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Towers of measurable functions

by

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Abstract. We formulate variants of the cardinals $\mathfrak{f}$, $\mathfrak{p}$ and $\mathfrak{t}$ in terms of families of measurable functions, in order to examine the effect upon these cardinals of adding one random real.

1. Introduction. Let $\mathbb{N}$ be the set of all nonnegative integers, and let $\mathcal{P}(\mathbb{N})$ denote the power-set of $\mathbb{N}$. We give $\mathcal{P}(\mathbb{N})$ a topology by identifying it with the Cantor set $2^{\mathbb{N}}$ endowed with the product topology. Define a relation "almost set inclusion" on $\mathcal{P}(\mathbb{N})$ by

$$A \subseteq^* B \quad \text{iff} \quad A \setminus B \text{ is finite},$$

where $A, B \subseteq \mathbb{N}$. And $A \supseteq^* B$ iff $B \subseteq^* A$. For a family $\mathcal{X} \subseteq \mathcal{P}(\mathbb{N})$, $A \subseteq \mathbb{N}$ is a *pseudo-intersection* of $\mathcal{X}$ iff $A \subseteq^* X$ for every $X \in \mathcal{X}$.

A family of (infinite) subsets of $\mathbb{N}$ is called a *filter base* iff every nonempty finite subfamily has an infinite intersection. Let $\mathfrak{p}$ be the smallest cardinality of a filter base which has no infinite pseudo-intersection. Let $\mathfrak{f}$ be the smallest cardinality of a filter base which is contained in an F_σ filter, and which has no infinite pseudo-intersection. (We always assume that a filter is *proper*, i.e. it contains no finite sets.) Let $\mathbb{N}^{[\infty]}$ denote the set of all infinite subsets of $\mathbb{N}$. A *tower* is a subfamily of $\mathbb{N}^{[\infty]}$ that is well-ordered by $\supseteq^*$. Let $\mathfrak{t}$ be the smallest cardinality of a tower with no infinite pseudo-intersection.

Let $\mathbb{N}^{\mathbb{N}}$ be the set of all functions from $\mathbb{N}$ into $\mathbb{N}$. We give $\mathbb{N}^{\mathbb{N}}$ the product topology. Define the relation $\le^*$ on $\mathbb{N}^{\mathbb{N}}$ by

$$f \le^* g \quad \text{iff} \quad \forall^\infty n \in \mathbb{N}\ f(n) \le g(n),$$

where $f, g \in \mathbb{N}^{\mathbb{N}}$. Let $\mathfrak{b}$ be the smallest cardinality of a subfamily of $\mathbb{N}^{\mathbb{N}}$ that is unbounded in $(\mathbb{N}^{\mathbb{N}}, \le^*)$. The reader can find an introduction to the cardinals $\mathfrak{p}$, $\mathfrak{t}$ and $\mathfrak{b}$ and other *small cardinals* in [vD84] and in [Bla99]. For further discussion on the cardinal $\mathfrak{f}$ see [Laf97].

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Let $\mathbb{N}^{[<\infty]}$ denote the set of all finite subsets of $\mathbb{N}$. We give $(\mathbb{N}^{[<\infty]})^{\mathbb{N}}$ the product topology where $\mathbb{N}^{[<\infty]}$ has the discrete topology.

Let I denote the unit interval $[0,1]$, and let μ denote the Lebesgue measure on I.

NOTATION. For any formula $\varphi(v_1,\dots,v_n)$ of the language of set theory with all free variables displayed, and for any functions $f_1,\dots,f_n$ from I into either $\mathcal{P}(\mathbb{N})$, $\mathbb{N}^{\mathbb{N}}$ or $(\mathbb{N}^{[<\infty]})^{\mathbb{N}}$, we define

$$\|\varphi(\dot{f}_1,\dots,\dot{f}_n)\| = \{x \in I : \varphi(f_1(x),\dots,f_n(x))\}.$$

For example, given $k \in \mathbb{N}$ and $f : I \to \mathcal{P}(\mathbb{N})$,

$$\|k \in \dot{f}\| = \{x \in I : k \in f(x)\}.$$

Any set A which is either in $\mathcal{P}(\mathbb{N})$, $\mathbb{N}^{\mathbb{N}}$ or $(\mathbb{N}^{[<\infty]})^{\mathbb{N}}$ may be viewed as a constant function on I. In this case, we suppress the dot to emphasize that A is a constant. For example, given $A \subseteq \mathbb{N}$ and $f : I \to \mathcal{P}(\mathbb{N})$,

$$\|A \subseteq \dot{f}\| = \{x \in I : A \subseteq f(x)\}.$$

If f is a (Lebesgue) measurable function on I, and if $\varphi(v)$ is a sufficiently "simple" formula, then $\|\varphi(\dot{f})\|$ will be a (Lebesgue) measurable set. For example, if $\varphi(v)$ is a *Borel notion*, i.e. if $\{x \in I : \varphi(x)\}$ is a Borel set, then $\|\varphi(\dot{f})\|$ is measurable for every measurable $f : I \to \mathcal{P}(\mathbb{N})$.

For any $F, G \subseteq I$, we write

(a) $F = 0$ if F is null,
(b) $F = 1$ if F has measure one,

and we let

(c) $F + G = F \cup G$,
(d) $F \cdot G = F \cap G$,
(e) $-F = I \setminus F$,
(f) $F \triangle G = (F \cdot (-G)) + ((-F) \cdot G)$.

Also, we abbreviate "$F \cdot (-G)$" by "$F - G$". We write

(g) $F \le G$ if $F - G = 0$,

and $F \equiv G$ if $F \le G$ and $G \le F$. The order of precedence is $\cdot$, $-$, $+$.

The reader familiar with Boolean algebras will recognize that we are using the notation for Boolean operations to represent unions, intersections and complements. This is justified in the context of forcing, because the poset for adding one random real is the Boolean algebra $\mathcal{R}$ defined by

$$\mathcal{R} = \{F \subseteq I : F \text{ is Lebesgue measurable}\}/\mathcal{N},$$

where $\mathcal{N}$ is the ideal of Lebesgue null subsets of I. We refer to $\mathcal{R}$ as the *random algebra*.

Therefore, while the proofs in Sections 4 and 5 make use of the above notation, none of these proofs involves forcing. However, the reader may, if he or she so wishes, correctly interpret these proofs by viewing all statements with the $\|\varphi(\dot{f})\|$ notation as statements in the forcing language of $\mathcal{R}$.

Now we define the corresponding notions in the realm of functions from I into $\mathcal{P}(\mathbb{N})$: We define a relation on the set of all functions from I to $\mathcal{P}(\mathbb{N})$ by

$$f \subseteq^* g \quad \text{iff} \quad \|\dot{f} \subseteq^* \dot{g}\| = 1,$$

where $f, g : I \to \mathcal{P}(\mathbb{N})$. For a family $\mathcal{F}$ of functions from I to $\mathcal{P}(\mathbb{N})$, $f : I \to \mathcal{P}(\mathbb{N})$ is a *pseudo-intersection* of $\mathcal{F}$ if $f \subseteq^* g$ for every $g \in \mathcal{F}$.

Definition 1.1. A function $f : I \to \mathcal{P}(\mathbb{N})$ is called *infinitary* if

$$\|\dot{f} \text{ is infinite}\| = 1.$$

Definition 1.2. A family $\mathcal{F}$ of functions from I into $\mathcal{P}(\mathbb{N})$ is *filtered* if for every nonempty finite subfamily $\mathcal{A} \subseteq \mathcal{F}$,

$$\bigcap_{f \in \mathcal{A}} f \text{ is infinitary}$$

(i.e. $\mu(\{x \in I : \bigcap_{f \in \mathcal{A}} f(x) \text{ is infinite}\}) = 1$).

Definition 1.3. A family $\mathcal{T}$ of functions from I into $\mathcal{P}(\mathbb{N})$ is a *tower* if

(i) f is infinitary for all $f \in \mathcal{T}$,
(ii) $\mathcal{T}$ is well-ordered by $\supseteq^*$.

Definition 1.4. Let $\mathfrak{p}_\mu$ be the smallest cardinality of a filtered family of measurable functions from I into $\mathcal{P}(\mathbb{N})$ which has no measurable infinitary pseudo-intersection from I into $\mathcal{P}(\mathbb{N})$.

Definition 1.5. Let $\mathfrak{t}_\mu$ be the smallest cardinality of a tower of measurable functions from I into $\mathcal{P}(\mathbb{N})$ which has no measurable infinitary pseudo-intersection from I into $\mathcal{P}(\mathbb{N})$.

We define a relation on the set of all functions from I into $\mathbb{N}^{\mathbb{N}}$ by

$$f \le^* g \quad \text{iff} \quad \|\dot{f} \le^* \dot{g}\| = 1.$$

A family $\mathcal{F}$ of functions from I into $\mathbb{N}^{\mathbb{N}}$ is called *bounded* if there is a $b \in \mathbb{N}^{\mathbb{N}}$ such that $\|\dot{f} \le^* b\| = 1$ for all $f \in \mathcal{F}$; such a function is called a *bound* for $\mathcal{F}$.

Definition 1.6. A function $S : I \to (\mathbb{N}^{[<\infty]})^{\mathbb{N}}$ is called a *slalom* if

$$\|\forall n \in \mathbb{N}\ |\dot{S}(n)| \le n\| = 1.$$

Definition 1.7. For functions $S : I \to (\mathbb{N}^{[<\infty]})^{\mathbb{N}}$, $X : I \to \mathcal{P}(\mathbb{N})$ and $f : I \to \mathbb{N}^{\mathbb{N}}$, we say that (S, X) *captures* f if

$$\|\forall^\infty n \in \dot{X}\ \dot{f}(n) \in \dot{S}(n)\| = 1.$$

Definition 1.8. Let $\mathfrak{f}_\mu$ be the smallest cardinality of a bounded family $\mathcal{F}$ of measurable functions from I into $\mathbb{N}^{\mathbb{N}}$ such that for every measurable slalom $S : I \to (\mathbb{N}^{[<\infty]})^{\mathbb{N}}$ and every measurable infinitary $X : I \to \mathcal{P}(\mathbb{N})$, (S, X) does not capture every $f \in \mathcal{F}$.

Some of the results of this paper are as follows:

Theorem 1.9. $\mathfrak{p}_\mu \geq \mathfrak{p}$.

Theorem 1.10. $\mathfrak{t}_\mu \geq \mathfrak{t}$.

Theorem 1.11. $\mathfrak{f}_\mu \geq \min\{\mathfrak{b}, \mathfrak{f}\}$.

Theorem 1.12. $\mathfrak{f}_\mu \leq \mathfrak{b}$.

The cardinal $\mathfrak{f}$ is less known than $\mathfrak{p}$ and $\mathfrak{t}$. We describe its relation to $\mathfrak{t}$ here.

Theorem 1.13 (Laflamme). $\mathfrak{t} \leq \mathfrak{f}$.

Proof. See [Laf97]. ∎

We also remark that due to a result of S. Shelah (see [BS96]) both of the inequalities $\mathfrak{b} < \mathfrak{f}$ and $\mathfrak{b} > \mathfrak{f}$ are consistent (see also [Laf97]).

The effect upon the cardinals $\mathfrak{p}$ and $\mathfrak{t}$—two of the most important *cardinal characteristics* of the continuum—of adding one random real was previously unknown. The same was true of the cardinal $\mathfrak{f}$. For example, it was unknown whether $\mathfrak{p}$ was preserved under the addition of a random real, i.e. whether $\kappa < \mathfrak{p}$ implies $\mathcal{R} \Vdash \check{\kappa} < \dot{\mathfrak{p}}$. In Section 2 we will see that $\mathfrak{p}_\mu$, $\mathfrak{t}_\mu$ and $\mathfrak{f}_\mu$ are precisely the values of $\mathfrak{p}$, $\mathfrak{t}$ and $\mathfrak{f}$, respectively, in the forcing extension via one random real. Thus Theorems 1.9 and 1.10 give a positive answer to the preservation of $\mathfrak{p}$ and $\mathfrak{t}$ under the addition of a single random real. And Theorems 1.11 and 1.12 give both lower and upper bounds for the random real value of the cardinal $\mathfrak{f}$.

