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Trees of visible components in the Mandelbrot set

by

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Abstract. We discuss the tree structures of the sublimbs of the Mandelbrot set $\mathcal{M}$, using internal addresses of hyperbolic components. We find a counterexample to a conjecture by Eike Lau and Dierk Schleicher concerning topological equivalence between different trees of visible components, and give a new proof to a theorem of theirs concerning the periods of hyperbolic components in various trees.

1. Introduction. In this paper we discuss the combinatorial tree structure of the Mandelbrot set $\mathcal{M}$. We construct a tree which disproves a conjecture in [LS] (here in 2.6) and give a new proof to a theorem in [LS] (here 2.7) which is a weak version of the conjecture. When finishing the preparation of this paper, the author was informed that Dierk Schleicher and Henk Bruin had found a counterexample independently and stated another partial version—stronger than 2.7—of the original conjecture which Karsten Keller [K2] had proved and which is given here as Theorem 2.8.

We use ideas introduced in [S1] and [LS] to describe the trees. An important concept is the *internal address* of a hyperbolic component $\mathcal{A}$, which lists the periods of certain components that are "on the way" from the main cardioid $\mathcal{C}_0$ to $\mathcal{A}$ (definition 2.2). [LS] presents a simple algorithm, described here in §3, which gives the internal address from the *kneading sequence* [Th], [BK] of the *external angles* of $\mathcal{A}$, or the angles of external rays landing at $\mathcal{A}$ in the parameter plane.

Recall that the (*dynamical*) *external ray* with angle θ of the filled Julia set K_c of the polynomial $P_c : z \mapsto z^2 + c$, denoted by $\mathcal{R}_\theta^c$, is the preimage of the radial line $\{re^{i2\pi\theta} : r > 1\}$ under the conformal *Böttcher map* from $\widehat{\mathbb{C}} \setminus K_c$ to the exterior of the closed unit disk; consult [CG] or [Be] for details. The *parameter rays* or *external rays of the Mandelbrot set*, $\mathcal{R}_\theta^{\mathcal{M}}$, are

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defined similarly as preimages of straight rays under the conformal mapping obtained by evaluating the Böttcher map in c itself. The polynomial P_c maps each dynamical ray to another ray doubling the angle (which we measure in full turns, i.e. $0 = 1 = 2\pi\,\mathrm{rad} = 360^\circ$), and the dynamical rays of any polynomial "look like straight rays" near infinity. This allows us to study the Mandelbrot and Julia sets combinatorially, replacing the dynamical plane by the unit circle, rays by angles, and the quadratic polynomial by the doubling modulo one map.

1.1. *Wakes and periods.* The main cardioid $\mathcal{C}_0$ only has one external angle, 0. Any other hyperbolic component $\mathcal{A}$ has exactly two external rays that land at its rootpoint separating its *wake* $W_{\mathcal{A}}$ from the rest of the parameter plane.

An angle θ in the circle $\mathbb{R}/\mathbb{Z}$ is periodic under doubling if and only if it is rational with odd denominator (in the reduced form). For example, $1/7 \mapsto 2/7 \mapsto 4/7 \mapsto 8/7 = 1/7$. The external angles of hyperbolic components are periodic. The *period* of a hyperbolic component $\mathcal{A}$ is denoted by $\operatorname{Per}\mathcal{A}$ and is defined as the period of the attracting orbit that each polynomial P_c with $c \in \mathcal{A}$ has. It has been proved (in [Mi], for instance) to be the same as the period of the external angles of $\mathcal{A}$ under the doubling modulo one map.

The *width* $|W|$ of the wake W of a hyperbolic component $\mathcal{C}$ with $\operatorname{Per}\mathcal{C} = k$ is the difference between its two external angles. Since the k-periodic angles are of the form $\theta = a/(2^k - 1)$ with $a \in \mathbb{N}$, we have $|W| = t/(2^k - 1)$ for some integer t. This t is *odd* because of the following

1.2. Remark. If two external rays of $\mathcal{M}$ with angles θ_1 and θ_2 land at the same point in $\partial\mathcal{M}$, then the number of periodic angles $\psi \in \,]\theta_1, \theta_2[$ with *any* period n is *even*. This is because rays with period n land in pairs at roots of hyperbolic components of period n, which then must be in the wake bounded by the given rays.

The *p/q-subwake* of W is the wake of the kq-periodic satellite component of $\mathcal{C}$ at internal angle p/q, and its width is

$$|W_{\mathcal{C}}^{p/q}| = |W|\frac{(2^k-1)^2}{2^{qk}-1} = \frac{t(2^k-1)}{2^{qk}-1}. \tag{1.3}$$

This is proved in [S1]. A different, more direct proof will appear in my Ph.D. thesis; it uses Milnor's *orbit portraits* [Mi], which will also be needed in the present paper.

1.4. *Orbit portraits.* If a dynamical ray $\mathcal{R}_\theta^c$ with a *rational* angle θ lands at a point of a periodic orbit $\mathcal{O} := \{x_1, x_2 := P_c(x), \dots, x_k := P_c^k(x)\}$ $(\subset J_c)$, then for each $x_i \in \mathcal{O}$ the collection A_i of all external angles of x_i is a finite, non-empty subset of $\mathbb{Q}/\mathbb{Z}$.

The collection $\{A_1, \ldots, A_k\} =: \Theta(\mathcal{O})$ is called the *orbit portrait* of $\mathcal{O}$. Every polynomial P_c has either zero, a finite number, or infinitely many different non-trivial orbit portraits, by which we mean ones with $\#A_i =: v > 1$. Each A_i cuts the circle into v intervals, and exactly one of all the kv intervals is the shortest. This is called the *characteristic interval* of the orbit portrait. The main theorem in [Mi] is:

1.5. Lemma. *If $[\theta_-, \theta_+]$ is the characteristic interval of any non-trivial orbit portrait Θ, then the parameter rays $\mathcal{R}^{\mathcal{M}}_{\theta_\pm}$ land at the same point $\widehat{c} \in \partial M$*

