Every reasonably sized matrix group
is a subgroup of $S_\infty$

by

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Abstract. Every reasonably sized matrix group has an injective homomorphism into the group $S_\infty$ of all bijections of the natural numbers. However, not every reasonably sized simple group has an injective homomorphism into $S_\infty$.

1. Introduction. If $X$ is any nonempty set, let $S(X)$ be the set of bijections of $X$. Let $S_\infty = S(\mathbb{N})$, where $\mathbb{N}$ is the set of natural numbers. Of course, we may think of $S_\infty$ as the set of all permutations of any countable set. It is natural to ask: to what extent is Cayley's Theorem true for $S_\infty$?

In 1960 S. M. Ulam ([11] and [12], page 58) asked “Can one show that the group $R$ of all rotations in the three-dimensional space is isomorphic (as an abstract group, not continuously, of course) to a subgroup of the group $S_\infty$ of all permutations of integers? Or, perhaps quite generally: is every Lie group isomorphic (as an abstract group) to a subgroup of the group $S_\infty$?”

These questions are obviously motivated by Problem 95 of the Scottish Book ([8]), due to Schreier and Ulam (November 1935), who asked and answered this question for $(\mathbb{R}, +)$.

The purpose of this paper is to prove the following theorem. It certainly answers Ulam’s first question and a large portion of the second.

**Theorem 1.** Let $n$ be a positive integer and $F$ be a field of arbitrary characteristic such that $\text{card}(F) \leq 2^{\aleph_0}$. Then there is an injective homomorphism of $\text{GL}(n, F)$ into $S_\infty$.

The proof is rather elementary, requiring what at best are minor perturbations of well known results in field theory. Note that since $\mathbb{N}$ can be decomposed into a countable number of countable subsets, the product of

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countably many copies of $S_\infty$ can be embedded into $S_\infty$. This implies the following corollary.

**Corollary 2.** Let $G = \prod_{k \geq 1} G_k$, where each $G_k$ is one of the groups described in Theorem 1 or $G_k = \{e\}$. Then there is an injective homomorphism of $G$ into $S_\infty$.

Recall that the rotation group in three dimensions $SO(3)$ has no subgroups of finite index. A simple consequence of Theorem 1 is that $SO(3)$ does have subgroups of countable index.

It is not clear just what groups have an injective homomorphism into $S_\infty$. However, the following theorem proves that not all reasonably sized groups have an injective homomorphism into $S_\infty$.

**Theorem 3.** There exists a simple group $G$ such that $\text{card}(G) = 2^{\aleph_0}$ and there is no injective homomorphism of $G$ into $S_\infty$.

The group $G$ of Theorem 3 can be taken to be $S_\infty/S_I$, where $S_I$ is the normal subgroup of $S_\infty$ consisting of all permutations which move only finitely many integers. Theorem 3 should be compared with the results of de Bruijn [3] who proves, for example, that $S_\infty$ can be embedded into $S_\infty/S_I$.  

2. **Proof of Theorem 1.** The bulk of the proof will be carried out in a sequence of simple and probably well known lemmas.

**Lemma 4.** Let $F_1$ and $F_2$ be two fields which have the same characteristic, are algebraically closed, and satisfy $\text{card}(F_1) = \text{card}(F_2) > \aleph_0$. Then $F_1$ and $F_2$ are isomorphic fields.

**Proof.** Let $P_j$ be the prime subfield of $F_j$ ($j = 1, 2$). $P_1$ is isomorphic to $P_2$ since $F_1$ and $F_2$ have the same characteristic. Let $B_j$ be a transcendence basis for $F_j$ over $P_j$, so that $F_j$ is the algebraic closure of $P_j(B_j)$. Since $F_j$ is uncountable so is $P_j(B_j)$ and hence $B_j$ is infinite. Thus $\text{card}(P_j(B_j)) = \text{card}(B_j)$ and $\text{card}(P_j(B_j)) = \text{card}(F_j)$ (Kaplansky [7], Theorem 65, p. 74). It follows that $\text{card}(B_1) = \text{card}(B_2)$. Hence, $P_1(B_1)$ and $P_2(B_2)$ are isomorphic fields and so are their algebraic closures $F_1$ and $F_2$.  

**Corollary 5.** Let $F$ be a field which is algebraically closed and satisfies $\text{card}(F) = 2^{\aleph_0}$. If $F$ has characteristic zero, then $F$ is algebraically isomorphic to the algebraic closure of the $q$-adic numbers $\mathbb{Q}_q$ for any prime $q$ or to the field of complex numbers $\mathbb{C}$. If $F$ has characteristic $p$, then $F$ is algebraically isomorphic to the algebraic closure of the field of formal Laurent series $F_p((x))$, where $F_p$ is the field of $p$ elements.

