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PCA sets and convexity

by

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Abstract. Three sets occurring in functional analysis are shown to be of class PCA (also called Σ^1_2) and to be exactly of that class. The definition of each set is close to the usual objects of modern analysis, but some subtlety causes the sets to have a greater complexity than expected. Recent work in a similar direction is in [1, 2, 10, 11, 12].

I. Extreme points and integrals. Suppose that S is a subset of a linear space; a classical problem in analysis concerns representation of elements of S by integrals over extreme points of S. The work of Choquet and Bishop–de Leeuw [13] relies on compactness; the theorem of Edgar [8, 6] relies on geometric properties of the space containing S and on set-theoretic ideas. Our interest is in the set $\iota(\mathrm{ex}\,K)$ of elements which can be represented by an integral over $\mathrm{ex}\,K$, where K is closed, bounded, convex, and separable. More generally, let $\mathcal{S}$ be a co-analytic set in K; then $y \in \iota(\mathcal{S})$ means that $y = \int x\,d\mu(x)$, where μ is a probability measure in K such that $\mu^*(S) = 1$. (We write $\mu^*(\mathcal{S})$ because S need not be a Borel set.)

Theorem 1. *The set $\iota(\mathcal{S})$ is a PCA set.*

Theorem 2. *For each PCA set Σ there is a closed, bounded, convex set K in c_0 such that Σ is homeomorphic to a closed subset of $\iota(\mathrm{ex}\,K)$.*

Proof of Theorem 1. We denote by $P^*(\mathcal{S})$ the set of probability measures occurring in the definition of $\iota(\mathcal{S})$. Let K_1 be a compact metric space containing K as a G_δ, so that $K_1\backslash\mathcal{S}$ is an analytic set in K_1. Thus $K_1\backslash\mathcal{S} = h(V)$, where h is a continuous function on some G_δ-set V. By a small adjustment we can assume that h is continuous on a compact metric space $Y_1 \supseteq V$ and that $h(Y_1) = K_1$.

Lemma 1. *$P^*(\mathcal{S})$ is a co-analytic set in $P(K_1)$.*

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P r o o f. Let $M^+(V)$ be the set of probability measures μ in Y_1 such that $\mu(V) > 0$. If $\mu \in M^+(V)$ then $(h^*\mu)(K_1 \setminus \mathcal{S}) \ge \mu(V) > 0$ whence $h^*\mu \notin P^*(\mathcal{S})$. Conversely, if $\lambda \notin P^*(\mathcal{S})$ and $\lambda(K_1) = 1$, then there is a compact set $Y_2 \subseteq V$ such that $\lambda(h(Y_2)) > \frac{1}{2}\lambda(K_1 \setminus \mathcal{S}) > 0$. Thus it is easy to see that $\lambda = h^*\mu$ for some μ such that $\mu(Y_2) > 0$ and so $\mu(V) > 0$. Thus $P^*(\mathcal{S})$ is the complement of $h^*(M^+(V))$. But $M^+(V)$ is a Borel set, in fact a set of type $G_{\delta\sigma}$, whence $h^*(M^+(V))$ is analytic and $P^*(\mathcal{S})$ is co-analytic.

Let $(x_m^*)_{n=1}^{\infty}$ be a total sequence of bounded linear functionals on the B-space containing K, and let K_1 be defined so that the functionals x_m^* can be extended continuously over K_1. Then $x \in \iota(\mathcal{S})$ if and only if there is a measure $\mu \in P^*(\mathcal{S})$ such that $x_m^*(x) = \int x_m^* \, d\mu$ for $m \ge 1$; from this it follows that $\iota(\mathcal{S})$ is a PCA set.

Proof of Theorem 2. Let $\Sigma = p(\mathcal{S})$ where $\mathcal{S}$ is a co-analytic set in a compact metric space X_1 (and p is continuous on X_1). Now [10, 12] there is a closed, bounded, convex set K_0 in $P(X_1) \oplus c_0$ such that $\operatorname{ex} K_0$ is just the set of points $(\delta_y, 0)$ with $y \in \mathcal{S}$. Here $P(X_1)$ is realized as a compact subset of c_0, so that $K_0 \subseteq c_0 \oplus c_0$. A similar observation applies to the set $M_1(X_1)$ of measures on X_1 of total variation at most 1. X_2 is defined to be $p(X_1)$, so that $X_2 \supseteq \Sigma$.