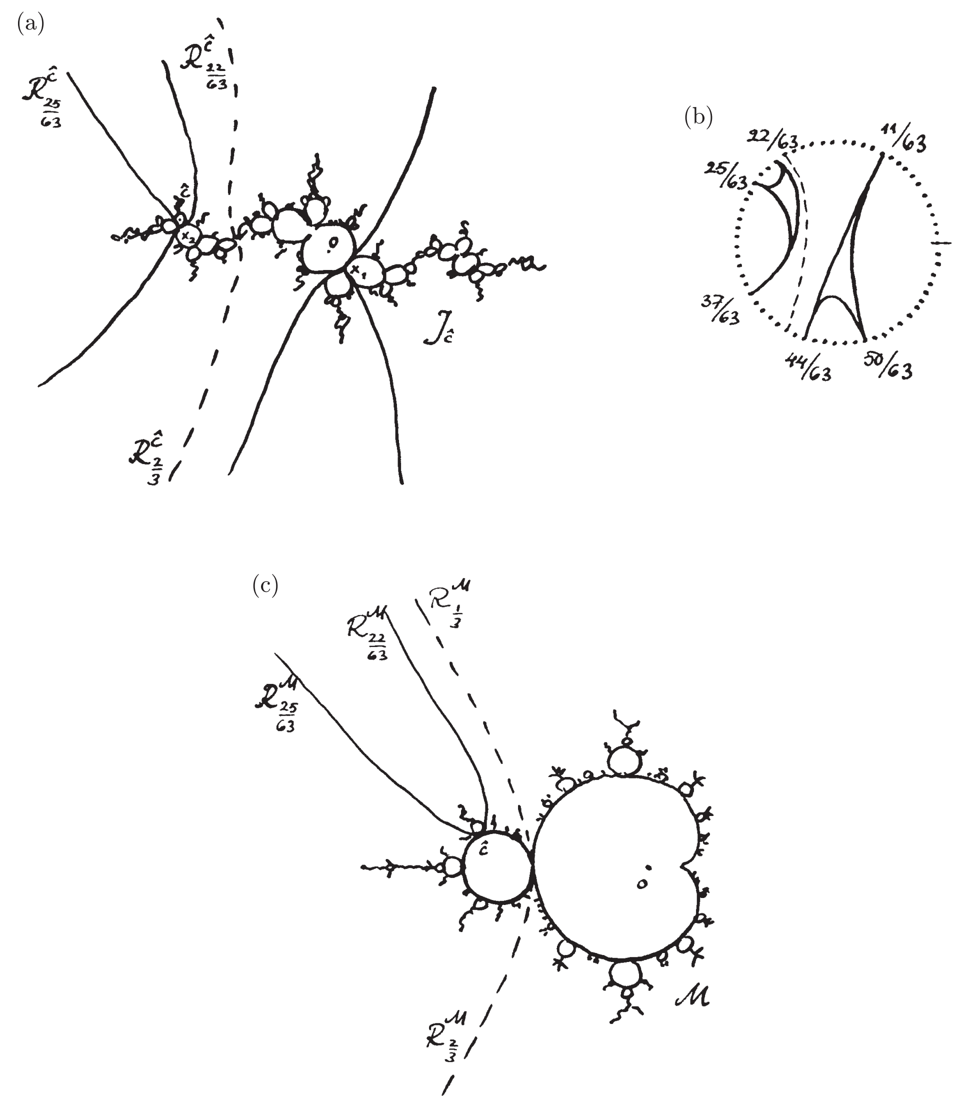

Fig. 1

bounding a wake W. A polynomial P_c has a repelling orbit with portrait Θ if and only if $c \in W$, and a parabolic orbit with portrait Θ if and only if $c = \widehat{c}$.

Example. Figure 1.a shows the dynamical plane of the polynomial $P_{\widehat{c}}$ with parameter $\widehat{c} = -1 + \frac{1}{4}e^{i2\pi/3} \in \partial\mathcal{M}$. A six-periodic cycle of rays is landing at a two-periodic parabolic orbit $x_\pm := -\frac{1}{2} \pm \frac{1}{2}\sqrt{1 - e^{i2\pi/3}}$. The corresponding orbit portrait $\{\{22/63, 25/63, 37/63\}, \{44/63, 50/63, 11/63\}\}$ is visualized by the diagram in Figure 1.b. Figure 1.c shows the parameter rays with the bounding angles of the characteristic interval $[22/63, 25/63]$ landing at $\widehat{c}$. Since $\widehat{c}$ is an interior point of the wake bounded by the rays with angles $1/3$ and $2/3$, $P_{\widehat{c}}$ also has a repelling fixed point $\frac{1}{2} + \frac{1}{2}\sqrt{5 - e^{i2\pi/3}}$ with orbit portrait $\{\{1/3, 2/3\}\}$ (the dotted lines in Figures 1.a, b, c).

1.6. Note. The open interval of angles of all external rays in the wake of a $\mathcal{C}$ is denoted by $I(\mathcal{C})$. Considered as just intervals on the unit circle, $\overline{I(\mathcal{C})}$ equals the characteristic interval of the corresponding orbit portrait. Since the wakes of any two components $\mathcal{A}$ and $\mathcal{B}$ are either strictly nested or disjoint (see 1.1), either $\overline{I(\mathcal{A})} \subset I(\mathcal{B})$ or $\overline{I(\mathcal{A})} \cap \overline{I(\mathcal{B})} = \emptyset$.

2. Internal addresses

2.1. *Partial ordering of components.* If a hyperbolic component (1) $\mathcal{A}$ is in the wake of another hyperbolic component $\mathcal{C}$, we write $\mathcal{C} \prec \mathcal{A}$. The set of components $\mathcal{B}$ such that $\mathcal{C} \prec \mathcal{B} \prec \mathcal{A}$ is called the *combinatorial arc* $]\mathcal{C}, \mathcal{A}[$

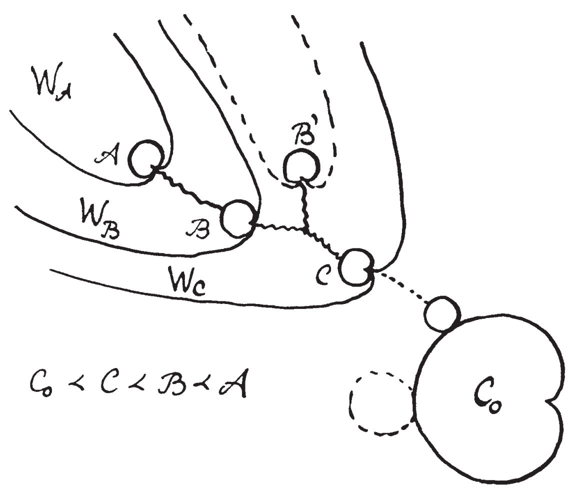

Fig. 2.a

(1) This paper concentrates on hyperbolic components, but similar concepts and results hold for Misiurewicz points.

(cf. Figure 2.a). Including the endpoints we write $[\mathcal{C}, \mathcal{A}]$. (There is actually also a *topological arc* connecting $\mathcal{A}$ to $\mathcal{C}$, cf. [S2], but the combinatorial arc is well defined even if we ignore this.) The arc, of course, contains infinitely many components $\mathcal{B}$; the internal address mentions just some of them.

2.2. *Internal address.* Denote the main cardioid by $\mathcal{C}_0$ and let $\mathcal{A}$ be any hyperbolic component. Let $\mathcal{B}_1$ be the component on the combinatorial arc $]\mathcal{C}_0, \mathcal{A}]$ which has the smallest period, n_1. Then let $\mathcal{B}_i$ be the component on $]\mathcal{B}_{i-1}, \mathcal{A}]$ with smallest period, n_i, for all integers i as long as $n_i \le \operatorname{Per} \mathcal{A}$ $=: k$. The sequence

$$1 \to n_1 \to n_2 \to \ldots \to k =: \mathbf{A}(\mathcal{A})$$

is called the *internal address* of $\mathcal{A}$. Any finite, strictly increasing sequence of integers starting with 1 is called an *abstract address*.

NOTE. *A priori*, one should worry about the uniqueness of the $\mathcal{B}_i$'s. If there were two components with the same period n_i, then n_{i+1} might depend on the choice of $\mathcal{B}_i$. But this is excluded by the following lemma of Lavaurs [La], which thus justifies the definition of the internal address and guarantees that the sequence is strictly increasing.

2.3. LEMMA. *If two hyperbolic components $\mathcal{C} \prec \mathcal{A}$ have periods equal to some n, then there is a component $\mathcal{B}$ with period less than n such that $\mathcal{C} \prec \mathcal{B} \prec \mathcal{A}$.*

Figure 2.b shows a diagram of all parameter rays ("modulo symmetry") with periods up to six and the hyperbolic components at which they land. The addresses of some components are written into the picture to illustrate the idea. An interesting question is: *Given an abstract address, is there a hyperbolic component (or several ones) with that sequence as its internal address?* Obviously, the component is not unique in general ([2]). For example, there are four components with address $1 \to 5 \to 6$, one behind each $p/5$-satellite of $\mathcal{C}_0$. There are also "non-existent components":

2.4. EXAMPLE. The sequence $1 \to 2 \to 4 \to 5 \to 6$ is not realized as an internal address. The six-periodic component with this address should be behind the five-periodic component at address $1 \to 2 \to 4 \to 5$, but all the four six-periodic components behind it are also behind the three-periodic component with address $1 \to 2 \to 3$ and therefore have a "3" in their addresses.

([2]) By equipping the internal address with the *internal angles* by which the arc $[\mathcal{C}_0, \mathcal{A}]$ leaves each component $\mathcal{B}_i$, we get the *angled internal address* which does specify $\mathcal{A}$ uniquely. The number of components sharing an address is the number of possible combinations of angles. All this is done in [LS].

