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Dynamics on Hubbard trees

by

Lluís Alsedà and **Núria Fagella** (Barcelona)

Abstract. It is well known that the Hubbard tree of a postcritically finite complex polynomial contains all the combinatorial information on the polynomial. In fact, an abstract Hubbard tree as defined in [23] uniquely determines the polynomial up to affine conjugation. In this paper we give necessary and sufficient conditions enabling one to deduce directly from the restriction of a quadratic Misiurewicz polynomial to its Hubbard tree whether the polynomial is renormalizable, and in this case, of which type. Moreover, we study dynamical features such as entropy, transitivity or periodic structure of the polynomial restricted to the Hubbard tree, and compare them with the properties of the polynomial on its Julia set. In other words, we want to study how much of the "dynamical information" about the polynomial is captured by the Hubbard tree.

1. Introduction. In this paper we deal with Hubbard trees of Misiurewicz polynomials, for the most part of degree two, and the dynamical properties of such a polynomial f when restricted to its Hubbard tree $H = H(f)$. A Hubbard tree and the restricted map catch the essence of the dynamics of f. Indeed, from the combinatorial information given by a map on an abstract Hubbard tree (satisfying certain conditions) one can obtain the affine class of the actual polynomial realizing the tree as its Hubbard tree. But since it is easier to deal with the dynamics on the tree, it is of interest to describe how one reads off properties of the polynomial f directly from the dynamics of the tree map $f|_{H(f)}$.

The main results of the paper (see Subsection 1.5) give necessary and sufficient conditions enabling one to deduce directly from $f|_{H(f)}$ whether the polynomial is renormalizable, and of which type. Renormalization is a very important concept in holomorphic dynamics; it is therefore of interest to have a purely combinatorial characterization of this notion. Our results also show that other dynamical properties of the polynomial on its Julia set, such as density of periodic points, total transitivity or maximal topological

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entropy, are not always inherited by $f|_{H(f)}$, even though the Hubbard tree is an invariant subset of the Julia set. Even more, we give a precise characterization of those polynomials for which these properties pass on to their Hubbard trees. This classification is strongly related to renormalization.

In what follows we introduce notation and summarize the basic facts which are necessary to state the main results. For more details and general background we refer to [7, 8, 12, 19, 16, 20, 25].

1.1. *Complex polynomials.* Let f be a complex polynomial and let $z \in \mathbb{C}$. We denote by $\mathcal{O}_f^+(z)$ the forward orbit of z under f, i.e. the set $\{f^n(z) \mid n \in \mathbb{N} \cup \{0\}\}$. If $f^k(z) = z$ for some $k \in \mathbb{N}$ then z is a *periodic point*. The smallest such k is called the *period* of z. If z is not periodic but $\mathcal{O}_f^+(z)$ is finite we say that z is *preperiodic*.

For a complex polynomial f of degree $d \geq 2$ the point at infinity is a superattracting fixed point. This allows us to define the *filled Julia set* of f, denoted by $K(f)$, as the complement of the basin of attraction of infinity. That is,

$$K = K(f) = \{z \in \mathbb{C} \mid \mathcal{O}_f^+(z) \text{ is bounded}\}.$$

The filled Julia set is clearly compact and totally invariant (that is, $f(K) = f^{-1}(K) = K$). The common boundary of $K(f)$ and of the basin of attraction of infinity is called the *Julia set* of f and is denoted by $J(f)$. The Julia set is also totally invariant and is the set of points where "chaotic dynamics" occurs.

We say that a point $\omega \in \mathbb{C}$ is a *critical point* of f if $f'(\omega) = 0$. Then its orbit is called a *critical orbit*. The behavior of the critical points under iteration determines in many ways the topology of $K(f)$ and the dynamics of f. As an example, let $C(f)$ denote the set of critical points of f. Then $K(f)$ is known to be connected if and only if $C(f) \subset K(f)$.

We are interested in the special case where the critical orbits are finite, since these are the polynomials for which Hubbard trees are well defined. We call these polynomials *postcritically finite* (or PCF for short) and they can be of three types. If all critical orbits are periodic then f is called a *center*. If the critical orbits are all preperiodic then f is called a *Misiurewicz polynomial*. In this case $K(f)$ has empty interior and hence $K(f) = J(f)$ (see [14] and the Appendix for some examples). Finally, a PCF polynomial could exhibit both types of critical orbits. In all cases $K(f)$ is connected and locally connected (see [14]). In this paper, we will be concerned with Misiurewicz polynomials since our questions have trivial answers when there is a periodic critical point. We also restrict ourselves to degree two in all what concerns renormalization. Hence, in this case, our polynomials are conjugate to one in the family $z^2 + c$, and thus, $\omega_0 = 0$ is the only critical point. Generalizations to higher degrees will be the object of a later paper.

1.2. *Renormalization*

Definition. Let $f(z) = z^2 + c$ be such that $J(f)$ is connected. For $n > 1$ we say that f^n is *renormalizable* (or that f is *renormalizable for* $n > 1$) if there exist open bounded sets U and V isomorphic to disks such that

(1) $\overline{U} \subset V$,

(2) $f^n(U) = V$ and $f^n : U \to V$ is proper of degree two, i.e. every point in V has two preimages under f in U counted with multiplicity,

(3) $f^{kn}(0) \in U$ for all $k \geq 0$.

We define the *small filled Julia set* of the renormalization, K_n, as the points that never leave U under iteration of f^n.

It follows from the Straightening Theorem (see [15]) that there exists a unique (up to affine conjugation) quadratic polynomial Q such that f^n and Q are hybrid equivalent, that is, there exists a quasiconformal conjugacy h between f^n and Q on neighborhoods of K_n and $K(Q)$ respectively, such that h maps K_n to $K(Q)$ and $\overline{\partial} h = 0$ on K_n.

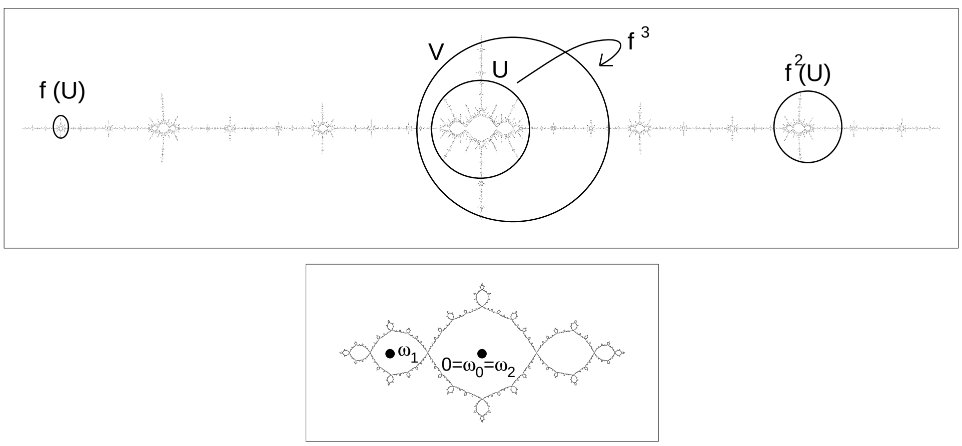

Fig. 1. The Julia set of f_c with $c = -1.772892\ldots$ The map f_c^3 is renormalizable and the small filled Julia set is quasiconformally homeomorphic to the filled Julia set of f_{-1} (lower figure). This is an example of renormalization of disjoint type.

Hence, in particular, the small filled Julia set K_n is homeomorphic to the filled Julia set of an actual quadratic polynomial (see Figure 1). We define the cycle

$$K_n(0) = K_n, \quad K_n(1) = f(K_n(0)), \ \ldots \ , K_n(n-1) = f(K_n(n-2)),$$

where $f(K_n(n-1)) = K_n(0)$. We call each $K_n(i)$ a *small filled Julia set* of the renormalization. Likewise, we define the *small Julia sets* as $J_n(i) = \partial K_n(i)$ for $i = 0, 1, \ldots, n-1$. McMullen [19] showed that these sets can intersect each other in at most one single point, which of course must be a fixed point of f^n.

Quadratic polynomials have only two fixed points. We call them α and β, where the latter is set to be the most repelling one. We denote the α- and β-fixed points of $J_n(i)$ (under f^n) by $\alpha_n(i)$ and $\beta_n(i)$ for $0 \le i \le n-1$. From the above it follows that, for a given n, the renormalization can only be of one of the following types:

(i) f^n is renormalizable of *disjoint type* if the small Julia sets are all disjoint.

(ii) f^n is renormalizable of *β-type* if all intersections among small Julia sets occur at their β-fixed points.

(iii) f^n is renormalizable of *α-type* or *crossed type* if all intersections among small Julia sets occur at their α-fixed points.

We remark that PCF polynomials are at most finitely many times renormalizable. However, each stage of the renormalization process can be of any of the three types.

1.3. *Hubbard trees.* Douady and Hubbard [14] introduced a combinatorial description of the dynamics of PCF polynomials by associating to each filled Julia set a tree, called the Hubbard tree. For Misiurewicz polynomials the Hubbard tree is defined as follows. Given a subset A of $J(f)$ we denote by $[A]$ the *convex hull* of A in $J(f)$, i.e. the smallest closed connected subset of $J(f)$ that contains A.

Definition. Let f be a Misiurewicz polynomial and let $\Omega(f)$ denote the *postcritical set* $\bigcup_{\omega\in C(f)} \mathcal{O}_f^+(\omega)$, which in this case is finite and contained in $J(f)$. We define the *Hubbard tree* of f as

$$H(f) = [\Omega(f)].$$

Recall that a *tree* is a topological space which is uniquely arcwise connected and homeomorphic to a union of finitely many copies of the closed unit interval. Note that each Hubbard tree is indeed a tree. To see this, one only has to show that for any two points $x, y \in J(f)$ there is a unique Jordan arc in $J(f)$ that joins x and y. The existence of this arc follows from the fact that $J(f)$ is connected and locally connected in $\mathbb{S}^2$; the uniqueness follows from the fact that $J(f)$ has empty interior and is contractible. Therefore, since $H(f)$ is the union of the Jordan arcs in $J(f)$ joining $x, y \in \Omega(f)$, it follows that $H(f)$ is a tree.