Let us remark on a related Theorem of Kunen:

Theorem 1.14 (Kunen). MA_κ(*σ-linked*) *implies* $\mathcal{R} \Vdash \mathrm{MA}_{\check{\kappa}}$(*σ-linked*).

Proof. See [Roi79] and [Roi88]. ∎

Recall that MA_κ(σ-linked) implies $\kappa < \mathfrak{p}$ (see e.g. [Bel81]). Thus Kunen's Theorem implies that it is consistent that $\kappa < \mathfrak{p}$ in the extension by one random real for any κ beneath the continuum. However, one should note that Kunen's Theorem does not imply the random real preservation of $\mathfrak{p}$. Moreover, by Bell's Theorem [Bel81], an equivalent reformulation of Theorem 1.9 is:

Theorem 1.15. MA_κ(*σ-centered*) *implies* $\mathcal{R} \Vdash \mathrm{MA}_{\check{\kappa}}$(*σ-centered*).

Section 3 is an extension of Section 2 where we observe that the objects in $V^{\mathcal{R}}$ under our consideration can be named with continuous functions. In Section 4 we prove Theorems 1.9–1.12. The proofs of Theorems 1.9 and 1.10 amount to taking a filtered family [tower] of measurable functions which is maximal in the sense that it has no measurable infinitary pseudo-intersection, and transforming it into a filter base [tower] of sets of integers while preserving its maximality. The proof of Theorem 1.11 is similar. In Section 5 we discuss the inequalities not covered by Theorems 1.9–1.12.

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2. Adding one random real

NOTATION. Fix $\mathcal{R}$-names $\dot{\mathfrak{b}}$, $\dot{\mathfrak{f}}$, $\dot{\mathfrak{p}}$ and $\dot{\mathfrak{t}}$ which are forced to be the values of $\mathfrak{b}$, $\mathfrak{f}$, $\mathfrak{p}$ and $\mathfrak{t}$, respectively. Let $\dot{r}$ be the canonical $\mathcal{R}$-name for the random real.

There is a canonical correspondence between names for reals in the random extension and measurable functions from I into $\mathcal{P}(\mathbb{N})$. We describe this by $\dot{x} \mapsto f_{\dot{x}}$, where $\mathcal{R} \Vdash f_{\dot{x}}(\dot{r}) = \dot{x}$ (see [Sco67] and [Abr80]). Note that for any formula $\varphi(x)$ and any Borel function $f : I \to \mathcal{P}(\mathbb{N})$, when we write $\mathcal{R} \Vdash \varphi(f(\dot{r}))$, we are implicitly identifying f with a name for the decoding of f in the forcing extension; moreover, since for every measurable $f : I \to \mathcal{P}(\mathbb{N})$, there is a Borel function $g : I \to \mathcal{P}(\mathbb{N})$ which agrees with f almost everywhere (i.e. $\|\dot{g} = \dot{f}\| = 1$), we see that $\mathcal{R} \Vdash \varphi(f(\dot{r}))$ makes sense whenever f is measurable. In the other direction we have $f \mapsto \dot{x}_f$, where $\mathcal{R} \Vdash f(\dot{r}) = \dot{x}_f \subseteq \check{\mathbb{N}}$.

It follows from the absoluteness of Borel notions that given any two names $\dot{x}$ and $\dot{y}$ for reals,

$$\mathcal{R} \Vdash \dot{x} \subseteq^* \dot{y} \quad \text{iff} \quad f_{\dot{x}} \subseteq^* f_{\dot{y}}. \tag{1}$$

Also, for any finite sequence $\dot{x}_1, \ldots, \dot{x}_n$ of names for reals,

$$\mathcal{R} \Vdash \bigcap_{k=1}^{n} \dot{x}_k \text{ is infinite} \quad \text{iff} \quad \bigcap_{k=1}^{n} f_{\dot{x}_k} \text{ is infinitary.} \tag{2}$$

The following two propositions are now immediate:

PROPOSITION 2.1. $\mathcal{R} \Vdash \dot{\mathfrak{p}} = \check{\mathfrak{p}}_\mu$.

PROPOSITION 2.2. $\mathcal{R} \Vdash \dot{\mathfrak{t}} = \check{\mathfrak{t}}_\mu$.

These two propositions in turn relate Theorems 1.9 and 1.10 to the effect on the cardinals $\mathfrak{p}$ and $\mathfrak{t}$, respectively, of adding one random real:

COROLLARY 2.3. $\mathcal{R} \Vdash \dot{\mathfrak{p}} \geq \check{\mathfrak{p}}$.

Corollary 2.4. $\mathcal{R} \Vdash \dot{\mathfrak{i}} \geq \check{\mathfrak{i}}$.

The relationship between $\mathfrak{f}$ and $\mathfrak{f}_\mu$ becomes a little clearer by considering a reformulation of $\mathfrak{f}$.

Definition 2.5. For $b : \mathbb{N} \to \mathbb{N}$, define

$$\mathbb{R}(b) = \prod_{n=0}^{\infty} b(n).$$

We give $\mathbb{R}(b)$ the product topology, and we endow $\mathbb{R}(b)$ with the product measure $\nu = \prod_{n=0}^{\infty} \nu_n$, where $\nu_n(\{l\}) = 1/b(n)$ for all $l < b(n)$.

Definition 2.6. Set

$$\mathfrak{f}_1 = \min\{|\mathcal{H}| : \exists b \in \mathbb{N}^{\mathbb{N}}\ \mathcal{H} \subseteq \mathbb{R}(b), \exists g \in \mathbb{N}^{\mathbb{N}}\ \lim_{n\to\infty} g(n) = \infty,$$
$$\forall X \in \mathbb{N}^{[\infty]}\ \forall S_n \in b(n)^{[\leq g(n)]}\ \exists h \in \mathcal{H}\ \exists^\infty n \in X\ h(n) \notin S_n\}.$$

Theorem 2.7 (Laflamme). $\mathfrak{f} = \mathfrak{f}_1$.

Proof. See [Laf97]. ∎

We consider a slightly simpler formulation:

Definition 2.8. Define

$$\mathfrak{f}_2 = \min\{|\mathcal{H}| : \exists b \in \mathbb{N}^{\mathbb{N}}\ \mathcal{H} \subseteq \mathbb{R}(b),$$
$$\forall X \in \mathbb{N}^{[\infty]}\ \forall S_n \in b(n)^{[\leq n]}\ \exists h \in \mathcal{H}\ \exists^\infty n \in X\ h(n) \notin S_n\}.$$

Essentially the same argument as in Laflamme's proof that $\mathfrak{f}_1 \leq \mathfrak{f}$ will show that $\mathfrak{f}_2$ is indeed a reformulation of $\mathfrak{f}$. But first we need a combinatorial description of F_σ filters:

Lemma 2.9 (Laflamme). *Let $\mathcal{F}$ be an F_σ filter and $g \in \mathbb{N}^{\mathbb{N}}$. Then there is an increasing sequence of integers $\langle k_n : n \in \mathbb{N}\rangle$ and sets $a_i^n \subseteq [k_n, k_{n+1})$ $[i < m_n]$ such that*

$$\forall n \in \mathbb{N}\ \forall s \in m_n^{[\leq g(n)]}\ \bigcap_{i\in s} a_i^n \neq \emptyset, \tag{1}$$

$$\forall X \in \mathcal{F}\ \forall^\infty n \in \mathbb{N}\ \exists i < m_n\ a_i^n \subseteq X. \tag{2}$$

Conversely, given $\langle k_n\rangle$, $\langle a_i^n : i < m_n\rangle$ and g satisfying (1) *and* (2) *and with* $\lim_{n\to\infty} g(n) = \infty$, *the set*

$$\{X \subseteq \mathbb{N} : \forall n \in \mathbb{N}\ \exists i < m_n\ a_i^n \subseteq X\}$$

generates an F_σ filter.

Proof. See [Laf97]. ∎

Proposition 2.10. $\mathfrak{f} = \mathfrak{f}_2$.

Proof. Trivially, $\mathfrak{f}_2 \geq \mathfrak{f}_1$. Hence, by Theorem 2.7, we need only show that $\mathfrak{f}_2 \leq \mathfrak{f}$. We take $\kappa < \mathfrak{f}_2$ and prove that $\kappa < \mathfrak{f}$. Suppose that $\langle X_\xi : \xi < \kappa\rangle$

is an enumeration of a filter base which is included in an F_σ filter, say $\mathcal{F}$. By Lemma 2.9, there exist $\{k_n\}$ and $a_i^n \subseteq [k_n, k_{n+1})$ $[i < m_n]$ such that

$$\forall s \in m_n^{[\le n]} \ \bigcap_{i\in s} a_i^n \ne \emptyset, \tag{3}$$

$$\forall X \in \mathcal{F}\ \forall^\infty n\ \exists i < m_n\ a_i^n \subseteq X. \tag{4}$$

Let $b \in \mathbb{N}^{\mathbb{N}}$ be given by $b(n) = m_n$ for all n. Then we can choose, for each $\xi < \kappa$, $h_\xi : \mathbb{N} \to \mathbb{R}(b)$ so that

$$\forall^\infty n\ a^n_{h_\xi(n)} \subseteq X_\xi. \tag{5}$$

Since $\kappa < \mathfrak{f}_2$, there exist $X \in \mathbb{N}^{[\infty]}$ and $S_n \in m_n^{[\le n]}$ such that

$$\forall \xi < \kappa\ \forall^\infty n \in X\ h_\xi \in S_n. \tag{6}$$

Therefore, $\bigcup_{n\in X} \bigcap_{i\in S_n} a_i^n$ is an infinite pseudo-intersection of $\langle X_\xi : \xi < \kappa\rangle$. ■

PROPOSITION 2.11. *Every measurable $f : I \to \mathbb{N}^{\mathbb{N}}$ is bounded* (*i.e. the family $\{f\}$ is bounded*).

Proof. The proposition is equivalent to the fact that $\mathcal{R}$ is $\mathbb{N}^{\mathbb{N}}$-bounding (see [BJ95]). ■

Using Propositions 2.10 and 2.11 and other considerations similar to (and including) (1) and (2), we see that:

PROPOSITION 2.12. $\mathcal{R} \Vdash \dot{\mathfrak{f}} = \check{\mathfrak{f}}_\mu$.

And Theorems 1.11 and 1.12 say the following about the effect on $\mathfrak{f}$ of adding a random real:

COROLLARY 2.13. $\mathcal{R} \Vdash \dot{\mathfrak{f}} \ge \min\{\check{\mathfrak{b}}, \check{\mathfrak{f}}\}$ *and* $\mathcal{R} \Vdash \dot{\mathfrak{f}} \le \check{\mathfrak{b}}$.

3. Continuous names

LEMMA 3.1. *For every measurable $F \subseteq 2^{\mathbb{N}}$ and every $\varepsilon > 0$, there is a clopen set B such that $\mu(B \triangle F) < \varepsilon$.*

Proof. See [Roy88]. ■

This means that the metric space $(\mathcal{R}, d)$, where

$$d(F, G) = \mu(F \triangle G), \tag{7}$$

is separable. We remark that the absence of a countable dense set in the metric space corresponding to a nonseparable measure algebra is the precise reason that none of the proofs of Theorems 1.9–1.11 will work for the addition of many random reals.

Observe that by considering the base 2 expansion of members of $2^{\mathbb{N}}$ after the decimal point, after removing a countable set from $2^{\mathbb{N}}$ we can identify it with I. Moreover, under this identification the standard product measure on

$2^{\mathbb{N}}$ agrees with the Lebesgue measure on I. Therefore, if we replace I with $2^{\mathbb{N}}$ in any of our definitions, then the corresponding results will hold when I is replaced with $2^{\mathbb{N}}$. For example, for any of the cardinals $\mathfrak{f}_\mu$, $\mathfrak{p}_\mu$ and $\mathfrak{t}_\mu$, if every instance of I in their definition is replaced with $2^{\mathbb{N}}$, then we get the same cardinal. Also notice that Lemma 3.1 says that *for every measurable* $F \subseteq I$ *and every* ε, *there is a finite union of rational intervals* B *such that* $\mu(B \bigtriangleup F) < \varepsilon$.

