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Free spaces

by

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Abstract. A space Y is called a free space if for each compactum X the set of maps with hereditarily indecomposable fibers is a dense G_δ-subset of $C(X,Y)$, the space of all continuous functions of X to Y. Levin proved that the interval I and the real line $\mathbb{R}$ are free. Krasinkiewicz independently proved that each n-dimensional manifold M $(n \geq 1)$ is free and the product of any space with a free space is free. He also raised a number of questions about the extent of the class of free spaces. In this paper we will answer most of those questions. We prove that each cone is free. We introduce the notion of a locally free space and prove that a locally free ANR is free. It follows that every polyhedron is free. Hence, 1-dimensional Peano continua, Menger manifolds and many hereditarily unicoherent continua are free. We also give examples that show some limits to the extent of the class of free spaces.

1. Introduction. All spaces in this paper are separable metric. A continuum (compact, connected, metric space) is *decomposable* if it is the union of two proper subcontinua. Otherwise, it is said to be *indecomposable*. A compactum (compact, metric space) is *hereditarily indecomposable* if each of its subcontinua is indecomposable.

Let I denote the closed unit interval. Let X be a space and $p \in C(X,I)$. We say that X is *folded relative to* p if $X = F_0 \cup F_{1/2} \cup F_1$ where $F_0, F_{1/2}$ and F_1 are closed sets in X such that:

(i) $F_0 \cap F_1 = \emptyset$,
(ii) $p^{-1}(0) \subset F_0,\ p^{-1}(1) \subset F_1$, and
(iii) $F_0 \cap F_{1/2} \subset p^{-1}((1/2,1])$ and $F_{1/2} \cap F_1 \subset p^{-1}([0,1/2))$.

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LEMMA 1.1 [K, 2.1]. *Let X be a space and $p \in C(X, I)$.*

(i) *If X is folded relative to p and $q \in C(Y, X)$ for some space Y, then Y is folded relative to the mapping $p \circ q$. In particular, each subset of X is folded relative to p.*

(ii) *If $F \subset X$ is folded relative to p, then there is a neighbourhood of F which is folded relative to p.*

(iii) *If $f \in C(X, Y)$ for some space Y and each fiber of f is folded relative to p, then there is an open cover $\mathcal{V}$ of Y such that $f^{-1}(V)$ is folded relative to p for each $V \in \mathcal{V}$.*

LEMMA 1.2 [K, 2.2]. *Let X be a compactum, and let $p \in C(X, I)$. Then X is not folded relative to p if and only if there exist continua A and B in X such that*:

(1) $A \cap B \neq \emptyset$,
(2) $A \cap p^{-1}(0) \neq \emptyset \neq B \cap p^{-1}(1)$,
(3) $A \subset p^{-1}([0, 1/2])$, $B \subset p^{-1}([1/2, 1])$.

If X and Y are compacta, let $C(X, Y; \mathrm{h.i.})$ be the set of maps of X to Y with all fibers hereditarily indecomposable. A space Y is said to be *free* if $C(X.Y; \mathrm{h.i.})$ is a dense G_δ-set in $C(X, Y)$ for each compactum X. Levin [L, 1.8] proved that I is free and independently Krasinkiewicz [K, 5.1] proved that each manifold of dimension at least 1 is free.

PROPOSITION 1.3 [K, 4.1]. *If the space Y is free, then $Y \times Z$ is free for each space Z.*

If X and Y are spaces, we denote by π_1 (resp. π_2) the first (resp. second) coordinate projection of $X \times Y$ onto X (resp. onto Y).

The main tool for checking whether a space is free is the following theorem of Krasinkiewicz.

THEOREM 1.4 [K, 4.5]. *A compactum Y is free if and only if the projection $\pi_1 : Y \times I \to Y$ can be approximated by mappings $g \in C(Y \times I, Y)$ with fibers folded relative to $\pi_2 : Y \times I \to I$.*

Krasinkiewicz [K] raised a number of questions about the extent of the class of free spaces. It is our purpose to answer most of those questions.

