the case of the Euclidean (or formally real) Jordan algebra $V = \mathrm{Sym}(n, \mathbb{R})$ of all $n \times n$ real symmetric matrices, these geodesics are eventually of the form $\alpha(t, A) := \exp tX_A \cdot \mathbf{0}$, $A \in V$, where $X_A = \left(\begin{smallmatrix} 0 & A \\ -A & 0 \end{smallmatrix}\right) \in \mathfrak{sp}(2n, \mathbb{R})$, the Lie algebra of the symplectic group $\mathrm{Sp}(2n, \mathbb{R})$. In [5], the authors classified closed geodesics and symmetric geodesics of these types on the real conformal compactification $\mathcal{M}$ of $V = \mathrm{Sym}(n, \mathbb{R})$. In Section 2, for convenience we give a brief review of these results and show some elementary facts.

It is known [1] that the conformal compactification $\mathcal{M}$ of $V = \mathrm{Sym}(n, \mathbb{R})$ is diffeomorphic to the Shilov boundary Σ_n of the symmetric tube domain $T_\Omega = V + i\Omega$, where Ω is the open convex cone of all positive definite $n \times n$ symmetric matrices. The main interest of this paper is to give a realization of these closed geodesics in the Shilov boundary Σ_2 as knots in $S^2 \times S^1$ and to characterize their symmetry properties.

Throughout this paper, all maps and spaces will be assumed to be in the piecewise-linear (PL) category. A *link* L of μ components in a connected 3-manifold M is (the image of) an embedding of μ disjoint 1-spheres into M. If $\mu = 1$, then L is called a *knot* in M. Two links L and L' are said to be *equivalent* if there exists an ambient isotopy $H : M \times [0, 1] \to$

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$M \times [0,1]$, $H(x,t) = h_t(x)$ ($t \in [0,1]$), such that h_0 is the identity on M and $h_1(L) = L'$.

A knot (or link) K in a connected 3-manifold M is said to have *period n of type* (X,Y) (or to be an *n-periodic knot of type* (X,Y)) if there is an n-periodic homeomorphism $h : (M,K) \to (M,K)$ such that the fixed point set, $\mathrm{Fix}(h)$, of h is homeomorphic to X and $\mathrm{Fix}(h) \cap K$ is homeomorphic to Y. If M is a homology 3-sphere, then P. A. Smith [9] proved that the set of fixed points of a periodic homeomorphism of M is $\emptyset, S^0, S^1$, or S^2. By the positive solution of the Smith conjecture [7], the possible types of non-trivial knots in S^3 are $(\emptyset,\emptyset), (S^0,\emptyset), (S^0,S^0), (S^1,\emptyset)$, (S^1,S^0), and (S^2,S^0) (cf. [3]).

In Section 3, we show that the closed geodesics on the conformal compactification $\mathcal{M}$ of $\mathrm{Sym}(2,\mathbb{R})$ are knots in the Shilov boundary Σ_2 which have period 2 of type both $(\emptyset,\emptyset)$ and $(S^1 \cup S^0, S^0)$.

In [4], [10], and [11], it was shown that $S^2 \times S^1$ admits exactly thirteen distinct involutions (up to conjugation) and the possible types of their fixed point sets are $\emptyset, S^0 \dot\cup S^0, S^1, S^1 \dot\cup S^1, S^1 \times S^1$, Klein bottle, $S^0 \dot\cup S^2$, or $S^2 \dot\cup S^2$, where $X \dot\cup Y$ denotes the disjoint union of X and Y.

In Section 4, we give an explicit description of an orientable double cover $S^2 \times S^1$ of the Shilov boundary Σ_2 and show that the knots in Σ_2 corresponding to the closed geodesics lift to knots or links of 2-components in $S^2 \times S^1$; we show that these knots or links in $S^2 \times S^1$ also have period 2 of types $(S^1 \times S^1, T(m,n)), (S^1 \times S^1, S^0 \dot\cup \ldots \dot\cup S^0)$, or $(S^1 \dot\cup S^1, \emptyset)$, where $T(m,n)$ denotes the torus knot of type (m,n) [8].

2. Geodesics on the conformal compactification of $\mathrm{Sym}(n,\mathbb{R})$**.** Let $M_n(\mathbb{R})$ denote the space of all $n \times n$ real matrices. A *symmetric* (respectively, *skew-symmetric*) matrix $A \in M_n(\mathbb{R})$ is one satisfying $A^t = A$ (respectively, $A^t = -A$), where A^t denotes the transpose of a matrix A. Let $\mathrm{Sym}(n,\mathbb{R})$ (respectively, $\mathrm{Skew}(n,\mathbb{R})$) be the space of all symmetric (respectively, skew-symmetric) $n \times n$ matrices. Let $A \in \mathrm{Sym}(n,\mathbb{R})$ have the spectral decomposition $A = \sum_{k=1}^n \lambda_k C_k$, where $\{C_k\}$ is a complete system of orthogonal projections. Then the *spectral norm* $|A|$ of A is defined by $|A| = \max\{|\lambda_1|,\ldots,|\lambda_n|\}$.

Let $(\cdot|\cdot)$ be the skew-symmetric form on $\mathbb{R}^{2n}$ defined by $(u|v) = \langle Ju \mid v\rangle$ for $u,v \in \mathbb{R}^{2n}$, where $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. Here, I stands for the $n \times n$ identity matrix. The *symplectic group* $\mathrm{Sp}(2n,\mathbb{R})$ on $\mathbb{R}^{2n}$ is the Lie group of all invertible transformations $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ satisfying one of the following equivalent conditions:

(1) g preserves $(\cdot|\cdot)$.
(2) $g^t J g = J$.
(3) $A^t C, B^t D$ are symmetric and $A^t D - C^t B = I$.

