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Wildness in the product groups

by

G. Hjorth (Los Angeles, CA)

Abstract. Non-abelian Polish groups arising as countable products of countable groups can be tame in arbitrarily complicated ways. This contrasts with some results of Solecki who revealed a very different picture in the abelian case.

0. Group trees. The class of all Polish (completely metrizable, separable) groups may be naturally divided into two classes.

0.1. Definition. A Polish group G is *tame* if whenever X is a Polish G-space (that is to say, G acts continuously on X) the orbit equivalence relation is Borel as a subset of $X \times X$. A Polish group that is not tame is *wild.*

On the one hand the wild groups include almost all groups of reasonable topological complexity—for instance: S_∞, the infinite permutation group in the topology of pointwise convergence; U_∞, the unitary group of Hilbert space; $\mathbb{R}^{\mathbb{N}}$ (the infinite product of $(\mathbb{R}, +)$ in the product group structure and topology); c_0; l^2. The main examples of tame groups are the locally compact ones. These were not quite the only known examples—in [11] it is also shown that

$$\bigoplus_{p \,\mathrm{prime}} (\mathbb{Z}(p^\infty)),$$

the infinite product of the subgroups of $\mathbb{R}/\mathbb{Z}$ generated by $\{p^{-n} : n \in \mathbb{N}\}$ for p a prime, is tame.

Thus there is a gap in the spectrum of examples.

On the one hand we have the wild groups that can give rise to enormously complicated actions. On the other hand the principal examples of tame

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groups give rise to actions that are only F_σ—they are not just tame but excessively timid. The equivalence relations that can be induced by the more exotic groups of the form $\bigoplus_{p\,\mathrm{prime}}(\mathbb{Z}(p^\infty))$ are not necessarily F_σ, but still have low Borel complexity.

This paper paints in the gap and shows that we may have tame Polish groups producing arbitrarily complicated Borel orbit equivalence relations.

0.2. Theorem. *For every $\alpha < \omega_1$ there is a tame Polish group G acting continuously on a Polish space X with the resulting orbit equivalence relation $E_G \subset X \times X$ not $\utilde{\Pi}^0_\alpha$ as a subset of $X \times X$.*

So we may have tame G with some E_G not $F_{\sigma\delta}$, or not $G_{\delta\sigma\delta}$, or not $G_{\sigma\delta\sigma\delta}$, and so on.

For this purpose it is necessary to construct different tame G for different levels in the Borel hierarchy. By the universal space construction of §2.6 of [2] or the more recent construction of [4], G is tame if and only if there is a *bound* on the Borel complexity of the possible E_G.

0.3. Theorem (Becker–Kechris). *A Polish group G is tame if and only if there is an $\alpha < \omega_1$ such that whenever G acts continuously on a Polish space X the induced orbit equivalence relation $E_G \subset X \times X$ is $\utilde{\Pi}^0_\alpha$ as a subset of $X \times X$.*

The proof of 0.2 gives more information. In particular it provides a negative answer to a question from [11], where Sławek Solecki had raised the ambitious and provocative question of whether we may be able to characterize tameness *algebraically* for a broad class of Polish groups.

0.4. Definition. A Polish group G is said to be a *product group* if there is a sequence $G_0, G_1, G_2, \ldots$ of countable discrete groups such that $G = \prod_{i\in\mathbb{N}} G_i$ in the product topology and the product group structure (of pointwise multiplication).

For *abelian* product groups, [11] *does* characterize wildness.

0.5. Definition (Solecki). Let p be prime and G a group. Then G is *p-compact* if there is no decreasing sequence of subgroups $G_k < G \times \mathbb{Z}_p$ such that for each k,

$$\mathbb{Z}_p = \{\overline{n} : (\exists g \in G)((g, \overline{n}) \in G_k)\}$$

but

$$\{\overline{0}\} = \Big\{\overline{n} : (\exists g \in G)\Big((g, \overline{n}) \in \bigcap_{k\in\mathbb{N}} G_k\Big)\Big\}.$$

0.6. Theorem (Solecki). *Let $G = \prod_{i\in\mathbb{N}} G_i$ be an abelian product group. Then G is wild if and only if there is some prime p such that infinitely many G_i are not p-compact.*

[11] also shows that for non-abelian $G = \prod_{i\in\mathbb{N}} G_i$ the existence of some p with infinitely many G_i not p-compact is a necessary condition for wildness and raises the question of whether this condition is sufficient. A positive answer would in particular show that any product group giving rise to sufficiently complicated Borel orbit equivalence relations (more precisely: the universal G-space $(\mathcal{T}_p)^{\mathbb{N}}$ of Lemma 2 of [11] has orbit equivalence relation that is not at a finite level of the Borel hierarchy) must necessarily be wild. Since the counterexamples from 0.2 all have the form $G = \prod_{i\in\mathbb{N}} G_i$ we obtain a negative solution.

None the less, since the constructions underlying 0.2 are lengthy and the original question from [11] allows a simpler counterexample, we go to trouble of showing directly in §1 that:

0.7. Theorem. *There is a tame product group $G = \prod_{i\in\mathbb{N}} G_i$ such that no G_i is 2-compact.*

The G from 0.7 is rank-2 solvable.

0.8. Set-theoretical notation. (i) The vector notation, $\vec{x}$, indicates a sequence (that may be either finite or infinite). If $\vec{x} = \langle x_0, x_1, \ldots, x_i, \ldots\rangle$ then for any $n \in \mathbb{N}$, I will use $(\vec{x})_n$ for x_n. If $\vec{x} = \langle x_0, x_1, \ldots, x_i, \ldots, x_{k-1}\rangle$ then $l(\vec{x}) = k$, the length of $\vec{x}$. Given $\vec{x} = \langle x_0, \ldots, x_{k-1}\rangle$ and $\vec{y} = \langle y_0, \ldots \ldots, y_i, \ldots\rangle$ the sequence $\vec{x}\vec{y}$ is the concatenation of $\vec{x}$ and $\vec{y}$—so that its ith term is x_i for $i < k$ but its $(k + j)$th term is y_j (for $j < l(\vec{y})$ if $l(\vec{y})$ is finite). I will extend this to $\vec{x}a = \vec{x}\langle a\rangle := \langle x_0, \ldots, x_{k-1}, a\rangle$ and $a\vec{x} = \langle a\rangle\vec{x} := \langle a, x_0, \ldots, x_{k-1}\rangle$ (here and later := indicates equality by definition of the terms already defined).

(ii) For me $\mathbb{N} = \{0, 1, \ldots\}$ begins with zero. As with standard set-theoretical notation, an ordinal is identified with its predecessors: $n = \{0, 1, \ldots, n-1\}$. Ord is the class of all ordinals—infinite and finite.

(iii) A set $S \subset X^{<\mathbb{N}}$ (:= finite sequences from X) is said to be a *tree* if it is closed under subsequences. We then define a ranking function, Rk_S, from S to the ordinals plus infinity:

(a) $\mathrm{Rk}_S(\vec{x}) = 0$ if $\vec{x}$ has no proper extensions in S;

(b) given that we have defined $\mathrm{Rk}_S(\vec{y})$ for all $\vec{y}$ strictly extending $\vec{x}$ we set $\mathrm{Rk}_S(\vec{x}) = \sup\{\alpha + 1 \in \mathrm{Ord} : (\exists\vec{y} \in S)(\vec{y} \supset \vec{x},\ \vec{y} \neq \vec{x},\ \mathrm{Rk}_S(\vec{y}) = \alpha)\}$;

(c) for $\mathrm{Rk}_S(\vec{x})$ not defined by transfinite iteration of the process in (b) we let $\mathrm{Rk}_S(\vec{x}) = \infty$.

The reader can find in 2.E and appendix B of [6] a proof that $\mathrm{Rk}_S(\vec{x}) = \infty$ if and only if there is an infinite branch $f : \mathbb{N} \to X$ such that $f \supset \vec{x}$ and for all $n \in \mathbb{N}$,

$$f|_n\ (:= f|_{\{0,1,\ldots,n-1\}}) \in S.$$

We let the *rank* of S be the sup of $\mathrm{Rk}_S(\vec{x})$ for $\vec{x} \in S$. By induction on α we may construct for every countable successor ordinal α a tree $S \subset \mathbb{N}^{<\mathbb{N}}$ with rank α.

(iv) For A a countable set we may identify 2^A $(:= \{f : A \to \{0,1\}\})$ with $\mathcal{P}(A)$, the set of all subsets of A. If we give $\{0,1\}$ the discrete topology and 2^A the resulting product topology, then 2^A is a compact Polish space.

At 1.9 we will need to use the following fact: If α is a countable ordinal and $\vec{x} \in \mathbb{N}^{<\mathbb{N}}$, then the set of $S \subset A$ such that $\mathrm{Rk}_S(\vec{x}) = \alpha$ is a Borel set in $2^{\mathbb{N}^{<\mathbb{N}}}$. This is also proved by transfinite induction on α.

0.9. Descriptive set-theoretical notation. (i) A *Polish space* is a separable space that admits a complete metric. A topological group is said to be a *Polish group* if it is Polish as a space. If G is a Polish group and X is a Polish space equipped with a continuous action of G on X one says that X is a *Polish G-space*. We then use E_G^X, or just E_G when X is indicated, to denote the *orbit equivalence relation*

$$x_0 E_G^X x_1 \Leftrightarrow (\exists g \in G)(g \cdot x_0 = x_1).$$

(ii) Given a Polish space X, the $\utilde{\Sigma}_1^0$ sets are the open sets, the $\utilde{\Pi}_1^0$ sets are the closed sets. For α a countable ordinal we say that a set A is $\utilde{\Sigma}_\alpha^0$ if

$$A = \bigcup_{i \in \mathbb{N}} B_i$$

with each $B_i \in \utilde{\Pi}_{\beta(i)}^0$ for some $\beta(i) < \alpha$. A set is $\utilde{\Pi}_\alpha^0$ if its complement is $\utilde{\Sigma}_\alpha^0$.

Thus we have a hierarchy, starting with $\utilde{\Sigma}_2^0 = F_\sigma$, $\utilde{\Pi}_2^0 = G_\delta$, $\utilde{\Sigma}_3^0 = G_{\delta\sigma}$, and so on. For each countable α, $\utilde{\Pi}_\alpha^0 \subset \utilde{\Sigma}_{\alpha+1}^0$ and $\utilde{\Sigma}_\alpha^0 \subset \utilde{\Pi}_{\alpha+1}^0$. Every Borel set appears at some point in this hierarchy, and $\utilde{\Sigma}_\alpha^0 \neq \utilde{\Pi}_\alpha^0$ for each countable α.

0.10. Group-theoretical notation. (i) For G and H being groups, and $\psi : G \to \mathrm{Aut}(H)$ a homomorphism from G to the automorphism group of H, we form the semidirect product along ψ, $H \ltimes_\psi G$, in the usual fashion. It has $H \times G$ as its underlying set. Multiplication is given by

$$(h_1, g_1) \cdot (h_2, g_2) = (h_1((\psi(g_1))(h_2)), g_1 g_2).$$

(ii) Given a collection of groups, $\{G_i : i \in \Lambda\}$, we define $\prod_{i \in \Lambda} G_i$ to be the infinite product in the following way. The underlying set is the collection of functions f with domain Λ and each $f(i) \in G_i$ (if Λ is $\mathbb{N}$ or a natural number, we may use the sequence notation $\vec{x}$ to indicate elements of $\prod_{i \in \Lambda} G_i$). Given f_1 and f_2 in $\prod_{i \in \Lambda} G_i$ we define the product by taking pointwise multiplication:

$$(f_1 \cdot f_2)(i) = f_1(i) \cdot f_2(i).$$

$\bigoplus_{i\in\Lambda} G_i$ is the subgroup consisting of elements with finite support—that is to say, those $f \in \prod_{i\in\Lambda} G_i$ such that $f(i)$ is the identity for all but finitely many i.

Thus for Λ finite there is no difference between $\prod_{i\in\Lambda} G_i$ and $\bigoplus_{i\in\Lambda} G_i$.

(iii) I use the notations $G \times H$ and $G \oplus H$ interchangeably to mean the product of these two groups. Given homomorphisms $\psi_1 : G_1 \to H_1$ and $\psi_2 : G_2 \to H_2$ we define $\psi_1 \times \psi_2$ (or $\psi_1 \oplus \psi_2$) in the natural fashion from $G_1 \times G_2$ to $H_1 \times H_2$ by

$$(g_1, g_2) \mapsto (\psi_1(h_1), \psi_2(h_2)).$$

(iv) For abelian groups it is customary to use additive notation: $+$ for the group operation, $-a$ for the inverse of a, 0 for the identity. $n \cdot a$ denotes $a + \ldots + a$ (n times).

(v) If I do not know a group to be abelian I will use $g \cdot h$ or just gh to indicate the group operation. I will use a^{-1} for the inverse and e for the identity. g^n denotes $g \cdot \ldots \cdot g$ (n times).

(vi) $\mathbb{Z}_p$ is the cyclic group of size p. Its elements are $\{\overline{0}, \overline{1}, \ldots, \overline{p-1}\}$.

(vii) (See [11]) A set

$$S \subset \bigcup_{N\in\mathbb{N}} \prod_{i\leq\mathbb{N}} G_i$$

that actually forms a tree in the sense of 0.8(iii) above is called a *group tree* if $S \cap \prod_{i\leq N} G_i$ is a subgroup of $\prod_{i\leq N} G_i$ whenever it is non-empty. S is said to be a *coset tree* if for all $N \in \mathbb{N}$ and $\vec{g}_1, \vec{g}_2, \vec{g}_3 \in S \cap \prod_{i\leq N} G_i$ we have

$$\vec{g}_1 \cdot (\vec{g}_2)^{-1} \cdot \vec{g}_3 \in S.$$

In slight contrast to 0.8(iii), I will say that a group tree S is *well founded as a group tree* if it is well founded off the identity, in the sense that whenever $\vec{g} \in S$ with some $(\vec{g})_i \neq e$ then $\mathrm{Rk}_S(\vec{g}) < \infty$.

Group trees have been used in logic at various places: not just in Solecki's construction from [11] to obtain wild abelian groups, but also by Makkai in [7] for showing that there are sentences σ with no uncountable models despite unboundedness in Scott ranks and also in [9] in refuting the Ehrenfeucht conjecture. The real theorem in this paper is in fact a purely combinatorial construction for group trees.

0.11. Theorem. *For every countable ordinal α there is a sequence $(G_i)_{i\in\mathbb{N}}$ of countable groups such that*:

(i) *there is a group tree $S \subset \bigcup_{N\in\mathbb{N}} \prod_{i\leq\mathbb{N}} G_i$ with some $\vec{g} \in S$ and*

$$\infty > \mathrm{Rk}_S(\vec{g}) > \alpha;$$

(ii) *but there is some further countable $\beta > \alpha$ such that whenever $T \subset$*

$\bigcup_{N\in\mathbb{N}}\prod_{i\le\mathbb{N}} G_i$ *is a group tree and* $\vec{h}\in T$ *with* $Rk_T(\vec{h})>\beta$ *then*

$$\mathrm{Rk}_T(\vec{h})=\infty.$$

The connection between this purely combinatorial result proved in §2 and the descriptive set-theoretic consequences in 0.2 are discussed in §3.

One can also obtain a Makkai type result from this construction. It is perhaps somewhat strained, but I will state it to give a general impression of how this paper compares with [7].