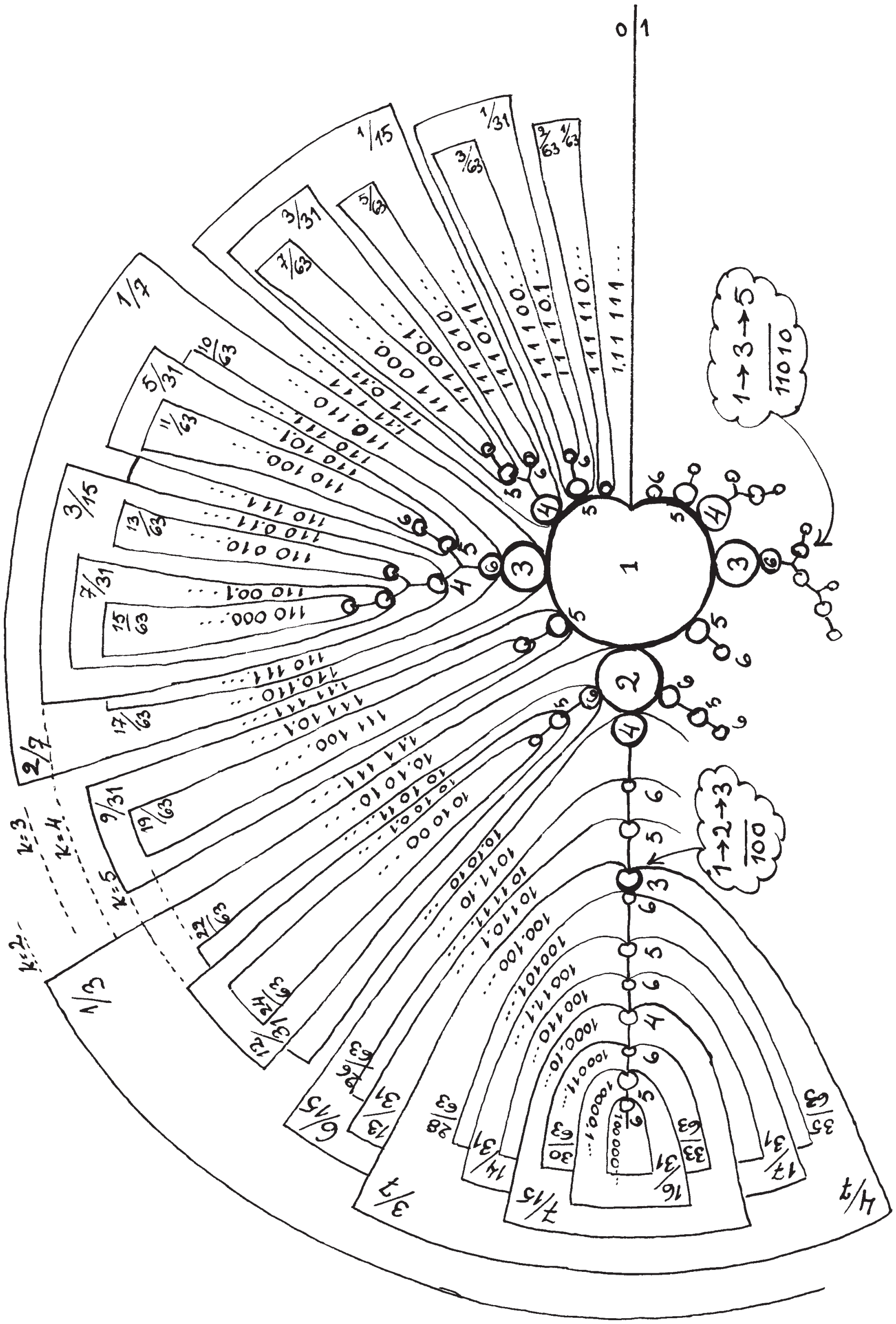

Fig. 2.b

2.5. *Visibility and trees.* A hyperbolic component $\mathcal{A}$ is said to be *visible* from $\mathcal{B}$ if $\mathcal{B} \prec \mathcal{A}$ or $\mathcal{A} \prec \mathcal{B}$ and all components on $]\mathcal{B}, \mathcal{A}[$ have periods greater than $\operatorname{Per} \mathcal{A}$. In particular, all components $\mathcal{B}_i$ that are mentioned in the internal address of some $\mathcal{A}$, are visible from $\mathcal{A}$, looking "down" towards the main cardioid (cf. definition 2.2). Looking "upwards", $\mathcal{B}_{i+1}$ is visible from $\mathcal{B}_i$ but not from $\mathcal{B}_{i-1}$, since $n_i < n_{i+1}$ and $\mathcal{B}_{i-1} \prec \mathcal{B}_i \prec \mathcal{B}_{i+1}$.

The *tree* $\mathcal{T}$ of visible components of $\mathcal{C}$ (cf. Figure 2.c) is the collection of hyperbolic components which are visible from $\mathcal{C}$ together with the topological and combinatorial structure induced by the embedding of $\mathcal{M}$ into the parameter plane. $\mathcal{C}$, in turn, is called the *stem component* of the tree $\mathcal{T}$. We call $\mathcal{T} \cap W_{\mathcal{C}}^{p/q} =: \mathcal{T}^{p/q}$ the *p/q-subtree* of $\mathcal{T}$. Each p/q-subtree of a given $\mathcal{T}$ obviously consists of only a finite number of visible components, since the p/q-satellite is "blocking the view" to all components except for the finitely many ones with periods less than kq.

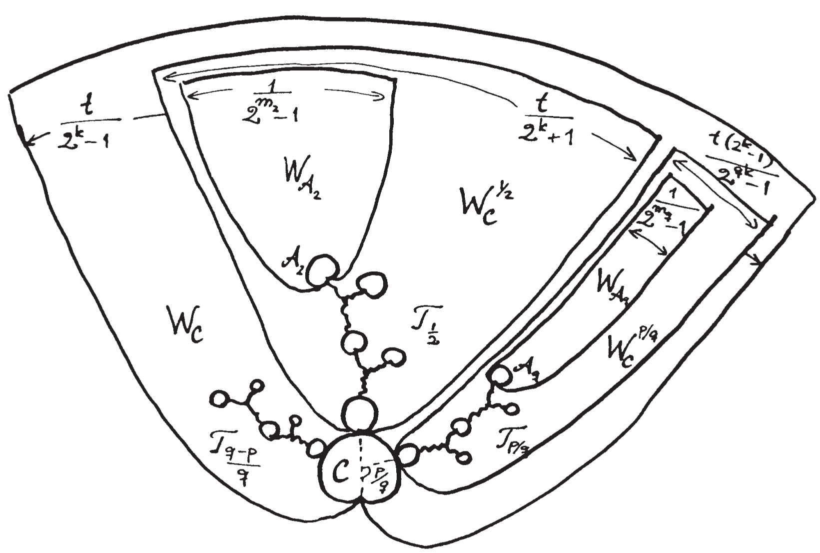

Fig. 2.c

Following [K2], we call two subtrees $\mathcal{T}_{p_1/q_1}$ and $\mathcal{T}_{p_2/q_2}$ *equivalent* if they "coincide" in the sense that there is a homeomorphism between them which maps each n-periodic hyperbolic component in $\mathcal{T}_{p_1/q_1}$ to a component in $\mathcal{T}_{p_2/q_2}$ with period $(q_1 - q_2)k + n$ preserving the embedding into the parameter plane.