The Lie algebra of $\mathrm{Sp}(2n,\mathbb{R})$ is given by

$$\mathfrak{sp}(2n,\mathbb{R}) = \left\{ \begin{pmatrix} X & Y \\ Z & -X^t \end{pmatrix} \,\middle|\, X \in M_n(\mathbb{R}),\ Y, Z \in \mathrm{Sym}(n,\mathbb{R}) \right\}.$$

It has a Cartan decomposition $\mathfrak{sp}(2n,\mathbb{R}) = \mathfrak{p} \oplus \mathfrak{k}$, where

$$\mathfrak{p} = \left\{ \begin{pmatrix} X & Y \\ Y & -X \end{pmatrix} \,\middle|\, X, Y \in \mathrm{Sym}(n,\mathbb{R}) \right\},$$

$$\mathfrak{k} = \left\{ \begin{pmatrix} X & Y \\ -Y & X \end{pmatrix} \,\middle|\, X \in \mathrm{Skew}(n,\mathbb{R}),\ Y \in \mathrm{Sym}(n,\mathbb{R}) \right\}.$$

Let $\tau = \left(\begin{smallmatrix} -I & 0 \\ 0 & I \end{smallmatrix}\right) \in \mathrm{GL}(2n,\mathbb{R})$ and let $\tau(g) = \tau \cdot g \cdot \tau$ for $g \in \mathrm{Sp}(2n,\mathbb{R})$. Then τ is an involution on $\mathrm{Sp}(2n,\mathbb{R})$. The differential $\mathrm{d}\tau$ of τ at the identity is given by

$$\mathrm{d}\tau \begin{pmatrix} X & Y \\ Z & -X^t \end{pmatrix} = \begin{pmatrix} X & -Y \\ -Z & -X^t \end{pmatrix}.$$

The Lie algebra $\mathfrak{sp}(2n,\mathbb{R})$ can be decomposed into the $(+1)$-eigenspace $\mathfrak{h}$ and the (-1)-eigenspace $\mathfrak{q}$ of $\mathrm{d}\tau$:

$$\mathfrak{sp}(2n,\mathbb{R}) = \mathfrak{h} \oplus \mathfrak{q} = \mathfrak{h} \oplus \mathfrak{n}^+ \oplus \mathfrak{n}^-, \qquad \mathfrak{q} = \mathfrak{n}^+ \oplus \mathfrak{n}^-,$$

where

$$\mathfrak{n}^+ = \left\{ \begin{pmatrix} 0 & Y \\ 0 & 0 \end{pmatrix} \,\middle|\, Y \in \mathrm{Sym}(n,\mathbb{R}) \right\},$$

$$\mathfrak{n}^- = \left\{ \begin{pmatrix} 0 & 0 \\ Z & 0 \end{pmatrix} \,\middle|\, Z \in \mathrm{Sym}(n,\mathbb{R}) \right\},$$

$$\mathfrak{h} = \left\{ \begin{pmatrix} X & 0 \\ 0 & -X^t \end{pmatrix} \,\middle|\, X \in M_n(\mathbb{R}) \right\}.$$

Let $N^\pm$ be the Lie subgroups of $\mathrm{Sp}(2n,\mathbb{R})$ corresponding to $\mathfrak{n}^\pm$ respectively. Then

$$N^+ = \left\{ \begin{pmatrix} I & A \\ 0 & I \end{pmatrix} \,\middle|\, A \in \mathrm{Sym}(n,\mathbb{R}) \right\} = \exp \mathfrak{n}^+,$$

$$N^- = \left\{ \begin{pmatrix} I & 0 \\ A & I \end{pmatrix} \,\middle|\, A \in \mathrm{Sym}(n,\mathbb{R}) \right\} = \exp \mathfrak{n}^-.$$

Let $H = \{g \in \mathrm{Sp}(2n,\mathbb{R}) \mid \tau(g) = g\}$. We observe that

$$H = \left\{ \begin{pmatrix} A & 0 \\ 0 & (A^{-1})^t \end{pmatrix} \,\middle|\, A \in \mathrm{GL}(n,\mathbb{R}) \right\}.$$

Theorem 2.1 (see [1]). *Let $P = HN^-$. Then P is a closed subgroup of $G := \mathrm{Sp}(2n,\mathbb{R})$ and the homogeneous space $\mathcal{M} := G/P$ is a compact real manifold with $V := \mathrm{Sym}(n,\mathbb{R})$ as an open dense subset. The embedding of*

V into $\mathcal{M}$ is given by

$$X \in V \mapsto \begin{pmatrix} I & X \\ 0 & I \end{pmatrix} \cdot P \in \mathcal{M}.$$

Furthermore, for $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}(2n, \mathbb{R})$ *and* $X \in V$ *with* $g \cdot X \in V$, *we have*

$$g \cdot X = (AX + B)(CX + D)^{-1}.$$

Let $A \in V = \mathrm{Sym}(n, \mathbb{R})$ and let $X_A := \left(\begin{smallmatrix} 0 & A \\ -A & 0 \end{smallmatrix}\right) \in \mathfrak{k}$. Then it is known [6] that the geodesic in $\mathcal{M}$ originating at the origin $\mathbf{0}$ with direction A is of the form

$$\alpha(t, A) := \exp tX_A \cdot \mathbf{0} = \begin{pmatrix} \cos tA & \sin tA \\ -\sin tA & \cos tA \end{pmatrix} \cdot \mathbf{0}.$$

The *period* of a non-constant closed geodesic $\alpha(t, A)$ is the smallest positive number t_0 satisfying $\alpha(t_0, A) = \mathbf{0}$.

