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Every reasonably sized matrix group is a subgroup of S_∞

by

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Abstract. Every reasonably sized matrix group has an injective homomorphism into the group S_∞ of all bijections of the natural numbers. However, not every reasonably sized simple group has an injective homomorphism into S_∞.

1. Introduction. If X is any nonempty set, let $S(X)$ be the set of bijections of X. Let $S_\infty = S(\mathbb{N})$, where $\mathbb{N}$ is the set of natural numbers. Of course, we may think of S_∞ as the set of all permutations of any countable set. It is natural to ask: to what extent is Cayley's Theorem true for S_∞?

In 1960 S. M. Ulam ([11] and [12], page 58) asked "Can one show that the group R of all rotations in the three-dimensional space is isomorphic (as an abstract group, not continuously, of course) to a subgroup of the group S_∞ of all permutations of integers? Or, perhaps quite generally: is every Lie group isomorphic (as an abstract group) to a subgroup of the group S_∞?" These questions are obviously motivated by Problem 95 of the Scottish Book ([8]), due to Schreirer and Ulam (November 1935), who asked and answered this question for $(\mathbb{R}, +)$.

The purpose of this paper is to prove the following theorem. It certainly answers Ulam's first question and a large portion of the second.

THEOREM 1. *Let n be a positive integer and F be a field of arbitrary characteristic such that* $\mathrm{card}(F) \le 2^{\aleph_0}$. *Then there is an injective homomorphism of* $\mathrm{GL}(n, F)$ *into* S_∞.

The proof is rather elementary, requiring what at best are minor perturbations of well known results in field theory. Note that since $\mathbb{N}$ can be decomposed into a countable number of countable subsets, the product of

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countably many copies of S_∞ can be embedded into S_∞. This implies the following corollary.

COROLLARY 2. *Let* $G = \prod_{k\ge 1} G_k$, *where each* G_k *is one of the groups described in Theorem* 1 *or* $G_k = \{e\}$. *Then there is an injective homomorphism of* G *into* S_∞.

Recall that the rotation group in three dimensions SO(3) has no subgroups of finite index. A simple consequence of Theorem 1 is that SO(3) does have subgroups of countable index.

It is not clear just what groups have an injective homomorphism into S_∞. However, the following theorem proves that not all reasonably sized groups have an injective homomorphism into S_∞.

THEOREM 3. *There exists a simple group* G *such that* $\mathrm{card}(G) = 2^{\aleph_0}$ *and there is no injective homomorphism of* G *into* S_∞.

The group G of Theorem 3 can be taken to be $S_\infty/S_{\mathrm f}$, where $S_{\mathrm f}$ is the normal subgroup of S_∞ consisting of all permutations which move only finitely many integers. Theorem 3 should be compared with the results of de Bruijn [3] who proves, for example, that S_∞ can be embedded into $S_\infty/S_{\mathrm f}$.

2. Proof of Theorem 1. The bulk of the proof will be carried out in a sequence of simple and probably well known lemmas.

LEMMA 4. *Let* F_1 *and* F_2 *be two fields which have the same characteristic*, *are algebraically closed*, *and satisfy* $\mathrm{card}(F_1) = \mathrm{card}(F_2) > \aleph_0$. *Then* F_1 *and* F_2 *are isomorphic fields.*

Proof. Let P_j be the prime subfield of F_j $(j = 1,2)$. P_1 is isomorphic to P_2 since F_1 and F_2 have the same characteristic. Let B_j be a transcendence basis for F_j over P_j, so that F_j is the algebraic closure of $P_j(B_j)$. Since F_j is uncountable so is $P_j(B_j)$ and hence B_j is infinite. Thus $\mathrm{card}(P_j(B_j)) = \mathrm{card}(B_j)$ and $\mathrm{card}(P_j(B_j)) = \mathrm{card}(F_j)$ (Kaplansky [7], Theorem 65, p. 74). It follows that $\mathrm{card}(B_1) = \mathrm{card}(B_2)$. Hence, $P_1(B_1)$ and $P_2(B_2)$ are isomorphic fields and so are their algebraic closures F_1 and F_2. ∎

COROLLARY 5. *Let* F *be a field which is algebraically closed and satisfies* $\mathrm{card}(F) = 2^{\aleph_0}$. *If* F *has characteristic zero*, *then* F *is algebraically isomorphic to the algebraic closure of the* q*-adic numbers* $\mathbb{Q}_q$ *for any prime* q *or to the field of complex numbers* $\mathbb{C}$. *If* F *has characteristic* p, *then* F *is algebraically isomorphic to the algebraic closure of the field of formal Laurent series* $F_p((x))$, *where* F_p *is the field of* p *elements.*