2.6. Translation Principle. *Let $\mathcal{C}$ be a k-periodic hyperbolic component and $\mathcal{T}$ its tree of visible components. Then the subtree $\mathcal{T}_{p_1/q_1}$ is equivalent to $\mathcal{T}_{p_2/q_2}$ for any p_1, p_2, q_1, and q_2.*

This principle is true at least for *narrow-waked components*, i.e. ones with $|W_{\mathcal{C}}| = 1/(2^k - 1)$, as proved by Lau and Schleicher in [LS]. Their Conjecture stated that the Translation Principle is true without any additional assumption—a counterexample to this will be presented here in §4. A weaker statement is true in general:

2.7. THEOREM. *Let $\mathcal{C}$ be a k-periodic hyperbolic component and m_2 the smallest period of hyperbolic components in its 1/2-wake $W_{\mathcal{C}}^{1/2}$. Then the minimal period, m_q, of components in any p/q-wake $W_{\mathcal{C}}^{p/q}$ is $(q-2)k + m_2$.*

This is the "Weak Translation Principle" in [LS] and is proved there by using properties of the *dynamical planes* of certain parameters c. In our proof (§5) the dynamics is "hidden behind" orbit portraits. As side-products we obtain Lemma 5.1, which gives m_2 in terms of k and $|W_{\mathcal{C}}|$, and 5.6, which says that certain wake-widths do not occur. The "Partial Translation Principle" proven in [K2] is:

2.8. THEOREM. *Let $\mathcal{T}$ be the tree of visible components of any hyperbolic component in $\mathcal{M}$. Then every subtree of $\mathcal{T}$ other than $\mathcal{T}_{1/2}$ is equivalent to $\mathcal{T}_{1/3}$ or to $\mathcal{T}_{2/3}$.*

3. Itineraries, kneading sequences and the algorithm. Here we use codings of angles and hyperbolic components by binary sequences, introduced in e.g. [At] and [BK], to prove Lemma 3.8 which is an important tool in proving 2.7, and present the algorithm from [LS] connecting internal addresses to kneading sequences which we need to construct the counterexample to 2.6.

3.1. *Itinerary and kneading sequence of angles.* Given angles φ, α in the circle $\mathbb{R}/\mathbb{Z}$, the *α-itinerary* $\mathrm{It}^\alpha(\varphi)$ of φ is defined as a sequence of ones, zeros and triangles as follows: the angles $\alpha/2$ and $(\alpha+1)/2$ cut the circle into two halves $H_1^\alpha \ni \alpha$ and $H_0^\alpha \ni 0 = 1$, and

$$It^\alpha(\varphi)_n := \begin{cases} \mathbf{0} & \text{if } 2^{n-1}\varphi \in H_0^\alpha, \\ \blacktriangle & \text{if } 2^{n-1}\varphi = \alpha/2, \\ \mathbf{1} & \text{if } 2^{n-1}\varphi \in H_1^\alpha, \\ \blacktriangledown & \text{if } 2^{n-1}\varphi = (\alpha+1)/2. \end{cases}$$

The *kneading sequence* of θ is its own θ-itinerary: $K(\theta) := \mathrm{It}^\theta(\theta)$ (cf. Figure 3.a).

We use $*$ as a joker symbol meaning either $\blacktriangle$ or $\blacktriangledown$, and overline to indicate that a word is repeated periodically. For example, $\mathrm{It}^{1/8}(1/7) = \overline{\mathbf{110}}$. The α-itinerary of a k-periodic angle φ is obviously also periodic for any α, with period (dividing) k. The converse statement is not true—even an irrational angle might have a periodic itinerary. Clearly, $K(\theta)$ contains a $*$

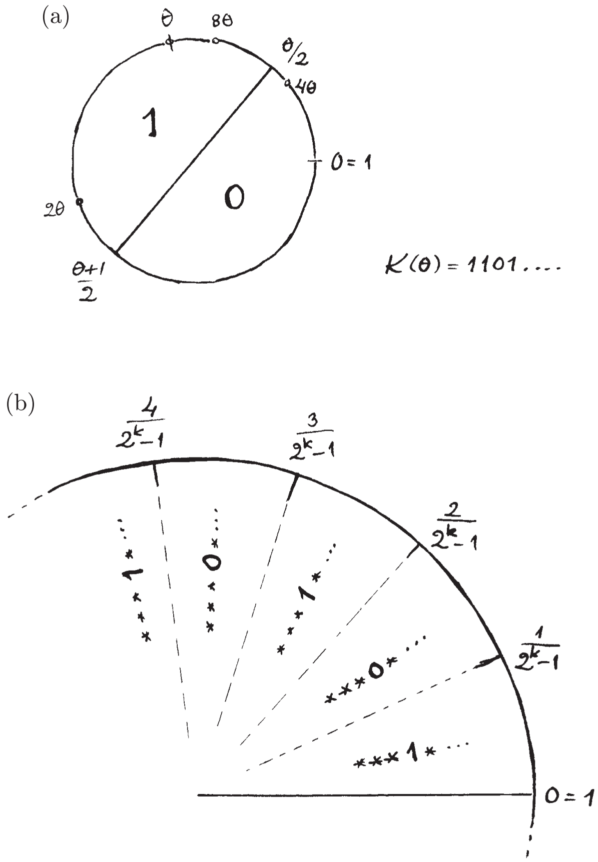

Fig. 3

if and only if θ is periodic under doubling, and then also $K(\theta)$ has the same exact period.

3.2. Lemma. *For a fixed k-periodic angle $\varphi \in \mathbb{R}/\mathbb{Z}$, $\mathrm{It}^{\alpha}(\varphi)_i \neq \mathrm{It}^{\beta}(\varphi)_i$ if and only if $\alpha < 2^{i \bmod k}\varphi < \beta \bmod 1$.*

Proof. Since $2^{i-1}\varphi$ equals either $2^i\varphi/2$ or $(2^i\varphi + 1)/2$, $\mathrm{It}^{\alpha}(\varphi)_{i+jk}$ changes for every integer j when α crosses the ith angle on the orbit of φ. ■

3.3. Lemma. *Let n be a fixed integer. For an angle $\theta = p/(2^n - 1)$ with any $p = 1, \ldots, 2^n - 2$ and any $0 < \varepsilon < 1/2^n$, $K(\theta - \varepsilon)_n = \mathrm{It}^{\theta+\varepsilon}(\theta)_n \neq \mathrm{It}^{\theta-\varepsilon}(\theta)_n = K(\theta + \varepsilon)_n$.*

Idea of proof. θ is periodic, and n is some multiple of its exact period. By halving the circle with respect to each angle $\theta - \varepsilon$, θ, and $\theta + \varepsilon$ in turn one checks to which side the $(n-1)$th iterate of the doubling map takes the angles. We have

$$2^{n-1}(\theta+\varepsilon)=\frac{\theta}{2}+2^{n-1}\varepsilon=\frac{\theta+\varepsilon}{2}+\frac{\varepsilon}{2}(2^n-1)\in H_1^{\theta+\varepsilon}\Rightarrow K(\theta+\varepsilon)_n=1$$

for an even numerator p, $K(\theta+\varepsilon)_n=0$ for an odd p, etc. ∎

3.4. Corollary. *When θ is moving counter-clockwise around the circle, $K(\theta)_n$ changes from $\mathbf{0}$ to $\mathbf{1}$ every time θ crosses an angle $p/(2^n-1)$ with even numerator p and from $\mathbf{1}$ to $\mathbf{0}$ every time it crosses one with odd p.* (See Figure 3.b.)

The first n entries of the binary sequences $K(\theta\pm\varepsilon), \mathrm{It}^{\theta\mp\varepsilon}(\theta)$ agree for any $\varepsilon\le 1/2^{2n}$ and every θ because there are no angles with periods $m\le n$ within distance $1/2^{2n}$ from the angle $a/(2^n-1)$;

$$\left|\frac{a}{2^n-1}-\frac{b}{2^m-1}\right|\ge\frac{1}{(2^n-1)(2^m-1)}>\frac{1}{2^{n+m}-1}\ge\frac{1}{2^{2n-1}-1}>\varepsilon. \tag{3.5}$$

Therefore the limit sequences

$$K^-(\theta):=\lim_{\varepsilon\searrow 0}K(\theta-\varepsilon)\quad\text{and}\quad K^+(\theta):=\lim_{\varepsilon\searrow 0}K(\theta+\varepsilon)$$

exist for every angle θ, periodic or not. Now Remark 1.2 allows us to define:

3.6. *Kneading sequences of hyperbolic components.* If $\theta_\pm$ are the external angles of a hyperbolic component $\mathcal{C}$, then its *kneading sequence* and *outside-kneading sequence* are, respectively, $K(\mathcal{C}):=K^+(\theta_-)=K^-(\theta_+)$ and $K_{\mathrm{out}}(\mathcal{C}):=K^-(\theta_-)=K^+(\theta_+)$.