0.12. Theorem. *For every countable ordinal* α *there are countable languages* $\mathcal{L}\supset\mathcal{L}'$ *and some* $\sigma\in\mathcal{L}'_{\omega_1\omega}$ *such that*:

(i) σ *has* (*up to isomorphism*) *exactly one* $\mathcal{L}'$ *model*, *and thus in particular it has no uncountable models*;

(ii) σ *has the form* $\bigwedge_i\bigvee_j\psi_{i,j}$ *where each* $\psi_{i,j}$ *is quantifier free*;

(iii) *there are* $\mathcal{L}$-*expansions of models of* σ *with Scott height greater than* α;

(iv) *there is some other countable* $\beta>\alpha$ *such that every* $\mathcal{L}$-*expansion of a model of* σ *has Scott height less than* β.

While in [7] one essentially finds:

0.13. Theorem (Makkai). *There are countable languages* $\mathcal{L}\supset\mathcal{L}'$ *and some* $\sigma\in\mathcal{L}'_{\omega_1\omega}$ *such that*:

(i) σ *has* (*up to isomorphism*) *exactly one* $\mathcal{L}'$ *model*;

(ii) σ *has the form* $\bigwedge_i\bigvee_j\psi_{i,j}$ *where each* $\psi_{i,j}$ *is quantifier free*;

(iii) *for all* $\alpha<\omega_1$ *there are* $\mathcal{L}$-*expansions of models of* σ *with Scott height greater than* α.

1. A counterexample. While all of this section is redundant, the methods of the general construction resemble those given below. In any case it seems desirable to present the narrow counterexample directly.

1.1. Definition. Let p be a prime. Let $(\prod_{\mathbb{N}}\mathbb{Z}_p)\ltimes\mathbb{Z}_2$ consist of pairs $(\vec{m},\bar{i})$ where $\vec{m}\in(\mathbb{Z}_p)^{\mathbb{N}}$, $\bar{i}\in\mathbb{Z}_2$. We view $(\mathbb{Z}_p)^{\mathbb{N}}$ as a group under the operation of pointwise addition, so that $(\vec{m}+\vec{m}')_n=(\vec{m})_n+(\vec{m}')_n$. Multiplication in $(\prod_{\mathbb{N}}\mathbb{Z}_p)\ltimes\mathbb{Z}_2$ is given by

$$(\vec{m},\bar{0})\cdot(\vec{m}',\bar{i}')=(\vec{m}+\vec{m}',\bar{i}'),\qquad(\vec{m},\bar{1})\cdot(\vec{m}',\bar{i}')=(\vec{m}-\vec{m}',\bar{1}+\bar{i}').$$

Thus we have taken the semidirect product of $\prod_{\mathbb{N}}\mathbb{Z}_p$ and $\mathbb{Z}_2$ along the homomorphism $\varphi:\mathbb{Z}_2\to\mathrm{Aut}(\prod_{\mathbb{N}}\mathbb{Z}_p)$ given by

$$(\varphi(\bar{0}))(\vec{m})=\vec{m},\qquad(\varphi(\bar{1}))(\vec{m})=-\vec{m}.$$

Thus $(\prod_{\mathbb{N}}\mathbb{Z}_p)\ltimes\mathbb{Z}_2$ is a group.

1.2. Definition. Let $(p_k)_{k\in\mathbb{N}}$ enumerate in increasing order the primes greater than 2. Let H'_k be the elements of $(\prod_{\mathbb{N}}\mathbb{Z}_p)\ltimes\mathbb{Z}_2$ with finite support—that is to say,

$$H'_k=\Big\{(\vec{m},\bar{i})\in\Big(\prod_{\mathbb{N}}\mathbb{Z}_p\Big)\ltimes\mathbb{Z}_2:(\exists N_0\in\mathbb{N})(\forall n>N_0)((\vec{m})_n=\overline{0})\Big\}.$$

In other words,

$$H'_k=\Big(\bigoplus_{\mathbb{N}}\mathbb{Z}_p\Big)\ltimes\mathbb{Z}_2.$$

For G a group and $g\in G$ let $o(g)$ denote the order of g—the least $n\in\mathbb{N}$ for which g^n is the identity (and let it be ∞ if no such n exists).

1.3. Lemma. *Let $(\vec{m},\overline{1})\in H'_k$. Then $o((\vec{m},\overline{1}))=2$.*

Proof. At once from the definitions of the group operations. ■

1.4. Lemma. *Let $(\vec{m},\overline{0})\in H'_k$. Then $o((\vec{m},\overline{0}))=p_k$.*

Proof. Since every non-identity element of $\prod_{\mathbb{N}}\mathbb{Z}_p$ has order p_k. ■

1.5. Lemma. *No H'_k is p-compact for $p=2$.*

Proof. Fix k. First we define two subgroups of H'_k. For $j\in\mathbb{N}$ define

$$\sigma^k_j\in\Big(\prod_{\mathbb{N}}\mathbb{Z}_{p_k}\Big)\ltimes\mathbb{Z}_2,\qquad \sigma^k_j=(\vec{m}(k,j),\overline{1}),$$

where

$$(\vec{m}(k,j))_n=\begin{cases}\overline{1} & \text{if } n=j,\\ \overline{0} & \text{otherwise.}\end{cases}$$

Let $\widehat{H}_k$ be the subgroup of H'_k generated by $\{\sigma^k_j: j\in\mathbb{N}\}$. We then let $\widehat{H}^0_k=\{(\vec{m},\overline{0}):(\vec{m},\overline{0})\in\widehat{H}_k\}$, the subgroup of $\widehat{H}_k$ consisting of elements that are trivial in the $\mathbb{Z}_2$ coordinate.

We also need a homomorphism from $\widehat{H}^0_k$ to $\mathbb{Z}_{p_k}$. For $(\vec{m},\overline{0})\in\widehat{H}^0_k$ and N_0 some (no matter which) element of $\mathbb{N}$ such that $(\vec{m})_n=\overline{0}$ for all $n>N_0$ we let

$$\pi_k((\vec{m},\overline{0}))=(\vec{m})_0+(\vec{m})_1+\ldots+(\vec{m})_{N_0}.$$

Clearly π_k is a homomorphism. The distinguishing feature of this homomorphism is its triviality.

Claim (1). *If $(\vec{m},\overline{0})\in\widehat{H}^0_k$ then $\pi_k((\vec{m},\overline{0}))=\overline{0}$.*

Proof. Write $(\vec{m},\overline{0})=\tau_1\ldots\tau_{2l}$ where each $\tau_i\in\{\sigma^k_j:j\in\mathbb{N}\}$. Inspecting the definitions of the group operations we see that each $\pi_k(\tau_{2i+1}\cdot\tau_{2i+2})$ equals $\overline{0}$. Then the claim follows from π_k being a homomorphism. $\blacksquare_{\text{Claim(1)}}$

Let $\vec{0}$ denote the element of $\prod_{\mathbb{N}}\mathbb{Z}_{p_k}$ that is zero at every coordinate—so $(\vec{0})_n=\overline{0}$ for all n.

CLAIM (2). $(\vec{0},\overline{1})$ *is not element of* $\widehat{H}_k$.

P r o o f. Otherwise we may write $(\vec{0},\overline{1})=(\vec{n},\overline{0})\cdot(\vec{m}(k,j),\overline{1})$ for some $\vec{m}(k,j)$. Then from Claim (1) we obtain $\pi_k((\vec{n},\overline{0}))=\overline{0}$, and hence for sufficiently large N_0,

$$(\vec{n})_0+(\vec{m}(k,j))_0+\ldots+(\vec{n})_{N_0}+(\vec{m}(k,j))_{N_0}=\overline{0},$$

with a contradiction. $\blacksquare_{\text{Claim(2)}}$

Finally then define $G_n<\mathbb{Z}_2\times\widehat{H}_k$ to be the set of all $(\bar{i},(\vec{m},\bar{i}))$ such that $(\vec{m})_j=\overline{0}$ for all $j\le n$. For each n the projection homomorphism $p_1:\mathbb{Z}_2\times\widehat{H}_k\twoheadrightarrow\mathbb{Z}_2$ has image $p_1[G_n]=\mathbb{Z}_2$. But by Claim (2),

$$\bigcap_{n\in\mathbb{N}}G_n=\{(\overline{0},(\vec{0},\overline{0}))\}.\ \blacksquare$$

1.6. LEMMA (implicit in Lemma 8 of [11]). *Let* (G_n) *be a sequence of countable torsion groups. Let*

$$(G_n)^{<\mathbb{N}}=\bigcup_{N\in\mathbb{N}}\prod_{n<N}G_n$$

and let $S\subset(G_n)^{<\mathbb{N}}$ *be a group tree. Let* $\mathrm{Rk}_S:S\to\omega_1\cup\infty$ *be the rank function, and for* $\sigma\in(G_n)^{<\mathbb{N}}$ *let* $l(\sigma)$ *indicate the unique* N *such that* $\sigma\in\prod_{n<N}G_n$. *Suppose* $\sigma\in S$ *and* p *is a prime with* $o(\sigma)=p^N$ *for some* $N\in\{1,2,3,\ldots\}$. *Then for all* $\tau\supset s$ *with* $\tau\in S$ *there exists* $\widehat{\tau}\in S$ *with*

$$o(\widehat{\tau})\in\{p^m:m=1,2,\ldots\},\qquad\widehat{\tau}\supset\sigma,\qquad l(\widehat{\tau})=l(\tau),\qquad\mathrm{Rk}_S(\widehat{\tau})\ge\mathrm{Rk}_S(\tau).$$

P r o o f. Choose q relatively prime to p with $o(\tau)=p^mq$ for some $m\in\{0,1,2,\ldots\}$. Let $\widehat{q}$ be divisible by q with $\widehat{q}\equiv1\bmod p^N$. Then setting $\widehat{\tau}=\tau^{\widehat{q}}$ completes the proof. $\blacksquare$

1.7. LEMMA. *Let* (H_n') *be as in* 1.2. *As above, let*

$$(H_n')^{<\mathbb{N}}=\bigcup_{N\in\mathbb{N}}\prod_{n<N}H_n',$$

let $S\subset(H_n')^{<\mathbb{N}}$ *be a group tree,* $\sigma\in S$, $o(\sigma)=p_k$ *for some* k, *and let* $\mathrm{Rk}_S:S\to\omega_1\cup\infty$ *be the rank function. Then* $\mathrm{Rk}_S(\sigma)\ge\omega$ *implies* $\mathrm{Rk}_S(\sigma)=\infty$.

P r o o f. Note that every $\tau\in(H_n')^{<\mathbb{N}}$ with order a power of p_k has $\tau(i)=e$ for all $i\ne k,i<l(\tau)$. Thus if $\mathrm{Rk}_S(\sigma)\ge\omega$, then it has extensions of all lengths, and therefore the element τ_n which takes value $\sigma(k)$ at k and e at $i<n$, $i\ne k$ will be an extension of σ in S. Thus S is ill founded below σ, and so $\mathrm{Rk}_S(\sigma)=\infty$. $\blacksquare$

1.8. LEMMA. *Let* (H_n') *be as in* 1.2. *Let* $S\subset(H_n')^{<\mathbb{N}}$ *be a group tree,* $\sigma\in S$, $o(\sigma)=2$. *Then* $\mathrm{Rk}_S(\sigma)\ge\omega\cdot2$ *implies* $\mathrm{Rk}_S(\sigma)=\infty$.

P r o o f. In light of 1.6 it suffices to show that no element σ with order a power of 2 has $\mathrm{Rk}_S(\sigma)=\omega\cdot 2$. Since the H'_n groups have no elements of order 2^l for $l>1$, we may suppose for a contradiction that $\mathrm{Rk}_S(\sigma)=\omega\cdot 2$ and $o(\sigma)=2$.

Let $k=l(\sigma)$. Then by 1.5 we may find cofinal strictly increasing λ_i with

$$\lambda_i\to\omega\cdot 2,$$

each $\lambda_i>\omega$, and $\tau_i\in\prod_{n<k+1}H'_n\cap S$ with

$$\mathrm{Rk}_S(\tau_i)=\lambda_i,\qquad \tau_i\supset\sigma,\qquad o(\tau_i)=2.$$

This in particular implies that

$$\mathrm{Rk}_S(\tau_i\cdot(\tau_j)^{-1})=\lambda_i$$

for $i<j$. We may also assume that each $\lambda_i>\omega$.

Inspecting the definition of the group H'_k we obtain, for $\tau_i\neq\tau_j$,

$$o((\tau_i\cdot(\tau_j)^{-1})(k))=p_k.$$

Thus we have some element of order p_k with rank exactly $\lambda_i>\omega$, $\lambda_i\neq\infty$, and therefore a contradiction to 1.7. ∎

The next result is implicit in Lemmas 2 and 6 of [11]. For the sake of completeness I will include a short, self-contained, but somewhat left handed, proof.

The idea of the lemma is that tameness is implied by there being a bound in the countable ordinals associated with a ranking function for group trees through a product group: If we can find a countable α such that no $\vec{g}$ in S has rank in the open interval (α,∞), then all the orbit equivalence relations are tame.

1.9. Lemma (Solecki). *Let α be a countable ordinal. Suppose*

$$\vec{G}^*=\prod_{i\in\mathbb{N}}\widehat{G}_i$$

is a product group with each $\widehat{G}_i$ countable. Suppose for every group tree

$$G\subset\bigcup_{N\in\mathbb{N}}\prod_{i\le N}\widehat{G}_i$$

and $\vec{g}\in G$,

$$\mathrm{Rk}_G(\vec{g})>\alpha\Rightarrow\mathrm{Rk}_G(\vec{g})=\infty.$$

Then $\vec{G}^*$ is tame.

P r o o f. Let X be a $\vec{G}^*$-Polish space. Let d be a complete metric for X. For each $x,y\in X$ let

$$S_{x,y}\subset\bigcup_{N\in\mathbb{N}}\prod_{i\le N}\widehat{G}_i$$

be the set of $\vec{g}$ such that for all $n \in \mathbb{N}$ there is some $\vec{h} \supset \vec{g}$ in $\prod_{i\in\mathbb{N}} \widehat{G}_i$ with

$$d(\vec{h}\cdot x, y) < 2^{-n}.$$

The function from $X \times X \to 2^{\bigcup_{N\in\mathbb{N}}\prod_{i\le N}\widehat{G}_i}$ given by $(x,y) \mapsto S_{x,y}$ is Borel (in the sense that for each $N \in \mathbb{N}$ and $\vec{g} \in \prod_{i\in\mathbb{N}} \widehat{G}_i$ the set of x, y with $\vec{g} \in S_{x,y}$ is Borel). Since $xE^X_G y$ if and only if $S_{x,y}$ has an infinite branch we will be done if we can show the set $\{(x,y) : (x,y) \in X \times X,\ S_{x,y}$ ill founded$\}$ to be Borel. This would certainly hold true if we can show that whenever $\mathrm{Rk}_{S_{x,y}}(\vec{g}) = \alpha + 1$ for some $\vec{g} \in S_{x,y}$ then $S_{x,y}$ is well founded.

So suppose that there is $\vec{g} \in S_{x,y}$ with $\mathrm{Rk}_{S_{x,y}}(\vec{g}) = \alpha + 1$ and that $f \in [S_{x,y}]$ is an infinite branch through $S_{x,y}$. If we let $G \subset \bigcup_{N\in\mathbb{N}}\prod_{i\le N}\widehat{G}_i$ be given by

$$G \cap \prod_{i\le N}\widehat{G}_i = f|_{N+1}^{-1}\cdot S_{x,y} \cap \prod_{i\le N}\widehat{G}_i$$

then we see that G is a group tree, since $G \cap \prod_{i\le N}\widehat{G}_i$ equals the set of $\vec{g}$ such that for all $n \in \mathbb{N}$ there is $\vec{h} \supset \vec{g}$ with

$$d(\vec{h}\cdot x, x) < 2^{-n}.$$

We can also see that the group tree $S_{x,y}$ is isomorphic (as a tree) to G under the isomorphism $\vec{h} \mapsto f|_{l(\vec{h})}\cdot \vec{h}$. Thus in particular $\mathrm{Rk}_G(f|_{l(\vec{g})}\vec{g}) = \alpha + 1$, contradicting the assumptions on $\vec{G}^*$. ∎

Solecki also proves a converse to 1.9, for which we will have no need.