If T is a tree and $x \in T$, the *valence* of x is defined to be the number of components of $T \setminus \{x\}$. A point of valence 1 is called an *endpoint* and a point of valence greater than 2 is called a *branching point*. We define the set of *vertices* of $H(f)$ as

$$V(f) = \Omega(f) \cup \{v \in H(f) \mid v \text{ is a branching point}\}.$$

The closure of the arc in $H(f)$ joining two consecutive vertices is called an *edge*. Note that any endpoint of $H(f)$ belongs to $\Omega(f)$ and, hence, $V(f)$ contains all points of $H(f)$ with valence different from 2.

It is easy to check that the Hubbard tree is a forward invariant subset of the Julia set (see Lemma 1.10 in [23]). The set of vertices is also forward invariant since $\Omega(f)$ is forward invariant and non-critical branching points must be mapped to branching points (because the map is a local homeomorphism). For these and other basic properties of Hubbard trees we refer to [23]; however in Section 2, we study the features that we use in proving the main results of this paper.

REMARK 1.1. Hubbard trees are in fact defined for any PCF polynomial (see [14, 23]). If f is not Misiurewicz, it follows from the definition that the Hubbard tree intersects the basins of attraction of points of $\Omega(f)$ in $K(f) \setminus J(f)$.

The interest of Hubbard trees lies in the fact that they contain all the combinatorial information on the polynomial. Indeed, Douady and Hubbard showed that if we retain the dynamics and the local degree of f on the set of vertices, the way the tree is embedded in the complex plane and a little bit of extra information (which we will not make precise here), then different PCF polynomials (not conjugate as dynamical systems) give rise to different Hubbard trees. A variation of the converse is also true and was proved in a general version by A. Poirier [23].

1.4. *Dynamical properties.* Next we are going to describe the basic properties of the Julia set with respect to periodic points, topological entropy and transitivity. The notion of topological entropy was introduced by Adler, Konheim and McAndrew in [1], to which we refer for a precise definition and basic properties (see also, for instance, [13] or [4]). In what follows, the topological entropy of a map f will be denoted by $h(f)$. Next we recall the definition of transitivity. Let $f : X \to X$ be a continuous map of a compact metric space. We say that f is (*topologically*) *transitive* if for any two non-empty open sets U and V in X, there is a positive integer k such that $f^k(U) \cap V \neq \emptyset$. It is well known (see [26]) that if either X has no isolated points or f is onto, then f is transitive if and only if it has a dense orbit (i.e. if there exists $x \in X$ such that $\mathcal{O}_f^+(x)$ is dense in X). When f^n is transitive for each $n \in \mathbb{N}$ then f is called *totally transitive* (see [5]).

The following proposition states some basic properties of the Julia set, all of which are well known (see for example [7]).

Proposition 1.2. *For a complex polynomial f of degree $d \geq 2$ the following statements hold.*

(a) *The periodic points of f are dense in $J(f)$.*
(b) *$f|_{J(f)}$ is totally transitive.*
(c) *The topological entropy of $f|_{J(f)}$ is $\log d$.*
(d) *$f|_{J(f)}$ has periodic points of each period except maybe period 2.*

In this context it is natural to ask the following question. Apart from the topological and combinatorial structure of the Julia set, what dynamical information can be deduced directly from the Hubbard tree? In this case, by dynamical information we mean dynamical features such as transitivity, topological entropy, density of periodic points and periodic structure. More precisely, which of the properties in Proposition 1.2 (if any) is still true when we replace $J(f)$ by $H(f)$? If the answer depends on the polynomial, what is a characterization of polynomials for which those properties hold? We will see that these features are generally not inherited by the Hubbard tree except in the case where no renormalization is possible.

1.5. *Main results.* The main results of this paper are the following.

Theorem A. *Let f be a Misiurewicz polynomial of degree 2 and let $H(f)$ be its Hubbard tree. Then the following statements are equivalent*:

(a) *f is renormalizable of disjoint type for some $n \geq 2$.*
(b) *The periodic points of f are not dense in $H(f)$.*
(c) *The map $f|_{H(f)}$ is not transitive.*
(d) *There exists an edge l of $H(f)$ such that $\bigcup_{n\geq 0} f^n(l) \neq H(f)$.*

Theorem B. *Let f be a Misiurewicz polynomial of degree 2 and let $H(f)$ be its Hubbard tree. Then the following statements are equivalent*:

(a) *f is non-renormalizable for any $n \geq 2$.*
(b) *$f|_{H(f)}$ is totally transitive.*
(c) *For any edge l of $H(f)$, there exists $n > 0$ such that $f^n(l) = H(f)$.*

We observe that Theorems A and B refer to sets of Misiurewicz polynomials with no intersection but not complementary to each other. The remaining Misiurewicz polynomials are characterized in the following corollary which follows trivially from negating Theorems A and B simultaneously (see Figure 2).

Corollary C. *Let f be a Misiurewicz polynomial of degree 2 and let $H(f)$ be its Hubbard tree. Then the following statements are equivalent*:

(a) *f is renormalizable but not of disjoint type.*
(b) *$f|_{H(f)}$ is transitive but not totally transitive.*

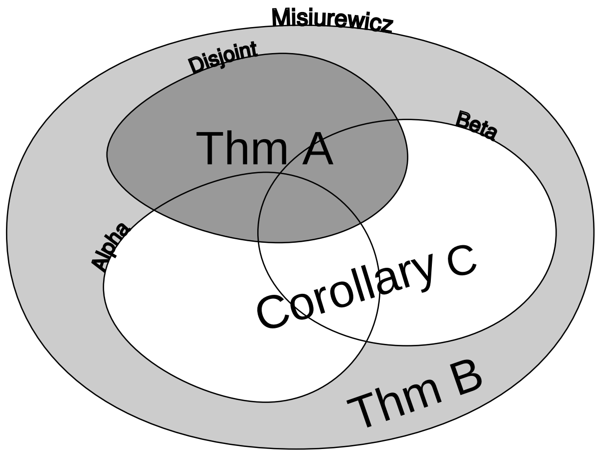

Fig. 2. This diagram shows which polynomials are described by Theorems A and B and Corollary C. Notice that "Disjoint" means "renormalizable of disjoint type for some $n \geq 2$" and likewise for "Alpha" and "Beta".

(c) *For any edge l of $H(f)$ we have $\bigcup_{n\geq 0} f^n(l) = H(f)$ but there exists an edge l^* of $H(f)$ such that $f^n(l^*) \neq H(f)$ for each $n \geq 0$.*

We remark that under the conditions of Theorem B or Corollary C the set of periodic points is dense in $H(f)$. Moreover, the results above give easy criteria to check if a polynomial f is renormalizable and, in that case, of which type. One only needs to construct the Hubbard tree and check the images of its edges. We also note that, in general, density of periodic points does not imply transitivity. It does though in the above cases.

To complete the picture arising from Theorem B, note that from [6] it follows that if $f|_{H(f)}$ is transitive then any of the conditions of Theorem B is equivalent to the fact that $\mathbb{N} \setminus \mathrm{Per}(f|_{H(f)})$ is finite ([1]), where $\mathrm{Per}(f)$ denotes the set of all $n \in \mathbb{N}$ such that f has a periodic point of period n. Even in our case, the converse of this fact is not true without the assumption that $f|_{H(f)}$ is transitive. Indeed, the map $f_{c_2}|_{H(f_{c_2})}$ from Example 2 in the Appendix is not transitive but it has periodic points of all periods.

The relationship between the entropy of f and that of $f|_{H(f)}$ is given in the following theorem.

Theorem D. *Let f be a Misiurewicz polynomial of degree $d \geq 2$ and let $H(f)$ be its Hubbard tree. Then $h(f|_{H(f)}) \leq \log d$ and equality holds if and only if $H(f) = J(f)$.*

([1]) Although this fact has been generalized in [6] to arbitrary transitive maps of trees, it was proved by the first time in a previous version of this paper for tree maps which are transitive restrictions of quadratic polynomials to their Hubbard trees.

We remark that if f is a Misiurewicz polynomial and $H(f) = J(f)$ then $J(f)$ is an interval. Indeed, if $J(f)$ is not an interval, then it contains a point z such that $J(f) \setminus \{z\}$ has more than two connected components. Therefore it contains infinitely many such points. Thus, since $H(f)$ contains finitely many branching points, we obtain $H(f) \subsetneq J(f)$. The exceptional cases where $J(f) = H(f)$ occur, for example, for the monic Chebyshev polynomials.

From [3, Theorem B] the following corollary of Theorem A follows immediately.

Corollary E. *Let f be a Misiurewicz polynomial of degree 2 which is not renormalizable of disjoint type, and let $H(f)$ be its Hubbard tree. Then $f|_{H(f)}$ is transitive and*

$$h(f|_{H(f)}) \geq \frac{\log 2}{\mathrm{End}(H(f))}$$

where $\mathrm{End}(H(f))$ *denotes the number of endpoints of $H(f)$.*

Hence if a Misiurewicz polynomial f of degree 2 is not renormalizable of disjoint type, the entropy of the map $f|_{H(f)}$ is always bounded below by a positive quantity which depends on each particular polynomial. One might then ask whether there exists a universal lower bound for the topological entropy of such polynomials. As shown in the following proposition, the answer to this question is negative even in the case of non-renormalizable polynomials (which are the ones that show the most "chaotic" behavior).