$K(\mathcal{C})$ has the same exact period, say k, as $\mathcal{C}$ (cf. [LS]). By 3.4, $K(\mathcal{C})_n\ne K_{\mathrm{out}}(\mathcal{C})_n$ if and only if $n=jk$ for some $j\in\mathbb{N}$. To prove 3.8 and 3.9, which we shall use in §5, we need one more lemma from [At]:

3.7. Lemma. *If $c\in\mathcal{R}^{\mathcal{M}}_\alpha$, then $\mathrm{It}^\alpha(\psi)=\mathrm{It}^\alpha(\varphi)\in\{\mathbf{0},\mathbf{1}\}^{\mathbb{N}}$ if and only if the dynamical rays $\mathcal{R}^c_\psi$ and $\mathcal{R}^c_\varphi$ land at the same point $x\in J_c$.*

3.8. Lemma. *Let θ and φ be two angles, periodic under doubling with periods equal to some $k\in\mathbb{N}$, and $0<\theta<\varphi<1$. If $K^-(\theta)=K^+(\varphi)$ and the interval $]\theta,\varphi[$ contains no angles in the cycles of either θ or φ, then the parameter rays $\mathcal{R}^{\mathcal{M}}_\theta$ and $\mathcal{R}^{\mathcal{M}}_\varphi$ land at the same point $\widehat{c}\in\partial\mathcal{M}$.*

Proof. When α moves left of θ and β right of φ, Lemma 3.2 says that $\mathrm{It}^\alpha(\theta)=\mathrm{It}^\beta(\varphi)$ until α or β hits some of the angles in the cycles of θ or φ. But by assumption, 3.3, and (3.5),

$$K^-(\theta)=\mathrm{It}^\alpha(\theta)=\mathrm{It}^\alpha(\varphi)=K^+(\varphi)\qquad\text{for every }\alpha\in\,]\theta,\varphi[.$$

3.7 implies that the dynamical rays $\mathcal{R}^c_\theta$ and $\mathcal{R}^c_\varphi$ for any parameter $c\in\mathcal{R}^{\mathcal{M}}_\alpha$ land at the same point, thus determining an orbit portrait Ψ (cf. 1.4) on whose characteristic interval the angle α lies. Since the interval $[\theta, \varphi]$ contains α but no angles of the portrait other than the endpoints, it is the characteristic interval of Ψ. The claim now follows from 1.5. ∎

3.9. COROLLARY. *Two parameter rays with angles* $\theta = t/(2^n - 1)$ *and* $\varphi = (t+1)/(2^n - 1)$ *either land at the same point in* $\partial\mathcal{M}$, *or else there is an angle with a period* $i < n$ *on* $[\theta, \varphi]$.

Proof. If both angles are exactly n-periodic but the rays do not land at the same point, then by 3.8, $K^-(\theta)_i \neq K^+(\varphi)_i$ for some i. Since these binary sequences are at most n-periodic, $i < n$. By 3.4 there must be exactly one i-periodic angle on $]\theta, \varphi[$. The other possibility is that the period of one of the angles strictly divides n. ∎

3.10. NOTE. A similar argument would yield another proof for Lavaurs' Lemma 2.3.

3.11. *Algorithm* [LS]. Given an angle θ with period k, we can now use kneading sequences to find the internal address $1 \to n_1 \to n_2 \to \ldots \to k = \mathbf{A}(\mathcal{A})$ (see definition 2.2) of the hyperbolic component $\mathcal{A}$ at whose root point the parameter ray $\mathcal{R}_\theta^{\mathcal{M}}$ lands. Moving from the main cardioid $\mathcal{C}_0$ towards $\mathcal{A}$ along the combinatorial arc we enter into nested wakes of hyperbolic components with various periods but never come out of any wake. By 3.4, every time we enter the wake of an n-periodic component $\mathcal{B} \in]\mathcal{C}_0, \mathcal{A}]$, the nth entry in the kneading sequence changes. Hence the number n_i must be the index at which the *first* difference between $K(\mathcal{A})$ and $K(\mathcal{B}_{i-1})$ occurs, for each i.

The first $k-1$ entries of $K(\mathcal{A})$ agree with those of θ; $K(\mathcal{A})|_k$ is determined by the fact that the kth entry must be changed since the last thing we do is enter the wake of $\mathcal{A}$.

EXAMPLE. $\theta = 11/63$ is 6-periodic since $63 = 2^6 - 1$. Next, $K(\theta) = \overline{\mathbf{11010\blacktriangledown}}$. Comparing this sequence first to **111111**… and to **110110**… we obtain $n_1 = 3$ and $n_2 = 5$. The first difference between **1101011010**… and **11010** $*$ **11010** $*$ … should occur at place $6 = n_3$, so $K(\mathcal{A}) = \overline{\mathbf{110100}}$ and $\mathbf{A}(\mathcal{A}) = 1 \to 3 \to 5 \to 6$. (See again Figure 2.a.)

This algorithm can be used to find the component with the smallest period on the combinatorial arc between any two hyperbolic components; we shall do that in §4. The algorithm works in both directions; any abstract address gives a periodic binary sequence which may or may not be the kneading sequence of some hyperbolic component ([3]).

([3]) Translating kneading sequences back to angles is more difficult; this will be done in [BS], which will also give a complete characterization of non-realizable abstract addresses using Hubbard trees. See also [Pe].

For each initial word $b_1 \dots b_{n-1}$ with $b_i \in \{\mathbf{0}, \mathbf{1}\}$ exactly one of the binary sequences $\overline{b_1 \dots b_{n-1}\mathbf{0}}$ and $\overline{b_1 \dots b_{n-1}\mathbf{1}}$, call it $\overline{B}$, produces an *abstract address* (cf. 2.2) ending with n. Obviously, $\overline{B}$ has period exactly n. The other one, $\overline{B'}$, either produces an infinite address skipping n (and thus is not the kneading sequence of any hyperbolic component, but may be the outside kneading sequence of one), or else the period k of $\overline{B'}$ strictly divides n.

For example, $\overline{\mathbf{101}}$ gives an infinite address $1 \to 2 \to 4 \to 5 \to 7 \to 8 \to 10 \to 11 \to \dots$, which lists the periods of a sequence of hyperbolic components "approaching" the primitive 3-periodic component at address $1 \to 2 \to 3$ whose kneading sequence is $\overline{\mathbf{100}}$; thus $\overline{\mathbf{101}}$ is the outside kneading sequence of this component.

3.12. Remark. If $\overline{a_1 \dots a_k} = \overline{A}$ produces a finite address which is realized by some hyperbolic component, then $\overline{A^{q-1}A'} =: \overline{B}$ is the kneading sequence of its qk-periodic p/q-satellite(s) sitting at the internal address $1 \to \dots \to k \to qk$. This is because no pair of rays separate their rootpoints, so $\overline{B'} = \overline{A}$. For example, the 1/3-satellite of the component at $1 \to 2 \to 3$ has kneading sequence $\overline{\mathbf{100100101}}$ (Proposition 5.4 in [LS]).

4. Counterexample. Now we construct a tree $\mathcal{T}$ which does not obey the Translation Principle. Figure 4 is a diagram of the 1/2- and 1/3-subtrees of $\mathcal{T}$. Choose $\theta = 25/127$, whose kneading sequence is $\overline{\mathbf{110111}\blacktriangledown}$, so the parameter ray $\mathcal{R}^{\mathcal{M}}_{25/127}$ lands at the root of a seven-periodic component $\mathcal{C}$ with internal address $1 \to 3 \to 6 \to 7$ and kneading sequence $\overline{\mathbf{1101110}}$ ([4]). The other external angle with kneading sequence $\overline{\mathbf{110111}\blacktriangle}$ is 34/127, so $|W_{\mathcal{C}}| = 9/127$. Formula (1.3) gives the widths of the subwakes:

$$|W_{\mathcal{C}}^{1/2}| = \frac{9}{127} \cdot \frac{127^2}{16383} = \frac{9}{129} \quad \text{and} \quad |W_{\mathcal{C}}^{1/3}| = \frac{9 \cdot 127}{2097151} = \frac{9}{16513}.$$

The smallest possible n such that $1/(2^n - 1) < 9/129$ is 4, so there is a four-periodic component $\mathcal{A}_4$ in $W_{\mathcal{C}}^{1/2}$. By 3.11, $K(\mathcal{A}_4) = \overline{\mathbf{1100}}$, and by comparing $K_{\mathrm{out}}(\mathcal{A}_4) = \overline{\mathbf{1101}}$ to $K(\mathcal{C})$ we find the smallest period of components on the combinatorial arc $]\mathcal{C}, \mathcal{A}_4[$, which is ten:

110111011**0**1110...	1101110110111**1**0...	1101110110111**0**110...
110111011**1**01...	110111011011**0**1...	1101110110111**1**10...