2. Cones. In this section we prove that each cone is free. We shall repeatedly use the construction in the next lemma which reproves that I is free.

LEMMA 2.1. *For each $\varepsilon > 0$ there exists $f : I \times I \to I$ such that f is ε-close to the first coordinate projection $\pi_1 : I \times I \to I$ and each fiber of f is folded relative to the second coordinate projection $\pi_2 : I \times I \to I$.*

Proof. Let n be a positive integer so that $4/(4n+3) < \varepsilon$. Let $x_i = i/(4n+3)$ for each $i = 0, 1, \ldots, 4n+3$.

For each $i = 0, 1, \ldots, 4n+3$ choose disjoint polygonal arcs $L_{x_i} \subset I \times I$ (see Figure 1) such that $L_{x_i} = a_i b_i c_i d_i e_i$ where

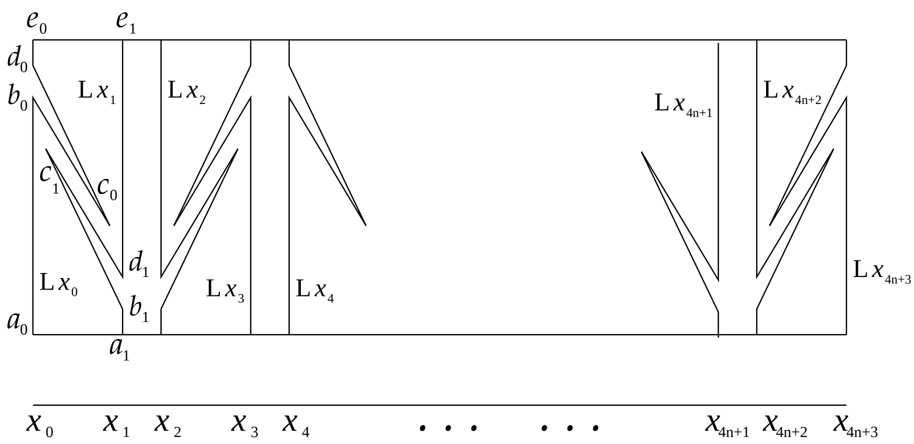

Fig. 1

(i) $\pi_1(a_i) = \pi_1(b_i) = \pi_1(d_i) = \pi_1(e_i) = x_i$;

(ii) $0 = \pi_2(a_i) < \pi_2(b_i) < \pi_2(d_i) < \pi_2(e_i) = 1$;

(iii) if $i = 0 \bmod 4$ then

$$x_i < \pi_1(c_{i+1}) < \frac{x_i + x_{i+1}}{2} < \pi_1(c_i) < x_{i+1},$$

$$\pi_2(d_{i+1}) < \pi_2(c_i) < \frac{1}{2} < \pi_2(c_{i+1}) < \pi_2(b_i),$$

$$\pi_2\left(L_{x_{i+1}} \cap \left(\left\{\frac{x_i + x_{i+1}}{2}\right\} \times I\right)\right) < \frac{1}{2} < \pi_2\left(L_{x_i} \cap \left(\left\{\frac{x_i + x_{i+1}}{2}\right\} \times I\right)\right);$$

(iv) if $i = 2 \bmod 4$ then

$$x_i < \pi_1(c_{i+1}) < \frac{x_i + x_{i+1}}{2} < \pi_1(c_i) < x_{i+1},$$

$$\pi_2(d_i) < \pi_2(c_{i+1}) < \frac{1}{2} < \pi_2(c_i) < \pi_2(b_{i+1}),$$

$$\pi_2\left(L_{x_i} \cap \left(\left\{\frac{x_i + x_{i+1}}{2}\right\} \times I\right)\right) < \frac{1}{2} < \pi_2\left(L_{x_{i+1}} \cap \left(\left\{\frac{x_i + x_{i+1}}{2}\right\} \times I\right)\right);$$

(v) L_1 (resp. $L_{x_{4n+2}}$) is the reflection of L_0 (resp. L_{x_1}) about the line $x = 1/2$ in the plane.