Proposition F. *There exists a family $\{g_n\}_{n\geq 1}$ of non-renormalizable quadratic Misiurewicz polynomials such that $h(g_n|_{H(g_n)})$ tends to zero as n tends to infinity.*

The remainder of the paper is organized as follows. Section 3 contains the proofs of Theorems A, B and D, and Proposition F, while Section 2 is meant to be a summary of the definitions and tools needed for those proofs. These preliminaries are distributed into independent subsections, according to the subject they belong to.

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2. Definitions and preliminaries

2.1. *Quadratic polynomials, renormalization and the Yoccoz puzzle.* Yoccoz puzzles (see for example [18, 21]) are a useful tool to deduce renormalization properties of quadratic polynomials. To define a Yoccoz puzzle we use an orthogonal set of coordinates in the basin of attraction of infinity: equipotentials and external rays. These coordinates are defined in the same way for polynomials of any degree $d \geq 2$ with a connected Julia set.

Let $f(z) = z^2 + c$ have a connected Julia set. Since the point at ∞ is a superattracting fixed point, the dynamics in its basin of attraction, $A(\infty)$, is very simple. One can find a holomorphic change of variables $\psi_f : \mathbb{C} \setminus \mathbb{D} \to A(\infty)$ (called the *Böttcher coordinates* at infinity) that conjugates $f|_{A(\infty)}$ to the map $z \mapsto z^2$ on the complement of the closed unit disk. This change is unique if we require the derivative at infinity to be one.

The image under ψ_f of a circle of radius $\exp(\eta) > 1$ in $\mathbb{C} \setminus \mathbb{D}$ is a simple closed curve in $A(\infty)$ called an *equipotential* of potential η. We denote the potential function defined this way by $G(z) := G_f(z)$. Thus an equipotential of potential η is mapped 2-to-1 under f to an equipotential of potential 2η (see Figure 3). If we parameterize the arguments of the unit circle between 0 and 1, the image under ψ_f of a ray of argument t is called an *external ray of argument* t and denoted by $R_f(t)$. Again, since ψ_f is a conjugacy, an external ray of argument t is mapped to an external ray of argument $2t$ (mod 1). Equipotentials and external rays give us orthogonal coordinates in the superattracting basin.

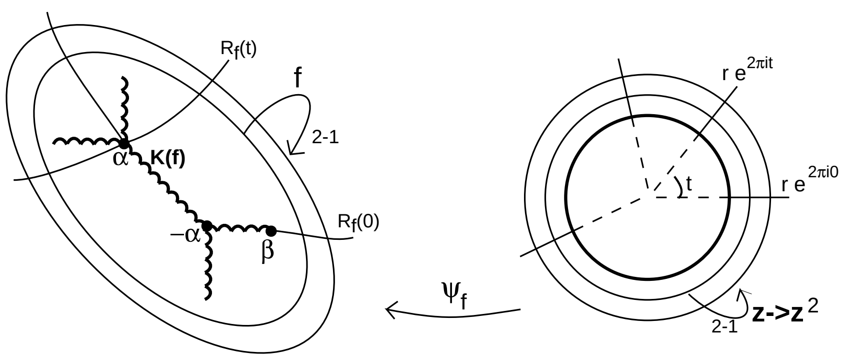


Fig. 3. Böttcher coordinates, equipotentials and external rays

From now on, we assume that both fixed points of f are repelling. Then one fixed point (to be called β) is the landing point of the external ray of argument zero and the other (called α) is the landing point of a cycle of q external rays where $q \geq 2$ (see for example [20, 22]).

Remark 2.1. By a simple combinatorial argument on external rays, using the fact that $\theta \mapsto 2\theta \pmod 1$ is order preserving, one can show that the points $0, f(0), \ldots, f^{q-1}(0)$ lie in different components of $K(f) \setminus \{\alpha\}$.

In what follows we define the Yoccoz puzzle construction and summarize its basic properties and applications, mainly following [21]. As always, let $\omega_0, \omega_1, \ldots$ be the critical orbit, where $\omega_i = f^i(0)$. Let $G(z)$ be the potential function and set $D = \{z \in \mathbb{C} \mid G(z) \leq 1\}$. This is a compact set isomorphic to a disk that contains the filled Julia set. The *Yoccoz puzzle of depth zero* consists of the pieces $P_0(\omega_0), P_0(\omega_1), \ldots, P_0(\omega_{q-1})$ obtained by cutting the region D along the q rays landing at α and labeling them so that the piece $P_0(\omega_i)$ contains the point ω_i. Each piece is a compact set whose boundary contains the α-fixed point, two segments of external rays and a piece of ∂D (see Figure 4).

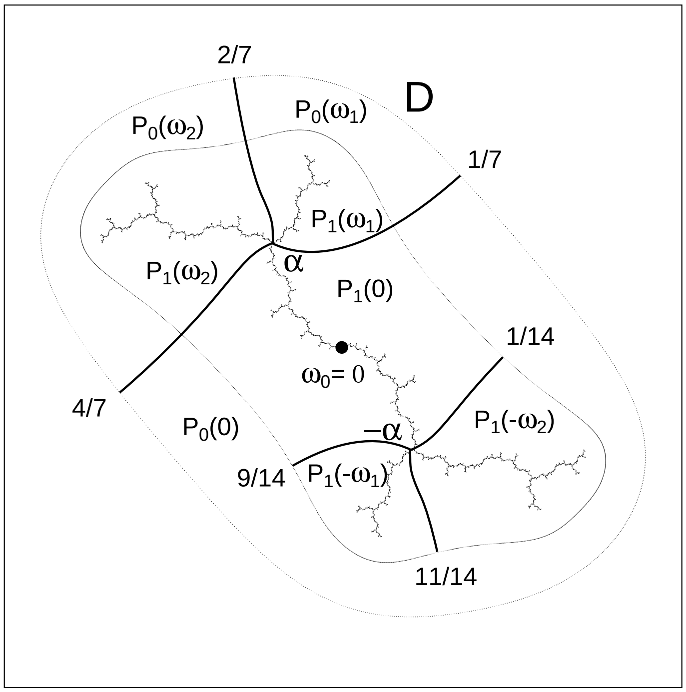


Fig. 4. Some pieces of the Yoccoz puzzle of f_{c_5} of Example 5

The puzzle pieces of depth $d > 0$ are defined by induction to be the connected components of $f^{-1}(P)$ where P ranges over all puzzle pieces of depth $d-1$. The puzzle pieces of depth d have disjoint interiors and each of them is contained in a unique piece of depth $d-1$. Any point $z \in K(f)$ which is not a preimage of α is contained in a unique puzzle piece at each depth, which we denote by $P_d(z)$.

In the next section we will use the following three lemmas to deduce renormalization of disjoint type, β-type and crossed type respectively. We always assume that f is a quadratic polynomial with a connected Julia set, with both fixed points repelling and q external rays landing at α.

LEMMA 2.2 (Lemma 2 of [21]). *Suppose the orbit of the critical point avoids the α-fixed point. If $P_d(0) = P_d(\omega_p)$ for all depths d and some $p > 1$ then f^p is renormalizable.*

LEMMA 2.3 (Lemma 3 of [21]). *If the critical orbit is entirely contained in*

$$P_1(\omega_0) \cup P_1(\omega_1) \cup \ldots \cup P_1(\omega_{q-1})$$

(*that is*, *the union of the puzzle pieces of depth one that touch the fixed point* α), *then f^q is renormalizable of disjoint type or β-type.*

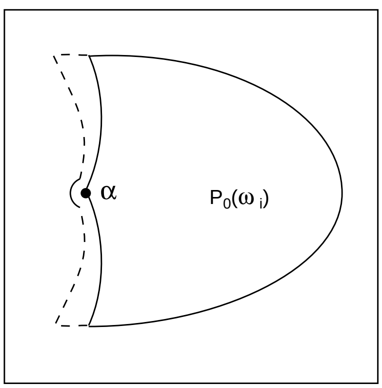
Our contribution in this section is the following lemma (motivated by an example of McMullen) which gives a criterion to deduce renormalization of crossed type.

LEMMA 2.4. *If there exists $n \in \mathbb{N}$ such that n divides q, $2n \leq q$ and $\{\omega_{nk}\}_{k\in\mathbb{N}}$ lies entirely in*

$$P_1(0) \cup P_1(\omega_n) \cup P_1(\omega_{2n}) \cup \ldots \cup P_1(\omega_{q-n}) \\ \cup P_1(-\omega_n) \cup P_1(-\omega_{2n}) \cup \ldots \cup P_1(-\omega_{q-n}),$$

then f^n is renormalizable of crossed type.

The proofs of the lemmas above make use of the so-called *thickened puzzle pieces*. Intuitively, a thickened puzzle piece $\widehat{P}_0(\omega_i)$ (for $0 \leq i \leq q$) is a slight enlargement of the puzzle piece $P_0(\omega_i)$ (see Figure 5). By the



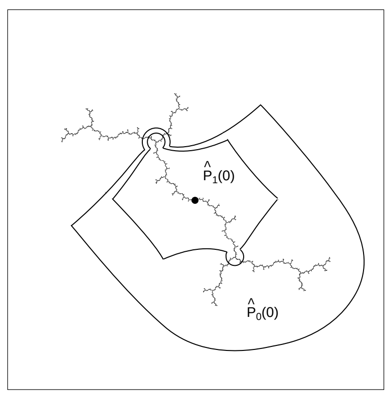


Fig. 5. Sketch of a puzzle piece $P_0(\omega_i)$ and its corresponding thickened puzzle piece $\widehat{P}_0(\omega_i)$

usual inductive procedure, the thickened puzzle pieces of depth $d > 0$ are the connected components of $f^{-1}(\widehat{P})$, where $\widehat{P}$ ranges over all the thickened puzzle pieces of depth $d-1$. The main virtue of these thickened pieces is the following: *If a puzzle piece $P_d(z)$ contains $P_{d+1}(z)$ then the corresponding thickened puzzle piece $\widehat{P}_d(z)$ contains $\widehat{P}_{d+1}(z)$ in its interior.* Indeed, in all the three lemmas above one can find puzzle pieces ($P_d(0)$ for d large enough in the case of Lemma 2.2 and $P_1(0)$ in the case of Lemma 2.3) that satisfy all the requirements in the definition of a renormalizable map, except that these pieces are contained, but not compactly contained, in their images under the appropriate iterate of f. The use of thickened pieces takes care of this problem.