Notice that $\mathfrak{p}$ and $\mathfrak{t}$ are cardinal characteristics of the object $\mathcal{P}(\mathbb{N})/\mathrm{fin}$. For example, $\mathfrak{t}$ is the smallest order type of a maximal well-ordered subset of $(\mathcal{P}(\mathbb{N})/\mathrm{fin}, \supseteq^*)$. Moreover, by considering the reformulation $\mathfrak{f}_2$, $\mathfrak{f}$ is a cardinal characteristic of the object $\mathbb{N}^{\mathbb{N}}/\mathrm{fin}$. We observe that for an $\mathcal{R}$-name for a member of one of these objects, the equivalence class can be named in a particularly nice manner.

DEFINITION 3.2. Let $\dot{x}$ be an $\mathcal{R}$-name for a member of $\mathcal{P}(\mathbb{N})/\mathrm{fin}$. A *lifting* of $\dot{x}$ is a function $f : 2^{\mathbb{N}} \to \mathcal{P}(\mathbb{N})$ such that

$$\|[\dot{f}] = \dot{x}\| = 1.$$

A lifting of an $\mathcal{R}$-name for a member of $\mathbb{N}^{\mathbb{N}}/\mathrm{fin}$ or $(\mathbb{N}^{[<\infty]})^{\mathbb{N}}/\mathrm{fin}$ is defined analogously.

LEMMA 3.3. *Every* $\mathcal{R}$*-name for a member of* $\mathcal{P}(\mathbb{N})/\mathrm{fin}$ *has a continuous lifting. Equivalently*, *for every measurable* $f : 2^{\mathbb{N}} \to \mathcal{P}(\mathbb{N})$ *there is a continuous* $g : 2^{\mathbb{N}} \to \mathcal{P}(\mathbb{N})$ *such that*

$$\|\dot{f} \equiv \dot{g} \bmod \mathrm{fin}\| = 1.$$

Proof. Let $\dot{x}$ be a given $\mathcal{R}$-name for a member of $\mathcal{P}(\mathbb{N})/\mathrm{fin}$. Then let $\dot{y}$ be an $\mathcal{R}$-name for a member of $\mathcal{P}(\mathbb{N})$ such that

$$\mathcal{R} \Vdash [\dot{y}] = \dot{x}. \tag{8}$$

Recall that there is a measurable $f_{\dot{y}} : 2^{\mathbb{N}} \to \mathcal{P}(\mathbb{N})$ such that $\mathcal{R} \Vdash f_{\dot{y}}(\dot{r}) = \dot{y}$ (see Section 2). Then from (1) we conclude that $f_{\dot{y}}$ is a lifting of $\dot{x}$.

For each $l \in \mathbb{N}$, let

$$A_l = \|l \in \dot{f}_{\dot{y}}\|.$$

Note that each A_l is a measurable subset of $2^{\mathbb{N}}$. Hence, by Lemma 3.1, there exists a clopen $B_l \subseteq 2^{\mathbb{N}}$ such that

$$\mu(A_l \bigtriangleup B_l) < 2^{-l}. \tag{9}$$

Define $g : 2^{\mathbb{N}} \to \mathcal{P}(\mathbb{N})$ by

$$g(x) = \{l \in \mathbb{N} : x \in B_l\}.$$

CLAIM 3.4. *g is continuous.*

Proof. For each $l \in \mathbb{N}$, let

$$V_l = \{A \subseteq \mathbb{N} : l \in A\} \quad \text{and} \quad V_{-l} = \{A \subseteq \mathbb{N} : l \notin A\}.$$

Since $\{V_l, V_{-l} : l \in \mathbb{N}\}$ is a subbasis of clopen sets for $\mathcal{P}(\mathbb{N})$, it suffices to prove that $g^{-1}(V_l)$ is clopen for all $l \in \mathbb{N}$. And

$$g^{-1}(V_l) = B_l, \tag{10}$$

which is clopen. ∎

It remains to show that $\|\dot{f}_{\dot{y}} \equiv \dot{g} \bmod \mathrm{fin}\| = 1$, which is equivalent to showing that

$$f_{\dot{y}} =^* g \tag{11}$$

(i.e. $f_{\dot{y}} \subseteq^* g$ and $g \subseteq^* f_{\dot{y}}$). First we prove that $f_{\dot{y}} \subseteq^* g$. This is equivalent to

$$\prod_{k=0}^{\infty} \sum_{l=k}^{\infty} \|l \in \dot{f}_{\dot{y}}\| \cdot \|l \notin \dot{g}\| = 0. \tag{12}$$

Fix $k \in \mathbb{N}$. Then by (9),

$$\mu\Big(\sum_{l=k}^{\infty} \|l \in \dot{f}_{\dot{y}}\| \cdot \|l \notin \dot{g}\|\Big) = \mu\Big(\sum_{l=k}^{\infty} B_l - A_l\Big) \le 2^{-k-1}. \tag{13}$$

Thus $\mu\big(\prod_{k=0}^{\infty} \sum_{l=k}^{\infty} \|l \in \dot{f}_{\dot{y}}\| \cdot \|l \notin \dot{g}\|\big) \le \inf_{k\to\infty} 2^{-k-1} = 0$, as needed. The proof that $g \subseteq^* f_{\dot{y}}$ is the same but with the roles of A_l and B_l interchanged. ∎

Analogous results for $\mathbb{N}^{\mathbb{N}}/\mathrm{fin}$ and $(\mathbb{N}^{[<\infty]})^{\mathbb{N}}/\mathrm{fin}$ can be proved in the same manner:

Lemma 3.5. *Every $\mathcal{R}$-name for a member of* $\mathbb{N}^{\mathbb{N}}/\mathrm{fin}$ [*resp.* $(\mathbb{N}^{[<\infty]})^{\mathbb{N}}/\mathrm{fin}$] *has a continuous lifting.*

Now the following reformulations of $\mathfrak{p}_\mu$, $\mathfrak{t}_\mu$ and $\mathfrak{f}_\mu$ follow easily:

Proposition 3.6. $\mathfrak{p}_\mu$ *is the smallest cardinality of a filtered family of continuous functions from $2^{\mathbb{N}}$ into $\mathcal{P}(\mathbb{N})$ which has no continuous infinitary pseudo-intersection from $2^{\mathbb{N}}$ into $\mathcal{P}(\mathbb{N})$.*

Proposition 3.7. $\mathfrak{t}_\mu$ *is the smallest cardinality of a tower of continuous functions from $2^{\mathbb{N}}$ into $\mathcal{P}(\mathbb{N})$ which has no continuous infinitary pseudo-intersection from $2^{\mathbb{N}}$ into $\mathcal{P}(\mathbb{N})$.*

Proposition 3.8. $\mathfrak{f}_\mu$ *is the smallest cardinality of a bounded family $\mathcal{F}$ of continuous functions from $2^{\mathbb{N}}$ into $\mathcal{P}(\mathbb{N})$ such that for every continuous slalom $S : 2^{\mathbb{N}} \to (\mathbb{N}^{[<\infty]})^{\mathbb{N}}$ and every continuous infinitary $X : 2^{\mathbb{N}} \to \mathcal{P}(\mathbb{N})$, (S, X) does not capture every $f \in \mathcal{F}$.*

4. The proofs. Here we give proofs of Theorems 1.9–1.12. Before starting we review the relations between the cardinals $\mathfrak{b}$, $\mathfrak{p}$ and $\mathfrak{t}$.

Theorem 4.1. $\aleph_1 \le \mathfrak{p} \le \mathfrak{t} \le \mathfrak{b}$.

Proof. See [vD84], [Bla99]. ■

We complete the preparation for the proofs with a basic fact about measurable functions.

Theorem 4.2 (Luzin). *If X is a second countable Hausdorff space and $f : I \to X$ is measurable, then for every $\varepsilon > 0$ there is a closed $K \subseteq I$ with $\mu(K) > 1 - \varepsilon$ such that $f \restriction K$ is continuous.*

Proof. See [Roy88]. ■

Proof of Theorem 1.9. We take an infinite $\kappa < \mathfrak{p}$ and prove that $\kappa < \mathfrak{p}_\mu$. Let $f_\xi : 2^{\mathbb{N}} \to \mathcal{P}(\mathbb{N})$ $[\xi < \kappa]$ be a filtered family of measurable functions. By Lemma 3.3 (see also Proposition 3.6), we can assume that f_ξ is continuous for all ξ. Since any finite intersection of continuous functions is also continuous, we may assume further without loss of generality that $\langle f_\xi : \xi < \kappa \rangle$ is closed under finite intersections, i.e. there is a map $\alpha : \kappa^{[<\aleph_0]} \to \kappa$ such that

$$f_{\alpha(\Gamma)} = \bigcap_{\xi \in \Gamma} f_\xi \quad \text{for all } \Gamma \in \kappa^{[<\aleph_0]}. \tag{14}$$

Notation. Let $\mathcal{B}$ be the family of all clopen subsets of $2^{\mathbb{N}}$. Let $\langle \mathcal{B}_i, q_i \rangle_{i=0}^{\infty}$ be an enumeration of all pairs $\langle \mathcal{C}, q \rangle$ such that $\mathcal{C} \in \mathcal{B}^{[<\aleph_0]}$ and $q : \mathcal{C} \to \mathbb{N}$.

For each $\xi < \kappa$ and $n \in \mathbb{N}$, define

$$A_\xi(n) = \Big\{ i \in \mathbb{N} : q_i(B) \ge n \text{ for all } B \in \mathcal{B}_i,\ \mu\Big(\sum \mathcal{B}_i\Big) \ge 1 - 2^{-n}, \\ B \le \|q_i(B) \in \dot{f}_\xi\| \text{ for all } B \in \mathcal{B}_i \Big\}.$$

Claim 4.3. *For every $\xi < \kappa$, $A_\xi(n) \ne \emptyset$ for all $n \in \mathbb{N}$.*

Proof. Fix $\xi < \kappa$ and $n \in \mathbb{N}$. The fact that f_ξ is infinitary is easily seen to be equivalent to the statement

$$\prod_{k=0}^{\infty} \sum_{l=k}^{\infty} \|l \in \dot{f}_\xi\| = 1. \tag{15}$$

Therefore there exists $p \in \mathbb{N}$ such that

$$\mu\Big(\sum_{l=n}^{p} \|l \in \dot{f}_\xi\|\Big) \ge 1 - 2^{-n}. \tag{16}$$

For each $l = n, \ldots, p$, put

$$B_l = \|l \in \dot{f}_\xi\|.$$

Since each B_l is clearly clopen, we can find $i \in \mathbb{N}$ such that $\mathcal{B}_i = \{B_n, \ldots, B_p\}$, and such that $q_i(B_l) = l$ for all $l = n, \ldots, p$. Then $i \in A_\xi(n)$. ■

For each $\xi < \kappa$, define $a_\xi : \mathbb{N} \to \mathbb{N}$ by

$$a_\xi(n) = \min A_\xi(n) \quad \text{for all } n \in \mathbb{N}.$$

By Theorem 4.1, $\kappa < \mathfrak{b}$. Hence there exists $a : \mathbb{N} \to \mathbb{N}$ such that a is a $\le^*$-bound for $\{a_\xi : \xi < \kappa\}$. For each $\xi < \kappa$, define

$$C_\xi = \{(n, i) \in \mathbb{N} \times \mathbb{N} : i \in A_\xi(n),\ i \le a(n)\}.$$