Let $m = p_{x_1} = q_{x_1} = r_{x_1} = (b_0 + d_0)/2$. For $0 < x \le x_1$ let J_x be the polygonal arc $p_x r_x q_x$ (see Figure 2) where

$$p_x = (4n+3)xm + (1 - (4n+3)x)b_0,$$
$$q_x = (4n+3)xm + (1 - (4n+3)x)d_0,$$
$$r_x = (4n+3)xm + (1 - (4n+3)x)c_0.$$

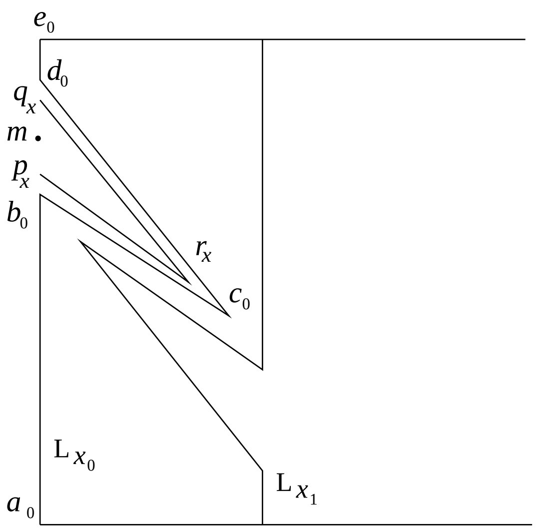

Fig. 2

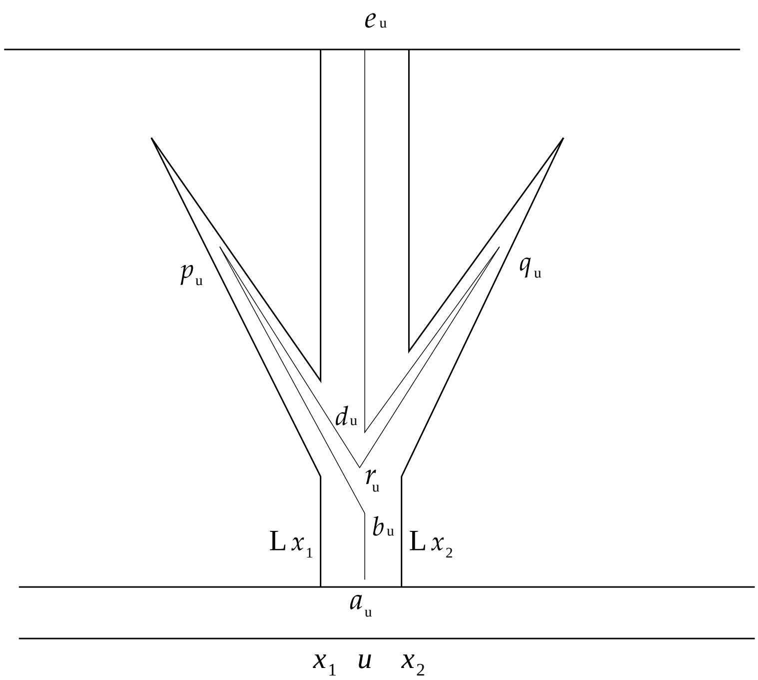

Fig. 3

(†) For $x \in (0, x_1)$ let L_x be an irreducible polygonal arc such that for $0 < x < y < x_1$, L_y separates $L_0 \cup L_x$ from L_{x_1} and $\bigcup_{x \in (0,x_1)} L_x$ is the component of $I \times I - L_0 \cup L_{x_1}$ with boundary $L_0 \cup L_{x_1}$ in $I \times I$.