Remark 2.5. Lemma 3 of [21] (Lemma 2.3 here) is included in a chapter where it is generally assumed that the critical orbit does not hit the fixed point α. To prove it though, one only needs to work with thickened pieces up to level one and hence this assumption is not necessary.

Proof of Lemma 2.4. Set

$$U' = P_n(0) \cup P_n(\omega_n) \cup P_n(\omega_{2n}) \cup \ldots \cup P_n(\omega_{q-n}) \cup P_n(-\omega_n) \cup P_n(-\omega_{2n}) \cup \ldots \cup P_n(-\omega_{q-n})$$

and

$$V' = P_0(\omega_n) \cup P_0(\omega_{2n}) \cup P_0(\omega_{3n}) \cup \ldots \cup P_0(0).$$

Let $L_i = P_0(\omega_i) \cap J(f)$ for $i = 0, 1, \ldots, q-1$, which are the connected components of $J(f) \setminus \{\alpha\}$. Then observe that $f(L_i) = L_{i+1}$ for $i = 1, \ldots, q-2$, $f(L_{q-1}) = L_0$ and $f(L_0) = J(f)$. Since $-\alpha \in L_0 \subset J(f)$, it follows that all kth preimages of $-\alpha$ for $k < q$ are contained in $L_0 \cup L_{q-1} \cup \ldots \cup L_{q-k}$. But any kth preimage of $-\alpha$ is a $(k+1)$th preimage of α and vice versa, hence this implies that the set $L_1 \cup \ldots \cup L_{q-n}$ does not contain any nth or earlier preimage of α. From this fact, it follows that the rays bounding $P_n(\omega_i)$ are the same as the ones bounding $P_0(\omega_i)$, for $i = n, 2n, \ldots, q-n$ (see Figure 6).

It is easy to check that $U' \subset V'$ and that f^n maps U' in V' with degree two. Moreover, the orbit of the critical point (under f^n) is contained in U'. To conclude that f^n is renormalizable we would need to see that U' is contained *in the interior* of V' and that the critical orbit (under f^n) is entirely contained *in the interior* of U' (which is not the case if the orbit hits the fixed point α). This is obviously not true but if we replace all puzzle pieces by thickened puzzle pieces then all requirements are satisfied. To see that the renormalization is of crossed type, observe that on the one hand, $K_n(0)$ contains all the iterates $0, \omega_n, \omega_{2n}, \ldots$ and hence it must contain the fixed point α. On the other hand, $K_n(1)$ contains $\omega_1, \omega_{n+1}, \omega_{2n+1}, \ldots$ and hence it also contains α. We conclude that $K_n(0) \cap K_n(1) = \{\alpha\}$ since two small filled Julia sets can intersect in at most at one point. ∎

2.2. *Hubbard trees.* Let f be a Misiurewicz polynomial of degree $d \geq 2$ and let $H(f)$ be its Hubbard tree. To ease notation, in the rest of this subsection, we set $H = H(f)$. The following proposition states some properties of

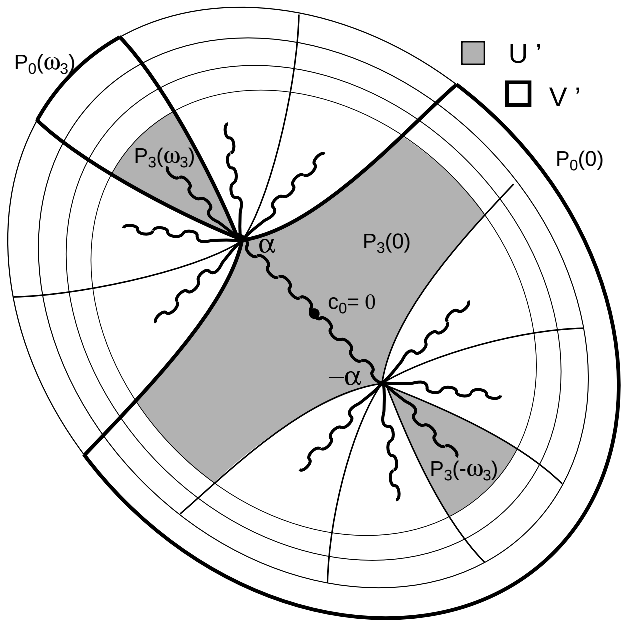


Fig. 6. Sketch of the construction in the proof of Lemma 2.4 for $n = 3$ and $q = 6$

Hubbard trees that we will need in the proofs of the main theorems. For a complete description we refer to [23]. In what follows, when speaking about the *interior* of $[x, y]$ we will mean the set $[x, y]\backslash\{x, y\}$ instead of the usual topological interior. Note that if $[x, y]\backslash\{x, y\}$ contains a point of valence larger than 2 then this is not an interior point of $[x, y]$ in the topological sense.

Proposition 2.6. *Let f be a Misiurewicz polynomial of degree $d \geq 2$ and let H be its Hubbard tree.*

(a) *If the interior of $[x, y]$ does not contain a critical point then f is one-to-one on $[x, y]$.*

(b) *The preperiodic points are dense in H.*

(c) *Let $x, y \in H$ be two preperiodic points. Then there exists $n > 0$ such that the interior of $[f^n(x), f^n(y)]$ contains a critical point.*

(d) *Given $x, y \in H$ there exists $n > 0$ such that $[f^n(x), f^n(y)]$ contains a whole edge of H.*

Moreover, *when $d = 2$*:

(e) *H has no invariant subtree.*

(f) *The fixed point α of f is a point of H of valence greater than one.*

P r o o f. Statement (a) is trivial. To show (b), let $x, y \in H$ and assume that $[x, y]$ contains no preperiodic point in its interior. Choose t in the interior of $[x, y]$ and choose $q \in J(f)$, periodic and sufficiently close to t so that q can be joined to H by an arc in $J(f)$ through a point in the interior of $[x, y]$ (this is possible because $J(f)$ is connected and locally connected, $H \subset J(f)$ and the periodic and branching points are dense in $J(f)$). Let p be the joining point in the interior of $[x, y]$. It follows that $J(f) \setminus \{p\}$ has more than two connected components. For a Misiurewicz polynomial, every such point is preperiodic (see Prop. 3.2 in [23]). Hence p is preperiodic, contradicting the assumption that $[x, y]$ contained no preperiodic point in its interior.

Statement (c) is Proposition 1.18 of [23] but we include its proof for completeness. Assume the conclusion is false. Then the interior of $[x, y]$ contains no preimage of a critical point and hence f^m is injective on $[x, y]$ for all $m > 0$. By taking high enough iterates we may assume that x and y are periodic. Let m be the least common multiple of the periods of x and y. Since there are a finite number of fixed points of f^m, we may assume that $[f^m(x), f^m(y)] = [x, y]$ does not contain any other such point. But both endpoints are repelling (since f is Misiurewicz) and f^m is a homeomorphism of $[x, y]$ onto itself. It follows that there must be another fixed point of f^m in the interior, contrary to what was assumed.

To show (d), take two different preperiodic points in the interior of $[x, y]$. This is possible by (b). By (c), there exists $k > 0$ such that $[f^k(x), f^k(y)]$ contains a vertex in its interior (recall that each critical point is a vertex by definition). If it contains two vertices we are done. Otherwise, let v be the unique vertex in the interior of $[f^k(x), f^k(y)]$ and apply the above procedure again to $[v, f^k(y)]$, to obtain $n > 0$ such that $[f^n(v), f^{n+k}(y)]$ contains a vertex v' in its interior. Then, since the set of vertices is forward invariant, we conclude that $[f^n(v), v']$ contains the desired edge.

To see (e) let $T \subseteq H$ be an invariant subtree of H (i.e. a non-empty, compact, connected, forward invariant subset of H). Applying (c) we find that T must contain the critical point and hence the critical orbit. Thus, since T is connected, the convex hull of the critical orbit, i.e. the Hubbard tree H, must be contained in T and we are done.

As we saw in the preceding section (see Remark 2.1), ω_0 and ω_1 belong to different components of $J(f) \setminus \{\alpha\}$. Hence, by definition, $\alpha \in [\omega_0, \omega_1] \subset H$. This proves (f). ■

2.3. *Transitive maps on trees.* This subsection summarizes some results and techniques for continuous maps on trees. The first proposition shows that a transitive non-totally transitive map gives a useful decomposition of the space. Its proof follows from a more general theorem of Blokh (for non-connected graphs) stated in [11] and proved in [10] (see also [5] for a version of this result for locally connected compact metric spaces).

PROPOSITION 2.7. *Let T be a tree and let $f : T \to T$ be transitive. Then either f is totally transitive or there exist closed, connected subsets X_0, $X_1, \ldots, X_{k-1}$ of T with non-empty interior and a fixed point y of f of valence larger than or equal to k such that*:

(a) $T = \bigcup_{i=0}^{k-1} X_i$,
(b) $X_i \cap X_j = \{y\}$ *for all* $i \neq j$,
(c) $f(X_i) = X_{i+1\,(\mathrm{mod}\,k)}$ *for* $i = 0, 1, \ldots, k-1$.