1.10. Lemma. *The product group $\prod_{n\in\mathbb{N}} H'_n$ is tame.*

Proof. By 1.9, it suffices to show that if $S \subset (H'_n)^{<\mathbb{N}}$ is a group tree, $\sigma \in S$, $\mathrm{Rk}_S(\sigma) \ge \omega\cdot 2$ then $\mathrm{Rk}_S(\sigma) = \infty$. So instead suppose $\tau \in S$ with $\mathrm{Rk}_S = \omega\cdot 2$. Note that τ must be torsion by the choice of the groups (H'_n). Let $q_1,\ldots,q_m$ enumerate the prime divisors of $o(\tau)$. Note then that we may find $\sigma_1,\ldots,\sigma_m$ such that each σ_j has the form $\tau^{N(j)}$ for some $N(j)$, each σ_j has order a power of q_j, and hence $o(\sigma_j) = q_j$ by choice of the (H'_n), and such that

$$\tau = \sigma_1\ldots\sigma_m.$$

Then we must have some σ_j with $\mathrm{Rk}_S(\sigma_j) = \omega\cdot 2$, a contradiction to 1.7 or 1.8. ∎

Thus $\prod_{n\in\mathbb{N}} H'_n$ is a tame group. By 1.5 there are infinitely many n with H'_n not 2-compact, and so it provides the counterexample.

2. The general construction

2.1. NOTATION. For each pair of primes $q < p$ with q dividing $p-1$ fix a non-trivial homomorphism

$$\varphi_{q,p} : \mathbb{Z}_q \hookrightarrow \mathrm{Aut}(\mathbb{Z}_p).$$

If G is a group and $g \in G$ has order p let us write

$$g^{\varphi_{q,p}(\bar{j})} \quad \text{or} \quad \varphi_{q,p}(\bar{j}) \cdot g$$

in place of g^k or $k \cdot g$ where $k \in \{0, 1, \ldots, p-1\}$ is such that $(\varphi_{p,q}(\bar{j}))(\overline{1}) = \overline{k}$. (The more literally correct alternative of $g^{(\varphi_{q,p}(\bar{j}))(\bar{1})}$ is plainly very cumbersome.)

For p prime, the automorphism group of $\mathbb{Z}_p$ has size $p-1$, and so by the Sylow theorems (see [5], p. 93) we may find some automorphism of $\mathbb{Z}_p$ with order q for every prime q dividing $p-1$. It should also be remarked that if we fix q there is no shortage of potential p: By Dirichlet's theorem (see [10], p. 129) for each prime q there will be infinitely many primes of the form $kq + 1$.

The next couple of lemmas make the point that we can generalize the construction of §1 to semidirect products of the form $(\prod \mathbb{Z}_p) \ltimes \mathbb{Z}_q$ for primes $q > 2$. This is not quite sufficient for the general construction, since we will also need to increase the algebraic *depth* of the group—obtaining a group tree consisting of rank n solvable groups for arbitrarily large n.

2.2. LEMMA. *Let p and q be primes, q dividing $p-1$. The element $(\overline{1}, \overline{1})$ has order q in $\mathbb{Z}_p \ltimes_{\varphi_{q,p}} \mathbb{Z}_q$.*

Proof. Clearly the order can be no smaller than q, so we need to show that $(\overline{1}, \overline{1})^q = e$, which amounts to claiming that

$$\overline{1} + (\varphi_{q,p}(\overline{1}))(\overline{1}) + (\varphi_{q,p}(\overline{2}))(\overline{1}) + \ldots + (\varphi_{q,p}(\overline{q-1}))(\overline{1}) = \overline{0}.$$

But note that this element is in the subgroup of $\mathbb{Z}_p$ fixed by $\varphi_{q,p}$—which, since p is prime and $\varphi_{q,p}$ non-trivial, must be just $\{\overline{0}\}$. ■

2.3. LEMMA. *Let p and q be as above. Let $\widehat{\mathcal{G}}(\langle p, q\rangle) = (\prod_{\mathbb{N}} \mathbb{Z}_p) \ltimes_{\vec{\varphi}} \mathbb{Z}_q$ be the semidirect product obtained along $\vec{\varphi} = \prod_{\mathbb{N}} \varphi_{q,p}$ so that for $(a_0, a_1, \ldots) = \vec{a} \in \prod_{\mathbb{N}} \mathbb{Z}_p$ and $\overline{n} \in \mathbb{Z}_q$ we have*

$$((\vec{\varphi}(\overline{n}))(\vec{a}))_i = (\varphi_{q,p}(\overline{n}))(a_i).$$

Let $\widehat{\mathcal{G}}^0(\langle p, q\rangle) < \widehat{\mathcal{G}}(\langle p, q\rangle)$ be the subgroup generated by elements of the form $(\vec{a}_k, \overline{1})$, where $(\vec{a}_k)_m = \overline{1}$ if $m = k$, and $(\vec{a}_k)_m = \overline{0}$ if $m \neq k$. Define $\pi : \widehat{\mathcal{G}}^0(\langle p, q\rangle) \to \mathbb{Z}_p$ by

$$\pi((\vec{a}, \overline{n})) = \sum_{i \in \mathbb{N}} (\vec{a})_i.$$

(Note: this *is* well defined, since all elements in $\widehat{\mathcal{G}}^0(\langle p,q\rangle)$ have finite support.) *Then for all* $(\vec{a},\overline{n}) \in \widehat{\mathcal{G}}^0(\langle p,q\rangle)$ *we have*

$$\pi((\vec{a},\overline{n})) = \overline{0} \quad \textit{if and only if} \quad \overline{n} = \overline{0}.$$

Proof. First consider the "if" direction. Here it suffices to consider the case where

$$(\vec{a},\overline{0}) = (\vec{a}_{k(1)},\overline{1})(\vec{a}_{k(2)},\overline{1})\ldots(\vec{a}_{k(q)},\overline{1}).$$

But then

$$\pi((\vec{a},\overline{0})) = \overline{1} + (\varphi_{q,p}(\overline{1}))(\overline{1}) + (\varphi_{q,p}(\overline{2}))(\overline{1}) + \ldots + (\varphi_{q,p}(\overline{q-1}))(\overline{1}),$$

which we saw in the course of 2.2 to be $\overline{0}$.

So we are left with "only if". Using the first part of the proof it suffices to show that for any k strictly between 0 and q we have

$$\overline{1} + (\varphi_{q,p}(\overline{1}))(\overline{1}) + (\varphi_{q,p}(\overline{2}))(\overline{1}) + \ldots + (\varphi_{q,p}(\overline{k-1}))(\overline{1}) \neq \overline{0},$$

which amounts to showing that it is not fixed by $\varphi_{q,p}(\overline{1})$, which in turn amounts to showing that $(\varphi_{q,p}(\overline{k}))(\overline{1}) \neq \overline{1}$. This follows since $(\varphi_{q,p}(\overline{k}))$ is not the identity for $\overline{k} \neq \overline{0}$ in $\mathbb{Z}_q$. ∎

2.4. Lemma. *Let p and q be as above. Let*

$$\widehat{\mathcal{H}}(\langle p,q\rangle) = (\mathbb{Z}_p \times \mathbb{Z}_p) \ltimes_\psi \mathbb{Z}_q$$

be the semidirect product obtained by taking $\psi = \varphi_{q,p} \times \varphi_{q,p}$, so that for $(a,b) \in \mathbb{Z}_p \times \mathbb{Z}_p$ and $\overline{n} \in \mathbb{Z}_q$ we have

$$(\psi(\overline{n}))(a,b) = ((\varphi_{q,p}(\overline{n}))(a), (\varphi_{q,p}(\overline{n}))(b)).$$

Let $\widehat{\mathcal{H}}^0(\langle p,q\rangle) < \widehat{\mathcal{H}}(\langle p,q\rangle)$ be the subgroup generated by the elements $((\overline{1},\overline{0}),\overline{1})$ and $((\overline{0},\overline{1}),\overline{1})$. Define $\sigma : \widehat{\mathcal{H}}^0(\langle p,q\rangle) \to \mathbb{Z}_p$ by

$$\sigma((a,b),\overline{n}) = a + b.$$

Then for all $((a,b),\overline{n}) \in \widehat{\mathcal{H}}^0(\langle p,q\rangle)$ we have

$$\sigma((a,b),\overline{n}) = \overline{0} \quad \textit{if and only if} \quad \overline{n} = \overline{0}.$$

Proof. Exactly as in 2.3. ∎

Before proceeding, some comment should be made on the underlying motive in the long mass of definitions that lie ahead. Of course any general discussion is likely to be vague and inexact, but it seems better to have a rough guide than none at all.

We want to iterate the construction of 2.3 and 2.4. Given primes $p > q > r$, with q dividing $p-1$, r dividing $q-1$, we wish to somehow conjoin a group resembling many products of $(\prod_{\mathbb{N}} \mathbb{Z}_p) \ltimes_{\vec{\varphi}} \mathbb{Z}_q$ with $\mathbb{Z}_r$. The hope here is that we would have many generating elements of order r, many of whose differences would have order q, which in turn by taking differences would give rise to elements of order p. This is the spirit of the construction below, though not quite the letter since it is unclear how we would say arrive at a non-trivial homomorphism from $\mathbb{Z}_r$ to $\mathrm{Aut}((\prod_{\mathbb{N}} \mathbb{Z}_p) \ltimes_{\vec{\varphi}} \mathbb{Z}_q)$. Instead we more or less replace $(\prod_{\mathbb{N}} \mathbb{Z}_p) \ltimes_{\vec{\varphi}} \mathbb{Z}_q$ with

$$\bigoplus_{0 \le i < r} \Big(\Big(\prod_{\mathbb{N}} \mathbb{Z}_p \Big) \ltimes_{\vec{\varphi}} \mathbb{Z}_q \Big)$$

and let $\mathbb{Z}_r$ be injected into the automorphisms of this group in the most obvious manner imaginable—simple rotation of the coordinates.

It remains then to define the analogue of the $\widehat{\mathcal{G}}^0$ and $\widehat{\mathcal{H}}^0$ groups, which we do at 2.7. The spirit of 2.3 is captured in a non-triviality lemma at 2.10. The process that enables us to form an entire tree of suitable groups is presented at 2.11–14.

2.5. Definition. A finite sequence $\vec{p} = \langle p_1, \ldots, p_n \rangle$ of primes is said to be *good* if each p_{i+1} divides p_i. With each good $\vec{p}$ we associate countable groups $\mathcal{G}(\vec{p})$ and $\mathcal{H}(\vec{p})$ defined by induction on $l(\vec{p})$ (:= the length of $\vec{p}$), starting with the case $l(\vec{p}) = 2$.

Base Case: $l(\vec{p}) = 2$, $\vec{p} = \langle p, q \rangle$. We then let

$$\mathcal{G}(\vec{p}) = \Big(\bigoplus_{i \in \mathbb{N}} \Big(\bigoplus_{0 \le j < q} \mathbb{Z}_p \Big) \Big) \ltimes_{\psi} \mathbb{Z}_q,$$

where ψ is obtained by rotating each copy of $\bigoplus_{0 \le j < q} \mathbb{Z}_p$—so for

$$(\vec{a}_0, \vec{a}_1, \vec{a}_2, \ldots) = \vec{a} \in \bigoplus_{i \in \mathbb{N}} \Big(\bigoplus_{0 \le j < q} \mathbb{Z}_p \Big)$$

and $\overline{n} \in \mathbb{Z}_q$ we define $(\psi(\overline{n}))(\vec{a}) = \vec{b} = (\vec{b}_0, \vec{b}_1, \ldots)$ by the specification

$$(\vec{b}_i)_j = (\vec{a}_i)_{j+n \bmod q}.$$

Analogously we define

$$\mathcal{H}(\vec{p}) = \Big(\bigoplus_{0 \le j < q} \mathbb{Z}_p \times \bigoplus_{0 \le j < q} \mathbb{Z}_p \Big) \ltimes_{\psi'} \mathbb{Z}_q,$$

where again ψ' is obtained as the product of the natural rotation homomorphisms from $\mathbb{Z}_q$ to $\mathrm{Aut}(\bigoplus_{0 \le j < q} \mathbb{Z}_p)$—so that for $(\vec{a}, \vec{b}) \in \bigoplus_{0 \le j < q} \mathbb{Z}_p \times \bigoplus_{0 \le j < q} \mathbb{Z}_p$ and $\overline{n} \in \mathbb{Z}_q$ we define $(\psi'(\overline{n}))((\vec{a}, \vec{b})) = (\vec{c}, \vec{d})$ where

$$(\vec{c})_j = (\vec{a})_{j+n \bmod q}, \qquad (\vec{d})_j = (\vec{b})_{j+n \bmod q}.$$

Inductive Step. Assume that we have defined $\mathcal{G}(\vec{p})$ and $\mathcal{H}(\vec{p})$ for some $\vec{p} = (p_1, \ldots, p_n)$, $n \ge 2$, and that the sequence $\vec{p}q := (p_1, \ldots, p_n, q)$ is good. We then define

$$\mathcal{G}(\vec{p}q) = \Big(\bigoplus_{0 \le i < q} \mathcal{G}(\vec{p}) \times \bigoplus_{0 \le i < q} \mathbb{Z}_{p_n} \Big) \ltimes_{\psi} \mathbb{Z}_q,$$

where ψ is the familiar rotation homomorphism, now defined by the requirement that if $\overline{n} \in \mathbb{Z}_q$ and $(\vec{a}, \vec{b}) \in \bigoplus_{0\le i<q} \mathcal{G}(\vec{q}) \times \bigoplus_{0\le i<q} \mathbb{Z}_{p_n}$, then $(\psi(\overline{n}))(\vec{a}, \vec{b}) = (\vec{c}, \vec{d})$, where

$$(\vec{c})_j = (\vec{a})_{j+n \bmod q}, \qquad (\vec{d})_j = (\vec{b})_{j+n \bmod q}.$$

We let

$$\mathcal{H}(\vec{p}q) = \Big(\bigoplus_{0\le i<q} \mathcal{H}(\vec{p}) \times \bigoplus_{0\le i<q} \mathbb{Z}_{p_n}\Big) \ltimes_{\psi'} \mathbb{Z}_q,$$

where ψ' is again generated by rotation, so that if $\overline{n} \in \mathbb{Z}_q$ and $(\vec{a}, \vec{b}) \in \bigoplus_{0\le i<q} \mathcal{G}(\vec{p}) \times \bigoplus_{0\le i<q} \mathbb{Z}_{p_n}$, then $(\psi'(\overline{n}))(\vec{a}, \vec{b}) = (\vec{c}, \vec{d})$, with

$$(\vec{c})_j = (\vec{a})_{j+n \bmod q}, \qquad (\vec{d})_j = (\vec{b})_{j+n \bmod q}.$$

For $\vec{p} = \langle p_0, p_1, \ldots, p_n\rangle$ the group $\mathcal{G}(\vec{p})$ has been carefully chosen so that given any infinite group $G_0 \subset \mathcal{G}(\vec{p})$ we may find an infinite sequence of distinct $h_i \in G_0$ with order p_0—in some sense, all the infinite growth of $\mathcal{G}(\vec{p})$ occurs in the $\aleph_0$ copies of $\mathbb{Z}_{p_0}$. We could think of elements of $\mathcal{G}(\vec{p})$ and $\mathcal{H}(\vec{p})$ as a tree of points chosen from $\bigoplus_{0\le i<p_{j+1}} \mathbb{Z}_{p_j}$ for $j < n$—the main difference between the two being that $\mathcal{G}(\vec{p})$ terminates with one final infinite split while the tree corresponding to $\mathcal{H}(\vec{p})$ is purely binary. We need some method of discussing the various nodes of these trees, and for this purpose it is necessary to define "projection" or "coordinate" functions.