For $x \in [x_1, x_{4n+2}]$ let $J_x = \emptyset$. Let $u = (x_1 + x_2)/2$. Let L_u be a polygonal arc which separates L_{x_1} from L_{x_2} with $L_u = a_u b_u p_u r_u q_u d_u e_u$ (see Figure 3) such that

(vi) $u = \pi_1(a_u) = \pi_1(b_u) = \pi_1(d_u) = \pi_1(e_u) = \pi_1(r_u)$,
(vii) $0 = \pi_2(a_u) < \pi_2(b_u) < \pi_2(r_u) < \pi_2(d_u) < \pi_2(e_u) = 1$,
(viii) $\pi_1(p_u) < x_1 < x_2 < \pi_1(q_u)$,
(ix) $\pi_2(p_u), \pi_2(q_u) > 1/2$.

(††) For each $x \in (x_1, u) \cup (u, x_2)$ let L_x be an irreducible polygonal arc such that for $x_1 < x < y < x_2$, L_y separates $L_{x_1} \cup L_x$ from L_{x_2} in $I \times I$ and $\bigcup_{x \in (x_1, x_2)} L_x$ is the component of $I \times I - L_{x_1} \cup L_{x_2}$ with boundary $L_{x_1} \cup L_{x_2}$ in $I \times I$.

For $x \in [x_{4n+2}, 1]$ let L_x (resp. J_x) be the reflection of L_{1-x} (resp. J_{1-x}) about the line $x = 1/2$ in the plane.

For $x \in (x_{4k-2}, x_{4k-1}) \cup (x_{4k}, x_{4k+1})$, $k = 1, \ldots, n$, let L_x be defined in a manner analogous to that in (†).

For $x \in (x_{4k-1}, x_{4k}) \cup (x_{4k+1}, x_{4k+2})$, $k = 1, \ldots, n$, let L_x be defined in a manner analogous to that in (††).

Define $f : I \times I \to I$ by $f^{-1}(x) = J_x \cup L_x$ for $x \in I$. Then f satisfies the required conditions.

THEOREM 2.2. *The cone over a space X is free.*

Proof. Let $Y = X \times I / X \times \{1\}$ be the cone and let $\varepsilon > 0$. Let $x_0 \in X$, $\xi : X \times I \to Y$ be the quotient map and let $\eta : Y \to \xi(\{x_0\} \times I)$ be given by $\eta(\xi(x,t)) = \xi(x_0, t)$. We construct a map $f : Y \times I \to Y$ such that f is ε-close to π_1 on $Y \times I$ and the fibers of f are folded relative to π_2.

For each $x \in X$ let $D_x = \xi(\{x\} \times I) \times I$. For the given $x_0 \in X$, $L_{\xi(x_0,t)}$ and $J_{\xi(x_0,t)}$ are defined as in Lemma 2.1 for each t. Let $f_0 : D_{x_0} \to \xi(\{x_0\} \times I)$ be defined as in Lemma 2.1 so that f_0 is $\varepsilon/2$-close to the projection of $D_{x_0} = \xi(\{x_0\} \times I) \times I$ onto $\xi(\{x_0\} \times I)$ and each fiber of f_0 is folded relative to π_2.

For $(x,t) \in X \times I$ define $L_{\xi(x,t)} = (\eta \times \mathrm{id}_I)^{-1}(L_{\xi(x_0,t)}) \cap D_x$ and $J_{\xi(x,t)} = (\eta \times \mathrm{id}_I)^{-1}(J_{\xi(x_0,t)}) \cap D_x$. Define $f : Y \times I \to Y$ by

$$f^{-1}(\xi(x,t)) = \begin{cases} L_{\xi(x,t)} \cup J_{\xi(x,t)} & \text{if } t < x_{4n+2}; \\ L_{\xi(x,t)} & \text{if } x \neq x_0 \text{ and } x_{4n+2} \le t < 1; \\ L_{\xi(x_0,t)} \cup (\bigcup_{x \in X} J_{\xi(x,t)}) & \text{if } x = x_0 \text{ and } x_{4n+2} \le t < 1; \\ \bigcup_{x \in X} (L_{\xi(x,1)} \cup J_{\xi(x,1)}) & \text{if } \xi(x,t) = \xi(x_0, 1). \end{cases}$$