Proof. We have

$$\mathrm{card}(\mathbb{Q}_q) = 2^{\aleph_0} \quad \text{and} \quad \mathrm{card}(F_p((x))) = 2^{\aleph_0}.$$

Now use Lemma 4 and Kaplansky [7], Theorem 65, p. 74. ∎

Recall the elementary fact that if F is a field, $|\cdot|$ is a nonarchimedian absolute value on F, $x, y \in F$, and $|y| < |x|$, then $|x+y| = |x|$ (Artin [1], Corollary 5.1).

LEMMA 6. *Let L be a field, $K \subset L$ a subfield such that L is algebraic over K, $|\cdot|$ a nonarchimedian absolute value on L, and $d(u,v) = |u-v|$ for all $u, v \in L$. Suppose that (K,d) is a separable metric space. Then (L,d) is a separable metric space.*

Proof. The key idea in this proof can be found in Artin [1], p. 45.

We can assume that L is algebraically closed, for $|\cdot|$ extends to be a nonarchimedian absolute value on the algebraic closure of K (Bourbaki [2], Proposition 9, p. 428).

Let F be a countable subfield of K which is d-dense in K. It suffices to prove that the roots of the monic polynomials in $F[x]$ are dense in L. Let $u \in L$, let $f(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 \in K[x]$ be the irreducible polynomial for u over K, let $C = \max\{1, |a_{n-1}|, \ldots, |a_1|, |a_0|\}$, and let $\varepsilon > 0$. Note that if $v \in L$ with $|v| > C \geq 1$ and if $0 \leq j \leq n-1$, then $|a_j v^j| \leq C|v|^j < |v|^n$, hence $|a_0 + a_1 v + \ldots + a_{n-1}v^{n-1}| \leq \max_{0\leq j\leq n-1} |a_j v^j| < |v|^n$, and therefore $|f(v)| = |v|^n > 0$. Since u is a root of $f(x)$, $|u| \leq C$. Choose $b_0, b_1, \ldots, b_{n-1} \in F$ such that $\max_{0\leq j\leq n-1} |a_j - b_j| < \varepsilon$. Let $g(x) = x^n + b_{n-1}x^{n-1} + \ldots + b_1 x + b_0 \in F[x]$. We have $g(x) = (x - v_1)\ldots(x - v_n)$, where $v_1, \ldots, v_n \in L$. Then $|(u - v_1)\ldots(u - v_n)| = |g(u)| = |f(u) - g(u)| \leq \max_{0\leq j\leq n-1} |(a_j - b_j)u^j| < \varepsilon C^n$. Hence, there exists at least one $1 \leq k \leq n$ such that $|u - v_k| \leq C\sqrt[n]{\varepsilon}$. ■

LEMMA 7. *Let F be a field which satisfies* $\mathrm{card}(F) \leq 2^{\aleph_0}$. *Then there is a nonarchimedian absolute value $|\cdot|$ on F with countable range and under which F becomes a separable metric space. If* $\mathrm{card}(F) > \aleph_0$, *then $|\cdot|$ is a nontrivial absolute value.*

Proof. That the range of $|\cdot|$ is countable is not essential for the proof of Theorem 1, but does seem to be of independent interest.

We may assume that F is algebraically closed and that $\mathrm{card}(F) = 2^{\aleph_0}$. If not, enlarge F to $F(B)$, where B is a set of transcendental elements so that $\mathrm{card}(F(B)) = 2^{\aleph_0}$, and let K be the algebraic closure of $F(B)$. Then $\mathrm{card}(K) = \mathrm{card}(F(B)) = 2^{\aleph_0}$ (Kaplansky [7], Theorem 65, p. 74) and replace F with K.

If F has characteristic 0, let q be a fixed prime. Corollary 5 implies that we may assume that F is the algebraic closure of $\mathbb{Q}_q$, which is complete under $|\cdot|$. The q-adic absolute value $|\cdot|_q$ extends in a unique manner to be an absolute value $|\cdot|$ on F (Bourbaki [2], Proposition 10, p. 429). The construction of $|\cdot|$ shows that the range of $|\cdot|$ is countable since the range of $|\cdot|_q$ is countable. Lemma 6 implies that (F,d) is a separable metric space since $(\mathbb{Q}_q, d)$ is a separable metric space ($\mathbb{Q}$ is dense in $\mathbb{Q}_q$) and F is an algebraic extension of $\mathbb{Q}_q$.