The functions appearing in the next definition are not in general homomorphisms. That is not the purpose of the definition. Rather we are interested in defining by induction on the length of $\vec{p}$ various analogues of the π map from 2.3.

2.6. Definition. For $\vec{p} = \langle p_1, \ldots, p_n\rangle$ we define by induction on n functions from the group $\mathcal{G}(\vec{p})$ to $\bigcup_{i\le n} \mathbb{Z}_{p_i} \cup \bigcup_{\vec{r}\subset\vec{p}} \mathcal{G}(\vec{r})$.

Base Case: $l(\vec{p}) = 2$, $\vec{p} = \langle p, q\rangle$. For $(\vec{a}, \overline{m}) \in \mathcal{G}(\vec{p})$, $\vec{a} = (\vec{a}_0, \vec{a}_1, \ldots)$, and $k \in \mathbb{N}$ we let

$$\pi_{\vec{p},k,\langle i\rangle}((\vec{a}, \overline{m})) = (\vec{a}_k)_i, \qquad \pi_{\vec{p},0,\mathrm{b},\emptyset}((\vec{a}, \overline{m})) = \pi_{\vec{p},0,\mathrm{b}}((\vec{a}, \overline{m})) = \overline{m}.$$

(Here the b is intended to be a formal symbol standing for "back"; it is *not* a variable ranging over $\mathbb{N}$.)

Inductive Step. Assume that we have completed the various definitions for $\mathcal{G}(\vec{p})$ for some good $\vec{p} = \langle p_1, \ldots, p_n\rangle$, $n \ge 2$, and assume that the sequence $\vec{p}q := \langle p_1, \ldots, p_n, q\rangle$ is good. Let

$$((\vec{a}, \vec{b}), \overline{m}) \in \mathcal{G}(\vec{p}q) = \Big(\bigoplus_{0\le i<q} \mathcal{G}(\vec{p}) \times \bigoplus_{0\le i<q} \mathbb{Z}_{p_n}\Big) \ltimes_{\psi} \mathbb{Z}_q,$$

$\vec{a} = \langle a_0, a_1, \ldots, a_{q-1} \rangle$, $\vec{b} = \langle k_0, k_1, \ldots, k_{q-1} \rangle$. Then for $l(\vec{u}) = l(\vec{p}) - 1$ and $k \in \mathbb{N}$ we define

$$\pi_{\vec{p}q,k,\vec{u}j}(((\vec{a}, \vec{b}), \overline{m})) = \pi_{\vec{p},k,\vec{u}}(a_j), \quad \pi_{\vec{p}q,0,\mathrm{m},\langle j \rangle}(((\vec{a}, \vec{b}), \overline{m})) = k_j,$$
$$\pi_{\vec{p}q,0,\mathrm{b},\emptyset}(((\vec{a}, \vec{b}), \overline{m})) = \pi_{\vec{p}q,0,\mathrm{b}}(((\vec{a}, \vec{b}), \overline{m})) = \overline{m}, \quad \pi_{\vec{p}q,0,\mathrm{f},\langle j \rangle}(((\vec{a}, \vec{b}), \overline{m})) = a_j,$$

while for $l(\vec{w}) = i + 1 < l(\vec{p}) - 1$,

$$\pi_{\vec{p}q,i+1,\mathrm{f},\vec{w}j}(((\vec{a}, \vec{b}), \overline{m})) = \pi_{\vec{p},i,\mathrm{f},\vec{w}}(a_j),$$
$$\pi_{\vec{p}q,i+1,\mathrm{m},\vec{w}j}(((\vec{a}, \vec{b}), \overline{m})) = \pi_{\vec{p},i,\mathrm{m},\vec{w}}(a_j),$$

and for $l(\vec{v}) = i < i + 1 < l(\vec{p}) - 1$,

$$\pi_{\vec{p}q,i+1,\mathrm{b},\vec{v}j}(((\vec{a}, \vec{b}), \overline{m})) = \pi_{\vec{p},i,\mathrm{b},\vec{v}}(a_j).$$

(Again b, f, and m are formal symbols, standing for “back”, “forward”, and “middle”.)

Now we iterate the construction of the groups $\widehat{\mathcal{G}}^0(\langle p, q \rangle)$. Inductively we define “descendants” of the group elements $(\vec{a}_k, \overline{1})$ from 2.3.

2.7. Definition. We define group elements in $\mathcal{G}(\vec{p})$ by induction on the length of $\vec{p}$.

Base Case: $l(\vec{p}) = 2$, $\vec{p} = \langle p, q \rangle$, $\vec{p}$ good. For each $k \in \mathbb{N}$ we let $g_{\vec{p},k} \in \mathcal{G}(\vec{p})$ be defined by

$$g_{\vec{p},k} = (\vec{a}, \overline{1}),$$

where $(\vec{a})_m$ is the identity for $m \neq k$ (i.e. each $((\vec{a})_m)_j = \overline{0}$) and

$$((\vec{a})_k)_j = (\varphi_{q,p}(\bar{j}))(\overline{1}) \quad \text{for each } j \in \{0, 1, \ldots, q-1\}.$$

Inductive Step: Suppose $\vec{p}q$ is good, $l(\vec{p}) \geq 2$, and we have established the above definitions for $\vec{p} = \langle p_1, \ldots, p_n \rangle$. Then let

$$g_{\vec{p}q,k} = ((\vec{a}, \vec{b}), \overline{1}) \in \mathcal{G}(\vec{p}q)$$

where $\vec{b} = 0$, the identity in $\bigoplus_{0 \leq i < q} \mathbb{Z}_{p_n}$ (that is to say, each $\pi_{\vec{p}q,0,\mathrm{m},\langle j \rangle}(g_{\vec{p}q,k}) = 0$), and each $(\vec{a})_j = (g_{\vec{p},k})^{\varphi_{q,p_n}(\bar{j})}$ (i.e. each $\pi_{\vec{p}q,0,\mathrm{f},\langle j \rangle}(g_{\vec{p}q,k}) = (g_{\vec{p},k})^{\varphi_{q,p_n}(\bar{j})}$); for each $i < l(\vec{p}) - 3$,

$$g_{\vec{p}q,i+1,\mathrm{r}} = ((\vec{a}, 0), \overline{1}) \quad \text{where each } (\vec{a})_j = (g_{\vec{p},i,\mathrm{r}})^{\varphi_{q,p_k}(\bar{j})}$$

(i.e. each $\pi_{\vec{p}q,0,\mathrm{f},\langle j \rangle}(g_{\vec{p}q,i+1,\mathrm{r}}) = (g_{\vec{p},i,\mathrm{r}})^{\varphi_{q,p_n}(\bar{j})}$);

$$g_{\vec{p}q,0,\mathrm{r}} = ((e, \vec{b}), \overline{1}) \in \mathcal{G}(\vec{p}q) \quad \text{where each } (\vec{b})_j = (\varphi_{q,p_n}(\bar{j}))(\overline{1})$$

(thus each $\pi_{\vec{p}q,0,\mathrm{m},\langle j\rangle}(g_{\vec{p}q,0,\mathrm{r}}) = \varphi_{q,p_n}(\bar{j})$). (The r is, in parallel to what has gone before, a formal symbol which can be thought of as standing for "right".)

We then define for each good $\vec{p}$ the group $\mathcal{G}^0(\vec{p}) < \mathcal{G}(\vec{p})$ to be the subgroup generated by the elements $\{g_{\vec{p},n}, g_{\vec{p},i,\mathrm{r}} : n \in \mathbb{N},\ i < l(\vec{p}) - 2\}$.

Before presenting the next lemma it might be helpful to isolate a simple part of this construction.

If q divides $p-1$ and we define

$$g = (\vec{a}, \bar{1}) \in \Big(\bigoplus_{0\le i<q} \mathbb{Z}_p \Big) \ltimes_\psi \mathbb{Z}_q$$

(where ψ is the rotating homomorphism from 2.5) by the specification that $(\vec{a})_j = (\varphi_{q,p}(\bar{1}))(\bar{j})$, then the subgroup generated by g may be naturally mapped into $\mathbb{Z}_p \ltimes_{\varphi_{q,p}} \mathbb{Z}_q$. The main point here is that

$$(\psi(\bar{i}))(\vec{a}) = (\varphi_{q,p}(\bar{i})) \cdot \vec{a}$$

(i.e. $k \cdot \vec{a} = \vec{a} + \ldots + \vec{a}$ [k times] where $(\varphi_{q,p}(\bar{i}))(\bar{1}) = \bar{k} \in \mathbb{Z}_p$)—and hence for any $\vec{b}$ in the subgroup of $\bigoplus_{0\le i<q} \mathbb{Z}_p$ generated by $\vec{a}$ and for any $\bar{i} \in \mathbb{Z}_q$ we have

$$(\psi(\bar{i}))(\vec{b}) = (\varphi_{q,p}(\bar{i})) \cdot \vec{b}.$$

Therefore a map such as $(\vec{b}, \bar{j}) \mapsto ((\vec{b})_0, \bar{j})$ defines an injective homomorphism from $\langle g\rangle$ to $\mathbb{Z}_p \ltimes_{\varphi_{q,p}} \mathbb{Z}_q$. In particular g has order q by Lemma 2.2. Similarly for $\vec{p} = \langle p, q\rangle$ and any $k \in \mathbb{N}$ we have $o(g_{\vec{p},k}) = q$.

This same consideration is relevant to determining the orders of the various generators in $\mathcal{G}^0(\vec{p})$. Notationally the general argument is more involved than the brief remark in the preceding paragraph only because these groups are significantly more complicated, and to even discuss them we require the ungainly coordinate functions from 2.6.

2.8. Lemma. *Let $\vec{p} = \langle p_1, \ldots, p_n\rangle$ be good.*

(A) *Each $g_{\vec{p},n}$ and $g_{\vec{p},i,r}$, has order p_n.*

(B) *Each $g \in \mathcal{G}^0(\vec{p})$ which is a power of one of the generators has the* property $P(\vec{p}, g)$, *defined by the following four equations*:

$$\begin{aligned}
\pi_{\vec{p},0,\mathrm{m},\langle l+1\rangle}(g) &= (\varphi_{p_n,p_{n-1}}(\bar{1}))(\pi_{\vec{p},0,\mathrm{m},\langle l\rangle}(g)),\\
\pi_{\vec{p},0,\mathrm{m},\langle 0\rangle}(g) &= (\varphi_{p_n,p_{n-1}}(\bar{1}))(\pi_{\vec{p},0,\mathrm{m},\langle p_n-1\rangle}(g)),\\
\pi_{\vec{p},0,\mathrm{f},\langle l+1\rangle}(g) &= (\varphi_{p_n,p_{n-1}}(\bar{1}))(\pi_{\vec{p},0,\mathrm{f},\langle l\rangle}(g)),\\
\pi_{\vec{p},0,\mathrm{f},\langle 0\rangle}(g) &= (\varphi_{p_n,p_{n-1}}(\bar{1}))(\pi_{\vec{p},0,\mathrm{f},\langle p_n-1\rangle}(g)).
\end{aligned}$$

(C) *Every* $g \in \mathcal{G}^0(\vec{p})$ *has the following* property $Q(\vec{p}, g)$:

$$\begin{aligned}\pi_{\vec{p},0,\mathrm{m},\langle l+1\rangle}(g) &= (\varphi_{p_n,p_{n-1}}(\bar{1}))(\pi_{\vec{p},0,\mathrm{m},\langle l\rangle}(g)),\\ \pi_{\vec{p},0,\mathrm{m},\langle 0\rangle}(g) &= (\varphi_{p_n,p_{n-1}}(\bar{1}))(\pi_{\vec{p},0,\mathrm{m},\langle p_n-1\rangle}(g)),\\ \pi_{\vec{p},1,\mathrm{b},\langle l+1\rangle}(g) &= (\varphi_{p_n,p_{n-1}}(\bar{1}))(\pi_{\vec{p},1,\mathrm{b},\langle l\rangle}(g)),\\ \pi_{\vec{p},1,\mathrm{b},\langle 0\rangle}(g) &= (\varphi_{p_n,p_{n-1}}(\bar{1}))(\pi_{\vec{p},1,\mathrm{b},\langle p_n-1\rangle}(g)).\end{aligned}$$

Proof. By simultaneous induction on $l(\vec{p})$.

The only role of (A) in supporting the proof of (B) is to ensure that $(\vec{a})_0 = ((\vec{a})_{p_n-1})^{\varphi_{p_n,p_{n-1}}(\bar{1})}$. For this we need that the order of $(\vec{a})_0$ is p_{n-1}. Logically the proof goes that we assume (A), (B), and (C) for $\langle p_1, \ldots, p_{n-1}\rangle$, then deduce (B) and (C), and only then (A) for $\langle p_1, \ldots, p_n\rangle$.

So suppose we have proved 2.8 for $\langle p_1, \ldots, p_{n-1}\rangle$ and we wish to extend it to $\vec{p} = \langle p_1, \ldots, p_n\rangle$. (In the case $n = 3$, which is the base of the induction, we rely not on the inductive hypothesis but on the remarks about $(\bigoplus \mathbb{Z}_{p_1}) \ltimes \mathbb{Z}_{p_2}$ from the paragraphs prior to the statement of 2.8.) First for (B), a moment's reflection indicates that if $g = (\vec{a}, \bar{i}), h = (\vec{b}, \bar{j})$, with $P(\vec{p}, g)$, $P(\vec{p}, h)$,

$$\vec{a}, \vec{b} \in \Big(\bigoplus_{0\le i<p_n} \mathcal{G}(\langle p_0, p_1, \ldots, p_{n-1}\rangle)\Big) \oplus \Big(\bigoplus_{0\le i<p_n} \mathbb{Z}_{p_n-1}\Big),$$

with $\vec{a}\vec{b} = \vec{b}\vec{a}$, then $P(\vec{p}, gh)$. Therefore using (A) for $\langle p_1, \ldots, p_{n-1}\rangle$ for the final clause of

$$\pi_{\vec{p},0,\mathrm{f},\langle p_n-1\rangle}(g) = (\varphi_{p_n,p_{n-1}}(\bar{1}))(\pi_{\vec{p},0,\mathrm{f},\langle 0\rangle}(g))$$

we conclude that each generator of $\mathcal{G}^0(\vec{p})$ has $P(\vec{p}, g)$ as do all its powers.

By a similar argument we obtain (C). The point is that (C) only makes reference to an abelian quotient of $(\bigoplus_{0\le i<p_n} \mathcal{G}(\langle p_0, p_1, \ldots, p_{n-1}\rangle)) \oplus (\bigoplus_{0\le i<p_n} \mathbb{Z}_{p_n-1})$, and thus the considerations marshaled in (B) continue marching through. (This is the reason why in (C) we pass to the coordinate functions $\pi_{\vec{p},0,\mathrm{m},\langle l\rangle}(g)$ and $\pi_{\vec{p},1,\mathrm{b},\langle l\rangle}(g)$.)