In particular, $f^k|_{X_i}$ *is transitive for all* $i \in \{0, 1, \ldots, k-1\}$.

The next result is proved by Blokh in [11] when the space is a graph, by using a spectral decomposition (and in [3] for other types of metric spaces).

PROPOSITION 2.8. *Let T be a tree and let $f : T \to T$ be transitive. Then the set of periodic points of f is dense.*

The rest of this subsection outlines a common technique to compute the topological entropy of tree maps which are "monotone" when restricted to each of its edges.

Let $f : T \to T$ be a tree map and let $P \subset T$ be a finite forward invariant set of f which contains all endpoints and branching points of f. The closure of a connected component of $T \setminus P$ will be called a *P-basic interval*. Notice that each P-basic interval is homeomorphic to a closed interval of the real line. The *f-graph* of P is the oriented generalized graph having the P-basic intervals as vertices, and arrows defined as follows. If K and L are P-basic intervals and K has m subintervals with pairwise disjoint interiors such that the f-image of each of these intervals contains L, then there are m arrows from K to L. The *transition matrix* of the f-graph of P is the matrix of size equal to the number of P-basic intervals such the the i,j-entry is the number of arrows from the vertex i to the vertex j. If M is such a matrix, let its largest eigenvalue (if it exists) be denoted by $\varrho(M)$. We note that, since M is a non-negative integral matrix, in view of the Perron–Frobenius Theorem (see [17]), $\varrho(M)$ exists and is in fact the spectral radius of M. The map f is called *P-monotone* if the image of each P-basic interval is homeomorphic to a closed interval of the real line and is monotone when considered as an interval map.

The next result gives the desired formula for the topological entropy of a P-monotone map. It can be proved in a similar way to [2, Theorems 4.4.3 and 4.4.5].

THEOREM 2.9. *Let $f : T \to T$ be a tree map and let $P \subset T$ be a finite forward invariant set of f which contains all endpoints and branching points* *of f. Let M denote the transition matrix of the f-graph of P. Then $h(f) \geq \log(\varrho(M))$. Moreover, if f is P-monotone then $h(f) = \max\{0, \log \varrho(M)\}$.*

REMARK 2.10. We note that, since $f|_{H(f)}$ is $V(f)$-monotone, then it is semiconjugate to the subshift of finite type with transition matrix equal to the transition matrix of the $f|_{H(f)}$-graph of $V(f)$. Consequently, our arguments admit an analogous formulation in terms of subshifts of finite type. For simplicity we have chosen to work with the terminology and notions outlined in this subsection.

3. Proofs of the main results. Let f be a quadratic Misiurewicz polynomial and set $H = H(f)$. For the proof of Theorem A we need the following three lemmas.

A closed subset $I \subsetneq H$ is called *proper* if $\mathrm{Int}(I) \neq \emptyset$. Note that the interior of the complement of any proper set is also non-empty.

LEMMA 3.1. *Let I be a proper, forward invariant subset of H. Then there exists a forward invariant set $E \subseteq I$ which is a finite union of edges of H (in particular E is also proper).*

Proof. Since the interior of I is non-empty, we can choose $x, y \in I$ such that $[x, y] \subset I$. By Proposition 2.6(d), there exists $n > 0$ such that $[f^n(x), f^n(y)] \subset f^n([x, y])$ contains a whole edge of H. Since I is invariant it follows that I contains an edge, which we denote by l. Then, since the set of vertices is invariant, the set $E = \bigcup_{n \geq 0} f^n(l)$ is obviously the union of a finite number of edges and is a proper, forward invariant subset of I. ∎

LEMMA 3.2. *Any proper, forward invariant set $E \subset H$ must contain the critical point of f. Moreover, E is not connected.*

Proof. The first statement follows from Proposition 2.6(b), (c). Moreover, if E were connected, it would be an invariant subtree of H (since it is proper and closed), contradicting Proposition 2.6(e). ∎

LEMMA 3.3. *Suppose f^n is renormalizable for some $n > 1$ and let $J_n(i)$ for $0 \leq i \leq n - 1$ be the small Julia sets. Then the set $J_n(i) \cap H$ has non-empty interior for all $0 \leq i \leq n - 1$.*

Proof. It suffices to show that $J_n(0) \cap H$ contains at least two points. Indeed, since $J_n(i) \cap H$ is connected and simply connected it follows that if $x, y \in J_n(0) \cap H$ with $x \neq y$, then $[x, y] \subset J_n(0) \cap H$. Moreover, $f^k([x, y]) \subset J_n(k) \cap H$ has non-empty interior for all $0 \leq k \leq n - 1$ because f is non-constant.

So, assume that $J_n(0) \cap H$ contains only one point. Since $0 \in J_n(0) \cap H$ it follows that this point must be $\omega = 0$. But $J_n(0)$ is invariant by f^n and so is H. Hence, $f^n(0) = 0$, which contradicts the fact that f is Misiurewicz. ∎

Now we are ready to prove Theorem A.

Proof of Theorem A. We will prove Theorem A by showing

$$\text{(a)}\Rightarrow\text{(b)}\Rightarrow\text{(c)}\Rightarrow\text{(d)}\Rightarrow\text{(a)}.$$

To show (a)$\Rightarrow$(b), assume that f^n is renormalizable of disjoint type for some $n > 1$. For $i = 0, 1 \dots, n-1$ let $J_n(i)$ be the small Julia sets and define $E_i = J_n(i) \cap H$; the latter sets have non-empty interior by Lemma 3.3. Set $E = \bigcup_{0 \le i \le n-1} E_i$. Then E is a closed, forward invariant subset of H. Moreover, $E \neq H$ because the E_i's are disjoint (since the renormalization is of disjoint type) and H is connected.

Since $f(E_i) = E_{i+1\,(\mathrm{mod}\,n)}$ and the E_i's are disjoint it follows that $\alpha \notin E$. Let C be the connected component of $H \setminus E$ that contains the fixed point α. Since $E \neq H$ and E is closed it follows that C is open (in H). If C is invariant then the closure of C is a proper invariant subtree, contradicting Proposition 2.6(e). Hence, $f(C) \neq C$. Note that $f(C)$ is open, connected and intersects C. Therefore C contains open sets whose image is contained in E. These sets cannot contain any periodic point of f, so the periodic points are not dense in H.

The fact that (b) implies (c) follows from Proposition 2.8.

To show (c)$\Rightarrow$(d), assume f is not transitive on H. Then there exist two open sets U and V in H such that $f^k(U) \cap V = \emptyset$ for all $k \in \mathbb{N}$. The set $I = \overline{\bigcup_{k\in\mathbb{N}} f^k(U)}$ is proper (since $V \cap I = \emptyset$) and forward invariant. By Lemma 3.1 we may assume that I contains an edge l. Then $\bigcup_{k\in\mathbb{N}} f^k(l) \subseteq I \subsetneq H$.

Finally we prove (d)$\Rightarrow$(a). Let $l \subset H$ be the edge such that $L := \bigcup_{n\in\mathbb{N}} f^n(l) \subsetneq H$. Since the set of vertices is forward invariant we deduce that L is a finite union of edges and hence closed. Also note that L is proper and forward invariant. Thus, by Lemma 3.2, L is disconnected and contains the critical point. Let $L_0, L_1, \dots, L_{p-1}$ be a cycle of (pairwise different) connected components of L such that $\omega_0 \in L_0$ and $f(L_i) = L_{i+1}$ for $i = 0, 1, \dots, p-2$. To see that such a cycle exists note that since L has finitely many connected components there must be a cycle among them. Also, by Proposition 2.6(c) this cycle has to contain the critical point. Without loss of generality we may assume that p is the smallest possible number with these properties and that $\omega_0 \in L_0$. Moreover, p must be larger than one, for L_0 cannot be an invariant subtree by Proposition 2.6(e). Therefore ω_0 and $\omega_p = f^p(\omega_0)$ lie inside L_0.

Now L, and in particular L_0, cannot contain any preimage of α. Indeed, that would imply that α belongs to all L_i's, contradicting the fact that they are disjoint. Hence the arc $[\omega_0, \omega_p]$ does not contain any preimage of α. This implies that the Yoccoz puzzle pieces defined in Section 2 satisfy

$P_d(\omega_p) = P_d(0)$ for all depths d. Indeed, pieces with disjoint interiors only have preimages of α as common boundary on the Julia set. By Lemma 2.2, f^p is renormalizable. It is easy to check that this renormalization is of disjoint type since the puzzle piece $\widehat{P}_d(0)$, for d large enough, does not contain L_i for any $i \neq 0$. ∎

Proof of Theorem B. Let (a′), (b′), (c′) denote the opposite statements to (a), (b) and (c). We will prove Theorem B by showing

$$(\mathrm{a}')\Rightarrow(\mathrm{c}')\Rightarrow(\mathrm{b}')\Rightarrow(\mathrm{a}').$$

To see that (a′)⇒(c′), suppose that f^n is renormalizable for some $n > 1$ and let $J_n(i)$, $0 \le i < n$, be the small Julia sets. Define $E_i = J_n(i) \cap H$ and let $E = \bigcup_i E_i$. Recall that the sets E_i have non-empty interior by Lemma 3.3. Since the union of the small Julia sets is invariant and so is H, it follows that E is invariant. Moreover, $E_i \cap E_j$ consists of at most one point, for all $i \neq j$ (see Subsection 2.1), and hence, $E_i \neq H$ for all i. By Proposition 2.6(d), E must contain an edge which we denote by l. If, say, $l \in E_0$, it follows that $f^k(l) \in E_{k\,(\mathrm{mod}\,n)}$ for all $k \ge 0$. Hence we have $f^k(l) \neq H$ for all $k \in \mathbb{N}$.