CLAIM 4.4. *$\langle C_\xi : \xi < \kappa\rangle$ is a filter base.*

Proof. Let $\Gamma \subseteq \kappa$ be a given nonempty finite subset. Take $k \in \mathbb{N}$. Find $n \ge k$ such that $a_{\alpha(\Gamma)}(n) \le a(n)$. Then $i = \min A_{\alpha(\Gamma)}(n) \le a(n)$, and hence $(n, i) \in C_{\alpha(\Gamma)}$. Since $\|l \in \dot{f}_{\alpha(\Gamma)}\| = \prod_{\xi\in\Gamma} \|l \in \dot{f}_\xi\|$, it follows from its definition that $A_{\alpha(\Gamma)}(n) \subseteq A_\xi(n)$ for all $\xi \in \Gamma$. And this implies that $(n, i) \in C_\xi$ for all $\xi \in \Gamma$. Thus $(\bigcap_{\xi\in\Gamma} C_\xi)\setminus(k\times\mathbb{N}) \ne \emptyset$, proving that $\bigcap_{\xi\in\Gamma} C_\xi$ is infinite. ■

Since $\kappa < \mathfrak{p}$, there exists an infinite $C \subseteq \mathbb{N} \times \mathbb{N}$ which is a pseudo-intersection of $\langle C_\xi : \xi < \kappa\rangle$. Moreover, we can insist that $C \subseteq C_0$. Let $D = \mathrm{dom}(C)$. Clearly, D is infinite. For each $n \in D$, choose i_n so that

$$(n, i_n) \in C. \tag{17}$$

Now we define $f : 2^{\mathbb{N}} \to \mathcal{P}(\mathbb{N})$ by

$$f(x) = \{q_{i_n}(B) : n \in D,\ B \in \mathcal{B}_{i_n},\ x \in B\}.$$

CLAIM 4.5. *f is continuous.*

Proof. It suffices to prove that $f^{-1}(V_l)$ is clopen for all $l \in \mathbb{N}$ (see the proof of Claim 3.4). And since $C \subseteq C_0$, we have

$$q_{i_n}(B) \ge n \quad \text{for all } n \in D \text{ and all } B \in \mathcal{B}_{i_n}. \tag{18}$$

Therefore,

$$f^{-1}(V_l) = \bigcup\{B : n \in D,\ B \in \mathcal{B}_{i_n},\ q_{i_n}(B) = l\}$$

is a clopen set. ■

CLAIM 4.6. *f is infinitary.*

Proof. From (18) it follows that if we take $k \in \mathbb{N}$, then

$$\|\dot{f} \setminus k \ne \emptyset\| \ge \sum_{n\in D\setminus k} \sum \mathcal{B}_{i_n}. \tag{19}$$

But clearly $\mu(\sum_{n\in D\setminus k} \sum \mathcal{B}_{i_n}) \ge \lim_{n\to\infty}(1-2^{-n}) = 1$. Thus $\|\dot{f}\setminus k \ne \emptyset\| = 1$ for all $k \in \mathbb{N}$, which implies that f is infinitary. ■

CLAIM 4.7. *$f \subseteq^* f_\xi$ for all $\xi < \kappa$.*

Proof. Fix $\xi < \kappa$. The statement $\|\dot{f} \subseteq^* \dot{f}_\xi\| = 1$ is equivalent to

$$\prod_{k=0}^{\infty} \sum_{l=k}^{\infty} \|l \in \dot{f}\| \cdot \|l \notin \dot{f}_\xi\| = 0. \tag{20}$$

Define $k:\mathbb{N}\to\mathbb{N}$ by $k(m)=0$ for $m\le\min(D)$, and

$$k(m)=\max_{n\in D\cap m}\max\operatorname{ran}(q_{i_n})+1\qquad\text{for } m>\min(D).$$

Subclaim. *We have*

$$(21)\quad \sum_{l=k(m)}^{\infty}\|l\in\dot f\|\cdot\|l\notin\dot f_\xi\|\le\sum_{n\in D\setminus m}\sum_{B\in\mathcal{B}_{i_n}}B-\|q_{i_n}(B)\in\dot f_\xi\|\quad \textit{for all } m.$$

Proof. Take $l\ge k(m)$. Suppose that $x\in\|l\in\dot f\|\cdot\|l\notin f_\xi\|$. Since $x\in\|l\in\dot f\|$, there exists $n\in D$ and $B\in\mathcal{B}_{i_n}$ such that $q_{i_n}(B)=l$ and $x\in B$. And then, since $x\notin\|l\in\dot f_\xi\|$, $x\in B-\|q_{i_n}(B)\in\dot f_\xi\|$. Moreover, as $l\ge k(m)$, $n\ge m$ as needed. ■

Now $C\subseteq^* C_\xi$, and hence there is an m such that $(n,i_n)\in C_\xi$ for all $n\in D\setminus m$. Therefore, by (21),

$$(22)\qquad \sum_{l=k(m)}^{\infty}\|l\in\dot f\|\cdot\|l\notin\dot f_\xi\|=0,$$

proving (20). ■

Claims 4.5–4.7 show that f is a continuous infinitary pseudo-intersection of $\langle f_\xi:\xi<\kappa\rangle$. Therefore, $\kappa<\mathfrak{p}_\mu$. ■

Proof of Theorem 1.10. We take $\kappa<\mathfrak{t}$ and prove that $\kappa<\mathfrak{t}_\mu$. Let $f_\xi:I\to\mathcal{P}(\mathbb{N})$ $[\xi<\kappa]$ be a tower of measurable functions such that the enumeration respects the well ordering of the tower, i.e. $\xi\le\eta\to f_\eta\subseteq^* f_\xi$.

For each $\xi<\eta<\kappa$, since $f_\eta\subseteq^* f_\xi$, $\sum_{k=0}^{\infty}\|\dot f_\eta\setminus k\subseteq\dot f_\xi\|=1$. And obviously the sets $\|\dot f_\eta\setminus k\subseteq\dot f_\xi\|$ are increasing with respect to k. Hence we can find an increasing function $H_{\xi\eta}:\mathbb{N}\to\mathbb{N}$ so that

$$(23)\qquad \mu(\|\dot f_\eta\setminus k\subseteq\dot f_\xi\|)\ge 1-\frac{1}{H_{\xi\eta}(k)}\qquad\text{for all } k\in\mathbb{N},$$

$$(24)\qquad \lim_{k\to\infty}H_{\xi\eta}(k)=\infty.$$

We construct $g_\eta,h_\eta:\mathbb{N}\to\mathbb{N}$ $[\eta<\kappa]$ by recursion on η so that for all $\eta<\kappa$,

$$(25)\qquad h_0(n)=n,\ g_0(n)=2^n\qquad\text{for all } n\in\mathbb{N},$$

$$(26)\qquad g_\eta\ge^* 2\cdot g_\xi\qquad\text{for all } \xi<\eta,$$

$$(27)\qquad h_\eta\ge^* h_\xi\qquad\text{for all } \xi\le\eta,$$

$$(28)\qquad g_\eta\le^* H_{\xi\eta}\circ h_\eta\qquad\text{for all } \xi<\eta.$$

Given $0<\eta<\kappa$, assume that $\langle g_\xi,h_\xi:\xi<\eta\rangle$ have been chosen satisfying the above conditions. By Theorem 4.1, $\eta<\mathfrak{b}$. Hence we can let $g_\eta:\mathbb{N}\to\mathbb{N}$

be a $\leq^*$-bound for $\{2\cdot g_\xi : \xi<\eta\}$. For each $\xi<\eta$, $\overline{H}_{\xi\eta}:\mathbb{N}\to\mathbb{N}$ given by

$$\overline{H}_{\xi\eta}(n)=\min\{k\in\mathbb{N}: H_{\xi\eta}(k)\geq n\}$$

is well defined by (24). For each $\xi<\eta$, define $\widetilde{h}_\xi:\mathbb{N}\to\mathbb{N}$ by

$$\widetilde{h}_\xi(n)=\max\{h_\xi(n),\overline{H}_{\xi\eta}(g_\eta(n))\}\quad\text{for all } n\in\mathbb{N}.$$

Let $h_\eta:\mathbb{N}\to\mathbb{N}$ be a $\leq^*$-bound for $\{\widetilde{h}_\xi:\xi<\eta\}$. Clearly, $h_\eta\geq^* h_\xi$ for all $\xi\leq\eta$. Fix $\xi<\eta$. Find $k\in\mathbb{N}$ such that $h_\eta(n)\geq\widetilde{h}_\xi(n)$ for all $n\geq k$. Then, since $H_{\xi\eta}$ is increasing, $H_{\xi\eta}(h_\eta(n))\geq H_{\xi\eta}(\overline{H}_{\xi\eta}(g_\eta(n)))\geq g_\eta(n)$ for all $n\geq k$.

NOTATION. Let $\mathcal{B}$ denote the family of all finite unions of rational intervals. Let $\langle B_i,q_i\rangle_{i=0}^\infty$ be an enumeration of all pairs $\langle\mathcal{C},q\rangle$ such that $\mathcal{C}\in\mathcal{B}^{[<\aleph_0]}$ and $q:\mathcal{C}\to\mathbb{N}$.

Define, for each $\xi<\kappa$ and $n\in\mathbb{N}$,

$$A_\xi(n)=\Big\{i\in\mathbb{N}: q_i(B)\geq h_\xi(n)\text{ for all } B\in\mathcal{B}_i,\ \mu\Big(\sum\mathcal{B}_i\Big)\geq 1-2^{-n},\ \mu\Big(\sum_{B\in\mathcal{B}_i}B-\|q_i(B)\in\dot{f}_\xi\|\Big)\leq\frac{1}{g_\xi(n)}\Big\}.$$

CLAIM 4.8. *For every $\xi<\kappa$, $A_\xi(n)\neq\emptyset$ for all $n\in\mathbb{N}$.*

Proof. Fix $\xi<\kappa$ and $n\in\mathbb{N}$. Since f_ξ is infinitary, there exists $p\in\mathbb{N}$ such that

$$\mu\Big(\sum_{l=h_\xi(n)}^{p}\|l\in\dot{f}_\xi\|\Big)\geq 1-2^{-n-1}. \tag{29}$$

For each $l=h_\xi(n),\dots,p$, choose $B_l\in\mathcal{B}$ such that

$$\mu(B_l\mathbin{\triangle}\|l\in\dot{f}_\xi\|)\leq\frac{1}{(p-h_\xi(n)+1)\cdot\max\{2^{n+1},g_\xi(n)\}}. \tag{30}$$

Then $\mu(\sum_{l=h_\xi(n)}^p B_l)\geq 1-2^{-n}$, and $\mu(\sum_{l=h_\xi(n)}^p B_l-\|l\in\dot{f}_\xi\|)\leq 1/g_\xi(n)$. Hence, if i is such that $\mathcal{B}_i=\{B_{h_\xi(n)},\dots,B_p\}$ and $q_i(B_l)=l$ for all $l=h_\xi(n),\dots,p$, then $i\in A_\xi(n)$. ∎

CLAIM 4.9. *For every $\xi<\eta<\kappa$, $A_\eta(n)\subseteq A_\xi(n)$ for all but finitely many $n\in\mathbb{N}$.*

Proof. Fix $\xi<\eta<\kappa$. Find $k\in\mathbb{N}$ such that

$$g_\eta(n)\geq 2\cdot g_\xi(n)\qquad\text{for all } n\geq k, \tag{31}$$

$$h_\eta(n)\geq h_\xi(n)\qquad\text{for all } n\geq k, \tag{32}$$

$$g_\eta(n)\leq H_{\xi\eta}(h_\eta(n))\qquad\text{for all } n\geq k. \tag{33}$$

Take $i \in A_\eta(n)$ for any $n \geq k$. Clearly, $q_i(B) \geq h_\xi(n)$ for all $B \in \mathcal{B}_i$. And

$$\begin{aligned}\sum_{B\in\mathcal{B}_i} B - \|q_i(B) \in \dot{f}_\xi\| &\leq \Big(\|\dot{f}_\eta \setminus h_\eta(n) \subseteq \dot{f}_\xi\| \cdot \sum_{B\in\mathcal{B}_i} B - \|q_i(B) \in \dot{f}_\xi\|\Big) \\ &\quad + (-\|\dot{f}_\eta \setminus h_\eta(n) \subseteq \dot{f}_\xi\|) \\ &\leq \Big(\sum_{B\in\mathcal{B}_i} B - \|q_i(B) \in \dot{f}_\eta\|\Big) \\ &\quad + (-\|\dot{f}_\eta \setminus h_\eta(n) \subseteq \dot{f}_\xi\|).\end{aligned}$$