Finally for (A), let $g = ((\vec{a}, \vec{b}), \bar{1})$ be a basis element of $\mathcal{G}^0(\vec{p})$. We deduce from (B) that every element in $\langle g\rangle$ has the form $((\vec{a}^j, \vec{b}^j), \bar{i})$. Therefore we can go ahead and define

$$\pi : \langle g\rangle \to \mathbb{Z}_{p_{n-1}} \ltimes_{\varphi_{p_n,p_{n-1}}} \mathbb{Z}_{p_n}$$

by the specification that

$$\pi((\vec{a}^j, \vec{b}^j), \bar{i}) = (\bar{j}, \bar{i})$$

and use (B) for $\langle p_1, \ldots, p_n\rangle$ and $o(\vec{a}) = p_{n-1}$ to guarantee this is well defined as a homomorphism. Then by 2.2 we conclude that g has order p_n. ∎

2.9. Lemma. *For any good $\vec{p} = \langle p_1, \ldots, p_n\rangle$ with length > 2, any $l < p_n$, and for any $g \in \mathcal{G}^0(\vec{p})$, we have $\pi_{\vec{p},0,\mathrm{b},\langle l\rangle}(g) \neq \overline{0}$ if and only if*

$$\pi_{\vec{p},1,\mathrm{b},\langle l\rangle}(g) + \pi_{\vec{p},0,\mathrm{m},\langle l\rangle}(g) \neq \overline{0}.$$

Proof. Using 2.8(C) for g having property $Q(\vec{p}, g)$ we have a homomorphism

$$\pi : \mathcal{G}^0(\vec{p}) \to (\mathbb{Z}_{p_{n-1}} \times \mathbb{Z}_{p_{n-1}}) \ltimes_{(\varphi_{p_n,p_{n-1}} \oplus \varphi_{p_n,p_{n-1}})} \mathbb{Z}_{p_n}$$

given by

$$g \mapsto ((\pi_{\vec{p},1,\mathrm{b},\langle l\rangle}(g), \pi_{\vec{p},0,\mathrm{m},\langle l\rangle}(g)), \pi_{\vec{p},0,\mathrm{b}}(g)).$$

This granted, the lemma is a corollary of 2.4. ∎

2.10. Lemma. *Let $\vec{p} = \langle p, q\rangle$ be good, $l < q$, and $g \in \mathcal{G}^0(\vec{p})$. Then $\pi_{\vec{p},0,\mathrm{b}}(g) \neq \overline{0}$ if and only if*

$$\sum_{n\in\mathbb{N}} \pi_{\vec{p},n,\langle l\rangle}(g) \neq \overline{0}.$$

Proof. This follows from 2.3 as 2.9 followed from 2.4. ∎

There is a clear sense in which the $\mathcal{H}(\vec{p})$ groups are just simpler than the $\mathcal{G}(\vec{p})$. We can think of each element of the former as being a finite branching tree of values in various $\mathbb{Z}_{p_i}$; the pattern of branching is precisely the same for the latter, except at the various end stages when there are an infinite number of copies of $\mathbb{Z}_{p_1}$. So it should be clear that there are many possible homomorphisms from $\mathcal{G}^0(\vec{p})$ to $\mathcal{H}(\vec{p})$. We will parameterize a subclass of the possibilities by $n \in \mathbb{N}$. Intuitively, the nth homomorphism will send the nth copy of $\bigoplus_{i<p_2} \mathbb{Z}_{p_1}$ to the leftmost copy of this group in $\mathcal{H}(\vec{p})$; all the other copies of $\bigoplus_{i<p_2} \mathbb{Z}_{p_1}$ will have their values summed up and sent to the rightmost copy in $\mathcal{H}(\vec{p})$.

This intuitive description is not yet quite accurate, since we multiplied the various copies of $\bigoplus_{i<p_2} \mathbb{Z}_{p_1}$ every time we increase the length of $\vec{p}$. But despite the inherent inaccuracy, the rough description should at least serve as a guide in what follows below.

We use these homomorphisms in describing the relatively high rank group trees in the eventual construction.

2.11. Definition. We define homomorphisms

$$\varphi_{\vec{p},n} : \mathcal{G}^0(\vec{p}) \to \mathcal{H}(\vec{p}), \qquad \widehat{\varphi}_{q\vec{p}} : \mathcal{G}^0(q\vec{p}) \to \mathcal{H}(\vec{p})$$

for $\vec{p} = \langle p_1, \ldots, p_N\rangle$ and $q\vec{p} = \langle q, p_1, \ldots, p_N\rangle$, both good; we do that by induction on N.

BASE CASE: $l(\vec{p}) = 2$, $\vec{p} = \langle p_1, p_2 \rangle$. The first of the two constructions is described easily enough with

$$\varphi_{\vec{p},n}((\vec{a},\overline{k})) = \Big(\Big((\vec{a})_n, \sum_{j\neq n}(\vec{a})_j\Big), \overline{k}\Big).$$

Now suppose that q is prime with $q\vec{p}$ good. Then for

$$g \in \Big(\Big(\bigoplus_{0\leq i<p_2} \mathcal{G}(\vec{p})\Big) \bigoplus \Big(\bigoplus_{0\leq i<p_2} \mathbb{Z}_{p_1}\Big)\Big) \ltimes \mathbb{Z}_{p_2}$$

with $g = ((\vec{a}, \vec{b}), \overline{k})$, each $(\vec{a})_i = (\vec{c}_i, \overline{l}_i)$, we let

$$\widehat{\varphi}_{q\vec{p}}(g) = ((\vec{l}, \vec{b}), \overline{k})$$

where $\vec{l}$ is defined by $(\vec{l})_i = \overline{l}_i$ for $0 \leq i < p_2$. (In other words, we discard the very tip of g inside the $\vec{a}$ where infinitely many copies of $\mathbb{Z}_q$ are arranged.)

INDUCTIVE STEP: Let $\vec{p} = \langle p_1, \ldots, p_N \rangle$ and suppose that q and r are primes with $q\vec{p}r := \langle q, p_1, \ldots, p_N, r\rangle$ good. Suppose we have defined the various $\varphi_{\vec{p},n}$ and $\widehat{\varphi}_{q\vec{p}}$ and we wish to extend the definition to the next stage.

Given

$$((\vec{a}, \vec{b}), \overline{k}) \in \mathcal{G}^0(\vec{p}r) \subset \Big(\Big(\bigoplus_{0\leq i<r} \mathcal{G}(\vec{p})\Big) \oplus \Big(\bigoplus_{0\leq i<r} \mathbb{Z}_{p_N}\Big)\Big) \ltimes \mathbb{Z}_r = \mathcal{G}(\vec{p}r),$$

we let

$$\varphi_{\vec{p}r,n}(((\vec{a}, \vec{b}), \overline{k})) = ((\vec{c}, \vec{b}), \overline{k})$$

where $\vec{c} \in \bigoplus_{0\leq i<r} \mathcal{H}(\vec{p})$ is given by

$$(\vec{c})_i = \varphi_{\vec{p},n}((\vec{a})_i) \quad \text{for each } i \in \{0, 1, \ldots, r-1\}.$$

Similarly given

$$((\vec{a}, \vec{b}), \overline{k}) \in \mathcal{G}^0(q\vec{p}r) \subset \Big(\Big(\bigoplus_{0\leq i<r} \mathcal{G}(q\vec{p})\Big) \oplus \Big(\bigoplus_{0\leq i<r} \mathbb{Z}_{p_N}\Big)\Big) \ltimes \mathbb{Z}_r = \mathcal{G}(q\vec{p}r),$$

we let

$$\widehat{\varphi}_{q\vec{p}r}(((\vec{a}, \vec{b}), \overline{k})) = ((\vec{c}, \vec{b}), \overline{k}) \in \mathcal{H}^0(\vec{p}r)$$

where $\vec{c} \in \bigoplus_{0\leq i<r} \mathcal{H}^0(\vec{p})$ is defined by $(\vec{c})_i = \widehat{\varphi}_{q\vec{p}}((\vec{a})_i)$ for each $i \in \{0, 1, \ldots$ $\ldots, r-1\}$.

2.12. LEMMA. *Let $\vec{p}q = \langle p_1, \ldots, p_N, q\rangle$ be good. If $((\vec{a}, \vec{b}), \overline{k}) \in \mathcal{G}^0(\vec{p}q)$ then $\vec{a} \in \bigoplus_{i<p_N} \mathcal{G}^0(\vec{p})$.*

Proof. Suppose we have some generators $g_1, \ldots, g_M$ for $\mathcal{G}^0(\vec{p}q)$, with each $g_j = ((\vec{a}_j, \vec{b}_j), \overline{1})$. Then by inspecting the definition of group multiplication and the definition of the generators we have

$$g_1 \ldots g_M = ((\vec{a}, \vec{b}), \overline{M})$$

where

$$\vec{a} = (\vec{a}_1)(\vec{a}_2)^{\varphi_{q,p_n}(\bar{1})} \dots (\vec{a}_j)^{\varphi_{q,p_n}(\bar{j}-\bar{1})} \dots (\vec{a}_M)^{\varphi_{q,p_n}(\overline{M}-\bar{1})}. \ \blacksquare$$

Note that the inclusion of 2.12 is natural with respect to the homomorphisms of 2.11: If $((\vec{a},\vec{b}),\bar{l}) \in \mathcal{G}^0(\vec{p}r)$ and $((\vec{c},\vec{d}),\bar{l}) \in \mathcal{G}^0(q\vec{p}r)$ with

$$\varphi_{\vec{p}r,n}((\vec{a},\vec{b}),\bar{l}) = \varphi_{q\vec{p}r}((\vec{c},\vec{d}),\bar{l})$$

then for each $i < r$,

$$\varphi_{\vec{p},n}((\vec{a})_i) = \varphi_{q\vec{p}}((\vec{c})_i).$$

This is a consequence of the inductive definitions of the homomorphisms: they are set just so.

2.13. Notation. For each successor ordinal $\alpha < \omega_1$ choose a well founded *tree* (i.e. a set closed under subsequences) $T_\alpha \subset \mathbb{N}^{<\mathbb{N}}$ consisting of increasing sequences such that:

(i) if $\langle r_0, r_1, \dots, r_N\rangle \in T_\alpha$ then $\langle r_N, r_{N-1}, \dots, r_0\rangle$ is good;

(ii) given $\langle r_0, r_1, \dots, r_N\rangle$, $\langle q_0, q_1, \dots, q_M\rangle$, both in T_α, with $r_i \neq q_i$ we have $r_{j_1} \neq q_{j_2}$ for all $j_1, j_2 \geq i$;

(iii) for any $\vec{r} \in T_\alpha$, if $\vec{r}$ is not terminal there are infinitely many $q \in \mathbb{N}$ with $\vec{r}q \in T_\alpha$;

(iv) the *rank* of T_α is α.

As remarked at the start of this section, Dirichlet promises for each prime r_N infinitely many prime q with r_N dividing $q-1$. In light of this it is routine to construct for each successor α a tree as above. The main point here is that we demand a *disjointness* condition on the sequences in (ii)—once $\vec{q}$ and $\vec{r}$ begin to disagree at some point they proceed to diverge totally.

2.14. Definition. Let $(s_i)_{i\in\mathbb{N}}$ enumerate $\mathbb{N}^{<\mathbb{N}}$ so that $s_i \subset s_j$ implies $i \leq j$. If $s_i = \langle p_n, \dots, p_0\rangle$ we let $(s_i)^* := \langle p_0, p_1, \dots, p_n\rangle$ be obtained from s_i by reversing the order in which the range appears. For $\alpha < \omega_1$ a successor and $i \in \mathbb{N}$, $l(s_i) \geq 2$, let

$$G_{i,\alpha} = \begin{cases} \mathcal{G}^0((s_i)^*) & \text{when } s_i \in T_\alpha, \\ \{0\} & \text{otherwise.} \end{cases}$$

For each $s_i \in T_\alpha$ that is non-terminal in T_α we let $\{t_{n,\alpha}(s_i) : n \in \mathbb{N}\}$ enumerate the immediate successors of s_i in T_α, so that each $t_{n,\alpha}(s_i)$ has the form $s_i q$ for some $q \in \mathbb{N}$.

We then define

$$\vec{\mathcal{G}}(T_\alpha) \subset \bigcup_{\mathbb{N}} \prod_{i\leq N} G_{i,\alpha}.$$

For $\vec{g} \in \prod_{i\leq N} G_{i,\alpha}$ we let $\vec{g}$ be in $\vec{\mathcal{G}}(T_\alpha)$ if and only if:

(i) for all $i < j \leq N$ with $t_{n,\alpha}(s_i) = s_j$, $(s_i)^* = \langle p_0, p_1, \ldots, p_l\rangle := \vec{p}$, $(s_j)^* = \langle q, p_0, \ldots, p_l\rangle$, we have

$$\widehat{\varphi}_{q\vec{p}}((\vec{g})_j) = \varphi_{\vec{p},n}((\vec{g})_i);$$

(ii) if $j < N$ with s_j terminal, then for every $n \in \mathbb{N}$ and $\vec{u}$ of length $l(s_j) - 1$ we have

$$\pi_{(s_j)^*,n,\vec{u}}((\vec{g})_j) = \overline{0}.$$

Thus we have built an infinite product group that at each level consists of some $\mathcal{G}^0(\vec{p})$ for good $\vec{p}$. Inside the group tree we have a further tree $\vec{\mathcal{G}}(T_\alpha)$, which we later see to be well founded off the identity but have rank at least α. The definition of $\vec{\mathcal{G}}(T_\alpha)$ is that at each stage when we extend an element we must respect the relationships suggested by the $\varphi_{\vec{p},n}$ and $\widehat{\varphi}_{q\vec{p}}$ homomorphisms; as soon as we reach a terminal $s_j = \langle p_l, \ldots, p_0\rangle$, the group elements in $\vec{\mathcal{G}}(T_\alpha) \cap \prod_{i\leq j} G_{i,\alpha}$ must be the identity at every copy of $\mathbb{Z}_{p_0}$ in order to have any extensions at all.

It should be remarked that $\vec{\mathcal{G}}(T_\alpha) \cap \prod_{i\leq N} G_{i,\alpha}$ is indeed a subgroup of $\prod_{i\leq N} G_{i,\alpha}$ for every N. To see that (i) is closed under the group operations in $\prod_{i\leq N} G_{i,\alpha}$ we observe that for any group H and homomorphisms $\pi_0, \pi_1 : H \to \widehat{H}$ the set $\{h \in H : \pi_0(h) = \pi_1(h)\}$ is a subgroup of H.

2.15. Definition. For $s_l = \langle p_N, p_{N-1}, \ldots, p_0\rangle \in T_\alpha$, $l(s_l) \geq 2$, and $n \in \mathbb{N}$ we define

$$\vec{g}_{s_l,n} \in \prod_{i\leq l} G_{i,\alpha}$$

by the following conditions:

(i) $\vec{g}_{s_l,n}(l) = g_{(s_l)^*,n}$ (as defined at 2.7); for $k < l$ the element $\vec{g}_{s_l,n}(k)$ does not depend on n:

(ii) if $s_k \subset s_l$, $k < l$, and n' is such that $t_{n'}(s_k) \subset s_l$, then $\vec{g}_{s_l,n}(k) = g_{(s_k)^*,n'}$;

(iii) if s_k does not include $\langle p_N, p_{N-1}\rangle$ then $\vec{g}_{s_l,n}(k) = e$;

(iv) if $s_k \supset \langle p_N, p_{N-1}, q\rangle$, $q \neq p_{N-2}$, then $\vec{g}_{s_l,n}(k) = g_{(s_k)^*,0,\mathrm{r}}$;

(v) if $s_k \supset \langle p_N, p_{N-1}, \ldots, p_{N-M}, q\rangle$, $q \neq p_{N-M-1}$, then $\vec{g}_{s_l,n}(k) = g_{(s_k)^*,M-1,\mathrm{r}}$.