Set $j = \left(\begin{smallmatrix} 0 & -I \\ I & 0 \end{smallmatrix}\right) \in \mathrm{Sp}(2n, \mathbb{R})$. Then j is an involution on $\mathcal{M}$ and for an invertible element $A \in V$ we have $j \cdot A = -A^{-1}$. A closed geodesic $\alpha(t, A)$ is said to be *symmetric* if it is invariant under the involution j on $\mathcal{M}$.

Let

$$E_c = \{r(p_1, \ldots, p_n) \in \mathbb{R}^n \mid r \geq 0,\ p_i \text{ integers}\},$$
$$E_s = \{r(p_1, \ldots, p_n) \in E_c \mid r > 0,\ p_i \text{ odd integers}\}$$

(in this setting, we always assume that the integers p_i have no common divisors), and let $A = \sum_{k=1}^{n} \lambda_k C_k$ be the spectral decomposition of A. Then $\alpha(t, A)$ is a closed geodesic if and only if $(\lambda_1, \ldots, \lambda_n) \in E_c$. If $A \neq 0$ and $(\lambda_1, \ldots, \lambda_n) = r(p_1, \ldots, p_n) \in E_c$, then π/r is the period of $\alpha(t, A)$ ([5], Theorem 4.2). Also, $\alpha(t, A)$ is a symmetric geodesic if and only if $(\lambda_1, \ldots, \lambda_n) \in E_s$ ([5], Theorem 4.4).

Now let Ω be the symmetric cone of positive definite $n \times n$ symmetric real matrices. Then the tube domain $T_\Omega := V + i\Omega$ can be realized as a bounded symmetric domain $\mathcal{D}$ in the complex plane $V^{\mathbb{C}} := V + iV$ as follows: Let

$$D(p) = \{Z \in V^{\mathbb{C}} \mid Z + iI \in \mathrm{GL}(n, \mathbb{C})\},$$
$$D(c) = \{W \in V^{\mathbb{C}} \mid I - W \in \mathrm{GL}(n, \mathbb{C})\},$$

and define for all $Z \in D(p)$ and $W \in D(c)$,

$$p(Z) = (Z - iI)(Z + iI)^{-1}, \qquad c(W) = i(I + W)(I - W)^{-1}.$$

Then $p : D(p) \to D(c)$ is a holomorphic bijection from $D(p)$ onto $D(c)$, and $c : D(c) \to D(p)$, called the *Cayley transform*, is its inverse. The closure of T_Ω in $V^{\mathbb{C}}$ is contained in $D(p)$. The image $\mathcal{D} := p(T_\Omega)$ of p is known as a bounded symmetric domain which is the open unit ball with respect to the spectral norm. We define Σ_n as the set of all invertible elements Z in $V^{\mathbb{C}}$

such that $Z^{-1} = \overline{Z}$. It is known that Σ_n is the Shilov boundary of $\mathcal{D}$, which is a compact connected $\frac{n(n+1)}{2}$-dimensional manifold, and is exactly equal to $\overline{p(V)}$ (for details, see [2]).

Let $\mathbf{c} = \{C_k\}_{k=1}^n$ be a complete system of orthogonal projections and let $V(\mathbf{c})$ be the subspace of V generated by C_k's. Then for $A = \sum_{k=1}^n \lambda_k C_k$,

$$p(A) = \sum_{k=1}^{n} \frac{\lambda_k - i}{\lambda_k + i} C_k. \tag{2.1}$$

Since $(\lambda_k - i)/(\lambda_k + i) \in S^1$ (the unit circle in $\mathbb{C}$) for $k = 1, \ldots, n$, we conclude that $\overline{p(V(\mathbf{c}))}$ is diffeomorphic to the n-torus $T^n = S^1 \times \ldots \times S^1$.

For a geodesic curve $\alpha(t, A)$ on $\mathcal{M}$, we let $\widehat{\alpha}(t, A) := p(\alpha(t, A))$ be the corresponding geodesic on Σ_n. From (2.1), we have the following

Proposition 2.2. *Let $A = \sum_{k=1}^n \lambda_k C_k$ be the spectral decomposition of A. Then*

$$\widehat{\alpha}(t, A) = \sum_{k=1}^{n} e^{i(\pi + 2\lambda_k t)} C_k.$$

P r o o f. For $t > 0$ with $\alpha(t, A) \in V$,

$$p(\alpha(t, A)) = \sum_{k=1}^{n} \frac{\tan \lambda_k t - i}{\tan \lambda_k t + i} C_k$$

and

$$\frac{\tan \lambda_k t - i}{\tan \lambda_k t + i} = \sin^2 \lambda_k t \cos^2 \lambda_k t - 2i \sin \lambda_k t \cos \lambda_k t = e^{i(\pi + 2\lambda_k t)}. \ \blacksquare$$

The symmetry $\widehat{j} := p \circ j \circ c$ on Σ_n corresponding to the symmetry j on $\mathcal{M}$ is the symmetry about the origin, i.e., $\widehat{j}(Z) = -Z$. Let J be the involution on Σ_n defined by $J(Z) = -\overline{Z}$. Then since $\overline{Z} = Z^{-1}$ for any $Z \in \Sigma_n$, this involution is just $J(Z) = -Z^{-1}$ and $J = j$ on $V = \mathrm{Sym}(n, \mathbb{R})$. By Proposition 2.2, we have