Let $x \in c_0$, $\lambda \in P(X_2)$, $\mu \in M_1(X_1)$. We define $(\mu, \lambda, x) \in K$ if there is some $\nu \in P(X_1)$ such that $\lambda = p^*(\nu)$, $-\nu \le \mu \le \nu$, and $(x, \nu) \in K_0$. This is a convex set, which can be realized as a closed convex set in c_0. Suppose (μ, λ, x) is an extreme point in K; clearly, $\mu = \nu$ or $\mu = -\nu$. Thus an extreme point of K_0 must take the shape $(\nu, p^*(\nu), x)$ or $(-\nu, p^*(\nu), x)$, where $(\nu, x) \in K_0$. These can be extreme only if (ν, x) is extreme in K_0. It is easy to see that the extreme points of K are just the elements $(\delta_y, p^*(\delta_y), 0)$, $(-\delta_y, p^*(\delta_y), 0)$ with $y \in \mathcal{S}$. Since $p(\mathcal{S}) = \Sigma$, the elements $(0, \delta_z, 0)$ with $z \in \Sigma$ are in the set $\iota(\operatorname{ex} K)$ as each is the average of two extreme points. Conversely, if $(0, \delta_w, 0)$ is in $\iota(\operatorname{ex} K)$, it is the resultant of an integral over certain elements (μ, λ, x). In this integral $\lambda = \delta_w$ a.e., and so, in view of the nature of $\operatorname{ex} K_0, w \in \Sigma$. Thus $(0, \delta_w, 0)$ is in $\iota(\operatorname{ex} K_0)$ if and only if $w \in \Sigma$, and this is the meaning of Theorem 2.

Introducing the interval $-\nu \le \mu \le \nu$, and averaging measures of opposite sign, allow us to "forget" the measures ν, thus increasing the complexity by one degree. This idea can be traced back to Jayne and Rogers [9], where it is used to pass from Borel sets to co-analytic sets.

Because there exist universal PCA sets Σ, when we have Theorem 2 for such a set, it follows for all PCA sets at once.

Representing PCA sets. It will be convenient to have at hand a representation of PCA sets. The main notion is this: A real function u on a product set $A \times B$ is of *type* $\mathcal{A}_1$ if there is some $a \in A$ such that $u(a,b) \neq 0$ for every $b \in B$; otherwise u is of *type* $\mathcal{A}_0$. Let X be an uncountable Polish space and F the metric space of increasing sequences of natural numbers.

(R) Let S be a PCA set in a Polish space M. Then there is a uniformly continuous function u to $[0,1]$ on $X \times F \times M$ such that the partial function $u(\cdot,\cdot,m)$ is of type $\mathcal{A}_1$ if and only if $m \in \mathcal{S}$.

To define u we begin with a co-analytic set $\mathcal{S}_1$ in $X \times M$ whose projection on M is $\mathcal{S}$. Then $(X \times M) \setminus \mathcal{S}_1$ is analytic, and is therefore the image $\psi(F)$ of a continuous map ψ on F. We write $\psi = (\psi_1, \psi_2)$ so that $\psi_1(F) \subseteq X$, $\psi_2(F) \subseteq M$. Let Γ be the set $\{(\psi_1(s), s, \psi_2(s)) : s \in F\}$, let v be the function which is distance to Γ (any metric can be used), and $u = v/(1+v)$. Since ψ is continuous, $u(x,s,m) = 0 \Leftrightarrow x = \psi_1(s)$, $m = \psi_2(s)$.

To verify that u has the necessary properties, we suppose that $m_0 \in \mathcal{S}$. Then there is an x_0 in X such that $(x_0, m_0) \in \mathcal{S}_1$. For every s in F, $(\psi_1(s), \psi_2(s)) \neq (x_0, m_0)$, so that $u(x_0, s, m_0) > 0$. Thus the partial function $u(\cdot,\cdot,m_0)$ is of type $\mathcal{A}_1$. Conversely, if $m_1 \notin \mathcal{S}$, then for every x_1 in X, $(x_1, m_1) \notin \mathcal{S}_1$. Thus there is an s in F such that $x_1 = \psi_1(s)$, $m_1 = \psi_2(s)$, and thus $\psi(x_1, s, m_1) = 0$. That is, the partial function $u(\cdot,\cdot,m_1)$ is not of type $\mathcal{A}_1$.