To show (c′)⇒(b′), suppose that there exists an edge l of H such that $f^n(l) \neq H$ for all $n \in \mathbb{N}$. If l is not contained in $\bigcup_{n\in\mathbb{N}} f^n(l)$ (which is a union of edges because the set of vertices is invariant), then f is not transitive (by definition) and we are done. Hence, assume there exists $t \ge 1$ such that $l \subset f^t(l)$. We will show that f^t is not transitive. Clearly we have an increasing sequence of sets $l \subset f^t(l) \subset f^{2t}(l) \subset f^{3t}(l) \subset \ldots$ such that, by hypothesis, $f^{kt}(l) \neq H$ for all $k \in \mathbb{N}$. Therefore, we have

$$\bigcup_{i=0}^{k} (f^t)^i(l) = f^{kt}(l) \neq H$$

for each $k \in \mathbb{N}$ and each of the sets $\bigcup_{i=0}^{k} (f^t)^i(l)$ is a union of edges. Therefore, $\bigcup_{i\in\mathbb{N}} (f^t)^i(l) \neq H$ because H has a finite number of edges. Consequently, f^t is not transitive by definition and so f is not totally transitive.

Finally we show (b′)⇒(a′). We note that if f is not transitive then it is renormalizable of disjoint type by Theorem A. So, we may assume that f is transitive. We divide the proof in two cases: the case where $-\alpha$ is not in the interior of H and the case where it is. As we shall see, this will correspond to f being renormalizable of β-type and crossed type respectively.

Case 1. If $-\alpha \notin \mathrm{Int}(H)$ it follows that the orbit of the critical point is entirely contained in the closure of the puzzle pieces of depth one that touch the fixed point α (see Section 2). Indeed, if $\omega_n \in \bigcup_{i=1}^{q-1} P_1(-\omega_i)$ for some n, then $[\omega_0, \omega_n]$ must contain $-\alpha$ in its interior. Hence we may apply Lemma 2.3 to conclude that f^q is renormalizable. We remark that this renormalization is of β-type for it cannot be of disjoint type since f is transitive (see Theorem A).

CASE 2. To deal with this case we need to introduce some notation. Let L_0 be the closure of the connected component of $J(f)\setminus\{\alpha,-\alpha\}$ that contains the critical point, and let $L_1,\ldots,L_{q-1}$ be the closures of the connected components of $J(f)\setminus\{\alpha\}$ that do not contain L_0 labeled in such a way that $f(L_i)=L_{i+1}$ for $0\le i<q-1$. Let also $L'_i=-L_i$ for $1\le i\le q-1$. Observe that $f(L'_i)=f(L_i)$. Since $J(f)$ is totally invariant, $f(L_{q-1})=L_0\cup\bigcup_{i=1}^{q-1}L'_i$. Clearly, $H\cap L_i$ is non-empty for all $0\le i\le q-1$ and since we are assuming that $-\alpha$ belongs to the interior of H, we see that $H\cap L'_i$ is non-empty for some $1\le i\le q-1$. Let $H_0=H\cap L_0$ and for $1\le i\le q-1$ let $H_i=H\cap L_i$ and $H'_i=H\cap L'_i$ (which might be empty).

Since $f|_H$ is transitive but not totally transitive, in view of Proposition 2.7 there exist closed, connected subsets $X_0,X_1,\ldots,X_{n-1}$ of H with non-empty interior such that $H=\bigcup_{i=0}^{n-1}X_i$, $X_i\cap X_j=\{\alpha\}$ for all $i\ne j$ and $f(X_i)=X_{i+1\,(\mathrm{mod}\,n)}$ for $i=0,1,\ldots,n-1$. We may also assume that X_0 is such that $0=\omega_0\in X_0$. Clearly, each connected component of $H\setminus\{\alpha\}$ is contained in some X_i. Observe that the connected components of $H\setminus\{\alpha\}$ are $H_0\cup\bigcup_{i=1}^{q-1}H'_i,H_1,\ldots,H_{q-1}$. Since $\omega_i=f^i(0)\in H_i\cap X_{i\,(\mathrm{mod}\,n)}$ for all $0\le i\le q-1$, we have $H_0\cup\bigcup_{i=1}^{q-1}H'_i\subset X_0$ and $H_i\subset X_{i\,(\mathrm{mod}\,n)}$ for $1\le i\le q-1$. Thus,

$$\begin{aligned}X_0&=H_0\cup H_n\cup H_{2n}\cup\ldots\cup H_{q-n}\cup\bigcup_{i=1}^{q-1}H'_i,\\ X_1&=H_1\cup H_{n+1}\cup H_{2n+1}\cup\ldots\cup H_{q-n+1},\\ &\vdots\\ X_{n-1}&=H_{n-1}\cup H_{2n-1}\cup H_{3n-1}\cup\ldots\cup H_{q-1},\end{aligned}$$

where all the H'_i except one could be empty. Therefore, in particular, n divides q. Note that since $f(X_i)=X_{i+1\,(\mathrm{mod}\,n)}$, $f(H'_i)=f(H_i)=H_{i+1}$ for $1\le i\le q-2$ and $f(H'_{q-1})=f(H_{q-1})=H_0\cup\bigcup_{i=1}^{q-1}H'_i$ it follows that $H'_i=\emptyset$ for all $i\not\equiv 0 \pmod n$. If $q=n$ then $H'_i=\emptyset$ for all i, a contradiction. Thus, $q\ge 2n$. Moreover, it follows easily that we are under the hypothesis of Lemma 2.4, and hence f^n is renormalizable of α-type. This ends the proof of the theorem. ■

Proof of Theorem D. By Proposition 1.2(c) and [2, Lemma 4.1.3] we get

$$h(f|_{H(f)})\le h(f|_{J(f)})=\log d,$$

because $H\subset J(f)$ and H is forward invariant. So, we only have to show that $h(f|_{H(f)})<\log d$ whenever $H\ne J(f)$. To this end we will use the techniques from Section 2.3 (see also [9], [2, Section 4.4]). Let M be the matrix of the f-graph of $V(f)$. From the fact that $V(f)$ contains the critical point of f and Proposition 2.6(a), it follows that f is $V(f)$-monotone. Hence, by

Theorem 2.9, $h(f|_{H(f)}) = \max\{0, \log(\varrho(M))\}$. Thus, it is enough to prove that if $H \neq J(f)$ then $\varrho(M) < d$. So, in what follows we assume that $H \subsetneq J(f)$ and, to simplify notation, we denote $f|_{H(f)}$ by φ.

We claim that for each $x \in H$ there exists a positive integer $n(x)$ such that $\operatorname{Card}(\varphi^{-n(x)}(x)) < d^{n(x)}$, where $\operatorname{Card}(\cdot)$ denotes the cardinality of a set. To prove the claim we assume the contrary. Then there exists $z \in H$ such that $\operatorname{Card}(\varphi^{-m}(z)) \geq d^m$ for each $m \in \mathbb{N}$. Note that $\varphi^{-m}(z) \subset f^{-m}(z)$ and $\operatorname{Card}(f^{-m}(z)) \leq d^m$ for each $m \in \mathbb{N}$. Hence, $f^{-m}(z) = \varphi^{-m}(z)$ for each $m \in \mathbb{N}$. On the other hand, since $H \subset J(f)$, it is closed, and the preimages of any point are dense in $J(f)$ (see for example [7]), it follows that

$$J(f) = \overline{\bigcup_{m=1}^{\infty} f^{-m}(z)} = \overline{\bigcup_{m=1}^{\infty} \varphi^{-m}(z)} \subset H;$$

a contradiction. This ends the proof of the claim.

From Proposition 2.6(a) and the fact that $V(f)$ is invariant it follows that if L is an edge of H and U is an open set (in H) contained in the interior of an edge then either $f(L) \cap U = \emptyset$ or $f(L) \supset U$. The inductive use of this fact shows that each point in the interior of an edge has the same number of preimages by φ^m for each $m \in \mathbb{N}$. So, by the above claim, for each edge L there exists a positive integer $n(L)$ such that for each x in the interior of L we have $\operatorname{Card}(\varphi^{-n(L)}(x)) < d^{n(L)}$. Note that for each $m \geq n(L)$ we have

$$\begin{aligned}
\operatorname{Card}(\varphi^{-m}(x)) &= \operatorname{Card}(\varphi^{-m+n(L)}(\varphi^{-n(L)}(x))) \\
&= \operatorname{Card}\Big(\bigcup_{y \in \varphi^{-n(L)}(x)} \varphi^{-m+n(L)}(y)\Big) \\
&\leq d^{m-n(L)} \operatorname{Card}(\varphi^{-n(L)}(x)) < d^{m-n(L)} d^{n(L)} = d^m.
\end{aligned}$$

Therefore, for all $\nu > \max n(L)$ where L ranges over all edges of H, we have $\operatorname{Card}(\varphi^{-\nu}(x)) < d^{\nu}$ for each $x \in H \setminus V(f)$. Consequently, by [2, Lemma 4.4.1], the sum of each column of M^{ν} is smaller than d^{ν}. So, there exists $\gamma < d$ such that the sum of each column of M^{ν} is smaller than γ^{ν}. Let v denote the vector of size r (where r denotes the order of the matrix M) with all entries 1. Clearly, $vM^{\nu} \leq \gamma^{\nu} v$. By induction we have $vM^{l\nu} \leq \gamma^{l\nu} v$ for each $l \in \mathbb{N}$. Hence, the sum of all entries of $M^{l\nu}$ is $vM^{l\nu}v' \leq \gamma^{l\nu} vv' = r\gamma^{l\nu}$, where v' denotes the transpose of v. Thus (see [24]),

$$\begin{aligned}
\varrho(M) &= \lim_{l\to\infty} \sqrt[l]{vM^l v'} = \lim_{l\to\infty} \sqrt[l\nu]{vM^{l\nu} v'} \leq \lim_{l\to\infty} \sqrt[l\nu]{r\gamma^{l\nu}} \\
&= (\lim_{l\to\infty} \sqrt[l\nu]{r})\gamma = \gamma < d.
\end{aligned}$$

This ends the proof of the theorem. ∎

Before proving Proposition F we recall the notion of an n-star. For $n \geq 3$, an *n-star* X_n is a tree which has a unique branching point denoted by b with valence n. The closure of a connected component of $X_n \setminus \{b\}$ will be called a *branch* of X_n.