**Proof.** We have

\[ \text{card}(\mathbb{Q}_q) = 2^{\aleph_0} \quad \text{and} \quad \text{card}(F_p((x))) = 2^{\aleph_0}. \]

Now use Lemma 4 and Kaplansky [7], Theorem 65, p. 74.
Recall the elementary fact that if $F$ is a field, $|\cdot|$ is a nonarchimedean absolute value on $F$, $x, y \in F$, and $|y| < |x|$, then $|x + y| = |x|$ (Artin [1], Corollary 5.1).

**Lemma 6.** Let $L$ be a field, $K \subset L$ a subfield such that $L$ is algebraic over $K$, $|\cdot|$ a nonarchimedean absolute value on $L$, and $d(u, v) = |u - v|$ for all $u, v \in L$. Suppose that $(K, d)$ is a separable metric space. Then $(L, d)$ is a separable metric space.

**Proof.** The key idea in this proof can be found in Artin [1], p. 45.

We can assume that $L$ is algebraically closed, for $|\cdot|$ extends to be a nonarchimedean absolute value on the algebraic closure of $K$ (Bourbaki [2], Proposition 9, p. 428).

Let $F$ be a countable subfield of $K$ which is $d$-dense in $K$. It suffices to prove that the roots of the monic polynomials in $F[x]$ are dense in $L$. Let $u \in L$, let $f(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 \in K[x]$ be the irreducible polynomial for $u$ over $K$, let $C = \max\{1, |a_{n-1}|, \ldots, |a_1|, |a_0|\}$, and let $\varepsilon > 0$. Note that if $v \in L$ with $|v| > C \geq 1$ and if $0 \leq j \leq n - 1$, then $|a_jv^j| \leq C|v|^j < |v|^n$, hence $|a_0 + a_1v + \ldots + a_{n-1}v^{n-1}| \leq \max_{0 \leq j \leq n-1} |a_jv^j| < |v|^n$, and therefore $|f(v)| = |v|^n > 0$. Since $u$ is a root of $f(x)$, $|u| \leq C$. Choose $b_0, b_1, \ldots, b_{n-1} \in F$ such that $\max_{0 \leq j \leq n-1} |a_j - b_j| < \varepsilon$. Let $g(x) = x^n + b_{n-1}x^{n-1} + \ldots + b_1x + b_0 \in F[x]$. We have $g(x) = (x - v_1)\ldots(x - v_n)$, where $v_1, \ldots, v_n \in L$. Then $|(u - v_1)\ldots(u - v_n)| = |g(u)| = |f(u) - g(u)| \leq \max_{0 \leq j \leq n-1} |(a_j - b_j)u^j| < \varepsilon|u|^n$. Hence, there exists at least one $1 \leq k \leq n$ such that $|u - v_k| \leq C \sqrt[\varepsilon]{u}$.

**Lemma 7.** Let $F$ be a field which satisfies $\text{card}(F) \leq 2^{\aleph_0}$. Then there is a nonarchimedean absolute value $|\cdot|$ on $F$ with countable range and under which $F$ becomes a separable metric space. If $\text{card}(F) > \aleph_0$, then $|\cdot|$ is a nontrivial absolute value.

**Proof.** That the range of $|\cdot|$ is countable is not essential for the proof of Theorem 1, but does seem to be of independent interest.

We may assume that $F$ is algebraically closed and that $\text{card}(F) = 2^{\aleph_0}$. If not, enlarge $F$ to $F(B)$, where $B$ is a set of transcendental elements so that $\text{card}(F(B)) = 2^{\aleph_0}$, and let $K$ be the algebraic closure of $F(B)$. Then $\text{card}(K) = \text{card}(F(B)) = 2^{\aleph_0}$ (Kaplansky [7], Theorem 65, p. 74) and replace $F$ with $K$.

If $F$ has characteristic 0, let $q$ be a fixed prime. Corollary 5 implies that we may assume that $F$ is the algebraic closure of $\mathbb{Q}_q$, which is complete under $|\cdot|$. The $q$-adic absolute value $|\cdot|_q$ extends in a unique manner to be an absolute value $|\cdot|$ on $F$ (Bourbaki [2], Proposition 10, p. 429). The construction of $|\cdot|$ shows that the range of $|\cdot|$ is countable since the range of $|\cdot|_q$ is countable. Lemma 6 implies that $(F, d)$ is a separable metric space.
since \((\mathbb{Q}_q, d)\) is a separable metric space (\(\mathbb{Q}\) is dense in \(\mathbb{Q}_q\)) and \(F\) is an algebraic extension of \(\mathbb{Q}_q\).