We use (R) in the case when X is a symmetric set in a Banach space, i.e. $X = -X$; in fact, X is the sphere in a space of dimension at least 2. We can define $\widetilde{u}$ on $X \times F \times M$ so that it is even with respect to X, i.e. $\widetilde{u}(x,s,m) = \widetilde{u}(-x,s,m)$. To attain this we add to $\mathcal{S}_1$ the set obtained from it by the map $(x,m) \mapsto (-x,m)$, which is co-analytic and has the same projection. We define u as before, and take finally $\widetilde{u} = \min(u(x,s,m),\ u(-x,s,m))$. The main point of the variant is this: when $m_0 \in \mathcal{S}$, then there is some x_0 such that (x_0, m) and $(-x_0, m)$ belong to $\mathcal{S}_1$. Then $u(x_0, s, m_0) > 0$ and $u(-x_0, s, m_0) > 0$ for every s, i.e. $\widetilde{u}(x_0, s, m_0) > 0$ for every s.

II. Norms and extreme points. Let X be a separable B-space with norm $|\cdot|$ and $\mathbb{N}(X)$ the set of all norms $\|\cdot\|$ equivalent to $|\cdot|$, i.e. satisfying $c_1\|x\| \le |x| \le c_2\|x\|$ for all x, with some constants $0 < c_1 \le c_2 < \infty$. Provided with the pointwise (product) topology, $\mathbb{N}(X)$ is not quite a metric space but each set $\{p \in \mathbb{N}(X) : k^{-1}p(x) \le |x| \le kp(x)\}$ is a compact metric space, and each set $\{p \in \mathbb{N}(X) : p(x) \le k|x|\}$ is a σ-compact metric space. (We shall gloss over this quibble.)

An interesting subset of $\mathbb{N}$ is the class $\mathcal{R}$ of rotund (strictly convex) norms; this chapter uses a device from a remarkable theorem of B. Bossard [3, 4, 5].

Theorem 3. *Let X be separable and infinite-dimensional. Then $\mathcal{R}$ is a true co-analytic subset of* $\mathbb{N}(X)$, *that is*, $\mathcal{R}$ *is not a Borel set.*

The rotundity property of a norm is just the fact that every element of the unit sphere defined by $\|\cdot\|$, i.e. the set $\{\|x\| = 1\}$, is an extreme point of that set. We denote by ε_0 those norms such that the unit sphere has at least one extreme point. When X has the Radon–Nikodým property (RNP) then every norm has this property [6] and no other spaces are known with this property.

Theorem 4. *The set ε_0 is of type PCA in $\mathbb{N}(X)$. When $X = c_0$ the following holds*: *For each set $\mathcal{S}$ of type PCA in a Polish space M, there is a continuous map h of M into $\mathbb{N}(c_0)$ such that*

(i) $h^{-1}(\varepsilon_0) = \mathcal{S}$,

(ii) *the map h is continuous into the uniform topology on $\mathbb{N}(X)$, i.e. the topology of uniform convergence on the unit ball of $|\cdot|$.*

The first assertion about ε_0 is elementary; to prove it we write p for elements in $\mathbb{N}(X)$. The function $p(x)$ defined on $\mathbb{N}(X) \times X$ is measurable, since the set $\{p(x) \le a\}$ is of type F_σ for each real a. Thus the subset of $\mathbb{N}(X) \times X \times X$ defined as

$$\{(p, y, z) : p(y + z) = p(y) = p(y - z),\ z \neq 0\}$$

is a Borel set. The set of pairs (p, y) such that $p(y) = 1$, and y is not an extreme point of the unit ball defined by p, is an analytic subset of $\mathbb{N}(X) \times X$. Thus ε_0 is the projection on $\mathbb{N}(X)$ of a co-analytic set, whence ε_0 is of class PCA.