2.16. Lemma. (A) *For $q\vec{p} = \langle q, p_0, \ldots, p_l\rangle$ good and $n, n' \in \mathbb{N}$,*

$$\widehat{\varphi}_{q\vec{p}}(g_{q\vec{p},n'}) = \varphi_{\vec{p},n}(g_{\vec{p},n}).$$

(B) *If $n \neq m$ and $l(\vec{p}) = 2 + i$, then*

$$\widehat{\varphi}_{q\vec{p}}(g_{q\vec{p},i,r}) = \varphi_{\vec{p},m}(g_{\vec{p},n}).$$

(C) *For $l(\vec{p}) > 2$ and $0 \leq i < l(\vec{p}) - 2$,*

$$\widehat{\varphi}_{q\vec{p}}(g_{q\vec{p},i,r}) = \varphi_{\vec{p},m}(g_{\vec{p},i,r}).$$

Proof. (A) Induction on $l(\vec{p})$, starting with the base case, $l(\vec{p}) = 2$. For the base case, unwinding the definitions yields

$$g_{\langle q,p_0,p_1\rangle,n'} = ((\langle g_{\langle q,p_0\rangle,n'}, (g_{\langle q,p_0\rangle,n'})^{\varphi_{p_1,p_0}(\overline{1})} \ldots\rangle, 0), \overline{1}),$$

with

$$g_{\langle q,p_0,p_1\rangle,n'} = (\vec{a}, \overline{1})$$

(where $\vec{a}$ is as in 2.7, taking 0 except at the n' coordinate) while

$$g_{\langle p_0,p_1\rangle,n} = (\vec{b}, \overline{1}),$$

where $\vec{b} = (b_0, b_1, b_2, \ldots)$. The requirement that $\widehat{\varphi}_{q\vec{p}}(g_{q\vec{p},n'}) = \varphi_{\vec{p},n}(g_{\vec{p},n})$ amounts to asserting that

$$\sum_{m\neq n} b_m = 0 \in \bigoplus_{j<p_1} \mathbb{Z}_{p_0},$$

and

$$b_n = \langle \overline{1}, (\varphi_{p_1,p_0}(\overline{1}))(\overline{1}), (\varphi_{p_1,p_0}(\overline{2}))(\overline{1}), \ldots\rangle \in \bigoplus_{j<p_1} \mathbb{Z}_{p_0},$$

which is just as in the definition in the base case of 2.7.

The inductive step follows since the inductive definitions of $g_{\vec{p},n}$ and $\widehat{\varphi}_{q\vec{p}}$ and $\varphi_{\vec{p},n}$ all parallel one another.

(B) By induction on $l(\vec{p})$ as in (A).

(C) Induction on i, with the base case of $i = 0$ following by inspection of the definitions. ∎

2.17. Corollary. *$\vec{g}_{s_l,n} \in \vec{\mathcal{G}}(T_\alpha)$ whenever $s_l \in T_\alpha$.*

Proof. 2.16 states exactly that the various levels of $\vec{g}_{s_l,n}$ respect the various homomorphisms used to define membership in $\vec{\mathcal{G}}(T_\alpha)$. ∎

Note then that for $t_n(s_k) \subset s_l$ we have $\vec{g}_{s_k,n} \subset \vec{g}_{s_l,m}$ for all $m \in \mathbb{N}$. Thus by transfinite induction we deduce that if $t_n(s_k) \subset s_l \in T_\alpha$ with $l(s_l) \geq 2$ then $\mathrm{Rk}_{\vec{\mathcal{G}}(T_\alpha)}(\vec{g}_{s_k,n}) \geq \mathrm{Rk}_{T_\alpha}(s_l)$. In particular for each $\alpha > 6$ we may find some $\vec{g} \in \vec{\mathcal{G}}(T_\alpha)$ with $\mathrm{Rk}_{\vec{\mathcal{G}}(T_\alpha)}(\vec{g}) \geq \alpha - 4$ and $\vec{g}(i) \neq e$ for some $i < l(\vec{g})$.

We have covered much of the distance towards showing that $\vec{\mathcal{G}}(T_\alpha)$ has well founded elements of relatively high rank. It is still necessary to show that the above $\vec{g}_{s_l,n}$ and the elements they generate do *not* have $\mathrm{Rk}_{\vec{\mathcal{G}}(T_\alpha)} = \infty$.

2.18. Lemma. *Each $\vec{\mathcal{G}}(T_\alpha)$ is well founded (off the identity).*

Proof. This is essentially proved by induction on α. I only say "essentially proved" since we take as our inductive hypothesis not exactly that each T_β for $\beta < \alpha$ is well founded, but rather that for each $\beta < \alpha$ any group tree S_β constructed according to the general requirements of 2.13 be well founded.

Suppose that $\vec{g} \in \vec{\mathcal{G}}(T_\alpha)$ has $\vec{g}(i) \neq e$ for some i. Then we can assume $\vec{g} \in \prod_{i' \leq i} G_{i',\alpha}$. Suppose $(s_i)^* = \vec{p} = \langle p_0, p_1, \ldots, p_N\rangle$, $N \geq 1$. Logically we prove this with a split in cases, (B1) being the main case to which the others reduce.

Case (A): $N = 1$. It suffices to show that $\vec{\mathcal{G}}(T_\alpha)$ is well founded below every $\vec{h}$ strictly extending $\vec{g}$.

Then 2.10 gives some l and n with $\pi_{\vec{p},n,\langle l\rangle}(\vec{g}(i)) \neq \overline{0}$. By 2.14(ii) we may assume s_i is not terminal in T_α. If we fix l and n with

$$\pi_{\vec{p},n,\langle l\rangle}(\vec{g}(i)) \neq \overline{0},$$

and we fix $s_{i'}$ with $t_n(s_i) = s_{i'}$, $s_{i'} = \langle p_1, p_0, r\rangle$, then for any $\vec{h} \supset \vec{g}$ in $\prod_{m\leq M} G_{m,\alpha}$ with $M \geq i'$ we have

$$\pi_{r\vec{p},0,\mathrm{f},\langle l\rangle}(\vec{h}(i')) \neq e$$

by the requirement that $\varphi_{\vec{p},n}(\vec{g}(i)) = \widehat{\varphi}_{r\vec{p}}(\vec{h}(i'))$, and then this case reduces to case (B1) below.

Case (B): Now assume that $N \geq 2$. Whether or not $\pi_{\vec{p},0,\mathrm{b}}(\vec{g}(i)) = \overline{0}$ we can use the assumption on $\vec{g}(i)$ and 2.9 to obtain some l with either $\pi_{\vec{p},0,\mathrm{f},\langle l\rangle}(\vec{g}(i)) \neq \overline{0}$ or $\pi_{\vec{p},0,\mathrm{m},\langle l\rangle}(\vec{g}(i)) \neq \overline{0}$. The first of these possibilities leads at once to case (B1).

Case (B1): $\pi_{\vec{p},0,\mathrm{f},\langle l\rangle}(\vec{g}(i)) \neq e$ for some l. Let S be the tree of $\{s \in \mathbb{N}^{<\mathbb{N}} : p_N s \in T_\alpha\}$. Then S has rank strictly less than α—say it is $\beta < \alpha$. Now if we go ahead and construct $\vec{\mathcal{G}}(S)$ according to the recipe of 2.14 then we can obtain a map $\pi_S : \vec{\mathcal{G}}(T_\alpha) \to \vec{\mathcal{G}}(S)$ given by the rule that

$$(\pi(\vec{h}))(j) = \pi_{\vec{p},0,\mathrm{f},\langle l\rangle}(\vec{h}(j'))$$

where j' is such that $p_N s_j = s_{j'}$. Each such $\pi(\vec{h})$ is in $\vec{\mathcal{G}}(S)$ by Lemma 2.12 and the remark it precedes. $(\pi(\vec{g}))(\widehat{i}) \neq e$ for $\widehat{i}$ chosen such that $p_N s_i = s_{\widehat{i}}$.

Then the inductive assumption on β, the rank of S, yields that $\vec{\mathcal{G}}(S)$ is well founded below $\pi(\vec{g})$, and hence $\vec{\mathcal{G}}(T_\alpha)$ is well founded below $\vec{g}$.

Case (B2): $\pi_{\vec{p},0,\mathrm{m},l}(\vec{g}(i)) \neq \overline{0}$ for some l. Let $s_j = \langle p_N, p_{N-1}\rangle$. Then for j' such that $s_{j'} = \langle p_N, p_{N-1}, p_{N-2}\rangle$ and n such that $t_n(s_j) = s_{j'}$ we see by 2.14(i) that

$$\sum_{k\neq n} \pi_{\langle p_{N-1},p_N\rangle,k,\langle l\rangle}(\vec{g}(j)) = \pi_{\langle p_{N-2},p_{N-1},p_N\rangle,0,\mathrm{m},\langle l\rangle}(\vec{g}(j')),$$

and then by induction on $l(s_{j''})$ we obtain, for j'' with $s_{j'} \subset s_{j''}$,

$$\pi_{(s_{j''})^*,0,\mathrm{m},\langle l\rangle}(\vec{g}(j)) = \pi_{(s_{j'})^*,0,\mathrm{m},\langle l\rangle}(\vec{g}(j')) = \pi_{\vec{p},0,\mathrm{m},\langle l\rangle}(\vec{g}(i)) \neq \overline{0},$$

and thus we obtain $\vec{\mathcal{G}}(T_\alpha)$ well founded below $\vec{g}|_{i+1}$ by (A) above. ∎

2.19. LEMMA. *Suppose that* $(g_n)_{n\in\mathbb{N}} \in \mathcal{G}^0(\vec{p})$, $\vec{p} = \langle p_0, p_1, \ldots, p_N\rangle$, p *a prime, each* g_n *having order* p. *Then*:

(A) $p = p_i$ *for some* i;

(B) *there is an infinite subset* $A \subset \mathbb{N}$ *such that* $o(g_n g_m^{-1}) \in \{1, p_0\}$ *for all* $m, n \in A$.

Proof. (A) follows from an examination of the definitions.

For (B), first for notational simplicity consider the case $N = 1$. Then each g_n has the form $(\vec{b}_n, \overline{k}_n)$, where $\vec{b}_n \in \bigoplus_{i\in\mathbb{N}}(\bigoplus_{0\le j<p_1} \mathbb{Z}_{p_0})$ and $\overline{k}_n \in \mathbb{Z}_{p_1}$. Thus we may find $\overline{k}$ with $\overline{k}_n = \overline{k}$ for infinitely many n, and let $A = \{n : \overline{k}_n = \overline{k}\}$.

The general case $N > 1$ is exactly similar. We go to infinite $A \subset \mathbb{N}$ so that all the coordinate functions of the form $\pi_{\vec{p},i,\mathrm{b},\vec{w}}(g_n)$ and $\pi_{\vec{p},i,\mathrm{m},\vec{w}}(g_n)$ are constant. ∎

2.20. LEMMA. *Suppose* $\vec{H} \subset \vec{\mathcal{G}}(T_\alpha)$ *is a group tree. Suppose that* s_i *is terminal in* T_α *with* $(s_i)^* = \langle p_0, p_1, \ldots, p_N\rangle$. *Let* $i' \ge i$ *and* $\vec{g} \in \prod_{j\le i'} G_{j,\alpha}$ *with* $o(\vec{g}) = p_0$ *and* $\mathrm{Rk}_{\vec{H}}(\vec{g}) \ge \omega$. *Then* $\mathrm{Rk}_{\vec{H}}(\vec{g}) = \infty$.

Proof. By the assumption on $\mathrm{Rk}_{\vec{H}}(\vec{g})$ we may find for each $k \in \mathbb{N}$ some $\vec{h}_k \supset \vec{g}$, $\vec{h}_k \in \vec{H}$ with

$$\vec{h}_k \in \prod_{j\le i'+k} G_{j,\alpha}.$$

We may assume $o(\vec{h}_k) = p_0$ (compare Lemma 8 of [11] or 1.6 above). Then by 2.19(A) and since p_0 does not appear in s_j for $j > i$ we have $\vec{h}_k(j) = e$ all $j > i$. Thus if we define $f(j) = \vec{g}(j)$ for $j \le i$ and $f(j) = e$ for $j > i$ then we obtain an infinite branch through $\vec{H}$ below $\vec{g}$. ∎

2.21. LEMMA. *Suppose* $\vec{H} \subset \vec{\mathcal{G}}(T_\alpha)$ *is a group tree. Suppose that* $\mathrm{Rk}_{T_\alpha}(s_i) < \beta$ *with* $(s_i)^* = \langle p_0, p_1, \ldots, p_N\rangle$. *Let* $i' \ge i$ *and* $\vec{g} \in \prod_{j\le i'} G_{j,\alpha}$ *with* $o(\vec{g}) = p_0$ *and* $\mathrm{Rk}_{\vec{H}}(\vec{g}) \ge \omega\cdot\beta$. *Then* $\mathrm{Rk}_{\vec{H}}(\vec{g}) = \infty$.

Proof. Note that the case $\beta = 1$ is taken care of at 2.20 above. Suppose as part of a transfinite induction that the lemma holds for all $s \supset (s_i)$, $\mathrm{Rk}_{T_\alpha}(s_i) < \beta$, $o(\vec{g}) = p_0$ and $\mathrm{Rk}_{\vec{H}}(\vec{g}) \ge \omega\cdot\beta$. For a contradiction assume $\mathrm{Rk}_{\vec{H}}(\vec{g}) \ne \infty$.

By assumption on rank of $\vec{g}$ we may find $k > i$ and an infinite sequence of

$$\vec{h}_n \in \prod_{j\le k} G_{j,\alpha} \cap \vec{H}$$

with

(i) $\mathrm{Rk}_{\vec{H}}(\vec{h}_n) \to \omega\cdot\beta$;

(ii) $o(\vec{h}_n) = p_0$;

(iii) $\mathrm{Rk}_{\vec{H}}(\vec{h}_n) < \omega\cdot\beta$;
(iv) $\mathrm{Rk}_{\vec{H}}(\vec{h}_n) < \mathrm{Rk}_{\vec{H}}(\vec{h}_{n+1})$.

Note that from (ii) and 2.19(A) we see that for each n and m if $\vec{h}_n(j)\neq e$ then $s_j\supset s_i$.

Now suppose that $(s_k)^* = \langle q_0,\ldots,q_M,p_0,p_1,\ldots,p_N\rangle$; necessarily $\mathrm{Rk}_{T_\alpha}(s_k)<\beta$. Applying 2.19 we obtain an infinite $A\subset\mathbb{N}$ such that for all $n,m\in A$ and $j\le k$,

$$o(\vec{h}_n(j)\vec{h}_m^{-1}(j)) = q_l \quad \text{for some } l\in\{0,1,\ldots,M\}.$$

We still have

$$\mathrm{Rk}_{\vec{H}}(\vec{h}_n\vec{h}_m^{-1})\to\omega\cdot\beta$$

as $n\neq m\to\infty$. For each $n,m\in A$ observe that $o(\vec{h}_n(j)\vec{h}_m^{-1}(j))$ is finite and has (a subset of) $q_M,q_{M-1},\ldots,q_0$ as its prime factors. Thus we may write

$$\vec{h}_n\vec{h}_m^{-1} = \vec{g}_{n,m,M}\vec{g}_{n,m,M-1}\cdots\vec{g}_{n,m,0}$$

where for $j\le M$ we have $o(\vec{g}_{n,m,j}) = q_j, 1$ and $\vec{g}_{n,m,j}$ is a power of $\vec{h}_n(j)\vec{h}_m^{-1}(j)$. Thus we may find some single j and $\vec{g}_{n,m,j}$ with

(i) $\mathrm{Rk}_{\vec{H}}(\vec{g}_{n,m,j}) > \omega\cdot\mathrm{Rk}_{T_\alpha}(s_j)$;
(ii) $o(\vec{g}_{n,m,j}) = q_j$;
(iii) $\mathrm{Rk}_{\vec{H}}(\vec{g}_{n,m,j}) < \omega\cdot\beta$.