Next, suppose that \(F\) has characteristic \(p\). In this case Corollary 5 implies that we may assume that \(F\) is the algebraic closure of the field of formal Laurent series \(F_p((x))\), where \(F_p\) is the field of \(p\) elements. \(F_p((x))\) is complete under a natural absolute value \(|\cdot|\) (Jacobson [6], Theorem 9.16, p. 577). As in the characteristic 0 case, the absolute value \(|\cdot|\) on the field \(F_p((x))\) extends in a unique manner to be an absolute value, also denoted by \(|\cdot|\), on \(F\). The construction of \(|\cdot|\) again shows that the range of \(|\cdot|\) is countable.

Lemma 6 again implies that \((F, d)\) is a separable metric space since \(F_p((x))\) is a separable metric space (the finite Laurent series with coefficients in \(F_p\) are dense in \(F_p((x))\)) and \(F\) is an algebraic extension of \(F_p((x))\).

Finally, if \(\text{card}(F) > \aleph_0\), then the nonarchimedean absolute value \(|\cdot|\) is nontrivial since \((F, d)\) is a separable metric space.

The argument given in the next lemma is inspired by that sketched in Serre [10], page LG 4.4, in case \(F\) is locally compact.

**Lemma 8.** Let \(F\) be a field, \(|\cdot|\) a nontrivial nonarchimedean absolute value on \(F\) under which \(F\) is a separable metric space, \(A = \{a \in F \mid |a| \leq 1\}\), and \(n \geq 2\). Then \(\text{SL}(n, A)\) is a proper subgroup of \(\text{SL}(n, F)\) of index \(\leq \aleph_0\).

**Proof.** Identify \(\text{SL}(n, F)\) with a subset of \(F^{n^2}\) by concatenating the rows of each element of \(\text{SL}(n, F)\) and give \(\text{SL}(n, F)\) the relative topology. \(\text{SL}(n, F)\) then certainly is a separable metric topological group.

Next, recall the elementary facts that \(A\) is a commutative ring with identity since \(|\cdot|\) is a nonarchimedean absolute value and that \(A\) is open in \(F\), for if \(a \in A\) and \(b \in F\) satisfies \(|b| < 1\), then \(|a + b| \leq \max(|a|, |b|) \leq 1\), and therefore the ball \(B(a, 1) \subset A\). Next, \(A^{n^2}\) is an open subset of \(F^{n^2}\) and thus \(\text{SL}(n, A) = \text{SL}(n, F) \cap A^{n^2}\) is an open subset of \(\text{SL}(n, F)\). Note that \(\text{SL}(n, A)\) is closed under multiplication since \(A\) is a ring. \(\text{SL}(n, A)\) is also closed under inversion by using Cramer’s Rule, again since \(A\) is a ring. Hence, \(\text{SL}(n, A)\) is an open subgroup of \(\text{SL}(n, F)\). Further, \(\text{SL}(n, A)\) is a proper subgroup of \(\text{SL}(n, F)\) since \(|\cdot|\) is a nontrivial absolute value.

The quotient topological space \(\text{SL}(n, F)/\text{SL}(n, A)\) is therefore discrete and separable and not just a single point. Hence,

\[1 < \text{card}(\text{SL}(n, F)/\text{SL}(n, A)) \leq \aleph_0.\]

**Corollary 9.** Use the notation of Lemma 8. Let \(\varphi : \text{SL}(n, F) \to S(\text{SL}(n, F)/\text{SL}(n, A))\) be the natural homomorphism. Then the kernel of \(\varphi\) is \(Z(\text{SL}(n, F))\), the center of \(\text{SL}(n, F)\). In particular, there is an injective homomorphism of \(\text{SL}(n, F)/Z(\text{SL}(n, F))\) into \(S_\infty\).
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Proof. If $x \in \text{SL}(n, F)$, then $\varphi(x)$ is the identity if and only if $x \in \bigcap_{y \in \text{SL}(n, F)} y \text{SL}(n, A) y^{-1} = N$, a normal subgroup of $\text{SL}(n, F)$. We infer that $N$ is a proper subgroup of $\text{SL}(n, F)$ by Lemma 8. Hence, $x \in Z(\text{SL}(n, F))$ by Dieudonně [4], pp. 38–39. On the other hand, $Z(\text{SL}(n, F)) \subset \text{SL}(n, A)$ since $Z(\text{SL}(n, F))$ consists of diagonal matrices. ■

Corollary 10. Use the notation of Lemma 8. Let $G \subset \text{SL}(n, F)$ be a subgroup such that $G \cap Z(\text{SL}(n, F)) = \{e\}$. Then there is an injective homomorphism of $G$ into $S_\infty$.