In the second part of Theorem 4, the space c_0 enters in two distinct places, so it seems best to write the details for a space $X = Y \oplus c_0$ with Y of infinite dimension. Choosing $Y = c_0$ we obtain the assertion for c_0. Theorem 2 does not depend on the norm $|\cdot|$, and we will assume that the norm of Y is *locally uniformly rotund* (LUR): whenever $(y_n) \subseteq Y$, $y_0 \in Y$, $|y_0| = |y_n| = 1$, and $\lim |y_0 + y_n| = 2$, then $\lim y_n = y_0$. Every separable space can be provided with an LUR norm (Kadec, 1950) [7].

Each element y_0 in the unit sphere of Y is then strongly exposed: let $f_0 \in Y^*$ be such that $|f_0| = 1$, $f_0(y_0) = 1$. Every sequence (y_n) in the unit sphere of Y such that $f_0(y_n) \to 1$ must converge to y_0.

Theorem 4 depends on a certain set in the unit sphere of c_0 which is homeomorphic to F but shares certain properties of compact sets. Let E_0, $E(n_1), E(n_1, n_2), E(n_1, n_2, n_3), \ldots$ be disjoint, infinite sets of positive integers, defined for $n_1 \ge 1$, $n_2 > n_1 \ge 1$, etc. Let $(n_k) \in F$; then $\tau(n_k)$ takes value

- 1 on the first n_1 elements of E_0,
- 2^{-1} on the first n_2 elements of $E(n_1)$,
- 2^{-2} on the first n_3 elements of $E(n_1, n_2)$,
- ...
- 0 elsewhere.

The map τ is continuous into the norm topology of c_0. The set $\tau(F)$ has compact closure in $\mathbb{R}^{\mathbb{N}}$ in the product topology; moreover, the product topology agrees with norm convergence in $\tau(F)$. Let $s_j = (n_k(j))$ be a sequence in F such that each sequence $n_1(j), \ldots, n_k(j)$ converges as $j \to \infty$ to a limit N_k finite or ∞. If $N_1 = \infty$, then the pointwise limit of $\tau(n_k(j))$ equals 1 on E_0, and 0 elsewhere. If $N_1 < \infty$ and $N_2 = \infty$ then the limit equals 1 on the first N_1 elements of E_0, 1/2 on all of $E(N_1)$, and 0 elsewhere, etc. Let H be the closure of $\tau(F)$ in the product topology in $\mathbb{R}^{\mathbb{N}}$, and $H^* = H \setminus \tau(F)$. Then H^* is a countable set $(v_r)_{r=1}^{\infty}$ such that no sum $\sum c_r v_r$ belongs to c_0 unless all $c_r = 0$, where $\sum |c_r| < \infty$. To explain this, we suppose that $v_1 = (\infty, \infty, \ldots)$. Then $v_1 = 1$ on the infinite subset E_0, while all of the remaining elements of H^* belong to c_0 on the set E_0. Hence $c_1 = 0$. Similarly $(1, \infty, \infty, \ldots)$ equals 2^{-1} on the infinite set $E(1)$, while all of the remaining elements of H^* belong to c_0 on $E(1)$; for example, $(2, \infty, \infty, \ldots)$ vanishes on an $E(1)$, etc. The property referred to above (stated pedantically) is proved in

LEMMA 2. *Suppose $S_1, S_2, \ldots$ is a decreasing sequence of closed subsets of $\tau(F)$, and suppose that $u \in \overline{\mathrm{co}}(S_r \cup -S_r)$ for each r. Then $u = \int x\, \lambda(dx)$, where λ is a signed Borel measure, of variation at most* 1, *concentrated in $\bigcap_{r=1}^{\infty} S_r \equiv S$.*