Therefore

$$\mu\Big(\sum_{B\in\mathcal{B}_i} B - \|q_i(B) \in \dot{f}_\xi\|\Big) \leq \frac{1}{g_\eta(n)} + \frac{1}{H_{\xi\eta}(h_\eta(n))} \leq \frac{2}{g_\eta(n)} \leq \frac{1}{g_\xi(n)},$$

which proves that $i \in A_\xi(n)$. ∎

For each $\xi < \kappa$, define $a_\xi : \mathbb{N} \to \mathbb{N}$ by

$$a_\xi(n) = \min A_\xi(n) \quad \text{for all } n \in \mathbb{N}.$$

By Theorem 4.1, $\kappa < \mathfrak{b}$. Hence there exists $a : \mathbb{N} \to \mathbb{N}$ such that a is a $\leq^*$-bound for $\{a_\xi : \xi < \kappa\}$. For each $\xi < \kappa$, define

$$C_\xi = \{(n,i) \in \mathbb{N}\times\mathbb{N} : i \in A_\xi(n),\ i \leq a(n)\}.$$

Claim 4.10. *$\langle C_\xi : \xi < \kappa\rangle$ is a tower.*

Proof. It follows from Claim 4.8 that each C_ξ is infinite. Fix $\xi < \eta < \kappa$. By Claim 4.9, there exists $k \in \mathbb{N}$ such that $A_\eta(n) \subseteq A_\xi(n)$ for all $n \geq k$. Then $C_\eta \setminus (k\times\mathbb{N}) \subseteq C_\xi$. Therefore $C_\eta \subseteq^* C_\xi$, because $C_\eta \cap (k\times\mathbb{N})$ is finite. ∎

Since $\kappa < \mathfrak{t}$, there exists an infinite $C \subseteq \mathbb{N}\times\mathbb{N}$ which is a pseudo-intersection of $\langle C_\xi : \xi < \kappa\rangle$. Moreover, we can insist that $C \subseteq C_0$. Let $D = \operatorname{dom}(C)$. Clearly, D is infinite. For each $n \in D$, choose i_n so that $(n, i_n) \in C$.

Now we define $f : I \to \mathcal{P}(\mathbb{N})$ by

$$f(x) = \{q_{i_n}(B) : n \in D,\ B \in \mathcal{B}_{i_n},\ x \in B\}.$$

Claim 4.11. *f is measurable.*

Proof. It suffices to prove that $f^{-1}(V_l)$ is measurable for all $l \in \mathbb{N}$ (see the proof of Claim 3.4). And

$$f^{-1}(V_l) = \bigcup\{B : n \in D,\ B \in \mathcal{B}_{i_n},\ q_{i_n}(B) = l\}$$

is an open set. (In fact, it is clopen.) ∎

Claim 4.12. *f is infinitary.*

Proof. Since $C \subseteq C_0$ and by (25), for all $n \in D$, $q_{i_n}(B) \geq h_0(n) = n$ for all $B \in \mathcal{B}_{i_n}$. Hence, if we take $k \in \mathbb{N}$, then

$$\|\dot{f} \setminus k \neq \emptyset\| \geq \sum_{n \in D \setminus k} \sum \mathcal{B}_{i_n}. \tag{34}$$

But clearly, $\mu(\sum_{n \in D \setminus k} \sum \mathcal{B}_{i_n}) \geq \lim_{n \to \infty}(1-2^{-n}) = 1$. Thus $\|\dot{f} \setminus k \neq \emptyset\| = 1$ for all $k \in \mathbb{N}$, as wanted. ∎

Claim 4.13. *$f \subseteq^* f_\xi$ for all $\xi < \kappa$.*

Proof. Fix $\xi < \kappa$. We need to show that $\prod_{k=0}^{\infty} \sum_{l=k}^{\infty} \|l \in \dot{f}\| \cdot \|l \notin \dot{f}_\xi\| = 0$. Choose $k : \mathbb{N} \to \mathbb{N}$ so that

$$k(m) \geq \max_{n \in D \cap m} \max \operatorname{ran}(q_{i_n}) + 1 \quad \text{for } m > \min(D), \tag{35}$$

$$\lim_{m \to \infty} k(m) = \infty. \tag{36}$$

Subclaim. *We have*

$$\sum_{l=k(m)}^{\infty} \|l \in \dot{f}\| \cdot \|l \notin \dot{f}_\xi\| \leq \sum_{n \in D \setminus m} \sum_{B \in \mathcal{B}_{i_n}} B - \|q_{i_n}(B) \in \dot{f}_\xi\| \quad \textit{for all } m. \tag{37}$$

Proof. Take $l \geq k(m)$. Suppose that $x \in \|l \in \dot{f}\| \cdot \|l \notin f_\xi\|$. Since $x \in \|l \in \dot{f}\|$, there exists $n \in D$ and $B \in \mathcal{B}_{i_n}$ such that $q_{i_n}(B) = l$ and $x \in B$. And then since $x \notin \|l \in \dot{f}_\xi\|$, $x \in B - \|q_{i_n}(B) \in \dot{f}_\xi\|$. Moreover, as $l \geq k(m)$, $n \geq m$ by (35). ∎

Now

$$\begin{aligned}\prod_{k=0}^{\infty} \sum_{l=k}^{\infty} \|l \in \dot{f}\| \cdot \|l \notin \dot{f}_\xi\| &= \prod_{m=0}^{\infty} \sum_{l=k(m)}^{\infty} \|l \in \dot{f}\| \cdot \|l \notin \dot{f}_\xi\| && \text{by (36)}\\ &\leq \prod_{m=0}^{\infty} \sum_{n \in D \setminus m} \sum_{B \in \mathcal{B}_{i_n}} B - \|q_{i_n}(B) \in \dot{f}_\xi\| && \text{by (37).}\end{aligned}$$

And thus $C \subseteq^* C_\xi$ and (25) imply that

$$\mu\Big(\prod_{k=0}^{\infty} \sum_{l=k}^{\infty} \|l \in \dot{f}\| \cdot \|l \notin \dot{f}_\xi\|\Big) \leq \inf_{m \to \infty} \sum_{n \in D \setminus m} 2^{-n} = 0. \ ∎$$

Claims 4.11–4.13 show that f is a measurable infinitary pseudo-intersection of $\langle f_\xi : \xi < \kappa \rangle$. Therefore, $\kappa < \mathfrak{t}_\mu$. ∎

Proof of Theorem 1.11. We take $\kappa < \min\{\mathfrak{b}, \mathfrak{f}\}$ and prove that $\kappa < \mathfrak{f}_\mu$. Let $f_\xi : 2^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ $[\xi < \kappa]$ be a bounded family of measurable functions, say $b \in \mathbb{N}^{\mathbb{N}}$ is a bound for the family. By Lemma 3.5, we may assume that every f_ξ is continuous. For each $\xi < \kappa$ and each $n \in \mathbb{N}$, define $A_\xi(n) : b(n) \to \mathcal{R}$

by

$$A_\xi(n)(l) = \|\dot f_\xi(n) = l\| \quad \text{for all } l < b(n).$$

NOTATION. Let $\mathcal{B}$ denote the family of all clopen subsets of $2^{\mathbb{N}}$.

Let $\{B_i\}_{i=0}^\infty$ be an enumeration of all members of $\mathcal{B}^{<\mathbb{N}}$ for which

$$(38) \qquad B_i(k) \cdot B_i(l) = 0 \quad \text{for all } k \neq l \text{ in } \operatorname{dom}(B_i).$$

By continuity, for each $\xi < \kappa$ and $n \in \mathbb{N}$, there is an $i \in \mathbb{N}$ such that $A_\xi(n) = B_i$. Hence we can define $a_\xi : \mathbb{N} \to \mathbb{N}$ so that

$$B_{a_\xi(n)} = A_\xi(n) \quad \text{for all } n \in \mathbb{N}.$$

Since $\kappa < \mathfrak{b}$, there exists $a : \mathbb{N} \to \mathbb{N}$ such that a is a $<^*$-bound for $\{a_\xi : \xi < \kappa\}$. Define for each $n \in \mathbb{N}$,

$$c_i^n = \{s \in a(n)^{[\leq n]} : i \in s\} \quad \text{for all } i < a(n).$$

Note that

$$(39) \qquad \forall n \in \mathbb{N}\ \forall s \in a(n)^{[\leq n]} \ \bigcap_{i \in s} c_i^n \neq \emptyset.$$

Therefore, by Lemma 2.9 the set

$$(40) \qquad \Big\{X \subseteq \bigcup_{n=0}^\infty \{n\} \times a(n)^{[\leq n]} : \forall^\infty n\ \exists i < a(n)\ \{n\} \times c_i^n \subseteq X\Big\}$$

generates an F_σ filter, say $\mathcal{F}$.

Define, for each $\xi < \kappa$,

$$C_\xi = \bigcup \big\{\{n\} \times c^n_{a_\xi(n)} : n \in \mathbb{N},\ a_\xi(n) < a(n)\big\}.$$

Then $\langle C_\xi : \xi < \kappa\rangle$ is contained in the set in (40), and hence in $\mathcal{F}$. Therefore, as $\kappa < \mathfrak{f}$, $\langle C_\xi : \xi < \kappa\rangle$ has an infinite pseudo-intersection, say C. Clearly, $D = \operatorname{dom}(C)$ is infinite. For each $n \in D$, choose $s_n \in a(n)^{[\leq n]}$ such that $(n, s_n) \in C$. And for each $n \notin D$, let $s_n \in a(n)^{[\leq n]}$ be arbitrary. Then for every $\xi < \kappa$, $s_n \in c^n_{a_\xi(n)}$ for all but finitely many $n \in D$. Thus

$$(41) \qquad \forall \xi < \kappa\ \forall^\infty n \in D\ a_\xi(n) \in s_n.$$

Define $S : 2^{\mathbb{N}} \to (\mathbb{N}^{[<\infty]})^{\mathbb{N}}$ by

$$S(x)(n) = \{l < b(n) : i \in s_n,\ x \in B_i(l)\}.$$

CLAIM 4.14. *S is a continuous function.*

Proof. For each $n \in \mathbb{N}$ and each $t \in \mathbb{N}^{[<\infty]}$, let

$$V_{n,t} = \{g \in (\mathbb{N}^{[<\infty]})^{\mathbb{N}} : g(n) = t\}.$$

Since $\{V_{n,t} : n \in \mathbb{N}, t \in \mathbb{N}^{[<\infty]}\}$ is a subbasis of clopen sets for $(\mathbb{N}^{[<\infty]})^{\mathbb{N}}$, it suffices to prove that $S^{-1}(V_{n,t})$ is clopen for all n and t. And

$$S^{-1}(V_{n,t}) = \bigcap_{l \in t \cap b(n)} \bigcup \{B_i(l) : i \in s_n\} \setminus \bigcup_{l \in b(n) \setminus t} \bigcup \{B_i(l) : i \in s_n\}$$

is a clopen set. ∎

CLAIM 4.15. *S is a slalom.*

Proof. Fix $n \in \mathbb{N}$. For each $i < a(n)$, define

$$S_i^n(x) = \{l < b(n) : x \in B_i(l)\} \quad \text{for all } x \in 2^{\mathbb{N}}.$$

By condition (38), $|S_i^n(x)| \le 1$ for all $x \in 2^{\mathbb{N}}$. Therefore, since $S(x)(n) = \bigcup_{i \in s_n} S_i^n(x)$ for all $x \in 2^{\mathbb{N}}$, and since $|s_n| \le n$, we have $|S(x)(n)| \le n$ for all x, which implies that S is a slalom. ∎

CLAIM 4.16. *(S, D) captures f_ξ for all $\xi < \kappa$.*

Proof. Fix $\xi < \kappa$. Since we are viewing D as a constant function on $2^{\mathbb{N}}$, it suffices to show that

$$\sum_{m=0}^{\infty} \prod_{n \in D \setminus m} \|\dot{f}_\xi(n) \in \dot{S}(n)\| = 1. \tag{42}$$

By (41), there is an $m \in \mathbb{N}$ such that

$$a_\xi(n) \in s_n \quad \text{for all } n \in D \setminus m, \tag{43}$$

and then since $B_{a_\xi(n)}(l) = A_\xi(l)$ for all $l < b(n)$, we have

$$\|\dot{f}_\xi(n) \in \dot{S}(n)\| = 1 \quad \text{for all } n \in D \setminus m. \tag{44}$$

This proves (42). ∎

By Claims 4.14–4.16, S is a continuous slalom such that (S, D) captures every member of the family $\langle f_\xi : \xi < \kappa \rangle$. Therefore, $\kappa < \mathfrak{f}_\mu$. ∎

REMARK. Note that in the proof of Theorem 1.11 we have proved more than stated in the theorem. We only needed to produce a measurable slalom S and a measurable infinitary X which captured the family, but in fact we found a constant function D in place of X.