Corollary 2.3. *If $\alpha(t, A)$ is a symmetric geodesic on $\mathcal{M}$ with $A = \sum_{k=1}^n r p_k C_k$ then*

$$\widehat{j}\widehat{\alpha}(t, A) = \widehat{\alpha}\left(t + \frac{\pi}{2r}, A\right),$$

$$J\widehat{\alpha}(t, A) = \begin{cases} \widehat{\alpha}\left(\dfrac{\pi}{2r} - t, A\right) & \text{if } 0 \le t \le \dfrac{\pi}{2r}, \\[2ex] \widehat{\alpha}\left(\dfrac{3}{2r}\pi - t, A\right) & \text{if } \dfrac{\pi}{2r} \le t \le \dfrac{\pi}{r}. \end{cases}$$

In particular, $\widehat{\alpha}(t, A)$ is invariant under both involutions $\widehat{j}$ and J on Σ_n.

Let $\mathrm{Fix}(J)$ denote the set of all fixed points of the involution J on Σ_n. The following lemma will be useful in what follows.

LEMMA 2.4. *Let* $P \in \mathrm{GL}(n,\mathbb{R})$ *be an orthogonal transformation and let* $A \in \mathrm{Sym}(n,\mathbb{R})$. *Then*:

(1) $\widehat{\alpha}(t, PAP^t) = P\widehat{\alpha}(t, A)P^t$.
(2) $P\mathrm{Fix}(J)P^t = \mathrm{Fix}(J)$.
(3) $\widehat{\alpha}(t, PAP^t) \cap \mathrm{Fix}(J) = P(\widehat{\alpha}(t, A) \cap \mathrm{Fix}(J))P^t$.

Proof. Let $A = \sum_{k=1}^n \lambda_k C_k$ be the spectral decomposition of A and let P be an orthogonal transformation. Then $\{PC_kP^t\}_{k=1}^n$ is a complete system of orthogonal projections. Hence (1) follows from Proposition 2.2, (2) follows from the fact that for any $Z \in \Sigma_n$, $J(PZP^t) = -(PZP^t)^{-1} = PJ(Z)P^t$, and (3) follows from (1) and (2). ■

3. Closed geodesics in the Shilov boundary Σ_2. From now on, we shall restrict our attention to the space $V = \mathrm{Sym}(2,\mathbb{R})$. Recall that the Shilov boundary Σ_2 of V is given by

$$\Sigma_2 = \left\{ Z \in \mathrm{Sym}(2,\mathbb{C}) \;\middle|\; Z = \begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix} \text{ is an invertible matrix with } \overline{Z} = Z^{-1} \right\}.$$

We identify $Z = \left(\begin{smallmatrix} z_1 & z_2 \\ z_2 & z_3 \end{smallmatrix}\right) \in \mathrm{Sym}(2,\mathbb{C})$ with $(z_1, z_2, z_3) \in \mathbb{C}^3$. Under this identification, the Shilov boundary Σ_2 of $\mathrm{Sym}(2,\mathbb{R})$ can be written as

$$\Sigma_2 = \{(z_1, z_2, z_3) \in \mathbb{C}^3 \mid\ z_1\overline{z}_2 + z_2\overline{z}_3 = 0,\ |z_1|^2 + |z_2|^2 = |z_2|^2 + |z_3|^2 = 1\}.$$

Let $A\ (\neq 0) \in \mathrm{Sym}(2,\mathbb{R})$ and let $\alpha(t, A)$ be the closed geodesic in $\mathcal{M}$ originating at the origin $\mathbf{0}$ with direction A. Let $A = \lambda_1 C_1 + \lambda_2 C_2$ be the spectral decomposition of A. Then, from Theorem 4.2 of [5], we know that $\lambda_1 = rp$ and $\lambda_2 = rq$ for some real $r > 0$ and coprime integers p and q. Set $E_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $E_2 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Note that $\mathbf{e} = \{E_1, E_2\}$ is a complete system of orthogonal projections. It is well known that the orthogonal group $\mathrm{SO}(2)$ acts transitively on the set of complete systems of orthogonal projections. Thus there exists a unique orthogonal matrix

$$P_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \in \mathrm{SO}(2) \qquad (\theta \in [0, \pi])$$

such that $C_i = P_\theta E_i P_\theta^t$ $(i = 1, 2)$, i.e., $A = P_\theta A_0 P_\theta^t$, where $A_0 := (rp)E_1 + (rq)E_2$. By Proposition 2.2 and Lemma 2.4, we obtain

$$\widehat{\alpha}(t, A) = \begin{pmatrix} \cos^2\theta\, e^{i(\pi+2rpt)} + \sin^2\theta\, e^{i(\pi+2rqt)} & \sin\theta\cos\theta\, (e^{i(\pi+2rqt)} - e^{i(\pi+2rpt)}) \\ \sin\theta\cos\theta\, (e^{i(\pi+2rqt)} - e^{i(\pi+2rpt)}) & \sin^2\theta\, e^{i(\pi+2rpt)} + \cos^2\theta\, e^{i(\pi+2rqt)} \end{pmatrix}.$$

DEFINITION 3.1. Let A $(\neq 0) \in \mathrm{Sym}(2,\mathbb{R})$ be a 2×2 symmetric matrix which has the spectral decomposition $A = (rp)C_1 + (rq)C_2$, where $r > 0$ is a real number and p and q are coprime integers, and let $\alpha(t,A)$ be the closed geodesic in the conformal compactification $\mathcal{M}$ of $\mathrm{Sym}(2,\mathbb{R})$ originating at $\mathbf{0}$ with direction A.