Proof. Let T_r be the pointwise closure of S_r in $\mathbb{R}^{\mathbb{N}}$. By standard limit theorems in measure theory, there is a signed measure λ, of variation at most 1, concentrated in $T \equiv \bigcap_{r=1}^{\infty} T_r$ such that $u(k) = \int x(k)\, \lambda(dx)$ for each integer $k = 1, 2, \ldots$ However, u is an element of c_0, whence λ can have no mass in H^*. Indeed, the integral of λ over $\tau(F)$ is in c_0, by the remark at the end of this paragraph. The remaining integral is a sum $\sum c_r v_r$, where c_r is the λ-measure of v_r, and so each $c_r = 0$, as explained above. Thus λ is concentrated in $T_r \cap \tau(F)$ for each r, that is, in S_r, and this proves the lemma. It is worthwhile remarking that every integral $\int x\, d\lambda(x)$ over $\tau(F)$ is a Bochner (strong) integral, so the sum *is* in $\overline{\mathrm{co}}(\tau(F) \cup -\tau(F))$.

Let θ be a uniformly continuous map of $S^1(Y) \times F$ into $[0, 1]$ which is even with respect to the first element, and let $\|\cdot\|_\theta$ be the norm on $Y \oplus c_0$ whose unit ball is the closed convex hull of the set $S(\theta) := S^1(Y) \cup S^1(c_0) \cup \{\pm\theta(y, s)y \pm \tau(s) : y \in S^1(Y),\ s \in F\}$.

LEMMA 3. *Let $y_0 \in S^1(Y)$, $x_0 \in c_0$, $x_0 \neq 0$. If $\|(y_0, x_0)\|_\theta \le 1$, then x_0 is in the closed convex hull of the set $\{\pm\tau(s) : \theta(y_0, s) = 1\}$.*

Proof. We apply Lemma 2 to the sequence of closed sets $F_r = \{\tau(s) : \theta(y_0, s) \geq 1 - r^{-1}\}$. Let f_0 be the bounded linear function on Y which exposes y_0 strongly, and g_0 a bounded linear functional on c_0 such that $g_0(x_0) = 1$. Let m be a natural number such that $m > \|g_0\|$. The linear functional $f_0 + m^{-1}g_0$ takes the value $1 + m^{-1}$ at (y_0, x_0), but its values on $S^1(Y)$ are at most 1, and on $S^1(c_0)$ less than 1. Hence it attains a value at least $1 + m^{-1} - m^{-2}$ on the third part of the set $S(\theta)$, i.e. at some element $\pm\theta(y_m, s_m)y_m \pm \tau(s_m)$, with $|y_m| \leq 1$ and $s_m \in F$. Then $|\theta(y_m, s_m)| \geq 1-m^{-1}$, $|f_0(y_m)| \geq 1-m^{-2}$, and $m^{-1}|g_0(\tau(s_m))| \geq m^{-1}-m^{-2}$. As $m \to \infty$, there is a choice of signs so that $\varepsilon_m y_m \to y_0$, and then $\theta(y_0, s_m) \to 1$, since θ is even and uniformly continuous. For large m, $\theta(y_0, s_m) \geq 1 - r^{-1}$ and $|g_0(\tau(s_m))| \geq 1-m^{-1}$. Thus x_0 is in the closed convex hull of $\pm F_r$ for every $r \geq 1$, and we can apply Lemma 2.

Thus the set mentioned at the conclusion of Lemma 3 is not empty. (Clearly, no conclusions can be drawn if $x_0 = 0$.)

Let X be the space of all sequences $x = (y, u_1, u_2, \ldots)$ with $y \in Y$, $u_n \in c_0$, and $\lim u_n = 0$. The norm is $\sup \|(y, u_n)\|_\theta \equiv \|x\|_\theta$, and clearly X is isomorphic to $Y \oplus c_0$.

LEMMA 4. (a) *Suppose there is some $y_0 \in S^1(Y)$ such that $\theta(y_0, s) < 1$ for each $s \in F$. Then $y_0 = (y_0, 0, 0, \ldots)$ is extreme in the unit ball of X.*

(b) *If no element y_0 of $S^1(Y)$ has the property defined in* (a), *then the unit ball of X has no extreme points.*

Proof. (a) Suppose that y_0 is an average of $(y_1, u_1, u_2, \ldots)$ and $(y_2, -u_2, -u_3, \ldots)$, each of these having norm 1. Then $|y_1| \leq 1$, $|y_2| \leq 1$, $2y_0 = y_1 + y_2$. Since the norm of Y is rotund, $y_1 = y_2 = y_0$. By Lemma 3, the inequalities $\|(y_0, u_n)\|_\theta \leq 1$ imply that each $u_n = 0$; thus y_0 is extreme.