The next lemma illustrates how analytic properties of measurable functions can lead to the existence of an object in $V^{\mathcal{R}}$ independently of the ground model. We will show that in $V^{\mathcal{R}}$ there always exists a bounded subfamily of $\mathbb{N}^{\mathbb{N}}$ of size $\mathfrak{b}$ which cannot be captured by any pair (S, X) where S is a slalom and X is infinite.

DEFINITION 4.17. For $h : I \to \mathbb{N}^{\mathbb{N}}$, a function $g \in \mathbb{N}^{\mathbb{N}}$ is called an *approximate lower bound* for h if the set $\{x \in I : h(x)(n) \ge g(n) \text{ for all } n \in \mathbb{N}\}$ has positive measure.

Notation. Let $\mathbb{N}^{\mathbb{N}}_{\nearrow}$ denote the subspace of $\mathbb{N}^{\mathbb{N}}$ of all strictly increasing functions.

Lemma 4.18. *Every measurable* $h : I \to \mathbb{N}^{\mathbb{N}}$ *for which* $\|\dot{h} \in \mathbb{N}^{\mathbb{N}}_{\nearrow}\| = 1$ *has an approximate lower bound in* $\mathbb{N}^{\mathbb{N}}_{\nearrow}$.

Proof. By Theorem 4.2, we can find a compact $K \subseteq I$ with positive measure such that

$$h \restriction K \text{ is continuous,} \tag{45}$$

$$h(x) \in \mathbb{N}^{\mathbb{N}}_{\nearrow} \quad \text{for all } x \in K. \tag{46}$$

By continuity and compactness, for each n, there is a finite sequence $A^0_n, \ldots, A^{m_n-1}_n$ of relatively clopen subsets of K and a finite sequence of integers $l^0_n, \ldots, l^{m_n-1}_n$ such that

$$h(x)(n) = l^i_n \quad \text{for all } x \in A^i_n \text{ and all } i < m_n, \tag{47}$$

$$A^i_n \neq 0 \quad \text{for all } i < m_n, \tag{48}$$

$$\sum_{i<m_n} A^i_n = K. \tag{49}$$

Define $g : \mathbb{N} \to \mathbb{N}$ by

$$g(n) = \min_{i<m_n} l^i_n \quad \text{for all } n \in \mathbb{N}.$$

Then by (47) and (49), K witnesses that g is an approximate lower bound for h, and it follows from (46) and (48) that $g \in \mathbb{N}^{\mathbb{N}}_{\nearrow}$. ∎

We define a notion of "complexity" for measurable subsets of $2^{\mathbb{N}}$:

Definition 4.19. For a measurable $G \subseteq 2^{\mathbb{N}}$ with positive measure, we define $c(G)$ to be the smallest integer n for which there exists a (finite) $T \subseteq 2^{<\mathbb{N}}$ such that

(i) $[t] \cdot [u] = 0$ for all $t \neq u$ in T,
(ii) $\mu(G - [T]) \leq \mu(G)/2$,
(iii) $\mu(G \cdot [t]) \geq 2^{-|t|-1}$ for all $t \in T$,
(iv) $n = \max_{t \in T} |t|$.

Example 4.20. $c([\langle 0,0,0,0,0\rangle] \cup [\langle 1,1,1,1,1\rangle]) = 4$.

Definition 4.21. Let $a : \mathbb{N} \to \mathbb{N}$ be the exponential function, i.e.

$$a(n) = 2^n \quad \text{for all } n \in \mathbb{N}.$$

We define an association $h \mapsto f_h$, between $\mathbb{N}^{\mathbb{N}}$ and the continuous functions from $2^{\mathbb{N}}$ into $\mathbb{R}(a)$, by

$$f_h(x)(n) \equiv x \restriction h(n) \bmod a(n) \quad \text{for all } x \in 2^{\mathbb{N}} \text{ and all } n \in \mathbb{N},$$

i.e.

$$f_h(x)(n) \equiv \sum_{i=0}^{h(n)-1} 2^{h(n)-i-1} \cdot x(i) \bmod a(n).$$

Thus a larger value of $h(n)$ gives a more rapidly oscillating function $f_h(\cdot)(n)$.

It will be convenient to deal indirectly with infinite sets via their enumerating functions. This is why we make the following auxiliary definition.

Definition 4.22. For $S : 2^{\mathbb{N}} \to (\mathbb{N}^{[<\infty]})^{\mathbb{N}}$, $E : 2^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ and $f : 2^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$, we say that (S, E) *captures* f if

$$\|\forall^\infty n \in \mathbb{N}\ \dot{f}(\dot{E}(n)) \in \dot{S}(\dot{E}(n))\| = 1.$$

Lemma 4.23. *Suppose that $E : 2^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is measurable, $b_* \in \mathbb{N}^{\mathbb{N}}_{\nearrow}$ is an approximate lower bound for E, $b^* \in \mathbb{N}^{\mathbb{N}}_{\nearrow}$ is an (upper) bound for E, $S : 2^{\mathbb{N}} \to \mathbb{R}(2^a)$ is a measurable slalom and that $h \in \mathbb{N}^{\mathbb{N}}_{\nearrow}$. Suppose further that (S, E) captures f_h. Then if $g : \mathbb{N} \to \mathbb{N}$ is given by*

$$g(n) = \max\{c(\|\dot{E}(n) = l\| \cdot \|\dot{S}(l) = s\|) : l \in [b_*(n), b^*(n)],\ s \in a(l)^{[\le l]}\},$$

then

$$\forall^\infty n \in \mathbb{N}\ g(n) \ge h(b_*(n)) - b^*(n).$$

Proof. Let

$$\delta = \mu(\|\forall n \in \mathbb{N}\ \dot{E}(n) \ge b_*(n)\|).$$

Since b_* is an approximate lower bound for E, $\delta > 0$. Therefore, as b^* is an upper bound for E, and by the definition of a slalom, for $k_0 \in \mathbb{N}$ sufficiently large,

$$\mu\Big(\sum_{l=b_*(n)}^{b^*(n)} \sum_{s\in a(l)^{[\le l]}} \|\dot{E}(n) = l\| \cdot \|\dot{S}(l) = s\|\Big) \ge \frac{\delta}{2} \quad \text{for all } n \ge k_0. \tag{50}$$

Since (S, E) captures h_f, $\sum_{k=0}^{\infty} \prod_{n=k}^{\infty} \|\dot{f}_h(\dot{E}(n)) \in \dot{S}(\dot{E}(n))\| = 1$. We can therefore find a $k_1 \in \mathbb{N}$ large enough so that

$$\mu\Big(\prod_{n=k_1}^{\infty} \|\dot{f}_h(\dot{E}(n)) \in \dot{S}(\dot{E}(n))\|\Big) \ge 1 - \frac{\delta}{8}. \tag{51}$$

And since $b_* \in \mathbb{N}^{\mathbb{N}}_{\nearrow}$, there is a $k_2 \in \mathbb{N}$ (e.g. $k_2 = 6$) such that

$$b_*(n) \ge 6 \quad \text{for all } n \ge k_2. \tag{52}$$

Take any $n \ge \max(k_0, k_1, k_2)$. Then observe that there is an $l \in [b_*(n), b^*(n)]$ and an $s \in a(l)^{[\le l]}$ such that

$$\mu(\|\dot{E}(n) = l\| \cdot \|\dot{S}(l) = s\| \cdot \|\dot{f}_h(l) \in s\|) \ge \frac{3}{4} \cdot \mu(\|\dot{E}(n) = l\| \cdot \|\dot{S}(l) = s\|) \ne 0. \tag{53}$$

For otherwise

$$\mu\Big(\sum_{l=b_*(n)}^{b^*(n)}\sum_{s\in a(l)^{[\le l]}}\|\dot E(n)=l\|\cdot\|\dot S(l)=s\|\cdot\|\dot f_h(l)\in s\|\Big) < \frac{3}{4}\cdot\mu\Big(\sum_{l=b_*(n)}^{b^*(n)}\sum_{s\in a(l)^{[\le l]}}\|\dot E(n)=l\|\cdot\|\dot S(l)=s\|\Big),$$

which would by (50) imply that $\mu(\|\dot f_h(\dot E(n))\in \dot S(\dot E(n))\|)<1-\delta/8$, contrary to (51).

Claim 4.24. $c(\|\dot E(n)=l\|\cdot\|\dot S(l)=s\|)>h(l)-l$.

Proof. Suppose that $T\subseteq 2^{<\mathbb N}$ is a collection of pairwise incompatible elements such that

$$[t]\cdot[u]=0 \quad \text{for all } t\ne u \text{ in } T, \tag{54}$$

$$\mu(\|\dot E(n)=l\|\cdot\|\dot S(l)=s\|-[T])\le \frac{1}{2}\cdot\mu(\|\dot E(n)=l\|\cdot\|\dot S(l)=s\|), \tag{55}$$

$$\mu([t]\cdot\|\dot E(n)=l\|\cdot\|\dot S(l)=s\|)\ge 2^{-|t|-1} \quad \text{for all } t\in T. \tag{56}$$

From (53)–(55), we can deduce that there exists $t\in T$ such that

$$\mu([t]\cdot\|\dot E(n)=l\|\cdot\|\dot S(l)=s\|\cdot\|\dot f_h(l)\in s\|) \ge \frac{1}{2}\cdot\mu([t]\cdot\|\dot E(n)=l\|\cdot\|\dot S(l)=s\|). \tag{57}$$

Therefore, by (56),

$$\mu([t]\cdot\|\dot E(n)=l\|\cdot\|\dot S(l)=s\|\cdot\|\dot f_h(l)\in s\|)\ge 2^{-|t|-2}. \tag{58}$$

Notation. For $T\subseteq 2^{<\mathbb N}$ and $t\in 2^{<\mathbb N}$, let $T_t=\{u\in T: u$ is compatible with $t\}$.

Subclaim. $|t|>h(l)-l$.