The first difference between the outside-sequence of this 10-periodic component $\mathcal{A}_{10}$ and $K(\mathcal{C})$ occurs at the 13th place. Repeating this once more

([4]) The first counterexample I found was the tree with a five-periodic stem component at address $1 \to 2 \to 4 \to 5$, the same as the one Dierk Schleicher and Henk Bruin had found independently. Other examples are trees at $1 \to 2 \to 6 \to 7$, $1 \to 2 \to 4 \to 6 \to 7$ and $1 \to 2 \to 4 \to 5 \to 7$.

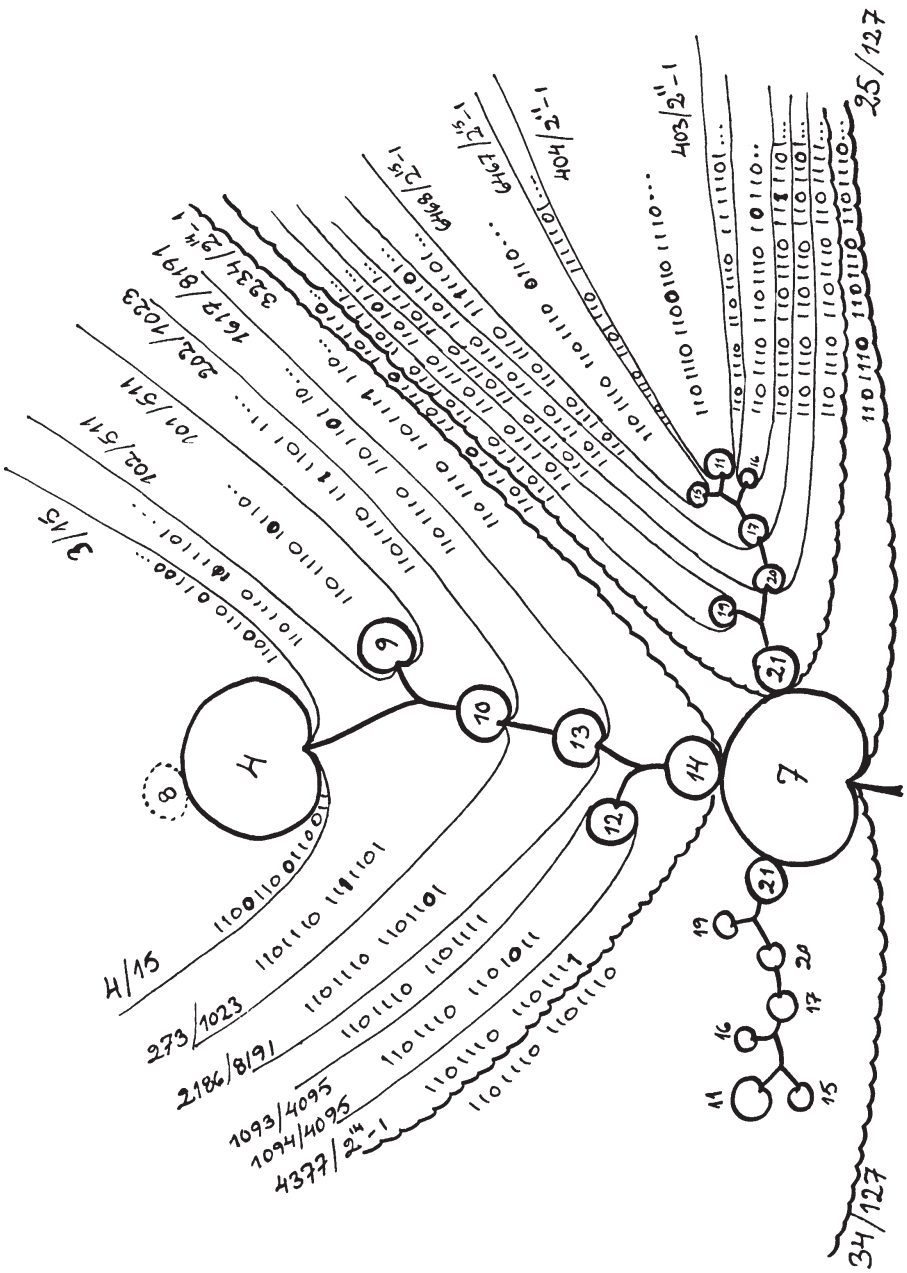

Fig. 4

we get a 14-periodic kneading sequence which, by 3.12, belongs to the 1/2-satellite of $\mathcal{C}$. Thus the "trunk" of the 1/2-subtree consists of $\mathcal{A}_4$, $\mathcal{A}_{10}$, $\mathcal{A}_{13}$, and $\mathcal{A}_{14}$. We must calculate the widths of these components to see if there are side-branches. Because

$$\frac{9}{127} \approx \frac{36.2}{511} \quad \text{and} \quad \frac{1}{15} \approx \frac{34.1}{511},$$

$W_{\mathcal{C}}$ contains 18 pairs of parameter rays—and thus 18 hyperbolic components—with period nine, 17 of which in $W_{\mathcal{A}_4}$, so there is a narrow-waked nine-periodic component $\mathcal{A}_9 \in W_{\mathcal{C}}^{1/2} \setminus W_{\mathcal{A}_4}$. Since we know that $\mathbf{A}(\mathcal{A}_9) = 1 \to 3 \to 6 \to 7 \to 9$ we can again compare the kneading sequences to find that $[\mathcal{C}, \mathcal{A}_4] \cap [\mathcal{C}, \mathcal{A}_9] = [\mathcal{C}, \mathcal{A}_{10}]$ and thus $|W_{\mathcal{A}_{10}}| = 71/1023$. Similarly we find $\mathcal{A}_{12}$ which branches off from $[\mathcal{C}, \mathcal{A}_{13}]$ and note that these six components make up the tree $\mathcal{T}_{1/2}$. In particular, of the nine components with periods dividing eight, one is $\mathcal{A}_4$ and all others are "invisible" behind it, so 8 does not appear in the tree at all.

In the same way, we construct the tree $\mathcal{T}_{1/3}$ (or $\mathcal{T}_{2/3}$), finding the main trunk with components $\mathcal{A}_{11}$, $\mathcal{A}_{17}$, $\mathcal{A}_{20}$, and $\mathcal{A}_{21}$ and two side-branches with narrow-waked components $\mathcal{A}_{16}$ and $\mathcal{A}_{19}$, in accordance with 2.6. But

$$\frac{9}{16513} \approx \frac{17.9}{2^{15}-1} \quad \text{and} \quad \frac{1}{2047} \approx \frac{16.0}{2^{15}-1},$$

so there is one visible 15-periodic component $\mathcal{A}_{15} \in W_{\mathcal{C}}^{1/3} \setminus W_{\mathcal{A}_{11}}$ which does not have an eight-periodic "partner" in $\mathcal{T}_{1/2}$ ($8 = (2-3)\cdot 7 + 15$) like the Translation Principle 2.6 states. By comparing its outside-kneading sequence $\overline{\mathbf{110111011011101}}$ to $\overline{\mathbf{1101110}}$ we find that $\mathcal{A}_{15}$ is in the end of its own branch like $\mathcal{A}_{11}$ and $\mathcal{A}_{16}$ are; therefore the trees $\mathcal{T}_{1/3}$ and $\mathcal{T}_{2/3}$ are not homeomorphic to $\mathcal{T}_{1/2}$.