This provides us with a contradiction to the inductive assumption that the lemma holds for all $s\supset s_l\in T_\alpha$. ∎

2.22. Corollary. *$\prod_{i\in\mathbb{N}}\mathcal{G}_{i,\alpha}$ is a tame group.*

Proof. By 1.9. ∎

3. The road home. The remainder of the journey is downhill, inasmuch as it simply consists in collecting various known facts.

I will make the link between the combinatorial properties of the group and the descriptive set theory of its actions by passing through the connections with countable model theory. It is not clear that this detour should be necessary, but it has precisely the convenience of requiring absolutely no new ideas.

For this section fix a countable ordinal κ. We will show that

$$\prod_{i\in\mathbb{N}} G_{i,\omega\cdot(\kappa+\omega+2)}$$

gives rise to continuous actions on Polish spaces where not every equivalence class is $\underset{\sim}{\Pi}^0_\kappa$. It will be convenient to suppress the parameter κ, and use G_i to denote $G_{i,\omega\cdot(\kappa+\omega+2)}$.

3.1. Definition. For $\mathcal{L}$ a countable relational language we let $\mathrm{Mod}(\mathcal{L})$ be the space of all $\mathcal{L}$-structures with underlying set $\mathbb{N}$. We equip this space with the zero-dimensional topology generated by quantifier free formulas, so that for $R \in \mathcal{L}$ and $n_1, \ldots, n_k$ we take

$$\{\mathcal{M} : \mathcal{M} \models R(n_1, \ldots, n_k)\} \quad \text{and} \quad \{\mathcal{M} : \mathcal{M} \models \neg R(n_1, \ldots, n_k)\}$$

as subbasic open sets.

$\mathrm{Mod}(\mathcal{L})$ in this topology is a compact Polish space.

3.2. Definition. From now on let $\mathcal{M}$ denote the "affine" structure of the group tree $\bigcup_{n\in\mathbb{N}} \prod_{i\le n} G_i$. More precisely, let $\mathcal{M}$ have in its language unary predicates (P_i) and binary relations $\{F_g : g \in \bigcup_{i\in\mathbb{N}} G_i\}$ such that:

(i) $\mathcal{M}$ is the disjoint union of $\{(P_i)^{\mathcal{M}} : i \in \mathbb{N}\}$;

(ii) for all $i \in \mathbb{N}$ and all $a, b \in (P_i)^{\mathcal{M}}$ there is a unique $g \in G_i$ with $\mathcal{M} \models F_g(a, b)$;

(iii) for all $i \in \mathbb{N}$, all $a \in (P_i)^{\mathcal{M}}$ and all $g \in G_i$ there is a unique $b \in (P_i)^{\mathcal{M}}$ with $\mathcal{M} \models F_g(a, b)$;

(iv) for all $i, j, k \in \mathbb{N}$ and all $a \in (P_i)^{\mathcal{M}}$, $b \in (P_j)^{\mathcal{M}}$ and $g \in G_k$, if $\mathcal{M} \models F_g(a, b)$ then $i = j = k$;

(v) for all $i \in \mathbb{N}$, all $a, b, c \in (P_i)^{\mathcal{M}}$ and all $g, h \in G_i$, if $\mathcal{M} \models F_g(a, b)$ and $\mathcal{M} \models F_h(b, c)$, then $\mathcal{M} \models F_{hg}(a, c)$.

For future reference let $\mathcal{L}_0$ be the language of $\mathcal{M}$.

Of course the F_g's are functions in disguise. There are minor technical advantages in restricting ourselves to relational languages, and so we only have the functions implicit.

It should be commented that we have not defined $\mathcal{M}$ uniquely, since there has been not a word said regarding the underlying set, or even how $\mathcal{M}$ is arranged over that set. 3.2 only defines $\mathcal{M}$ up to isomorphism, and this suffices for the construction below.

This method of taking the "affine structure" of the group tree appears already in [9].

3.3. Definition. Fixing a sequence of $a_i \in (P_i)^{\mathcal{M}}$ we let $\pi_i : G_i \cong (P_i)^{\mathcal{M}}$ be given by

$$\pi_i(g) = b \quad \text{where } b \text{ is unique such that } \mathcal{M} \models F_g(a_i, b).$$

We then define $\Psi : \prod_{i\in\mathbb{N}} G_i \to \mathrm{Sym}(\mathcal{M})$. Given $\vec{g} = (g_0, g_1, g_2, \ldots) \in \prod_{i\in\mathbb{N}} G_i$ we let $\Psi(\vec{g})$ be as follows: For any $b \in \mathcal{M}$ we let i_b be such that $b \in (P_{i_b})^{\mathcal{M}}$, then we let $h \in G_{i_b}$ be such that $\mathcal{M} \models F_h(a_{i_b}, b)$, and we let $(\Psi(\vec{g}))(b)$ be the unique c such that $\mathcal{M} \models F_{hg^{-1}}(a_{i_b}, c)$. (In other words, we define $\Psi(\vec{g})$ to be the unique automorphism that sends each a_i to the d with $F_{g^{-1}}(a_i, d)$.)

Let $\mathcal{L} \supset \mathcal{L}_0$ be the extension of $\mathcal{L}_0$ that contains for each n an n-ary relation R_n. Given

$$S \subset \bigcup_{N\in\mathbb{N}} \prod_{i\le\mathbb{N}} G_i$$

we define $\mathcal{N}_S$ to be the expansion of $\mathcal{M}$ such that

$$\mathcal{N}_S \models R_n(b_0, b_1, \ldots, b_{n-1})$$

if and only if $b_i \in (P_i)^m$ and $(\pi_0(b_0), \pi_1(b_1), \ldots, \pi_{n-1}(b_{n-1})) \in S$.

For $N \in \mathbb{N}$ and $S \subset \bigcup_{N\in\mathbb{N}} \prod_{i\le\mathbb{N}} G_i$ we define $\mathcal{N}_{S,N}$ to be the set

$$\bigcup_{i\le N} (P_i)^{\mathcal{M}}$$

equipped with the predicates $\{P_i : i \le N\}$, relations $\{F_g : g \in \bigcup_{i\le N} G_i\}$ and $\{R_i : i \le N\}$ with each $(P_i)^{\mathcal{N}_{S,N}} = (P_i)^{\mathcal{M}}$ for $i \le N$, each $(F_g)^{\mathcal{N}_{S,N}} = (F_g)^{\mathcal{M}}$ for $g \in \bigcup_{i\le N} G_i$, and each $(R_i)^{\mathcal{N}_{S,N}} = (R_i)^{\mathcal{N}_S}$ for $i \le N$. Similarly we define $\mathcal{M}_N$ to be the restriction of $\mathcal{N}_{\emptyset,N}$ to $\mathcal{L}_0 = \{P_i, F_g : i \in \mathbb{N},\ g \in \bigcup_{i\in\mathbb{N}} G_i\}$.

By analogy with the above we may define $\Psi_N : \prod_{i\le N} G_i \to \mathrm{Sym}(\mathcal{M}_N)$ by the requirement that for each $i \le N$ and $h \in G_i$ and $\vec{g} = (g_0, g_1, \ldots, g_N) \in \prod_{i\le N} G_i$,

$$(\Psi_N(\vec{g}))(\pi_i(h)) = \pi_i(h(g_i)^{-1}).$$

We are most interested in the case where S forms a group tree.

3.4. Lemma. (A) *Ψ_N defines an isomorphism between $\prod_{i\le N} G_i$ and the automorphism group of $\mathcal{M}_N$.*

(B) *Ψ defines an isomorphism between $\prod_{i\in\mathbb{N}} G_i$ and the automorphism group of $\mathcal{M}$.*

Proof. (A) Every automorphism of $\mathcal{M}$ must have the form $\Psi_N(\vec{g})$ in virtue of respecting the relations of the form $F_h(\cdot,\cdot)$. Conversely, any permutation of $\mathcal{M}_N$ that fixes setwise each $(P_i)^{\mathcal{M}_N}$ and preserves each $(F_h)^{\mathcal{M}_N}$ will be an automorphism.

(B) Now (B) follows as a corollary to (A) since the structure of $\mathcal{M}$ is just the union of the $\mathcal{M}_N$'s. ∎

3.5. Lemma. *Let $S \subset \bigcup_{M\in\mathbb{N}} \prod_{i\le M} G_i$ be a group tree.*

(A) *Let $N \in \mathbb{N}$. Suppose that $S \cap \prod_{i\le N} G_i$ is non-empty. Then*

$$\mathrm{Aut}(\mathcal{N}_{S,N}) = \Psi_N\Big[S \cap \prod_{i\le N} G_i\Big].$$

(B) *Suppose $S \cap \prod_{i\le N} G_i$ is non-empty for every $N \in \mathbb{N}$. Then*

$$\mathrm{Aut}(\mathcal{N}_S) = \Psi[[S]].$$

Proof. (A) Suppose that $\vec{g} \in \prod_{i \le N} G_i$. Then for all $M \le N$ and

$$\vec{h} = (h_0, h_1, \ldots, h_M) \in \prod_{i \le M} G_i \cap S$$

we have

$$\begin{aligned} \mathcal{N}_{N,S} \models R_M((\Psi_N(\vec{g}))(\pi_0(h_0)), (\Psi_N(\vec{g}))(\pi_1(h_1)), \ldots, (\Psi_N(\vec{g}))(\pi_M(h_M))) \\ \Leftrightarrow (h_0(g_0)^{-1}, h_1(g_1)^{-1}, \ldots, h_M(g_M)^{-1}) \in S \Leftrightarrow \vec{g} \in S, \end{aligned}$$

and thus membership in S is a necessary and sufficient condition for $\Psi_N(\vec{g})$ to move each $(R_M)^{\mathcal{N}_S}$ into $(R_M)^{\mathcal{N}_S}$. Conversely, by considering $(\vec{g})^{-1} \in \prod_{i \le N} G_i$ we find that membership in S is necessary and sufficient to fix each $(R_M)^{\mathcal{N}_S}$ setwise. In light of 3.4 we then obtain the lemma.

(B) (Here $[S]$ is the set of all infinite branches through S and $\Psi[[S]]$ is the image of this set under Ψ.) This now follows from (A). ∎

3.6. Definition. For $\mathcal{A}$ a model and $\vec{a}$ a finite sequence from $\mathcal{A}$ we define $\varphi_\alpha^{\vec{a},\mathcal{A}}$ by induction on the ordinal α:

$$\varphi_0^{\vec{a},\mathcal{A}} = \bigwedge\{\psi(\vec{x}) : \psi(\vec{x}) \text{ quantifier free, } \vec{a} \models \psi(\vec{a})\},$$

$$\varphi_{\alpha+1}^{\vec{a},\mathcal{A}} = \bigwedge\{(\exists \vec{y})\varphi_{\alpha+1}^{\vec{a}\vec{b},\mathcal{A}}(\vec{x},\vec{y}) : \vec{b} \in \mathcal{A}^{<\mathbb{N}}\} \wedge (\forall \vec{y}) \bigvee\{\varphi_{\alpha+1}^{\vec{a}\vec{b},\mathcal{A}}(\vec{x},\vec{y}) : \vec{b} \in \mathcal{A}^{<\mathbb{N}}\}.$$

For λ a limit we set

$$\varphi_\lambda^{\vec{a},\mathcal{A}} = \bigwedge_{\alpha<\lambda} \varphi_\alpha^{\vec{a},\mathcal{A}}.$$

As discussed in [1], since for $\alpha < \beta$,

$$\varphi_\beta^{\vec{a},\mathcal{A}} = \varphi_\beta^{\vec{b},\mathcal{A}} \Rightarrow \varphi_\alpha^{\vec{a},\mathcal{A}} = \varphi_\alpha^{\vec{b},\mathcal{A}}$$

we must necessarily come to an ordinal δ such that for all $\vec{a}, \vec{b} \in \mathcal{A}$,

$$(\exists \gamma)(\varphi_\gamma^{\vec{a},\mathcal{A}} \neq \varphi_\gamma^{\vec{b},\mathcal{A}}) \Leftrightarrow \varphi_\delta^{\vec{a},\mathcal{A}} \neq \varphi_\delta^{\vec{b},\mathcal{A}}.$$

The least such δ is the *Scott height* of $\mathcal{A}$ and is denoted by $\alpha(\mathcal{A})$. Moreover, $\varphi_{\alpha(\mathcal{A})+2}^{\emptyset,\mathcal{A}} =: \varphi^{\mathcal{A}}$ is the (canonical) *Scott sentence of* $\mathcal{A}$; $\varphi_{\alpha(\mathcal{A})+2}^{\vec{a},\mathcal{A}} =: \varphi^{\vec{a},\mathcal{A}}$ (or just $\varphi^{\vec{a}}$ when context indicates $\mathcal{A}$) is the *Scott sentence of* $\vec{a}$ (in $\mathcal{A}$); each $\varphi_\alpha^{\vec{a},\mathcal{A}}$ is the αth approximation of the Scott sentence for $\vec{a}$.

A formula in $\mathcal{L}_{\infty\omega}$ (the infinitary language obtained from $\mathcal{L}$ by admitting arbitrarily large conjunctions and disjunctions, as well as the usual negation operation and $\exists$ and $\forall$ quantifiers) is said to be $\underset{\sim}{\Sigma}_1^0$ if it has the form $\bigvee_{i\in I}\psi_i$ with each ψ_i quantifier free. A formula is $\underset{\sim}{\Pi}_1^0$ if it has the form $\bigwedge_{i\in I}\psi_i$ with each ψ_i quantifier free.

Inductively we say a formula of the form $\bigvee_{i\in I}\psi_i$ is $\underset{\sim}{\Sigma}^0_\alpha$ if each ψ_i is $\underset{\sim}{\Pi}^0_{\beta(i)}$ for some $\beta(i)<\alpha$, and a formula $\bigwedge_{i\in I}\psi_i$ is $\underset{\sim}{\Pi}^0_\alpha$ if each ψ_i is $\underset{\sim}{\Sigma}^0_{\beta(i)}$ for some $\beta(i)<\alpha$.