Next, suppose that F has characteristic p. In this case Corollary 5 implies that we may assume that F is the algebraic closure of the field of formal Laurent series $F_p((x))$, where F_p is the field of p elements. $F_p((x))$ is complete under a natural absolute value $|\cdot|$ (Jacobson [6], Theorem 9.16, p. 577). As in the characteristic 0 case, the absolute value $|\cdot|$ on the field $F_p((x))$ extends in a unique manner to be an absolute value, also denoted by $|\cdot|$, on F. The construction of $|\cdot|$ again shows that the range of $|\cdot|$ is countable. Lemma 6 again implies that (F, d) is a separable metric space since $F_p((x))$ is a separable metric space (the finite Laurent series with coefficients in F_p are dense in $F_p((x))$) and F is an algebraic extension of $F_p((x))$.

Finally, if $\mathrm{card}(F) > \aleph_0$, then the nonarchimedian absolute value $|\cdot|$ is nontrivial since (F, d) is a separable metric space. ∎

The argument given in the next lemma is inspired by that sketched in Serre [10], page LG 4.4, in case F is locally compact.

LEMMA 8. *Let F be a field, $|\cdot|$ a nontrivial nonarchimedian absolute value on F under which F is a separable metric space, $A = [a \in F \mid |a| \le 1]$, and $n \ge 2$. Then $\mathrm{SL}(n, A)$ is a proper subgroup of $\mathrm{SL}(n, F)$ of index $\le \aleph_0$.*

Proof. Identify $\mathrm{SL}(n, F)$ with a subset of F^{n^2} by concatenating the rows of each element of $\mathrm{SL}(n, F)$ and give $\mathrm{SL}(n, F)$ the relative topology. $\mathrm{SL}(n, F)$ then certainly is a separable metric topological group.

Next, recall the elementary facts that A is a commutative ring with identity since $|\cdot|$ is a nonarchimedian absolute value and that A is open in F, for if $a \in A$ and $b \in F$ satisfies $|b| < 1$, then $|a + b| \le \max(|a|, |b|) \le 1$, and therefore the ball $B(a, 1) \subset A$. Next, A^{n^2} is an open subset of F^{n^2} and thus $\mathrm{SL}(n, A) = \mathrm{SL}(n, F) \cap A^{n^2}$ is an open subset of $\mathrm{SL}(n, F)$. Note that $\mathrm{SL}(n, A)$ is closed under multiplication since A is a ring. $\mathrm{SL}(n, A)$ is also closed under inversion by using Cramer's Rule, again since A is a ring. Hence, $\mathrm{SL}(n, A)$ is an open subgroup of $\mathrm{SL}(n, F)$. Further, $\mathrm{SL}(n, A)$ is a proper subgroup of $\mathrm{SL}(n, F)$ since $|\cdot|$ is a nontrivial absolute value.

The quotient topological space $\mathrm{SL}(n, F)/\mathrm{SL}(n, A)$ is therefore discrete and separable and not just a single point. Hence,

$$1 < \mathrm{card}(\mathrm{SL}(n, F)/\mathrm{SL}(n, A)) \le \aleph_0. \ \blacksquare$$

COROLLARY 9. *Use the notation of Lemma 8. Let $\varphi : \mathrm{SL}(n, F) \to S(\mathrm{SL}(n, F)/\mathrm{SL}(n, A))$ be the natural homomorphism. Then the kernel of φ is $Z(\mathrm{SL}(n, F))$, the center of $\mathrm{SL}(n, F)$. In particular, there is an injective homomorphism of $\mathrm{SL}(n, F)/Z(\mathrm{SL}(n, F))$ into S_∞.*

P r o o f. If $x \in \mathrm{SL}(n,F)$, then $\varphi(x)$ is the identity if and only if $x \in \bigcap_{y\in \mathrm{SL}(n,F)} y\,\mathrm{SL}(n,A)y^{-1} = N$, a normal subgroup of $\mathrm{SL}(n,F)$. We infer that N is a proper subgroup of $\mathrm{SL}(n,F)$ by Lemma 8. Hence, $x \in Z(\mathrm{SL}(n,F))$ by Dieudonné [4], pp. 38–39. On the other hand, $Z(\mathrm{SL}(n,F)) \subset \mathrm{SL}(n,A)$ since $Z(\mathrm{SL}(n,F))$ consists of diagonal matrices. ∎

COROLLARY 10. *Use the notation of Lemma* 8. *Let* $G \subset \mathrm{SL}(n,F)$ *be a subgroup such that* $G \cap Z(\mathrm{SL}(n,F)) = \{e\}$. *Then there is an injective homomorphism of* G *into* S_∞.