(b) Let $x = (y_1, u_1, u_2, \ldots)$ have norm 1. If $|y_1| < 1$, then x cannot be an extreme point. For we would have $|y_1| + |u_n| < 1$ for large n, so there would be some $v \neq 0$ in c_0 such that $\|(y_1, u_n + v)\|_\theta < 1$, $\|(y_1, u_n - v)\|_\theta < 1$. Thus $|y_1|$ must be 1, and each u_n is the resultant $\int \tau(z)\, d\lambda_n(z)$ of an integral over the set Σ defined as $\{s \in F : \theta(y_1, s) = 1\}$; the variation of λ_n is at most 1. Moreover, since Σ is not empty, λ_n must have variation exactly 1. The elements $\tau(z)$ of c_0 have the value 1 at the first member of the set $E(0)$, so that $\|\int \tau\, d\lambda_n\| \geq |\lambda_n(F)|$.

But this implies that $\lambda_n(F) \to 0$ so that for large n the measures λ_n^+ and λ_n^- are different from 0. From this and the inequality on $\int \tau\, d\lambda_n$, we see that x cannot be extreme.

To complete the proof of Theorem 4, we make use of the representation (R) of the previous section, taking for X the unit sphere $S^1(Y)$ of Y. We map an element m of M to the norm $\|\cdot\|_\theta$, where θ is the partial function $1 - u(\cdot, \cdot, m)$ defined on $S^1(Y) \times F$. The symmetry of θ on $S^1(Y)$ is obtained in a remark to (R), and the continuity in point (ii) follows from the uniform continuity of u.

III. Extreme points, redux. Let X be a separable B-space and $\mathcal{E}$ the set of extreme points of its unit ball. Then NA denotes the set of linear functionals that attain their norm on the unit ball, and NAE those that attain their norm on $\mathcal{E}$. When X^* is provided with the w^*-topology, NA is analytic and NAE is a PCA (Σ^1_2) set.

THEOREM. *The space c_0 can be provided with a norm $|\cdot|$ so that* NAE *is then a complete PCA set.*

Completeness of NAE will be established in the same form as in previous sections, via a map φ which is continuous into the norm of c_0^*. We observe that NAE is analytic if $\mathcal{E}$ is a Borel set and also in certain other cases. For if X has the Radon–Nikodým property (RNP), as ℓ^1 clearly has, then NAE $=$ NA. It seems likely, on the basis of [9, 10, 12], that ℓ^1 can be normed so that $\mathcal{E}$ is not a Borel set.

We write $\|\cdot\|$ for a norm on X, the classical one for c_0, but in fact this norm plays almost no rôle in the proof. Let K be a closed, bounded, convex set in X, let B be the unit ball for the norm $\|\cdot\|$, and let $|\cdot|$ be the norm whose unit ball is $B^\sim = \overline{\mathrm{co}}\big(\frac{1}{2}B \cup K \cup -K\big)$. Henceforth $\mathcal{E}$, NA, and NAE refer to this norm. We introduce the following condition on functionals f in X^*:

$({}_*{}^*{}_*)$ $\quad f \ge 0$ on K and $\sup f(K) > \|f\|/2$.

Then $f \in$ NA (for the norm $|\cdot|$) if and only if f attains its norm on K; and if $f \in$ NAE then f must attain its norm on $\mathrm{ex}\, K$. Conversely, always subject to $({}_*{}^*{}_*)$, if f attains its norm at an element x_0 of $\mathrm{ex}\, K$ then $x_0 \in \mathcal{E}$, because $f \le 0$ on $-K$ and $f \le \|f\|/2 < f(x_0)$ on $\frac{1}{2}B$. Thus $f \in$ NAE.

Next we summarize the conclusions of [10, 12], beginning with a compact metric space M, a co-analytic subset $\mathcal{S}$ of M, and the convex set $P(M)$ of probability measures. We represent K at first as a closed, bounded, convex subset of $P(M) \oplus B$, where B is the unit ball of c_0; M has the following properties:

(i) K contains the set $P(M) \oplus (0)$.