3.7. Theorem (Scott). *If $\mathcal{A},\mathcal{B}$ are countable models then*

$$(\mathcal{A},\vec{a})\cong(\mathcal{B},\vec{b})\Leftrightarrow\varphi^{\vec{a},\mathcal{A}}=\varphi^{\vec{b},\mathcal{B}}.$$

Proof. By a *back and forth construction*, based on the idea that if $\varphi^{\vec{a},\mathcal{A}}=\varphi^{\vec{b},\mathcal{B}}$ and $c\in\mathcal{A}$ then we may find $d\in\mathcal{B}$ with $\varphi^{\vec{a}c,\mathcal{A}}=\varphi^{\vec{b}d,\mathcal{B}}$. The details may be found in [1]. ∎

3.8. Lemma. *If $\mathcal{A},\mathcal{B}$ are models, ψ a $\underset{\sim}{\Pi}^0_\alpha$ formula, $\vec{a}$ a finite sequence from $\mathcal{A}$, and $\vec{b}$ a finite sequence from $\mathcal{B}$, then*

$$\varphi^{\vec{a},\mathcal{A}}_\alpha=\varphi^{\vec{b},\mathcal{B}}_\alpha\Rightarrow(\mathcal{A}\models\psi(\vec{a})\Leftrightarrow\mathcal{B}\models\psi(\vec{b})).$$

Proof. By induction on α, resembling the proof of 3.7. Again a detailed proof can be found in [1]. ∎

3.9. Theorem (Vaught). *Let $\mathcal{L}_1$ be a countable language. Let $B\subset\mathrm{Mod}(\mathcal{L}_1)$ be $\underset{\sim}{\Pi}^0_\alpha$ and invariant (i.e. for $\mathcal{A}',\mathcal{B}'\in\mathrm{Mod}(\mathcal{L}_1)$ with $\mathcal{A}'\cong\mathcal{B}'$ we have $\mathcal{A}'\in B$ if and only if $\mathcal{B}'\in B$). Then there is a $\underset{\sim}{\Pi}^0_\alpha$ formula ψ such that for $\mathcal{A}\in\mathrm{Mod}(\mathcal{L}_1)$ we have $\mathcal{A}\in B$ if and only if $\mathcal{A}\models\psi$.*

Proof. The main idea of the proof presented in [12] is to show by induction on α that for A a $\underset{\sim}{\Sigma}^\alpha_0$ set and B a $\underset{\sim}{\Pi}^\alpha_0$ set and $E_{n_1,\ldots,n_k}=\{\pi\in S_\infty:\pi(n_1)=n_1,\ \pi(n_2)=n_2,\ldots,\pi(n_k)=n_k\}$ there is a $\underset{\sim}{\Sigma}^\alpha_0$ formula $\psi_{A,\vec{n}}$ and a $\underset{\sim}{\Pi}^\alpha_0$ formula $\psi_{B,\vec{n}}$ such that

$$\begin{aligned}\{\mathcal{A}\in\mathrm{Mod}(\mathcal{L}_1):\mathcal{A}\models\psi_{A,\vec{n}}(n_1,\ldots,n_k)\}\\ =\{\mathcal{A}\in\mathrm{Mod}(\mathcal{L}_1):(\exists^*\pi\in E_{n_1,\ldots n_k})(\pi\cdot\mathcal{A}\in A)\}\end{aligned}$$

and

$$\begin{aligned}\{\mathcal{A}\in\mathrm{Mod}(\mathcal{L}_1):\mathcal{A}\models\psi_{B,\vec{n}}(n_1,\ldots,n_k)\}\\ =\{\mathcal{A}\in\mathrm{Mod}(\mathcal{L}_1):(\forall^*\pi\in E_{n_1,\ldots n_k})(\pi\cdot\mathcal{A}\in B)\}.\end{aligned}$$

(Here $\exists^*$ and $\forall^*$ are the categoricity operators: "there exist non-meagerly many" and "there exist comeagerly many".) ∎

3.10. Lemma. *Let $\mathcal{L}_1$ be a countable language and $\mathcal{A},\mathcal{B}\in\mathrm{Mod}(\mathcal{L}_1)$. Let $B\subset\mathrm{Mod}(\mathcal{L}_1)$ be $\underset{\sim}{\Pi}^0_\alpha$ and invariant. If $\varphi^{\emptyset,\mathcal{A}}_\alpha=\varphi^{\emptyset,\mathcal{B}}_\alpha$ then*

$$\mathcal{A}\in B\Leftrightarrow\mathcal{B}\in B.$$

Proof. From 3.9 we find that there is a $\underset{\sim}{\Pi}^0_\alpha$ formula ψ such that $B=\{\mathcal{C}\in\mathrm{Mod}(\mathcal{L}_1):\mathcal{C}\models\psi\}$. Now the lemma follows by 3.8. ∎

Something like the following has been proved in [8]. In any case [2] has undertaken a more general analysis along these lines for arbitrary group actions.

3.11. Lemma (Nadel). *Let $\mathcal{L}_1$ be a countable language and $\mathcal{A} \in \mathrm{Mod}(\mathcal{L}_1)$. The Scott height of $\mathcal{A}$ provides the following lower bound on the Borel complexity*:

$$\{\mathcal{B} \in \mathrm{Mod}(\mathcal{L}_1) : \mathcal{B} \cong \mathcal{A}\} \in \underset{\sim}{\Pi}^0_\alpha \quad \textit{implies} \quad \alpha(\mathcal{A}) \le \alpha + \omega.$$

Proof. By 3.10,

$$\{\mathcal{B} \in \mathrm{Mod}(\mathcal{L}_1) : \mathcal{B} \cong \mathcal{A}\} = \{\mathcal{B} \in \mathrm{Mod}(\mathcal{L}_1) : \mathcal{B} \models \varphi_\alpha^{\emptyset,\mathcal{A}}\}.$$

Thus if we take $F \subset (\mathcal{L}_1)_{\infty,\omega}$ to be the fragment generated by $\varphi_\alpha^{\emptyset,\mathcal{A}}$ and if we let $T \subset F$ be the theory of $\mathcal{A}$ in that fragment, then there are $\le \aleph_0$ (complete) types over F consistent with T. In particular, T has an atomic model, $\mathcal{B}_0$, which is necessarily isomorphic to $\mathcal{A}$ by the above and necessarily has $\alpha(\mathcal{B}_0) \le \alpha + \omega$ by atomicity. ∎

3.12. Notation. Let $\mathrm{Exp}(\mathcal{M})$ be the set $\{\mathcal{N} \in \mathrm{Mod}(\mathcal{L}) : \mathcal{N}$ is (isomorphic to) an expansion of $\mathcal{M}\}$.

By definition $\mathrm{Exp}(\mathcal{M})$ is an invariant subspace of $\mathrm{Mod}(\mathcal{L})$. Since the isomorphism type of $\mathcal{M}$ is a $\underset{\sim}{\Pi}^0_2$ subset of $\mathrm{Mod}(\mathcal{L})$ we see that $\mathrm{Exp}(\mathcal{M})$ is a Polish space in the topology generated by first order logic.

3.13. Lemma (Becker–Kechris). (A) *There is a Polish $\prod_{i\in\mathbb{N}} G_i$-space X and a continuous map $\theta : \mathrm{Exp}(\mathcal{M}) \to X$ such that for all $\mathcal{N}_0, \mathcal{N}_1 \in \mathrm{Exp}(\mathcal{M})$,*

$$\mathcal{N}_0 \cong \mathcal{N}_1 \Leftrightarrow \theta(\mathcal{N}_0) E^X_{\prod_{i\in\mathbb{N}} G_i} \theta(\mathcal{N}_1).$$

(B) *Conversely, for any Polish $\prod_{i\in\mathbb{N}} G_i$-space Y there is a countable language $\mathcal{L}_1 \supset \mathcal{L}_0$ and $\varrho : Y \to \mathrm{Mod}(\mathcal{L}_1)$ such that*:

(i) *ϱ is Borel, and in fact Baire class* 1 (*the inverse image under ϱ of an open set will be $\underset{\sim}{\Sigma}^0_2$*);

(ii) *for all $y_0, y_1 \in Y$,*

$$y_0 E^Y_{\prod_{i\in\mathbb{N}} G_i} y_1 \Leftrightarrow \varrho(y_0) \cong \varrho(y_1);$$

(iii) *for all $y \in Y$ the model $\varrho(\mathcal{M})$ is isomorphic to an expansion of $\mathcal{M}$.*

Proof. (A) (In the statement of the theorem, $E^X_{\prod_{i\in\mathbb{N}} G_i}$ is the orbit equivalence relation induced by the action of $\prod_{i\in\mathbb{N}} G_i$.) We let X be the space of all *actual* $\mathcal{L}$-expansions of $\mathcal{M}$ with the topology generated by quantifier free formulas. The usual proof that $\mathrm{Mod}(\mathcal{L})$ is Polish yields that X is Polish.

Then given $\mathcal{N} \in \mathrm{Mod}(\mathcal{L})$ and $i \in \mathbb{N}$ we let c_i be the first element in $\mathcal{N}$ (under the usual ordering of $\mathbb{N}$, its underlying set) such that $\mathcal{N} \models P_i(c_i)$.

Then define $\pi_{\mathcal{N}} : \mathbb{N} \to \mathcal{M}$ by the specification that for each i, remembering the definition of 3.3,

$$\pi_{\mathcal{N}}(c_i) = a_i$$

and then for each $g \in G_i$ and $c \in (P_i)^{\mathcal{N}}$ with $\mathcal{N} \models F_g(c_i, c)$ we let

$$\pi_{\mathcal{N}}(c) = a$$

where $a \in \mathcal{M}$ is the unique element of $\mathcal{M}$ with $F_g(a_i, d)$.

From this we obtain an expansion of $\mathcal{M}$ isomorphic to $\mathcal{N}$ by requiring that for each n,

$$\theta(\mathcal{N}) \models R_n(\pi_{\mathcal{N}}(b_0, \ldots, b_{n-1})) \Leftrightarrow \mathcal{N} \models R_n(b_0, \ldots, b_{n-1}).$$

The map θ is continuous since each finite part of $\theta(\mathcal{N})$ depends solely on finitely many coordinates of $\mathcal{N}$.

(B) Since we will not use this direction, we do not prove it. It is actually implicit in Theorem 2.7.3 and Section 7.4 of [2]. ∎

More or less the following is performed for the product group $(\mathbb{Z})^{\mathbb{N}}$ in [7]. The proof there adapts without trouble to the present context.

3.14. Lemma (Makkai). *Suppose $S \subset \bigcup_{N\in\mathbb{N}} \prod_{i\le N} G_i$ is a group tree. Suppose $N \in \mathbb{N}$, α is an ordinal greater than 1, and $\vec{g} = (g_0, g_1, \ldots, g_N) \in \prod_{i\le N} G_i$ with $\mathrm{Rk}_S(\vec{g}) \ge \omega\cdot\alpha$. Suppose $\vec{b} = (b_0, b_1, \ldots, b_N)$, $\vec{c} = (c_0, c_1, \ldots, c_N)$, with each $b_i, c_i \in (P_i)^{\mathcal{N}_S}$, and $\mathcal{N}_S \models F_{g_i}(b_i, c_i)$. Then $\varphi_\alpha^{\vec{b},\mathcal{N}_S} = \varphi_\alpha^{\vec{c},\mathcal{N}_S}$.*

P r o o f. If $\mathrm{Rk}_S(\vec{g}) \ge 1$, in other words $\vec{g} \in S$, then 3.5(B) certainly gives that $\vec{a}$ and $\vec{b}$ have the same quantifier free type. This much granted, the rest of the lemma follows routinely by induction on α. (The reason for the erosion—needing $\mathrm{Rk}_S(\vec{g}) \ge \omega\cdot\alpha$ instead of the more natural $\mathrm{Rk}_S(\vec{g}) \ge \alpha$—is that in showing $\varphi_\alpha^{\vec{b},\mathcal{N}_S} = \varphi_\alpha^{\vec{c},\mathcal{N}_S}$ we need to show that for *all* M and all $b \in (P_M)^{\mathcal{M}}$ and all $\beta < \alpha$ we may find c with $\varphi_\beta^{\vec{b}b,\mathcal{N}_S} = \varphi_\beta^{\vec{c}c,\mathcal{N}_S}$; if we only needed to consider the case of $M = N+1$ then we could obtain the conclusion of the lemma assuming $\mathrm{Rk}_S(\vec{g}) \ge \alpha$.) ∎

3.15. Corollary. *There is $\mathcal{N}_S \in \mathrm{Exp}(\mathcal{M})$ with $\alpha(\mathcal{M}) > \kappa + \omega$.*

P r o o f. As remarked after 2.17, if we take $S = \vec{\mathcal{G}}(T_{\omega\cdot(\kappa+\omega+2)})$ then we may find $\vec{g} = (g_0, g_1, \ldots, g_N) \in S$ with

$$\omega\cdot(\kappa+\omega+1) < \mathrm{Rk}_S(\vec{g}) < \infty.$$

Then if we take $\vec{b} = (b_0, b_1, \ldots, b_N)$ such that $\mathcal{M} \models F_{g_i}(a_i, b_i)$ for each i then for $\vec{a} = (a_0, a_1, \ldots, a_N)$ we have

$$\varphi_{\kappa+\omega}^{\vec{a},\mathcal{N}_S} = \varphi_{\kappa+\omega}^{\vec{b},\mathcal{N}_S}.$$

On the other hand since $\mathrm{Rk}_S(\vec{g}) < \infty$ there is no infinite branch extending $\vec{g}$, and so by 3.5 there is no automorphism of $\mathcal{N}_S$ which sends $\vec{a}$ to $\vec{b}$, and thus by 3.7,

$$\varphi_{\alpha(\mathcal{N}_S)}^{\vec{a},\mathcal{N}_S} = \varphi_{\alpha(\mathcal{N}_S)}^{\vec{b},\mathcal{N}_S}.$$

Hence $\alpha(\mathcal{N}_S)$ (the Scott height of $\mathcal{N}_S$) will have rank at least $\kappa+\omega+1$. ∎

3.16. Corollary. *Let X be as in* 3.13(A). *Then $E^X_{\prod_{i\in\mathbb{N}} G_i}$ is not $\underset{\sim}{\Pi}^0_\alpha$.*

Proof. By 3.15 and 3.11 we see that there are models $\mathcal{N}_S \in \mathrm{Exp}(\mathcal{M})$ with $\{\mathcal{N}\in \mathrm{Exp}(\mathcal{M}) : \mathcal{M}\cong\mathcal{N}\}$ not $\underset{\sim}{\Pi}^0_\kappa$. Since there is a continuous reduction of $\cong|_{\mathrm{Exp}(\mathcal{M})}$ to $E^X_{\prod_{i\in\mathbb{N}} G_i}$ we conclude that there is an orbit in X that is not $\underset{\sim}{\Pi}^0_\alpha$. ∎

So actually we deduce that not every orbit in X is $\underset{\sim}{\Pi}^0_\alpha$.

4. Recap. Given κ, the construction of §2 enabled us to produce a product Polish group

$$\prod_{i\in\mathbb{N}} G_{i,\omega\cdot(\kappa+\omega+2)}$$

which:

(i) allows group trees $\vec{\mathcal{G}}(T_{\omega\cdot(\kappa+\omega+2)})$ with well founded elements of rank greater than $\omega\cdot(\kappa+\omega+1)$ (2.17 and 2.18); but

(ii) is tame (2.22).

In §3, (i) was exploited to show that the group $\prod_{i\in\mathbb{N}} G_{i,\omega\cdot(\kappa+\omega+2)}$ gives rise to continuous actions where the orbit equivalence relation is not $\underset{\sim}{\Pi}^0_\kappa$. Here the main method was in showing that high rank group trees give rise to models $\mathcal{N}_S$ which

(iii) have relatively high Scott ranks (3.15); and hence

(iv) have isomorphism types of relatively high Borel complexity (3.11); and

(v) are all expansions of a model $\mathcal{M}$ with automorphism group naturally isomorphic to $\prod_{i\in\mathbb{N}} G_{i,\omega\cdot(\kappa+\omega+2)}$ (3.4); and hence

(vi) show that a corresponding Polish $\prod_{i\in\mathbb{N}} G_{i,\omega\cdot(\kappa+\omega+2)}$-space X must have orbits of high Borel complexity (3.13).

Thus we are finished: For any $\kappa<\omega_1$ there is a product group of the form $\vec{G}=\prod_{i\in\mathbb{N}} G_i$ which is tame but has orbit equivalence relations that are not $\underset{\sim}{\Pi}^0_\kappa$.

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6363 MSB
Mathematics
University of California at Los Angeles
Los Angeles, CA90095-1555, U.S.A.
E-mail: greg@math.ucla.edu

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