Proof. For each $i<a(l)$, define

$$U_i=\{u\in 2_t^{h(l)}: u\equiv i \bmod a(l)\}.$$

Suppose towards a contradiction that $|t|\le h(l)-l$. Then, since $|2_t^{h(l)}|=2^{h(l)-|t|}\ge 2^l=a(l)$, a pigeon hole argument yields

$$|U_i|\le\frac{2}{a(l)}\cdot|2_t^{h(l)}| \quad \text{for all } i<a(l). \tag{59}$$

Now let p be the cardinality of the set $\{u\in 2_t^{h(l)}:[u]\cdot\|\dot f_h(l)\in s\|\ne 0\}$. Since $[u]\cdot\|\dot f_h(l)\in s\|\le\|i\in s\|$ for all $u\in U_i$ and all $i<a(l)$, and since

$|s| \leq l$, by (59),

$$p \leq \sum_{i \in s} |U_i| \leq |s| \cdot \max_{i < a(l)} |U_i| \leq \frac{2l}{2^l} \cdot 2^{h(l)-|t|}. \tag{60}$$

Combining formulas (52) and (60) yields $\mu([t] \cdot \|\dot{f}_h(l) \in s\|) < 2^{-|t|-2}$, contrary to (58). This proves the Subclaim and hence also Claim 4.24. ∎

Since h is an increasing function, with Claim 4.24 we have shown that $g(n) \geq h(b_*(n)) - b^*(n)$ for all $n \geq \max(k_0, k_1, k_2)$. This proves Lemma 4.23. ∎

Proof of Theorem 1.12. We take $\kappa < \mathfrak{f}_\mu$ and prove that $\kappa < \mathfrak{b}$. Let $\mathcal{H} \subseteq \mathbb{N}^{\mathbb{N}}$ with $|\mathcal{H}| = \kappa$. We may assume without loss of generality that $\mathcal{H} \subseteq \mathbb{N}^{\mathbb{N}}_{\nearrow}$. By the assumption on κ, there is a measurable infinitary $X : 2^{\mathbb{N}} \to \mathcal{P}(\mathbb{N})$ and a measurable slalom $S : 2^{\mathbb{N}} \to (\mathbb{N}^{[<\infty]})^{\mathbb{N}}$ such that (S, X) captures f_h for all $h \in \mathcal{H}$. We can assume without loss of generality that $\operatorname{ran}(S) \subseteq \mathbb{R}(2^a)$. Let $e_X : 2^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ be defined so that $e_X(x)$ is the strictly increasing enumeration of $X(x)$ for all $x \in I$. Then clearly (S, e_X) captures f_h for all $h \in \mathcal{H}$. It is also clear that e_X is measurable, and hence by Lemma 4.18, e_X has an approximate lower bound $b_* \in \mathbb{N}^{\mathbb{N}}_{\nearrow}$. And by Proposition 2.11, e_X has an (upper) bound $b^* \in \mathbb{N}^{\mathbb{N}}_{\nearrow}$. Let $g : \mathbb{N} \to \mathbb{N}$ be as defined in Lemma 4.23.

Define $f : \mathbb{N} \to \mathbb{N}$ by

$$f(n) = g(n) + b^*(n) \quad \text{for all } n.$$

Then since $b_*(n) \geq n$ for all n, by Lemma 4.23, $h \leq^* f$ for all $h \in \mathcal{H}$. Therefore, $\kappa < \mathfrak{b}$. ∎

Remark. Unlike Theorems 1.9–1.11, the proof of Theorem 1.12 does not rely on the fact that $\mathcal{R}$ is separable. However, if $\mathcal{B}$ is a nonseparable measure algebra, then $V^{\mathcal{B}} \models \operatorname{non}(\mathcal{N}) = \aleph_1$, because any $\aleph_1$ of the random reals added by $\mathcal{B}$ has positive outer measure (see [BJ95]). And (the known fact that) $\mathfrak{f} \leq \operatorname{non}(\mathcal{N})$ follows easily from Proposition 2.10. Thus these two facts show that $V^{\mathcal{B}} \models \mathfrak{f} \leq \aleph_1 \leq \mathfrak{b}$, and we see that Theorem 1.12 is not interesting in the case of large measure algebras.

5. The other inequalities. Theorems 1.9–1.12 do not give a complete description of the effect of adding a random real on the cardinals $\mathfrak{f}$, $\mathfrak{p}$ and $\mathfrak{t}$. For example, one may ask "does ZFC $\vdash \mathfrak{t}_\mu \leq \mathfrak{t}$?". The analogous question for the addition of one Cohen real can readily be answered due to the following, nearly trivial fact. Letting $\mathcal{C}$ denote the poset for adding one Cohen real, we have:

Lemma 5.1. *Suppose that $\mathcal{T} \subseteq \mathbb{N}^{[\infty]}$ is a tower. Then if*

$$\mathcal{C} \Vdash \text{“}\check{\mathcal{T}} \text{ has an infinite pseudo-intersection”},$$

then $\mathcal{T}$ does have an infinite pseudo-intersection.

Proof. See [Hir00]. ∎

Furthermore, if the Cohen poset is replaced with the Hechler poset in Lemma 5.1, then the result still holds (see [BD85]). However, as we shall see in Example 5.5, the analogous result fails for the random algebra.

DEFINITION 5.2. For $b \in \mathbb{N}^{\mathbb{N}}$, define

$$\mathcal{S}(b) = \left\{ s \in \mathbb{R}(2^b) : \forall n \in \mathbb{N}\ s(n) \neq \emptyset,\ \sum_{n=0}^{\infty} \frac{|s(n)|}{b(n)} < \infty \right\}.$$

For $s \in \mathcal{S}(b)$, define $\Sigma(s), \Pi(s) \subseteq \mathbb{R}(b)$ by

$$\Sigma(s) = \{f \in \mathbb{R}(b) : \exists^\infty n \in \mathbb{N}\ f(n) \in s(n)\},$$
$$\Pi(s) = \{f \in \mathbb{R}(b) : \forall^\infty n \in \mathbb{N}\ f(n) \in s(n)\}.$$

Define ideals

$$\mathcal{N}(b) = \{A \subseteq \mathbb{R}(b) : s \in \mathcal{S}(b),\ A \subseteq \Sigma(s)\},$$
$$\mathcal{E}(b) = \{A \subseteq \mathbb{R}(b) : s \in \mathcal{S}(b),\ A \subseteq \Pi(s)\}.$$

PROPOSITION 5.3. *For any $b \in \mathbb{N}^{\mathbb{N}}$, $\mathcal{E}(b)$ is σ-complete.*

Proof. Let A_i $[i \in \mathbb{N}]$ be a sequence of elements of $\mathcal{E}(b)$, say there exists $s_i \in \mathcal{S}(b)$ $[i \in \mathbb{N}]$ such that $A_i \subseteq \Pi(s_i)$ for all $i \in \mathbb{N}$. Choose a strictly increasing sequence $\langle k_i : i \in \mathbb{N}\rangle$ such that

$$\sum_{n=k_i}^{\infty} \sum_{j=0}^{i} \frac{|s_j(n)|}{b(n)} \le \frac{1}{i^2}. \tag{61}$$

Define $s \in \mathbb{R}(2^b)$ by

$$s(n) = \bigcup_{j=0}^{i} s_j(n) \quad \text{for all } k_i \le n < k_{i+1} \text{ and all } i \in \mathbb{N}.$$

Clearly, $s \in \mathcal{S}(b)$.

Given $i \in \mathbb{N}$, take $f \in A_i$. Find $l \in \mathbb{N}$ such that $f(n) \in s_i(n)$ for all $n \ge l$. Then $f(n) \in s(n)$ for all $n \ge \max(k_i, l)$. Hence $f \in \Pi(s)$. ∎

PROPOSITION 5.4. *For any $b \in \mathbb{N}^{\mathbb{N}}$, $\nu(\mathcal{N}(b)) = 0$* (see Definition 2.5).

Proof. Take $s \in \mathcal{S}(b)$. Then

$$\nu(\mathbb{R}(b) \setminus \Sigma(s)) = \sup_{m\to\infty} \prod_{n=m}^{\infty} \frac{|b(n) - s(n)|}{b(n)} = \sup_{m\to\infty} \prod_{n=m}^{\infty} 1 - \frac{|s(n)|}{b(n)}.$$

Therefore $\nu(\mathbb{R}(b) \setminus \Sigma(s)) = 1$ by the Cauchy criterion for products. ∎

EXAMPLE 5.5. (CH) *A tower for which the random algebra adds an infinite pseudo-intersection.* Fix $b \in \mathbb{N}^{\mathbb{N}}$ with $b(n) \ge n^2$ (so that $\mathcal{S}(b) \neq \emptyset$).

Let s_ξ $[\xi < \omega_1]$ be an enumeration of $\mathcal{S}(b)$. By Proposition 5.3, we can recursively choose $t_\eta \in \mathcal{S}(b)$ such that for all $\eta < \omega_1$,

$$\bigcup_{\xi<\eta} \Pi(t_\xi) \cup \Pi(s_\eta) \subseteq \Pi(t_\eta). \tag{62}$$

For each $\eta < \omega_1$, define $X_\eta \subseteq \mathbb{N} \times \mathbb{N}$ by

$$X_\eta = \bigcup_{n=0}^{\infty} \{n\} \times (b(n) \setminus t_\eta(n)).$$

CLAIM 5.6. *$\langle X_\eta : \eta < \omega_1 \rangle$ is a tower.*

Proof. Obviously, each X_η is infinite. Fix $\xi < \eta < \omega_1$. Suppose by way of contradiction that $X_\eta \setminus X_\xi$ is infinite. Then there exists $f \in \mathbb{R}(b)$ such that

$$\forall n \in \mathbb{N}\ f(n) \in t_\xi(n), \tag{63}$$

$$\exists^\infty n \in \mathbb{N}\ f(n) \notin t_\eta(n). \tag{64}$$

This implies that $f \in \Pi(t_\xi)$ while $f \notin \Pi(t_\eta)$, which contradicts the fact that $\Pi(t_\xi) \subseteq \Pi(t_\eta)$. ∎

CLAIM 5.7. *$\langle X_\eta : \eta < \omega_1 \rangle$ has no infinite pseudo-intersection.*

Proof. Suppose to the contrary that $X \subseteq \mathbb{N} \times \mathbb{N}$ is an infinite pseudo-intersection. Clearly, $D = \operatorname{dom}(X)$ is infinite. For each $n \in D$, choose $i_n \in b(n)$ such that $(n, i_n) \in X$. Let $f \in \mathbb{R}(b)$ be such that

$$f(n) = i_n \quad \text{for all } n \in D. \tag{65}$$

Then $f \notin \Pi(t_\eta)$ for all $\eta < \omega_1$. This contradicts the fact that $\bigcup_{\eta<\omega_1} \Pi(t_\eta) = \mathbb{R}(b)$. ∎

Let $\dot{r}$ name the random real in $\mathbb{R}(b)$.

CLAIM 5.8. $\mathcal{R} \Vdash \forall \eta < \omega_1\ \dot{r} \subseteq^* \check{X}_\eta$.

Proof. This follows from Proposition 5.4, because $\dot{r} \notin \Sigma(t_\eta)$ implies that $\dot{r} \subseteq^* \check{X}_\eta$. This proves the claim, and hence completes Example 5.5. ∎

5.1. *F_σ filters.* Now we work towards a partial analogue of Lemma 5.1 for the random real.

The following is a well known consequence of the fact that there are no (ω, κ^*)-gaps for $\kappa < \mathfrak{b}$:

LEMMA 5.9. *Let $\kappa < \mathfrak{b}$. Suppose that $\langle a_n^\xi : n \in \mathbb{N} \rangle \subseteq \mathbb{R}^+$ $[\xi < \kappa]$ is a family of sequences such that $\lim_{n\to\infty} a_n^\xi = 0$ for all $\xi < \kappa$. Then there exists a sequence $\langle a_n : n \in \mathbb{N} \rangle \subseteq \mathbb{Q}^+$ such that*

(1) $\lim_{n\to\infty} a_n = 0$,
(2) $\forall \xi < \kappa\ \forall^\infty n\ a_n^\xi < a_n$.