We endow each branch of X_n with a linear ordering such that b is the smallest point on the branch while the endpoint is the largest one.

Proof of Proposition F. Let g_n be the sequence of quadratic Misiurewicz polynomials which are the endpoints of the main vein in the limbs $M_{1/n}$ (in Example 12 the Julia set of g_2 is shown). Recall that, for those polynomials, the critical point falls at the β-fixed point after $n+1$ iterations. Hence, the Hubbard tree is an $(n+1)$-star with the point b being the α-fixed point of g_n (see Figure 7). Let Ω denote the orbit of the critical point, i.e., $\Omega = \{0 = \omega_0, \omega_1, \ldots, \omega_{n+1} = \beta\}$. Observe that, if we set $H_n = H(g_n)$ and $\varphi_n = g_n|_{H_n}$, then φ_n is injective on the closure of each connected component of $H_n \setminus (\Omega \cup \{\alpha\})$. Thus, following the notation in Subsection 2.3, φ_n is $(\Omega \cup \{\alpha\})$-monotone.

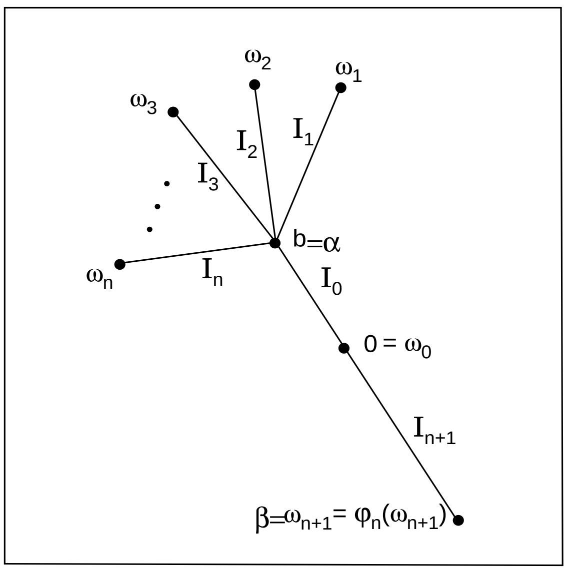


Fig. 7. H_n and the set $\Omega \cup \{\alpha\}$

To prove the proposition we have to show that $\lim_{n\to\infty} h(\varphi_n) = 0$, for which we shall use Theorem 2.9. We start by computing the φ_n-graph of $\Omega \cup \{\alpha\}$. To this end we label the closures of the connected components of $H_n \setminus (\Omega \cup \{\alpha\})$ according to the largest endpoint. That is, let $[x, y]$ be the closure of a connected component of $H_n \setminus (\Omega \cup \{\alpha\})$. Clearly, $[x, y]$ is contained in a branch of H_n. So we may suppose that $x < y$. Hence, $y = \omega_i$

for some $i \in \{0, 1, \dots, n+1\}$. Then $[x, y]$ will be denoted by I_i. With this notation the φ_n-graph of $\Omega \cup \{\alpha\}$ is

$$\begin{array}{ccccc} & & I_{n+1} & \rightarrow & I_{n+1} \\ & \nearrow & & \searrow & \\ I_1 \to I_2 \to \dots \to I_n & & \downarrow & & I_1 \\ & \searrow & & \nearrow & \\ & & I_0 & & \end{array}$$

Let M_n denote the transition matrix of the φ_n-graph of $\Omega \cup \{\alpha\}$. To compute its characteristic polynomial we use the "rome" method from [9] (see also [2, Section 4.4]). Indeed, we take I_1 and I_{n+1} as a "rome" and we find that the characteristic polynomial of M_n is $(-1)^n x(x^{n+1} - x^n - 2)$. Note that $\varrho(M_n)$ is the unique point larger than 1 where x^n intersects the curve $2/(x-1)$. Since $x^n < x^m$ for all $m > n$, we see that $1 < \varrho(M_m) < \varrho(M_n)$ and $\lim_{n\to\infty} \varrho(M_n) = 1$. Therefore, by Theorem 2.9 it follows that

$$\lim_{n\to\infty} h(\varphi_n) = \lim_{n\to\infty} \max\{0, \log \varrho(M_n)\} = \lim_{n\to\infty} \log \varrho(M_n) = 0.$$

This ends the proof of the proposition. ∎

Appendix: Examples. In this appendix we show some examples (without proofs) of renormalizable polynomials of each type and we study their properties from the point of view of Theorems A and B and Corollary C. In what follows f_{c_i} will denote the polynomial $z^2 + c_i$.

Example 1 (Theorem A; $\mathbb{N} \setminus \mathrm{Per}(f)$ infinite). Let $c_1 = -1.430357\ldots$ This parameter value is the last point of the period two copy of the Mandelbrot set on the real axis. The Julia set and Hubbard tree of the polynomial f_{c_1} are shown in Figure 8. Observe that condition (d) of Theorem A is satisfied since

$$E := \bigcup_{n>0} f_{c_1}^n([\omega_1, \omega_5]) = [\omega_1, \omega_3] \cup [\omega_4, \omega_2] \subsetneq H(f_{c_1}).$$

Hence, the periodic points are not dense in $H(f_{c_1})$, $f_{c_1}|_{H(f_{c_1})}$ is not transitive and f_{c_1} is renormalizable of disjoint type. In fact, this example was chosen so that f_{c_1} is renormalizable of disjoint type for $n = 2$ and of β-type for $n = 4$. Indeed, the set E has two disjoint components which are the Hubbard trees of the small Julia sets for $f_{c_1}^2$. This map restricted to the rightmost component of E (which contains the critical point) is conjugate to $z^2 - 1.543689\ldots$ while $f_{c_1}^4$ restricted to $[\omega_4, \omega_6]$ is conjugate to $z^2 - 2$.

Concerning the periods, it is not difficult to see that any periodic point in $H(f_{c_1})$ different from α must have period a multiple of two and, hence, $\mathbb{N} \setminus \mathrm{Per}(f_{c_1})$ is infinite. On the other hand, from Theorem 2.9 (see also [9]), it follows that $h(f_{c_1}|_{H(f_{c_1})}) = \frac{1}{4} \log 2$.

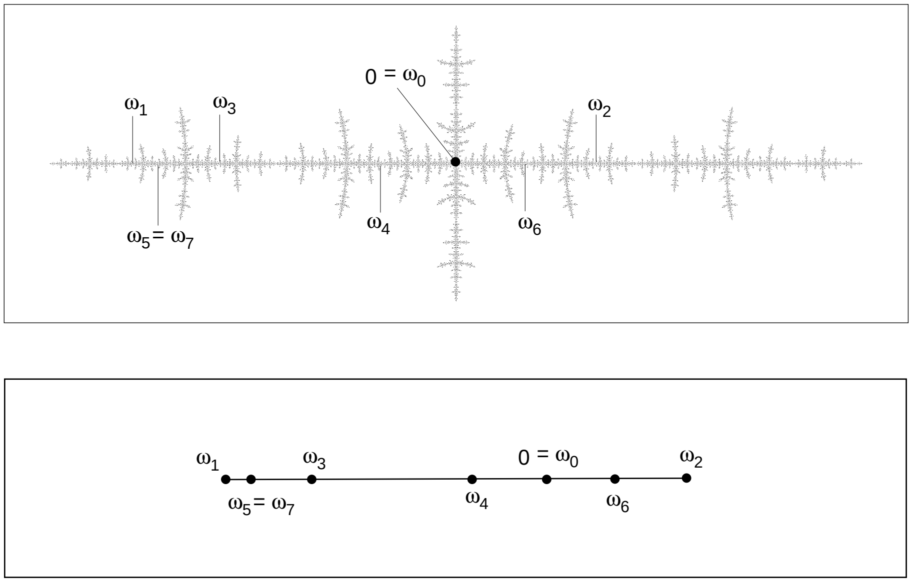


Fig. 8. The Julia set and Hubbard tree of f_{c_1}, where $c_1 = -1.430357\ldots$

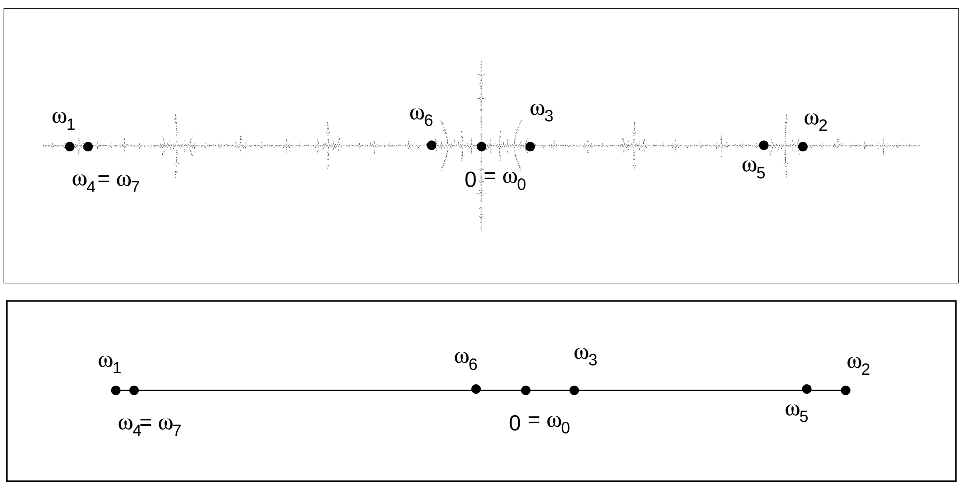
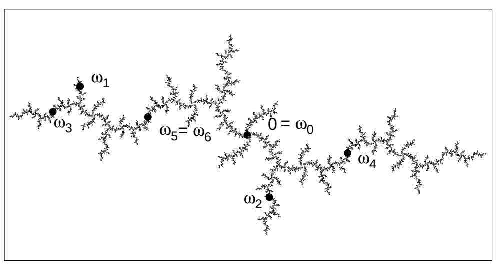