5. Treetops. This section is devoted to proving Theorem 2.7. If $\mathcal{A}_q$ is the component with the smallest period, m_q, in some p/q-sublimb of a given k-periodic stem component $\mathcal{C}$, there can be no components $\mathcal{B}$ *visible* from $\mathcal{C}$ such that $\mathcal{C} \prec \mathcal{A}_q \prec \mathcal{B}$; in other words, the branch of the tree in question must terminate at $\mathcal{A}_q$ (cf. Figure 2.c).

For each $q \geq 2$, we must find the smallest number m_q such that the wake of $\mathcal{A}_q$ (which is necessarily *narrow* because $3/(2^{m_q}-1) > 1/(2^{m_q-1}-1)$) is narrow enough to fit into $|W_{\mathcal{C}}^{p/q}|$. We start by finding the number m_2 in terms of $|W_{\mathcal{C}}|$ and then show that $m_q = (q-2)k + m_2$. We shall first find pretty easily that there are two possibilities for the number m_q: it is either what 2.7 claims or one more. To rule out the latter possibility we shall show that in that case the stem component $\mathcal{C}$ would have a wake with an impossible width.

5.1. Lemma. *If $\mathcal{C}$ is a k-periodic hyperbolic component whose wake has width $|W_{\mathcal{C}}| = t/(2^k-1)$, where $t = 2^s + r$ with maximal integer s, then the* *smallest period of components in the* 1/2*-subwake of* $W_{\mathcal{C}}$ *is* $m_2 = k+1$ *if* $t = 1$, *and* $m_2 = k - s$ *if* $t \geq 3$.

We want the minimal m_q such that

$$\frac{1}{2^{m_q}-1} < \frac{t}{2^k-1} \cdot \frac{(2^k-1)^2}{2^{qk}-1} = \frac{t(2^k-1)}{2^{qk}-1} = |W_{\mathcal{C}}^{p/q}|,$$

and so

$$\frac{1}{2^{m_q}-1} < \frac{t(2^k-1)}{2^{qk}-1} < \frac{1}{2^{m_q-1}-1}. \tag{5.2}$$

In particular, m_2 is the unique integer such that

$$\frac{1}{2^{m_2}-1} < \frac{t(2^k-1)}{2^{2k}-1} = \frac{t}{2^k+1} < \frac{1}{2^{m_2-1}-1}.$$

5.3. Remark. For any m, no component with period different from m can have wake with width exactly $1/(2^m-1)$; otherwise this wake would contain a single m-periodic ray.

Beginning of proof of 5.1. If the wake of $\mathcal{C}$ is narrow ($t = 1$), we require that (cf. (5.2))

$$\frac{1}{2^{m_2}-1} < \frac{1}{2^k+1} < \frac{1}{2^{m_2-1}-1}.$$

Therefore $2^{m_2} > 2^k + 2 > 2^{m_2-1}$ and hence $m_2 > k \geq m_2 - 1$, so

$$m_2 = k+1. \tag{5.1.1}$$

In the non-narrow case ($t > 1$), $t = 2^s + r$ for some integers $s \geq 1$ and odd $r < 2^s$. Since we require that $\frac{1}{2^{m_2}-1} < \frac{2^s+r}{2^k+1} < \frac{1}{2^{m_2-1}-1}$ (cf. (5.2)),

$$2^{m_2} - 1 > \frac{2^k+1}{2^s+r} > \frac{2^k+1}{2^{s+1}} > 2^{k-s-1} \Rightarrow m_2 \geq k - s,$$

$$2^{m_2-1} - 1 < \frac{2^k+1}{2^s+r} < \frac{2^k+1}{2^s} < 2^{k-s} + \frac{1}{2} \Rightarrow m_2 - 1 \leq k - s,$$

and thus

$$k - s \leq m_2 \leq k - s + 1. \tag{5.1.2}$$

At this point we have proved Lemma 5.1 in the case of a narrow wake. Assuming $m_2 \neq k - s + 1$ for the moment, we are done with Theorem 2.7:

If $W_{\mathcal{C}}$ is narrow, then by (5.2) and (5.1.1), $m_q = (q-1)k + 1 = (q-2)k + m_2$, because

$$\frac{1}{2^{(q-1)k}-1} > \frac{1}{2^{(q-1)k} - \frac{1}{2^k}} = \frac{2^k}{2^{qk}-1}$$
$$> \frac{2^k-1}{2^{qk}-1} = \frac{1}{2^{(q-1)k} + \ldots + 1} > \frac{1}{2^{(q-1)k+1}-1}.$$

If $W_{\mathcal{C}}$ is not narrow, then $m_2 < k$ so that $2^{m_2} - 2^k < 0$. By (5.2), $t > (2^k+1)/(2^{m_2}-1)$, and thus

$$
\begin{aligned}
|W_{\mathcal{C}}^{p/q}| = \frac{t(2^k-1)}{2^{qk}-1} &> \frac{2^k+1}{(2^{m_2}-1)(2^{(q-1)k}+\ldots+2^k+1)} \\
&= \frac{2^k+1}{2^{(q-1)k+m_2}+(2^{m_2}-2^k)(2^{(q-2)k}+\ldots+2^k+1)-1} \\
&> \frac{2^k+1}{2^{(q-1)k+m_2}-1} > \frac{1}{2^{(q-2)k+m_2}-1},
\end{aligned}
$$

so $m_q \le (q-2)k + m_2$.

On the other hand,

$$
\begin{aligned}
\frac{t(2^k-1)}{2^{qk}-1} &\le \frac{2^{s+1}-1}{2^{(q-1)k}+\ldots+2^k+1} \\
&< \frac{2^{s+1}}{2^{(q-1)k}} < \frac{1}{2^{(q-1)k-s-1}-1} \Rightarrow m_q \ge (q-1)k - s.
\end{aligned}
$$

Therefore $m_q = (q-1)k - s = (q-2)k + m_2$ if $m_2 = k - s$, and we have shown

5.4. Observation. Lemma 5.1 implies Theorem 2.7.

It remains to be shown that $\boxed{m_2 = k - s}$, or, by (5.1.2), that $m_2 \neq k - s + 1$. For some values of t, the wake of a component with period $k - s$ *is* too wide to fit into $W_{\mathcal{C}}^{p/q}$.

Example. Consider a seven-periodic hyperbolic component $\mathcal{C}$ with wake-width $17/127$. Then

$$
|W_{\mathcal{C}}^{1/2}| = \frac{17}{129} = \frac{2^4+1}{2^7+1} < \frac{1}{2^{7-4}-1} = \frac{1}{7},
$$

so $m_2 = 4 = k - s + 1$ (here $k = 7$ and $s = 4$). Since $17/127 < 1/7$, $\mathcal{C}$ would have to be either in the wake of one of the three 3-periodic components $\mathcal{N}_3$ or between (with respect to natural order of angles on the circle) two such wakes.

In the first case, since $17/127 > 1/9 = |W_{\mathcal{N}_3}|$, $\mathcal{C}$ would be on the combinatorial arc between $\mathcal{N}_3$ and its 1/2-satellite, which is obviously impossible. In the latter case, $I(\mathcal{C})$ (cf. 1.6) would be contained in one of the intervals $]0,1/7[$, $]2/7,1/3[$, $]1/3,3/7[$, $]4/7,2/3[$, $]2/3,5/7[$, $]6/7,1[$. If it were the first or last one of these, $\mathcal{C}$ would have to be in the wake of some other satellite of the main cardioid; but they all have widths at most $1/15 < 17/127$. All other intervals above are even shorter than this, so there just is no room for $\mathcal{C}$ anywhere.

In general (m_2 and s being as above), we must prove the following two lemmas:

5.5. LEMMA. *For any k-periodic component $\mathcal{C}$, the condition $m_2 = k - s + 1$ implies*

$$\frac{1}{2^{k-s}} < |W_{\mathcal{C}}| < \frac{1}{2^{k-s} - 1}.$$

5.6. LEMMA. *For any $n \in \mathbb{N}$, no hyperbolic component $\mathcal{C}$ (of any period) such that*

$$\frac{1}{2^n + 1} < |W_{\mathcal{C}}| < \frac{1}{2^n - 1}$$

can exist anywhere in the parameter plane.