Proof. For each $k \in \mathbb{N}$, define

$$A_k = \left\{(n,i) \in \mathbb{N} \times \mathbb{N} : i \geq \frac{n}{k+1}\right\}.$$

For each $\xi < \kappa$, define

$$B_\xi = \{(n,i) \in \mathbb{N} \times \mathbb{N} : i \leq n \cdot a_n^\xi\}.$$

Then $A_k \cap B_\xi$ is finite for all $k \in \mathbb{N}$ and all $\xi < \kappa$. Hence there exists an infinite $A \subseteq \mathbb{N} \times \mathbb{N}$ such that $A_k \subseteq^* A$ for all $k \in \mathbb{N}$, and $A \cap B_\xi =^* \emptyset$ for all $\xi < \kappa$. Assume without loss of generality that $A_0 \subseteq A$. Then $\langle a_n : n \in \mathbb{N} \rangle \subseteq \mathbb{Q}^+$ given by

$$a_n = \frac{\min\{i \in \mathbb{N} : (n,i) \in A\}}{n} \quad \text{for all } n \in \mathbb{N}$$

is well defined, and as needed. ∎

THEOREM 5.10. *Suppose that* $\mathcal{X} \subseteq \mathbb{N}^{[\infty]}$ *with* $|\mathcal{X}| < \mathfrak{b}$. *Then if*

$$\mathcal{R} \Vdash \text{“}\check{\mathcal{X}} \text{ has an infinite pseudo-intersection”},$$

then $\mathcal{X}$ *is contained in an* F_σ *filter.*

Proof. Let $\mathcal{X} \subseteq \mathbb{N}^{[\infty]}$ with cardinality $\kappa < \mathfrak{b}$, and let X_ξ $[\xi < \kappa]$ enumerate $\mathcal{X}$. Suppose that $f : I \to \mathcal{P}(\mathbb{N})$ is a measurable infinitary pseudo-intersection of $\mathcal{X}$. Let $e_f : I \to \mathbb{N}^{\mathbb{N}}$ be the measurable function where $e_f(x)$ is the strictly increasing enumeration of $f(x)$ for all x. Let $K \subseteq I$ be a compact set with positive measure such that

$$f \restriction K \text{ is continuous,} \tag{66}$$

$$f(x) \text{ is infinite for all } x \in K. \tag{67}$$

Using (66), (67) and compactness, we can obtain an $S : \mathbb{N} \to \mathbb{N}^{[<\infty]}$ such that for all $n \in \mathbb{N}$,

$$\sum_{l \in S(n)} \|\dot{e}_f(n) = l\| \geq K, \tag{68}$$

and $\|\dot{e}_f(n) = l\| \neq 0$ for all $l \in S(n)$. Then as in the proof of Lemma 4.18,

$$\lim_{n\to\infty} \min S(n) = \infty. \tag{69}$$

For each $\xi < \kappa$ and each $n \in \mathbb{N}$, define

$$E_\xi(n) = \{l \in S(n) : l \notin X_\xi\} \quad \text{and} \quad W_\xi(n) = \sum_{l \in E_\xi(n)} K \cdot \|\dot{e}_f(n) = l\|.$$

Clearly,

$$\lim_{n\to\infty} \mu(W_\xi(n)) = 0 \quad \text{for all } \xi < \kappa. \tag{70}$$

Hence by Lemma 5.9, there exists $h : \mathbb{N} \to \mathbb{Q}^+$ such that

$$\lim_{n\to\infty} h(n) = 0, \tag{71}$$

$$\forall \xi < \kappa\ \forall^\infty n\ h(n) > \mu(W_\xi(n)). \tag{72}$$

Define $g : \mathbb{N} \to \mathbb{N}$ by

$$g(n) = \left\lfloor \frac{1}{h(n)} \right\rfloor \quad \text{for all } n \in \mathbb{N}.$$

For each $n \in \mathbb{N}$, let $\langle a_n^i : i < m_n \rangle$ be a 1-1 enumeration of all $F \subseteq S(n)$ such that

$$\mu\Big(\sum_{l\in F} K \cdot \|\dot{e}_f(n) = l\|\Big) > 1 - h(n). \tag{73}$$

Notice that

$$\forall n \in \mathbb{N}\ \forall s \in m_n^{[g(n)]}\ \bigcap_{i\in s} a_n^i \neq \emptyset, \tag{74}$$

$$\forall \xi < \kappa\ \forall^\infty n \in \mathbb{N}\ \exists i < m_n\ a_n^i \subseteq X_\xi. \tag{75}$$

Therefore $\mathcal{X}$ is contained in the F_σ filter generated by

$$\{X \subseteq \mathbb{N} : \forall^\infty n \in \mathbb{N}\ \exists i < m_n\ a_n^i \subseteq X\}, \tag{76}$$

and this filter is proper by (69). ∎

The following corollary is immediate:

COROLLARY 5.11. *Suppose that* $\mathcal{X} \subseteq \mathbb{N}^{[\infty]}$ *with* $|\mathcal{X}| < \min\{\mathfrak{b}, \mathfrak{f}\}$. *Then if*

$$\mathcal{R} \Vdash \text{“}\check{\mathcal{X}} \text{ has an infinite pseudo-intersection”},$$

then $\mathcal{X}$ *has an infinite pseudo-intersection.*

We can apply the proof of Theorem 5.10 to obtain an example of a tower contained in an F_σ filter for which the random algebra adds an infinite pseudo-intersection.

COROLLARY 5.12. *Suppose that* $V \models \mathrm{CH}$ *and that* d *is either a Hechler real or a Laver real over* V (*see* [BD85] *or* [BJ95], *respectively*). *Then*, *in* $V[d]$, *there is a nonextendible tower* $\mathcal{T}$ *contained in an* F_σ *filter such that*

$$\mathcal{R} \Vdash \text{“}\check{\mathcal{T}} \text{ has an infinite pseudo-intersection”}.$$

Proof. In V: Let $\mathcal{T}$ be the tower constructed in Example 5.5. Then there is a measurable infinitary $f : I \to \mathcal{P}(\mathbb{N} \times \mathbb{N})$ which is a pseudo-intersection of the tower of (constant) functions $\mathcal{T}$. As noted in Section 2, we can insist that f is a Borel function.

In $V[d]$: Identifying f with its Borel code, by the absoluteness of Borel notions, we still have

$$\|\dot{f} \subseteq^* T\| = 1 \quad \text{for all } T \in \mathcal{T}, \tag{77}$$

and therefore the random algebra extends the tower.

Next we show that $\mathcal{T}$ has no infinite pseudo-intersection. First suppose that d is a Hechler real. Then we are done by the result mentioned at the beginning of the section. Now suppose that d is a Laver real. Then by the Laver property (see [BJ95]), we still have

$$\bigcup_{\eta<\omega_1} \Pi(t_\eta) = \mathbb{R}(b), \tag{78}$$

and the proof of Claim 5.7 goes through, as needed.

It remains to show that $\mathcal{T}$ is contained in an F_σ filter. However, the real d can play the same role that the cardinal $\mathfrak{b}$ did in the proof of Theorem 5.10, and hence we are done. (Note that a Laver real is also a dominating real (see [BJ95]).) ∎

5.2. *Sufficient conditions for* $\mathfrak{t}_\mu \le \mathfrak{t}$

LEMMA 5.13. (1) $\mathfrak{t}_\mu \le \mathfrak{t}$ *implies* $\mathfrak{p}_\mu \le \mathfrak{p}$,
(2) $\mathfrak{f}_\mu \le \mathfrak{f}$ *implies* $\mathfrak{t}_\mu \le \mathfrak{t}$.

Proof. (1). We take $\kappa < \mathfrak{p}_\mu$ and show that $\kappa < \mathfrak{p}$. Let $\mathcal{X}$ be a filter base of cardinality κ. Since $\kappa < \mathfrak{p}_\mu$, $\mathcal{X}$ has an infinitary measurable pseudo-intersection from I into $\mathcal{P}(\mathbb{N})$. Hence, $\mathcal{R} \Vdash$ "$\check{\mathcal{X}}$ has an infinite pseudo-intersection". It follows from Proposition 2.11 that $\kappa < \mathfrak{p}_\mu \le \mathfrak{b}$. Moreover, since every tower of functions is also filtered, we have $\mathfrak{p}_\mu \le \mathfrak{t}_\mu$. Hence by Theorem 1.13, $\kappa < \mathfrak{p}_\mu \le \mathfrak{t}_\mu \le \mathfrak{t} \le \mathfrak{f}$. Thus by Corollary 5.11, $\mathcal{X}$ has an infinite pseudo-intersection, proving that $\kappa < \mathfrak{p}$.

(2) can be proved similarly. ∎

NOTATION. Let $\mathcal{M}$ denote the meager ideal.

THEOREM 5.14. *Any of the following conditions implies that* $\mathfrak{t}_\mu \le \mathfrak{t}$, *and hence also* $\mathfrak{p}_\mu \le \mathfrak{p}$.

(1) $\mathfrak{t} = \mathfrak{b}$.
(2) $\mathfrak{t} < \mathfrak{f}$.
(3) $\mathfrak{t} = \operatorname{add}(\mathcal{M})$.
(4) $\mathfrak{t} < \operatorname{cov}(\mathcal{N})$.
(5) $2^{\mathfrak{t}} > \mathfrak{c}$.
(6) *Define*

$$\mathfrak{t}_1 = \min\{\mathcal{T} \subseteq \mathbb{N}^{[\infty]} : \mathcal{T} \textit{ is a tower which is not contained in an } F_\sigma \textit{ filter}\}.$$

Trivially, $\mathfrak{t}_1 \ge \mathfrak{t}$. *A sufficient condition is* $\mathfrak{t} = \mathfrak{t}_1$.

Proof. The first statement follows form the fact that $\mathcal{R} \Vdash \dot{\mathfrak{b}} = \check{\mathfrak{b}}$ (this is so by Proposition 2.11). And (2) follows from (1) and Theorem 5.10. Statement (3) follows from the fact that $\mathcal{R} \Vdash \operatorname{add}(\mathcal{M}) = \operatorname{add}(\mathcal{M})^V$ (see [BRS96]), and the inequality $\mathfrak{t} \le \operatorname{add}(\mathcal{M})$ (see [PS87]).

Now we prove (4). Let $\langle A_\xi : \xi < \mathfrak{t}\rangle$ enumerate a tower with no infinite pseudo-intersection. Suppose towards a contradiction that $f : I \to \mathcal{P}(\mathbb{N})$ is an infinitary pseudo-intersection of $\langle A_\xi : \xi < \mathfrak{t}\rangle$. Then

$$(79)\qquad \mu(\|\dot f \text{ is finite}\|) = 0,$$

$$(80)\qquad \mu(\|\dot f \not\subseteq^* A_\xi\|) = 0 \quad \text{for all } \xi < \mathfrak{t}.$$

By the assumption on $\mathrm{cov}(\mathcal{N})$, the union of $\|\dot f$ is finite$\|$ and $\|\dot f \not\subseteq^* A_\xi\|$ $[\xi < \mathfrak{t}]$ does not cover I. But if we pick an $x \in I$ outside this union, then $f(x)$ is an infinite pseudo-intersection of $\langle A_\xi : \xi < \mathfrak{t}\rangle$, giving a contradiction.

To see that (5) holds, note that there is a sequence $\mathcal{T}_z$ $[z \in 2^{\mathfrak{t}}]$ of towers of length $\mathfrak{t}$ with the property that

$$(81)\qquad z_0 \neq z_1 \quad \text{implies} \quad \exists A \in \mathcal{T}_{z_0}\ \exists B \in \mathcal{T}_{z_1}\ A \cap B \text{ is finite}$$

(see [vD84, §3], [Bla99]). On the other hand, there are only continuum many measurable functions from I into $\mathcal{P}(\mathbb{N})$ modulo the equivalence relation $f \sim g$ iff $\|\dot f = \dot g\| = 1$, because we can choose a Borel function from each equivalence class. Therefore, if $2^{\mathfrak{t}} > \mathfrak{c}$, we cannot find measurable infinitary pseudo-intersections for all of the towers $\mathcal{T}_z$ $[z \in 2^{\mathfrak{t}}]$.

To prove the last statement, note that by (1) we may assume that $\mathfrak{t} < \mathfrak{b}$, and then the result follows from Theorem 5.10. ∎

We conclude with a conjecture.

Conjecture 5.15. *The following statements are consistent with* ZFC:

(1) $\mathfrak{f}_\mu > \mathfrak{f}$,
(2) $\mathfrak{p}_\mu > \mathfrak{p}$,
(3) $\mathfrak{t}_\mu > \mathfrak{t}$.

We should mention that it is at least conceivable that research along the lines of Conjecture 5.15 could lead to a solution to the famous open problem of Rothberger: "does ZFC $\vdash \mathfrak{p} = \mathfrak{t}$?" (see [Bla99] and [Vau90]). For suppose we are able to obtain a model where say $\mathfrak{p}_\mu = \aleph_2$ yet $\mathfrak{t}_\mu = \aleph_3$. Then regardless of whether $\mathfrak{p} = \mathfrak{t}$ in this model, adding one random real to this model yields $\mathfrak{p} < \mathfrak{t}$.

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