(1) $\kappa(A_0)$ is the knot in Σ_2 defined by $\kappa(A_0) = \{\widehat{\alpha}(t,A_0) \in \Sigma_2 \mid 0 \le t \le \pi/r\}$, i.e., $\kappa(A_0) = \{(e^{i(\pi+2ps)}, 0, e^{i(\pi+2qs)}) \in \Sigma_2 \mid 0 \le s \le \pi\}$.

(2) $\kappa(A)$ is the knot in Σ_2 defined by $\kappa(A) = \{\widehat{\alpha}(t,A) \in \Sigma_2 \mid 0 \le t \le \pi/r\}$, i.e., $\kappa(A) = \{(z_1(s), z_2(s), z_3(s)) \in \Sigma_2 \mid 0 \le s \le \pi\}$, where

$$z_1(s) = -\cos^2\theta\, e^{i(2ps)} - \sin^2\theta\, e^{i(2qs)},$$
$$z_2(s) = \sin\theta\cos\theta\,(e^{i(2ps)} - e^{i(2qs)}),$$
$$z_3(s) = -\sin^2\theta\, e^{i(2ps)} - \cos^2\theta\, e^{i(2qs)},$$

and $\theta \in (0,\pi)$ satisfies $C_i = P_\theta E_i P_\theta^t$ $(i = 1,2)$.

PROPOSITION 3.2. *Let A $(\neq 0) \in \mathrm{Sym}(2,\mathbb{R})$. Then the knots $\kappa(A_0)$ and $\kappa(A)$ in Σ_2 are equivalent.*

Proof. Let $A = \lambda_1 C_1 + \lambda_2 C_2$ be the spectral decomposition of A and let $\theta \in [0,\pi]$ be such that $C_i = P_\theta E_i P_\theta^t$ $(i = 1,2)$. If $\theta = 0$, then the assertion is obvious. Suppose that $\theta \neq 0$. For $s \in [0,\theta]$, let $h_s : \Sigma_2 \to \Sigma_2$ be a homeomorphism of Σ_2 defined by $h_s(Z) = P_s Z P_s^t$ for $Z \in \Sigma_2$, with Σ_2 viewed as a subspace of $\mathrm{Sym}(2,\mathbb{C})$. Then it is clear that h_0 is the identity on Σ_2 and, by Lemma 2.4, $h_\theta(\kappa(A_0)) = \kappa(A)$. Furthermore, the map $H : \Sigma_2 \times [0,\theta] \to \Sigma_2 \times [0,\theta]$ defined by $H(Z,s) = (h_s(Z), s)$ is obviously an ambient isotopy between $\kappa(A_0)$ and $\kappa(A)$. This completes the proof. ■

THEOREM 3.3. *Let $A \in \mathrm{Sym}(2,\mathbb{R})$ be a 2×2 symmetric matrix such that $\alpha(t,A)$ is a symmetric geodesic in $\mathcal{M}$. Then the corresponding knot $\kappa(A)$ in Σ_2 has period 2 of type both $(\emptyset,\emptyset)$ and $(S^1 \cup S^0, S^0)$.*

Proof. Let $\widehat{\alpha}(t,A)$ be the symmetric geodesic corresponding to $\alpha(t,A)$. By Corollary 2.3, it is invariant under both $\widehat{j}$ and J. Since $\widehat{j}$ has no fixed points, it is obvious that $\kappa(A)$ has period 2 of type $(\emptyset,\emptyset)$ in Σ_2. Observe that the set of fixed points of the involution J on Σ_2 is

$$\mathrm{Fix}(J) = \{(i\cos\theta, i\sin\theta, -i\cos\theta) \mid \theta \in \mathbb{R}\} \cup \{\pm(i,0,i)\} \cong S^1 \cup S^0,$$

where S^k $(k = 0,1)$ denotes the k-sphere.

To prove that $\kappa(A)$ is of type $(S^1 \cup S^0, S^0)$ it suffices from Lemma 2.4 and Proposition 3.2 to show that $\kappa(A_0)$ meets $\mathrm{Fix}(J)$ in exactly two points. Let $A = \lambda_1 C_1 + \lambda_2 C_2$ be the spectral decomposition. Then $A_0 = \lambda_1 E_1 + \lambda_2 E_2$ and, by Theorem 4.4 of [5] and time reparametrization, we may assume that λ_1 and λ_2 are relatively prime odd integers. Then the possible points in $\kappa(A_0) \cap \mathrm{Fix}(J)$ are of the form $\pm(i,0,-i), \pm(i,0,i)$ since

$\kappa(A_0) = (e^{i(\pi+2\lambda_1 t)}, 0, e^{i(\pi+2\lambda_2 t)})$ $(0 \le t \le \pi)$. In several steps, we prove that $e^{i(\pi+2\lambda_1 t)} = e^{i(\pi+2\lambda_2 t)} = i$ for some $0 < t < \pi$ if and only if $(\lambda_1, \lambda_2) = (4m-1, 4n-1)$ or $(4m-3, 4n-3)$ for some $m, n \in \mathbb{Z}$.

Suppose that $e^{i(\pi+2\lambda_1 t)} = e^{i(\pi+2\lambda_2 t)} = i$ for some $0 < t < \pi$. Then

STEP 1. $t = \frac{4m-1}{4\lambda_1}\pi = \frac{4n-1}{4\lambda_2}\pi$ for some $m, n \in \mathbb{Z}$.