(ii) The extreme points of K are the elements $(\delta_y, 0)$, with $y \in \mathcal{S}$.

Next we replace $P(M)$ by a representation in B: we map each measure μ in $P(M)$ to a sequence $L\mu = (\langle \mu, g_k\rangle)_{k=1}^\infty$ where $(g_k)_{k=1}^\infty$ is a total sequence in $C(M)$ and $\sup |g_k| = o(1)$. Henceforth we construe K as a convex subset of $c_0 \oplus c_0 \sim c_0$.

The linear functionals we use in the theorem act on the first factor in $c_0 \oplus c_0$, i.e. on the factor in which $P(M)$ is represented. Suppose $f = (b_k)_{k=1}^\infty$

is a sequence in $\ell_1 = c_0^*$. Its norm as a functional on $c_0 \oplus c_0$ is of course $\sum |b_k|$, whereas its value at the sequence $L\mu$ is $\sum_{k=1}^{\infty} b_k \langle \mu, g_k \rangle$. Hence $({}_*{}^*{}_*)$ is true provided $\sum b_k g_k \ge 0$ everywhere in M and $\sup \sum b_k g_k > \sum |b_k|/2$. If these conditions are satisfied, then $f \in \mathrm{NAE}$ if and only if $\sum b_k g_k$ attains its supremum (on M) in the subset $\mathcal{S}$.

We now specify that M is the circle of length 2π,

$$g_{2k+1} = (k+1)^{-1/3} \cos kt, \quad k = 0, 1, 2, \ldots,$$
$$g_{2k} = (k+1)^{-1/3} \sin kt, \quad k = 1, 2, \ldots$$

Then every function u in the class $\mathrm{Lip}^1(M)$ admits exactly one expansion $\sum_{k=1}^{\infty} b_k g_k$ with $\sum |b_k| < \infty$, and therefore there is a functional, written $\alpha(u)$, such that $\langle \alpha(u), L\mu \rangle \equiv \int u\, d\mu$. These assertions are consequences of Parseval's formula and Cauchy's inequality. In fact, $\|\alpha(u)\| \le \sup |u| + c \operatorname{ess\,sup} |u'|$, with a certain constant c; a bit more work yields an upper bound $c\delta^{-5} \sup |u| + c\delta \operatorname{ess\,sup} |u'|$, for all $\delta \in (0, 1)$.

We can find co-analytic sets $\mathcal{S}$ which can be mapped continuously onto any PCA set, for example the set WF of trees with no infinite branch. Since the set of trees is 0-dimensional we can place $\mathcal{S} = \mathrm{WF}$ in the arc $(\pi/4, \pi/2)$ of M. Let Σ be a PCA set in a metric space N of diameter at most 1, so that $\Sigma = h(\mathcal{S})$, a continuous image of $\mathcal{S}$. We define F on $M \times N$ by

$$F(t, y) = \inf\{|t - s| + d(h(s), y)) : s \in \mathcal{S}\}$$

when $t \in M$, $y \in N$. Then F is jointly continuous on M, $0 \le F \le 1 + \pi < 5$, and $|F(t_1, y) - F(t_2, y)| \le |t_1 - t_2|$. If $y \in \Sigma$ then $F(s, y) = 0$ for some $s \in \mathcal{S}$, and the converse is true because h is continuous on $\mathcal{S}$. Since $|\sin s| > 1/2$ for each s in $\mathcal{S}$, the remark above remains true for the function $G(t, y) \equiv |\sin t| F(t, y)$. Let $\eta > 0$ be a small constant. We define a map ψ from N into ℓ^1 as follows. We apply α to the partial function $1 - \eta G(\cdot, y)$. Each of these functions on M has supremum 1 and is positive if $0 < \eta < 1/5$. When η is small enough, the resulting functional satisfies $({}_*{}^*{}_*)$; $\psi(y)$ belongs to NAE if and only if $1 - \eta G(s, y) = 1$ for some s in $\mathcal{S}$, that is, $y \in \Sigma$. The continuity of ψ is a consequence of the refined inequalities written above.

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