Proof of 5.5. Assume that the smallest period in the 1/2-subwake of $\mathcal{C}$ is $k - s + 1$, and $t = 2^s + r$ like above. Then, by (5.2),

$$\frac{1}{2^{k-s+1} - 1} < |W_{\mathcal{C}}^{1/2}| = \frac{t}{2^k + 1} < \frac{1}{2^{k-s} - 1}.$$

The second inequality implies

$$|W_{\mathcal{C}}| = \frac{t}{2^k - 1} < \frac{1}{2^{k-s} - 1}$$

because "$|W_{\mathcal{C}}| = 1/(2^{k-s} - 1)$" would contradict 5.3, and if $|W_{\mathcal{C}}| > 1/(2^{k-s} - 1)$, then there would be a $(k - s)$-periodic hyperbolic component on the combinatorial arc between $\mathcal{C}$ and its 1/2-satellite, which is impossible. On the other hand,

$$|W_{\mathcal{C}}| = \frac{t}{2^k - 1} = \frac{2^s + r}{2^k - 1} = \frac{1}{2^{k-s} - (1 + 2^{k-s}r)/t} > \frac{1}{2^{k-s}}. \quad \blacksquare$$

Beginning of proof of 5.6. Assume

$$\frac{1}{2^n + 1} < |W_{\mathcal{C}}| < \frac{1}{2^n - 1}.$$

We shall show that there is no room for the interval $I(\mathcal{C})$ anywhere on the circle $\mathbb{R}/\mathbb{Z}$. Since $W_{\mathcal{C}}$ cannot contain *just one* ray with angle period (dividing) n, $I(\mathcal{C})$ must be contained in one of the intervals

$$I_n^p := \left] \frac{p}{2^n - 1}, \frac{p+1}{2^n - 1} \right[, \qquad \text{where } p \in \mathbb{Z}_{2^n - 1}.$$

Corollary 3.9 directly implies that there are three disjoint possibilities:

(a) $I_n^p = I(\mathcal{N})$ for some narrow-waked n-periodic component $\mathcal{N}$,

(b) I_n^p contains one angle with period $m < n$,

(c) the period of one of the endpoints of I_n^p strictly divides n and I_n^p contains no angle with period less than n.

Obviously, $\mathcal{C}$ cannot fit into any narrow wake $W_{\mathcal{N}}$ because by assumption $|W_{\mathcal{C}}| > 1/(2^n+1) = |W_{\mathcal{N}}^{1/2}|$, so (a) is ruled out.

If $I(\mathcal{C}) \subset I_n^p$, then its distance to either of the two angles in ∂I_n^p is less than the difference of the lengths of these two intervals:

$$\frac{1}{2^n-1} - \frac{t}{2^k-1} < \frac{1}{2^n-1} - \frac{1}{2^n+1} = \frac{2}{2^{2n}-1} < \frac{1}{2^{2n-1}-1}.$$

But by (3.5) the difference between an endpoint of I_n^p and any angle with period $m < n$ is more than that, so $W_{\mathcal{C}}$ would have to contain a single m-periodic ray, which is impossible. Thus (b) is also ruled out.

We are left with (c), so assume the period h of $p/(2^n-1)$ is a proper divisor of $n = qh$. The ray $\mathcal{R}^{\mathcal{M}}_{p/(2^n-1)}$ lands at the root of some h-periodic hyperbolic component $\mathcal{H}$. Because I_n^p contains no angle with period less than n, the first $n-1$ digits in the kneading sequences of $p/(2^n-1)$ and $(p+1)/(2^n-1)$ must agree by 3.4. Hence the other ray, $\mathcal{R}^{\mathcal{M}}_{(p+1)/(2^n-1)}$, lands at the rootpoint of a hyperbolic component $\mathcal{N}$ with period n whose outside-kneading sequence consists of identical h-blocks.

SUBLEMMA. *$\mathcal{N}$ is the $1/q$-satellite component of $\mathcal{H}$.*

Proof. *A priori*, the exact period of $K_{\text{out}}(\mathcal{N})$ could be either h or some proper divisor j of $h = ij$; the latter case turns out to be impossible.

If the exact period of $K_{\text{out}}(\mathcal{N})$ is h, then by the Lau–Schleicher Algorithm 3.11, the internal address of $\mathcal{N}$ is $1 \to \ldots \to h \to qh = n$. Thus there must be an h-periodic hyperbolic component $\widetilde{\mathcal{H}}$ on the combinatorial arc between the main cardioid and $\mathcal{N}$, such that $K_{\text{out}}(\mathcal{N}) = K(\widetilde{\mathcal{H}})$. By 3.12, $\mathcal{N}$ is a satellite of $\widetilde{\mathcal{H}}$. Hence $\mathcal{H}$ must be $\widetilde{\mathcal{H}}$, because otherwise, by 1.6, there are three possibilities for the geometric arrangement of these two components:

(i) $\mathcal{H} \prec \widetilde{\mathcal{H}}$,
(ii) $\overline{W_{\mathcal{H}}} \cap \overline{W_{\widetilde{\mathcal{H}}}} = \emptyset$,
(iii) $\mathcal{H} \succ \widetilde{\mathcal{H}}$.

(i) and (ii) are impossible because the boundaries of the wakes must be at a distance at least $1/(2^h-1) > 1/(2^n-1)$ apart. We are assuming (c), i.e. that no pair of rays with a period less than n separates $\mathcal{N}$ from $\mathcal{H}$, so no such pair can separate $\mathcal{H}$ from $\widetilde{\mathcal{H}}$ either. But this contradicts Lavaurs' Lemma 2.3. Thus $\mathcal{N}$ must be some s/q-satellite component of $\mathcal{H}$; obviously, $s = 1$.

If the exact period of $K_{\text{out}}(\mathcal{N})$ were some proper divisor j of $h = ij$, then (by the same argument as above) both $\mathcal{H}$ and $\mathcal{N}$ would be satellites of a j-periodic component $\mathcal{J}$. Their internal angles have denominators $h/j = i$ and $n/j = qi$, respectively. The difference of these internal angles must be $1/(qi)$; when the circle $\mathbb{R}/\mathbb{Z}$ is divided into qi equal intervals, each of them except $]0, 1/(qi)[$ and $](qi-1)/(qi), 1[$ must contain exactly one angle with denominator $qi-1$. Therefore there is another satellite of $\mathcal{J}$ whose two external angles are between $p/(2^n-1)$ and $(p+1)/(2^n-1)$ and have period (dividing) $j(qi-1)=n-j$, which contradicts (c). ■

End of proof of 5.6. Since the $1/q$-subwake of $W_{\mathcal{H}}$ is at a distance $u/(2^{qh}-1)$ from $\partial I(\mathcal{H})$ if $|W_{\mathcal{H}}|=u/(2^h-1)$, now $W_{\mathcal{H}}$ must be narrow, i.e. $u=1$.

If $I(\mathcal{C})\subset I_n^p$, then $W_{\mathcal{C}}$ would have to be contained in some other l/r-subwake of $\mathcal{H}$ with $r>q$; but the width of such a subwake is at most

$$\frac{2^h-1}{2^{(q+1)h}-1}<\frac{1}{2^{qh}+1}<|W_{\mathcal{C}}|.$$

On the other hand,

$$\frac{1}{2^{qh}-1}-\frac{2^h-1}{2^{(q+1)h}-1}<\frac{2}{2^{(q+1)h}-1}<\frac{1}{2^{qh}+1}<|W_{\mathcal{C}}|,$$

so any subwake is too narrow to contain $W_{\mathcal{C}}$ but too wide to leave any room for it. Now (c) is ruled out as well.

Thus we conclude that $I(\mathcal{C})$ cannot be contained in *any* of the intervals I_n^p with $0\le p\le 2^n-2$, so there cannot exist a hyperbolic component $\mathcal{C}$ with this wake-width. ■

End of proof of 5.1. In particular, no k-periodic component $\mathcal{C}$ can have a wake with width in $]1/2^{k-s},1/2^{k-s}-1[$, so 5.5 implies that $m_2\neq k-s+1$. By (5.1.2), $m_2=k-s$. ■

Because of 5.4, we have now proved 2.7. ■

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