STEP 2. $4m-1 = \lambda_1 k, 4n-1 = \lambda_2 k$ for some $k \in \mathbb{Z}$ because λ_1 and λ_2 are relatively prime.

STEP 3. $k = 1$ or $k = 3$. Since $4m-1$ and λ_1 are odd integers, k must be an odd integer. Note that $t \in (0, \pi)$. By Step 1, k is 1 or 3.

If $k = 1$, then $\lambda_1 = 4m-1$ and $\lambda_2 = 4n-1$. If $k = 3$, then $\lambda_1 = (4m-1)/3$ and $\lambda_2 = (4n-1)/3$. In this case we may write $((4m-1)/3, (4n-1)/3)$ as $(4m'-3, 4n'-3)$.

The converse argument is easily followed by taking $t = \pi/4$ (respectively, $t = 3\pi/4$) for $\lambda_1 = (4m-1, 4n-1)$ (respectively, $\lambda_2 = (4m-3, 4n-3)$).

Similarly, we have $e^{i(\pi+2\lambda_1 t)} = i$ and $e^{i(\pi+2\lambda_2 t)} = -i$ for some $0 < t < \pi$ if and only if $(\lambda_1, \lambda_2) = (4m-1, 4n-3)$ or $(4m-3, 4n-1)$ for some $m, n \in \mathbb{Z}$.

Furthermore, note that if $e^{i(\pi+2\lambda_1 t)} = e^{i(\pi+2\lambda_2 t)} = i$ for some $0 < t < \pi$, then $e^{i(\pi+2\lambda_1 t')} = e^{i(\pi+2\lambda_2 t')} = -i$ for some $0 < t' < \pi$ (in this case, $t' = \pi/4$ or $t' = 3\pi/4$). Similarly, if $e^{i(\pi+2\lambda_1 t)} = i$ and $e^{i(\pi+2\lambda_2 t)} = -i$ for some $0 < t < \pi$, then $e^{i(\pi+2\lambda_1 t')} = -i, e^{i(\pi+2\lambda_2 t')} = i$ for some $0 < t' < \pi$.

Finally, by observing that the sets

$$\begin{aligned} X &:= \{(4m-1, 4n-1) \in E_s \mid m, n \in \mathbb{Z}\} \\ &\quad \cup \{(4m-3, 4n-3) \in E_s \mid m, n \in \mathbb{Z}\}, \\ Y &:= \{(4m-1, 4n-3) \in E_s \mid m, n \in \mathbb{Z}\} \\ &\quad \cup \{(4m-3, 4n-1) \in E_s \mid m, n \in \mathbb{Z}\} \end{aligned}$$

are disjoint and cover all pairs of coprime odd integers, we complete the proof. ∎

REMARK 3.4. Each symmetric geodesic $\widehat{\alpha}(t, A)$ meets $\mathrm{Fix}(J)$ at $t = \pi/(4r)$ and $t = 3\pi/(4r)$.

4. Covering links of the closed geodesics. Let $N = S^1 \times [0,1]$ be an annulus in $\mathbb{R}^3$ and let Φ be the map from $N \times S^1$ to Σ_2 defined by

$$\Phi(e^{i\phi}, r, e^{i\psi}) = (\sqrt{1-r^2}\,e^{i\phi}, re^{i\psi}, -\sqrt{1-r^2}\,e^{i(2\psi-\phi)})$$

for $(e^{i\phi}, r, e^{i\psi}) \in S^1 \times [0,1] \times S^1$. Note that $\Phi(e^{i\phi}, 0, e^{i\psi}) = (e^{i\phi}, 0, -e^{i(2\psi-\phi)})$ and $\Phi(e^{i\phi}, 1, e^{i\psi}) = (0, e^{i\psi}, 0)$. Now let $(z_1, z_2, z_3) \in \Sigma_2$ with $z_1 = re^{i\phi} \in \mathbb{C}$. Then $0 \le r \le 1$ and for some $\psi \in [0, 2\pi]$, we have the following.

(4-1) If $r = 0$, then $(z_1, z_2, z_3) = (0, e^{i\psi}, 0)$ and

$$\Phi^{-1}(z_1, z_2, z_3) = \{(e^{i\phi}, 1, e^{i\psi}) \mid 0 \le \phi \le 2\pi\}.$$

(4-2) If $r = 1$, then $(z_1, z_2, z_3) = (e^{i\phi}, 0, e^{i\psi})$ and

$$\Phi^{-1}(z_1, z_2, z_3) = (e^{i\phi}, 0, \pm e^{i(\pi+\phi+\psi)/2}).$$

(4-3) If $0 < r < 1$, then $(z_1, z_2, z_3) = (re^{i\phi}, \sqrt{1-r^2}e^{i\psi}, -re^{i(2\psi-\phi)})$ and

$$\Phi^{-1}(z_1, z_2, z_3) = \left(\frac{1}{|z_1|}z_1, \sqrt{1-|z_1|^2}, \frac{1}{\sqrt{1-|z_1|^2}}z_2\right) = (e^{i\phi}, \sqrt{1-r^2}, e^{i\psi}).$$

This shows that Σ_2 is an identification space of $N \times S^1 = S^1 \times [0,1] \times S^1$. In fact, this observation gives us the following