Proof. There is an injective homomorphism of $G$ into the quotient group $\text{SL}(n, F)/Z(\text{SL}(n, F))$. Now use Corollary 9. ■

We are now set to complete the proof of Theorem 1. We can assume that $\text{card}(F) = 2^{\aleph_0}$ and that $F$ is algebraically closed by the proof of Lemma 7. The same lemma implies that there is a nontrivial nonarchimedian absolute value $|·|$ on $F$ under which $F$ is a separable metric space. Define an injective homomorphism $\varphi : \text{GL}(n, F) \to \text{SL}(n+2, F)$ as follows: if $x \in \text{GL}(n, F)$, let

$$\varphi(x)_{i,j} = \begin{cases} 
\det(x)^{-1} & \text{for } i = j = n + 1, \\
\varphi(x)_{i,j} = 1 & \text{for } i = j = n + 2, \\
\varphi(x)_{i,j} = 0 & \text{otherwise}.
\end{cases}$$

Then $\varphi(\text{GL}(n, F)) \cap Z(\text{SL}(n+2, F)) = \{e\}$ since $Z(\text{SL}(n+2, F))$ consists of scalar multiples of the identity matrix. Hence, Corollary 10 implies that there is an injective homomorphism of $\varphi(\text{GL}(n, F))$, and therefore of $\text{GL}(n, F)$, into $S_\infty$. ■

3. Proof of Theorem 3. If $F \subset \mathbb{N}$ is a nonempty finite set, define $U(F) = \{ \pi \in S_\infty \mid \pi(x) = x \text{ for every } x \in F\}$. Each $U(F)$ is a subgroup of $S_\infty$ of countable index. There is a unique Hausdorff topological group topology on $S_\infty$ such that the $U(F)$'s form a basis for the topology of $S_\infty$ at the identity. It is simple to check that $S_\infty$ is a complete separable metric topological group in this topology.

Lemma 11. Let $H$ be a topological group such that every subgroup of at most countable index is open and let $K$ be a topological group such that the open subgroups of at most countable index form a basis at $e$ in $K$. Then every group homomorphism $\varphi : H \to K$ is continuous.

Proof. Let $U$ be an open subgroup of $K$ of at most countable index. Then $\varphi^{-1}(U)$ is a subgroup of $H$ which is of at most countable index and therefore is open. Since such $U$'s form a basis at $e$ in $K$, $\varphi$ is continuous at $e$ in $H$, and therefore $\varphi$ is continuous. ■

Corollary 12. Every group homomorphism $\psi : S_\infty \to S_\infty$ is continuous.
Proof. The open subgroups of countable index form a basis for the topology of $S_\infty$ at $e$. On the other hand, subgroups of countable index in $S_\infty$ are open by Dixon et al. [5], Theorem 1, p. 580. Now use Lemma 11.

We are now ready to complete the proof of Theorem 3. Let $S_I$ be the normal subgroup of $S_\infty$ consisting of all permutations which move only finitely many integers. $S_\infty/S_I$ is known to be a simple group (Schreier and Ulam [9], Satz 1, p. 135). Suppose that $\varphi : S_\infty/S_I \to S_\infty$ is an injective homomorphism. Let $\pi : S_\infty \to S_\infty/S_I$ be the natural surjective quotient mapping and let $\psi = \varphi \circ \pi$. Then $\psi$ is a group homomorphism. $\psi$ is continuous by Lemma 12. But $S_I$ is in the kernel of $\psi$ and $S_I$ is dense in $S_\infty$. Hence $\psi$ is trivial and therefore $\varphi$ is trivial. Contradiction. So there is no injective homomorphism of $S_\infty/S_I$ into $S_\infty$.

4. Remark. The referee has pointed out that the separability of the space $(F, d)$ and the countability of the index of the subgroup $\text{SL}(n, A)$ in $\text{SL}(n, F)$ can be proved algebraically in the special cases $F = \mathbb{Q}_p$ and $F = \mathbb{F}_p((x))$, which allows for an alternative purely algebraic proof of Theorem 1.

References


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