Then f is a map within ε of π_1 on $Y \times I$ and with fibers folded relative to π_2. By Theorem 1.4, Y is free.

3. Locally free spaces. A space Y is *locally free* if there is an open cover of Y by open sets which are free spaces. Clearly, every free space is locally free. We show that among ANRs the converse is also true.

For completeness we first state the following simple proposition.

PROPOSITION 3.1. *Each open subspace of a free space is free.*

THEOREM 3.2. *An ANR Y is free if and only if it is locally free.*

Proof. Let X be a compactum, $f \in C(X,Y)$ and $\varepsilon > 0$. For $y \in Y$ there exists a neighbourhood U_y of y in Y such that U_y is free. Let $0 < \varepsilon_y \le \varepsilon$ be so that $B(y, 3\varepsilon_y)$, the open $3\varepsilon_y$-ball about y, is contained in U_y.

Since Y is an ANR there is a $\delta > 0$ with $\delta < \varepsilon_y$ such that for each closed subset A of X and $g : A \to Y$ which is δ-close to $f|_A$, g has a continuous extension $\overline{g} : X \to Y$ which is ε_y-close to f (see [M, Theorem 5.1.3]).

Since U_y is free and $f^{-1}(\overline{B(y, 2\varepsilon_y)})$ is compact there exists $\phi \in C(f^{-1}(\overline{B(y, 2\varepsilon_y)}), U_y; \text{h.i.})$ with

$$d(\phi, f|_{f^{-1}(\overline{B(y,2\varepsilon_y)})}) < \delta.$$

Then ϕ has a continuous extension $\overline{\phi} : X \to Y$ which is ε_y-close to f. Hence, $\phi^{-1}(z) = \overline{\phi}^{-1}(z)$ for $z \in \overline{B(y, \varepsilon_y)}$ since $\overline{\phi}(x) \notin \overline{B(y, \varepsilon_y)}$ for $x \in X - f^{-1}(\overline{B(y, 2\varepsilon_y)})$.

We have shown that

$$H_y = \{g \in C(X,Y) \mid g^{-1}(z) \text{ is hereditarily indecomposable for each } z \in \overline{B(y, \varepsilon_y)}\}$$

is dense in $C(X,Y)$. By [K, 3.2], H_y is a G_δ-set in $C(X,Y)$. Since we need only finitely many $B(y, \varepsilon_y)$ to cover $f(X)$ it follows by the Baire Category Theorem that $C(X, Y; \text{h.i.})$ is a dense G_δ-set in $C(X,Y)$.

4. Some classes of free spaces

THEOREM 4.1. *Each locally finite polyhedron P without isolated points is free.*

Proof. Each point in the locally finite polyhedron P has a neighbourhood which is either a cone or the product of some set by an open interval. By Theorem 2.2 and Proposition 1.3, P is locally free. By Theorem 3.2, P is free.

The next result says that if a space can be suitably approximated by free spaces then it is free.

THEOREM 4.2. *Let Y be a metric space. If for each $\varepsilon > 0$ there exists a compact free space Z and mappings $r : Y \to Z$ and $\phi : Z \to Y$ such that ϕ is light and $\phi \circ r$ is within ε of the identity on Y then Y is free.*

Proof. Let $f \in C(X,Y)$ for some compactum X and let $\varepsilon > 0$. Let Z be a compactum such that there exist $r : Y \to Z$ and a light map $\phi : Z \to Y$ so that $\phi \circ r$ is within $\varepsilon/2$ of the identity on Y. Let $\delta > 0$ be so that if a and

b are within δ in Z then $\phi(a)$ and $\phi(b)$ are within $\varepsilon/2$ in Y. Let $g : X \to Z$ be a mapping with hereditarily indecomposable fibers such that $r \circ f$ and g are within δ. Then $\phi \circ g : X \to Y$ has hereditarily indecomposable fibers since ϕ is light. Also $\phi \circ g$ is within ε of f.