THEOREM 4.1. *The Shilov boundary Σ_2 of* $\mathrm{Sym}(2, \mathbb{R})$ *is homeomorphic to the non-orientable closed* 3*-manifold obtained from the solid torus $D^2 \times S^1$ by identifying (w, z) with $(w, -z)$ for each (w, z) in the boundary $\partial D^2 \times S^1$ of the solid torus.*

Now let $S^2 = \{(\sqrt{1-r^2}e^{i\phi}, r) \in \mathbb{C} \times \mathbb{R} \,|\, 0 \le \phi \le 2\pi,\ -1 \le r \le 1\}$ be the unit sphere in $\mathbb{R}^3$ and let $\widehat{N} = S^1 \times [-1, 1]$ be an annulus in $\mathbb{R}^3$. Let $f : \widehat{N} \to S^2$ be the map defined by $f(e^{i\phi}, r) = (\sqrt{1-r^2}e^{i\phi}, r)$ for $(e^{i\phi}, r) \in \widehat{N}$ and let $g : \widehat{N} \to N$ be defined by $g(e^{i\phi}, r) = (e^{i\phi}, |r|)$ for $(e^{i\phi}, r) \in \widehat{N}$. It is easy to see that S^2 is an identification space of $\widehat{N}$ and g is a 2-fold branched covering projection with branch set $S^1 \times \{0\} \subset N$. The preimage of this branch set by g is $S^1 \times \{0\} \subset \widehat{N}$. Let $\Psi : S^2 \times S^1 \to \Sigma_2$ be the map defined by $\Psi = \Phi \circ (g \times \mathrm{Id}_{S^1}) \circ (f \times \mathrm{Id}_{S^1})^{-1}$, where Id_{S^1} denotes the identity map of S^1:

$$\begin{array}{ccc} \widehat{N} \times S^1 & \xrightarrow{f \times \mathrm{Id}_{S^1}} & S^2 \times S^1 \\ \Big\downarrow {\scriptstyle g \times \mathrm{Id}_{S^1}} & & \Big\downarrow {\scriptstyle \Psi} \\ N \times S^1 & \xrightarrow{\Phi} & \Sigma_2 \end{array} \tag{4.1}$$

We observe that for $(\sqrt{1-r^2}e^{i\phi}, r, e^{i\psi}) \in S^2 \times S^1$,

$$\Psi(\sqrt{1-r^2}e^{i\phi}, r, e^{i\psi}) = (\sqrt{1-r^2}e^{i\phi}, |r|e^{i\psi}, -\sqrt{1-r^2}e^{i(2\psi-\phi)}).$$

Then it is not difficult to see that the map Ψ is a 2-fold covering projection and hence $S^2 \times S^1$ is an orientable double cover of the Shilov boundary Σ_2 of $\mathrm{Sym}(2, \mathbb{R})$.

Let $\widehat{\kappa}(A_0) := \Psi^{-1}(\kappa(A_0))$ and $\widehat{\kappa}(A) := \Psi^{-1}(\kappa(A))$. From (4-1), (4-2), (4-3), and (4.1) we obtain a certain class of knots and links in $S^2 \times S^1$ as follows.

DEFINITION 4.2. Let $A\ (\neq 0) \in \mathrm{Sym}(2, \mathbb{R})$ be a 2×2 symmetric matrix which has the spectral decomposition $A = (rp)C_1 + (rq)C_2$, where $r > 0$ is a real number and p and q are coprime integers.

(1) $\widehat{\kappa}(A_0) = \{(e^{i(\pi+2ps)}, 0, \pm e^{i(3\pi/2+(p+q)s)}) \in S^2 \times S^1 \mid 0 \le s \le \pi\}$.

(2) $\widehat{\kappa}(A) = \{a(s) \in S^2 \times S^1 \mid 0 \le |z_1(s)| < 1, 0 \le s \le \pi\} \cup$
$$\left\{\Psi^{-1}(z_1(s), 0, z_3(s)) \in S^2 \times S^1 \;\middle|\; s = \frac{k\pi}{p-q} \text{ for } k \in \mathbb{Z} \text{ with } 0 \le \frac{k}{p-q} \le 1\right\},$$
where
$$a(s) = \left(z_1(s), \pm\sqrt{1-|z_1(s)|^2}, \frac{1}{\sqrt{1-|z_1(s)|^2}} z_2(s)\right)$$
with
$$z_1(s) = -\cos^2\theta\, e^{i(2ps)} - \sin^2\theta\, e^{i(2qs)},$$
$$z_2(s) = \sin\theta\cos\theta\, (e^{i(2ps)} - e^{i(2qs)}),$$
$$z_3(s) = -\sin^2\theta\, e^{i(2ps)} - \cos^2\theta\, e^{i(2qs)},$$
and $\theta \in (0,\pi) - \{\pi/2\}$ satisfying $C_i = P_\theta E_i P_\theta^t$ $(i = 1,2)$.

THEOREM 4.3. *Let* $A \in \mathrm{Sym}(2,\mathbb{R})$ *be a* 2×2 *symmetric matrix which has the spectral decomposition* $A = (rp)C_1 + (rq)C_2$, *where* $r > 0$ *is a real number and* p, q *are coprime integers. Let* $\theta \in [0,\pi]$ *be such that* $A = P_\theta A_0 P_\theta^t$.

(1) *If both* p *and* q *are odd integers*, *or equivalently*, *the geodesic* $\alpha(t, A)$ *in* $\mathcal{M}$ *is symmetric*, *then* $\widehat{\kappa}(A_0)$ *is a link of* 2-*components in* $S^2 \times S^1$ *which has period* 2 *of type* $(S^1 \times S^1, T(|p|, |p+q|/2) \,\dot\cup\, T(|p|, |p+q|/2))$.