P r o o f. There is an injective homomorphism of G into the quotient group $\mathrm{SL}(n,F)/Z(\mathrm{SL}(n,F))$. Now use Corollary 9. ∎

We are now set to complete the proof of Theorem 1. We can assume that $\mathrm{card}(F) = 2^{\aleph_0}$ and that F is algebraically closed by the proof of Lemma 7. The same lemma implies that there is a nontrivial nonarchimedian absolute value $|\cdot|$ on F under which F is a separable metric space. Define an injective homomorphism $\varphi : \mathrm{GL}(n,F) \to \mathrm{SL}(n+2,F)$ as follows: if $x \in \mathrm{GL}(n,F)$, let $\varphi(x)_{i,j} = x_{i,j}$ for $1 \le i,j \le n$,

$$\varphi(x)_{i,j} = \begin{cases} \det(x)^{-1} & \text{for } i=j=n+1,\\ \varphi(x)_{ij} = 1 & \text{for } i=j=n+2,\\ \varphi(x)_{ij} = 0 & \text{otherwise.} \end{cases}$$

Then $\varphi(\mathrm{GL}(n,F)) \cap Z(\mathrm{SL}(n+2,F)) = \{e\}$ since $Z(\mathrm{SL}(n+2,F))$ consists of scalar multiples of the identity matrix. Hence, Corollary 10 implies that there is an injective homomorphism of $\varphi(\mathrm{GL}(n,F))$, and therefore of $\mathrm{GL}(n,F)$, into S_∞. ∎

3. Proof of Theorem 3. If $F \subset \mathbb{N}$ is a nonempty finite set, define $U(F) = [\pi \in S_\infty \mid \pi(x) = x$ for every $x \in F]$. Each $U(F)$ is a subgroup of S_∞ of countable index. There is a unique Hausdorff topological group topology on S_∞ such that the $U(F)$'s form a basis for the topology of S_∞ at the identity. It is simple to check that S_∞ is a complete separable metric topological group in this topology.

LEMMA 11. *Let* H *be a topological group such that every subgroup of at most countable index is open and let* K *be a topological group such that the open subgroups of at most countable index form a basis at* e *in* K. *Then every group homomorphism* $\varphi : H \to K$ *is continuous.*

P r o o f. Let U be an open subgroup of K of at most countable index. Then $\varphi^{-1}(U)$ is a subgroup of H which is of at most countable index and therefore is open. Since such U's form a basis at e in K, φ is continuous at e in H, and therefore φ is continuous. ∎

COROLLARY 12. *Every group homomorphism* $\psi : S_\infty \to S_\infty$ *is continuous.*

Proof. The open subgroups of countable index form a basis for the topology of S_∞ at e. On the other hand, subgroups of countable index in S_∞ are open by Dixon *et al.* [5], Theorem 1, p. 580. Now use Lemma 11. ∎

We are now ready to complete the proof of Theorem 3. Let $S_{\rm f}$ be the normal subgroup of S_∞ consisting of all permutations which move only finitely many integers. $S_\infty/S_{\rm f}$ is known to be a simple group (Schreier and Ulam [9], Satz 1, p. 135). Suppose that $\varphi : S_\infty/S_{\rm f} \to S_\infty$ is an injective homomorphism. Let $\pi : S_\infty \to S_\infty/S_{\rm f}$ be the natural surjective quotient mapping and let $\psi = \varphi \circ \pi$. Then ψ is a group homomorphism. ψ is continuous by Lemma 12. But $S_{\rm f}$ is in the kernel of ψ and $S_{\rm f}$ is dense in S_∞. Hence ψ is trivial and therefore φ is trivial. Contradiction. So there is no injective homomorphism of $S_\infty/S_{\rm f}$ into S_∞. ∎

4. Remark. The referee has pointed out that the separability of the space (F, d) and the countability of the index of the subgroup $\mathrm{SL}(n, A)$ in $\mathrm{SL}(n, F)$ can be proved algebraically in the special cases $F = \overline{\mathbb{Q}}_p$ and $F = \overline{F}_p((x))$, which allows for an alternative purely algebraic proof of Theorem 1.

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