Theorem 4.2 is of special interest when the space Z can be chosen to be a retract of Y.

A *Menger manifold* (resp. *Nöbeling manifold*) is a space modelled on the Menger cube M^n_{2n+1} (resp. Nöbeling space N^n_{2n+1}) [HW].

COROLLARY 4.3. *Every Menger manifold and every Nöbeling manifold are free.*

Proof. Menger manifolds and Nöbeling manifolds admit small retractions to polyhedra (see [B] and [CKT]).

COROLLARY 4.4. *Every* 1*-dimensional Peano continuum is free.*

Proof. Every 1-dimensional Peano continuum admits a small retraction to a finite graph [Ku, II, p. 258].

In the next section we shall see that this corollary fails in higher dimensions.

A continuum X is said to be *unicoherent* if whenever $X = A \cup B$ where A and B are subcontinua then $A \cap B$ is also a subcontinuum. We say X is *hereditarily unicoherent* if each of its subcontinua is unicoherent.

A *dendroid* is an arcwise connected, hereditarily unicoherent continuum. A dendroid X is said to be *smooth* if there exists $p \in X$ such that for each convergent sequence $\{a_i\}$ converging to a in X the sequence of arcs $\{a_i p\}$ from a_i to p converges to ap, the arc from a to p.

A *fan* is a dendroid with at most one ramification point.

COROLLARY 4.5. *Each smooth dendroid and each fan are free.*

Proof. By work of Fugate [F1-2] each smooth dendroid and each fan retract to a finite tree.

COROLLARY 4.6. *The* $\sin(1/x)$*-continuum S and Knaster's dyadic indecomposable chainable continuum* [Ku, II, p. 205] *are free.*

Proof. Each of these continua admits a small retraction to an arc.

In the next section we shall construct examples based on the $\sin(1/x)$-continuum S. For that reason it is essential to give another geometric construction showing that S is free.

4.7. *Alternative proof that S is free.* Let $K = \{0\} \cup \{1/k\}_{k=1}^{\infty}$ in the real line. Let $\varepsilon > 0$. Let $f : I \times I \to I$ be defined as in Lemma 2.1 so that f is within ε of π_1 and the fibers of f are folded relative to π_2.

Define $g : (I \times K) \times I \to I \times K$ by $g((x,t),s) = (f(x,s),t)$. Let $S = I \times K/\sim$ where $(x,t) \sim (y,s)$ iff $(x,t) = (y,s)$ or $\{(x,t),(y,s)\} = \{(1,1/(2k)),(1,1/(2k-1))\}$ or $\{(x,t),(y,s)\} = \{(0,1/(2k+1)),(0,1/(2k))\}$ for some k. Let $\pi : I \times K \to S$ be the quotient map. Note that S is a $\sin(1/x)$-continuum. Then g can be used to construct in an obvious way a map $\overline{g} : S \times I \to S$ so that $\overline{g}$ is within ε of π_1 on $S \times I$ and each fiber of $\overline{g}$ is folded relative to π_2 on $S \times I$. To make $\overline{g}$ continuous one can let $\overline{g}(J_{\pi(x,1/(2k))}) = \overline{g}(J_{\pi(x,1/(2k-1))}) \in \pi(I \times \{1/(2k)\})$ for $(4n+2)/(4n+3) \le x \le 1$ and $\overline{g}(J_{\pi(x,1/(2k))}) = \overline{g}(J_{\pi(x,1/(2k+1))}) \in \pi(I \times \{1/(2k+1)\})$ for $0 \le x \le 1/(4n+3)$ where n is chosen as in Lemma 2.1.