Fig. 9. The Julia set and Hubbard tree of f_{c_2}, where $c_2 = -1.790327\ldots$

Example 2 (Theorem A; $\mathbb{N} \setminus \mathrm{Per}(f)$ finite). Let $c_2 = -1.790327\ldots$ The polynomial f_{c_2}, whose Julia set and Hubbard tree are shown in Figure 9, can be found at the end of the small period three copy of the Mandelbrot set on the real axis. For this example condition (d) of Theorem A is satisfied with $l = [\omega_1, \omega_4]$. Hence, the same conclusions as in the previous example follow. In this case f_{c_2} is renormalizable of disjoint type for $n = 3$. Indeed, $E := \bigcup_{n \geq 0} f_{c_2}^n(l) = [\omega_1, \omega_4] \cup [\omega_6, \omega_3] \cup [\omega_5, \omega_2]$ has three disjoint components which are the Hubbard trees of the small Julia sets. The third iterate of f_{c_2} restricted to the middle component (which contains the critical point) is conjugate to z^2-2.

On the other hand, by using standard arguments (see [9]) one can check that there is an invariant Cantor set C in $H(f_{c_2})\setminus E$ which contains periodic points of all periods. Now, $h(f_{c_2}|_{H(f_{c_2})}) = \log\frac{\sqrt{5}+1}{2}$.

EXAMPLE 3 (Corollary C; β-type). Let $c_3 = -1.222863\ldots + i\,0.316882\ldots$ The Julia set and the Hubbard tree of f_{c_3} are shown in Figure 10. This example falls in the category of Corollary C. Indeed, it is easy to check that the union of the images of any edge is the whole Hubbard tree. However, since $T_0 = [\omega_5, \omega_2, \omega_4]$ and $T_1 = [\omega_5, \omega_1, \omega_3]$ are mapped to each other cyclically it follows that the nth iterate of any edge is different from $H(f_{c_3})$. Therefore we deduce that $f_{c_3}|_{H(f_{c_3})}$ is transitive but not totally transitive (it is easy to check that $f^2_{c_3}|_{H(f_{c_3})}$ is not transitive) and f_{c_3} is renormalizable but not of disjoint type. In fact, it follows from the proof of Theorem B that the polynomial is renormalizable of β-type for $n=2$, since $-\alpha$ does not belong to the interior of $H(f_{c_3})$. This example was chosen so that the second iterate of f_{c_3}, restricted to the small Julia set, is conjugate to the polynomial in Example 5. The tree T_0 is, under $f^2_{c_3}$, the Hubbard tree of the renormalized map. Note that the small Julia set $K_2(0)$ and its image meet at their corresponding β-fixed points ($\beta_2(0) = \beta_2(1)$), which is the α-fixed point of f_{c_3}.



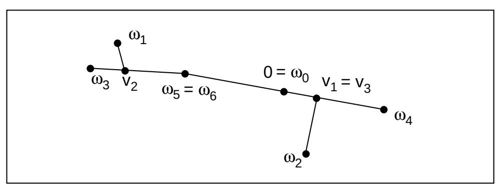


Fig. 10. The Julia set and Hubbard tree of f_{c_3}, where $c_3 = -1.222863\ldots + i\,0.316882\ldots$

In this example all periods different from one are multiples of two, and

$$h(f_{c_3}|_{H(f_{c_3})}) = \log 1.302160040\ldots$$

EXAMPLE 4 (Corollary C; α-type). Let $c_4 = 0.419643\ldots + i\,0.606291\ldots$ In Figure 11 the Julia set and Hubbard tree of f_{c_4} are shown. As in the previous example, f_{c_4} falls in the category of Corollary C. In this case, f_{c_4} is renormalizable of crossed type for $n=2$ since $-\alpha$ does belong to the interior of $H(f_{c_4})$. Now, the arc $[\omega_2, \omega_4]$ is the small Julia set $J_2(0)$, homeomorphic to the Julia set of z^2-2. The two sets $J_2(0)$ and $J_2(1)=f(J_2)$ "cross" at their α-fixed points, which is also the α-fixed point of f_{c_4}.

Just as above, all periods different from one are multiples of two, and $h(f_{c_4}|_{H(f_{c_4})})=\frac{1}{2}\log 2$.

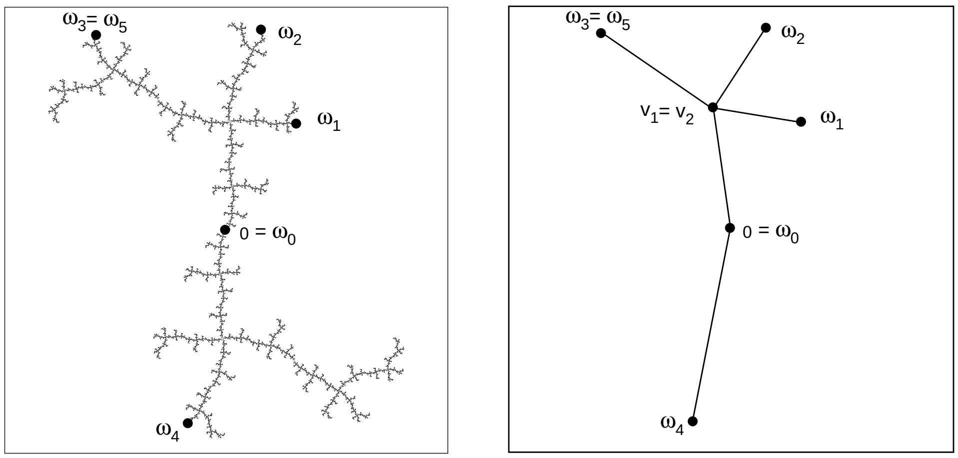


Fig. 11. The Julia set and Hubbard tree of f_{c_4}, where $c_4=0.419643\ldots+i\,0.606291\ldots$

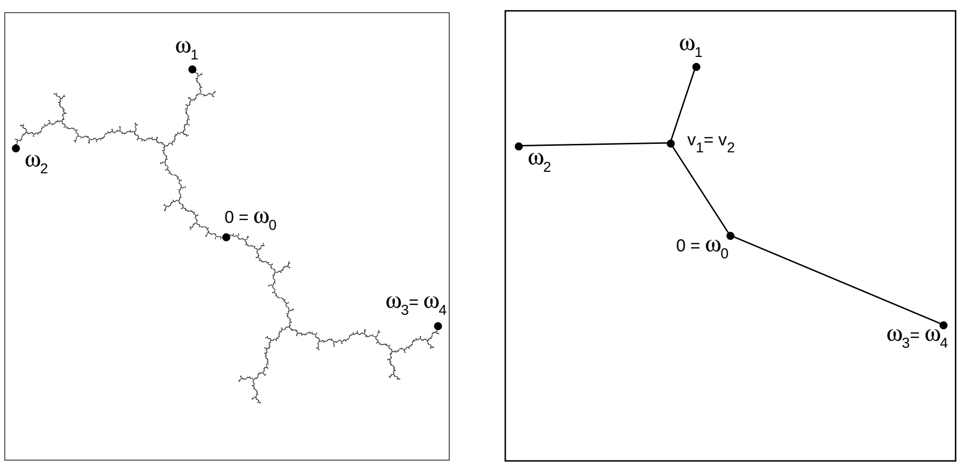


Fig. 12. The Julia set and Hubbard tree of f_{c_5}, where $c_5=-0.228155\ldots+i\,1.115142\ldots$

Example 5 (Theorem B). Let $c_5=-0.228155\ldots+i1.115142\ldots$ (see Figure 12). Clearly condition (c) of Theorem B is satisfied for this example since all edges eventually map to the whole tree (note that $l=[\omega_0,\omega_3]$ satisfies $f_{c_5}(l)\supsetneq l$). Hence, $f_{c_5}|_{H(f_{c_5})}$ is totally transitive, the periodic points are dense in $H(f_{c_5})$, $\mathbb{N} \setminus \mathrm{Per}(f_{c_5})$ is finite (in fact it is easy to check that $\mathrm{Per}(f_{c_5}) = \mathbb{N} \setminus \{2\}$) and the polynomial is not renormalizable for any $n > 1$. This last property is easily checked in this particular example. Indeed, as shown in [19], a small Julia set $K_n(i)$ cannot contain the β-fixed point of the original polynomial, since that would make $K_n(i)$ for $i > 0$ intersect $K_n(0)$ in more than one point. In particular, when some iterate of the critical point hits the β-fixed point (as in this example), the polynomial is not renormalizable.

Concerning the entropy, we note that the map f_{c_5} is conjugate to $f_{c_3}^2|_{T_0}$. So,

$$h(f_{c_5}|_{H(f_{c_5})}) = 2h(f_{c_3}|_{T_0}) = 2 \log 1.302160040\ldots$$

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Departament de Matemàtiques
Facultat de Ciències (Edifici Cc)
Universitat Autònoma de Barcelona
08193 Bellaterra, Barcelona, Spain
E-mail: alseda@mat.uab.es

Departament de Matemàtica Aplicada i Anàlisi
Universitat de Barcelona
Gran Via 585
08007 Barcelona, Spain
E-mail: fagella@maia.ub.es

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