(2) *If one of* p *and* q *is an even integer*, *or equivalently*, *the geodesic* $\alpha(t, A)$ *in* $\mathcal{M}$ *is not symmetric*, *then* $\widehat{\kappa}(A_0)$ *is a knot in* $S^2 \times S^1$ *which has period* 2 *of type* $(S^1 \times S^1, T(|p|, |p+q|))$.

(3) *If* $\theta \ne 0, \pi/2, \pi$, *then* $\widehat{\kappa}(A)$ *is a link in* $S^2 \times S^1$ *which has period* 2 *of type* $(S^1 \times S^1, 2|p-q|$ *points*).

Proof. Let $h : S^2 \times S^1 \to S^2 \times S^1$ be an involution of $S^2 \times S^1$ defined by $h(\sqrt{1-r^2}e^{i\phi}, r, e^{i\psi}) = (\sqrt{1-r^2}e^{i\phi}, -r, e^{i\psi})$. Then $\mathrm{Fix}(h) = \{(e^{i\phi}, 0, e^{i\psi}) \in S^2 \times S^1 \mid 0 \le \phi, \psi \le 2\pi\} \cong S^1 \times S^1$. Now recall that
$$\widehat{\kappa}(A_0) = \{(e^{i(\pi+2ps)}, 0, e^{i(\pi/2+(p+q)s)}) \cup (e^{i(\pi+2ps)}, 0, e^{i(\pi+\pi/2+(p+q)s)}) \mid 0 \le s \le \pi\}.$$
It is clear that $h(\widehat{\kappa}(A_0)) = \widehat{\kappa}(A_0)$ and $\mathrm{Fix}(h) \cap \widehat{\kappa}(A_0) = \widehat{\kappa}(A_0)$.

(1) If both p and q are odd integers, then $p+q$ must be an even integer. Hence
$$\widehat{\kappa}(A_0) = \{(e^{i(\pi+p)t}, 0, e^{i(\pi/2+(p+q)t/2)}) \,\dot\cup\, (e^{i(\pi+p)t}, 0, e^{i(\pi+\pi/2+(p+q)t/2)}) \mid 0 \le t \le 2\pi\}.$$
This implies that $\widehat{\kappa}(A_0)$ is the disjoint union of two torus knots of type $(|p|, |p+q|/2)$.

(2) If one of p and q is an even integer, then $p+q$ must be an odd integer because $(p,q)=1$. So

$$\begin{aligned}\widehat{\kappa}(A_0) &= \{(e^{i(\pi+2ps)}, 0, e^{i(\pi/2+(p+q)s)}) \\ &\qquad \cup (e^{i(\pi+2ps)}, 0, e^{i(\pi+\pi/2+(p+q)s)}) \mid 0 \le s \le \pi\} \\ &= \{(e^{i(\pi+p)t}, 0, e^{i(\pi+\pi/2+(p+q)t)}) \mid 0 \le t \le 2\pi\}.\end{aligned}$$

Hence $\widehat{\kappa}(A_0)$ is the torus knot of type $(|p|, |p+q|)$.

(3) Obviously, $h(\widehat{\kappa}(A)) = \widehat{\kappa}(A)$. Let $a(s) := (z_1(s), z_2(s), z_3(s)) \in \widehat{\kappa}(A)$. Then

$$\begin{aligned}a(s) \in \mathrm{Fix}(h) \cap \widehat{\kappa}(A) &\Leftrightarrow |z_1(s)| = 1 \\ &\Leftrightarrow s = \frac{k\pi}{p-q} \text{ for } k \in \mathbb{Z} \text{ with } 0 \le \frac{k}{p-q} \le 1.\end{aligned}$$

Hence the set $\mathrm{Fix}(h) \cap \widehat{\kappa}(A)$ consists of $2|p-q|$ points. This completes the proof. ∎

THEOREM 4.4. *Let $A \in \mathrm{Sym}(2,\mathbb{R})$ be a 2×2 symmetric matrix which has the spectral decomposition $A = (rp)C_1 + (rq)C_2$, where $r > 0$ is a real number and both p and q are coprime odd integers. Then $\widehat{\kappa}(A_0)$ is a link in $S^2 \times S^1$ which has period 2 of type $(S^1 \dot\cup S^1, \emptyset)$.*

Proof. Let $h : S^2 \times S^1 \to S^2 \times S^1$ be an involution of $S^2 \times S^1$ defined by $h(\sqrt{1-r^2}e^{i\phi}, r, e^{i\psi}) = (-\sqrt{1-r^2}e^{i\phi}, r, e^{i\psi})$ for any $(\sqrt{1-r^2}e^{i\phi}, r, e^{i\psi}) \in S^2 \times S^1$. Then $\mathrm{Fix}(h) = \{(0,1,e^{i\psi}), (0,-1,e^{i\psi}) \in S^2 \times S^1 \mid 0 \le \psi \le 2\pi\} \cong S^1 \dot\cup S^1$ and $\mathrm{Fix}(h) \cap \widehat{\kappa}(A_0) = \emptyset$. Now let $b(s) := (e^{i(\pi+2ps)}, 0, e^{i(\pi+2qs)}) \in \widehat{\kappa}(A_0)$. It is easy to check that

$$h(b(s)) = \begin{cases} b(\pi/2 + s) & \text{for } 0 \le s \le \pi/2, \\ b(s - \pi/2) & \text{for } \pi \le s \le \pi. \end{cases}$$

This completes the proof. ∎

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