5. Miscellaneous examples. We give some examples which show that the property of being a free space is not well-behaved under unions. Also, we show that this property has little to do with good homotopy properties.

EXAMPLE 5.1. The $\sin(1/x)$-continuum S with a sticker L adjoined at the end of the limit segment of S is free.

To see this simply reflect onto $L \times I$ the decomposition of the limit disk in $S \times I$ induced by the map f in 4.7. Use this decomposition on $L \times I$ to extend f continuously to the new space in the obvious way (see Figure 4).

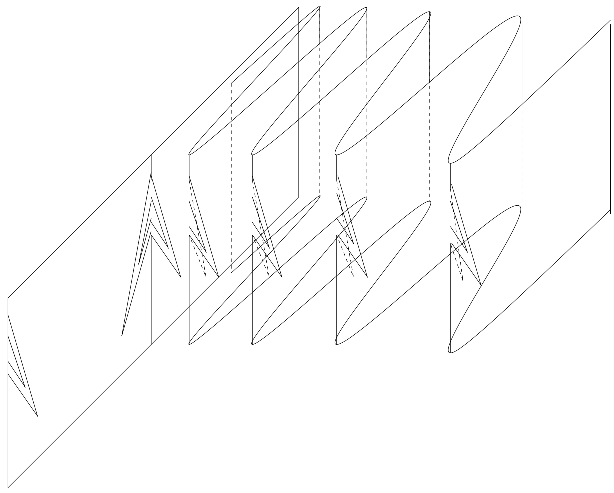

Fig. 4

The next example shows that the property of being a free space is not preserved in general even under very nice unions. In particular, it shows that the wedge of two free spaces need not be free, and the union of two free spaces with free intersection need not be free.

EXAMPLE 5.2. The wedge of two $\sin(1/x)$-continua A and B joined at a common endpoint of the limit segments in A and in B is not free.

To see this let $0 \in A\cap B$ and suppose $f : (A\cup B)\times I \to A\cup B$ is a mapping close to π_1. Using path components it is easy to see that $f(A\times I) \subset A$ and $f(B\times I) \subset B$ so $f(\{0\}\times I) \subset A\cap B = \{0\}$. Thus the fiber $f^{-1}(0)$ is not folded relative to π_2 by Lemma 1.2.

We can modify Example 5.2 to get for each $n \geq 2$ a C^n, LC^{n-2}, free continuum of dimension n whose wedge with a homeomorphic copy of itself is not free. In Example 5.3 we do this for $n = 2$. It is clear how to generalize this example for arbitrary n.

EXAMPLE 5.3. Let W_1 be the circular cone (with vertex removed) $x^2 = y^2 + z^2,\ 0 < x \leq 1$, in $\mathbb{R}^3$. Define

$$V_1 = \{(x, y + \sin(1/x) + 1, x + z) \in \mathbb{R}^3 \mid (x, y, z) \in W_1\},$$
$$P_1 = [0,1] \times [0,2] \times \{0\},$$
$$D_1 = \{(1, y + \sin 1 + 1, 1 + z) \mid y^2 + z^2 \leq 1\}.$$

Then V_1 is a circle of $\sin(1/x)$-curves identified along their limit segments. Let $Y_1 = P_1 \cup V_1$ and $X_1 = P_1 \cup V_1 \cup D_1$. Then Y_1 is a circle of $\sin(1/x)$-curves with limit segments identified and lying on a disk. Alternatively, Y_1 is a $\sin(1/x)$-tube lying on a disk. We obtain X_1 from Y_1 by capping the end of the tube in Y_1 by a disk. Then Y_1 is C^0 and LC^0 and X_1 is C^2 and LC^0. See Figure 5.

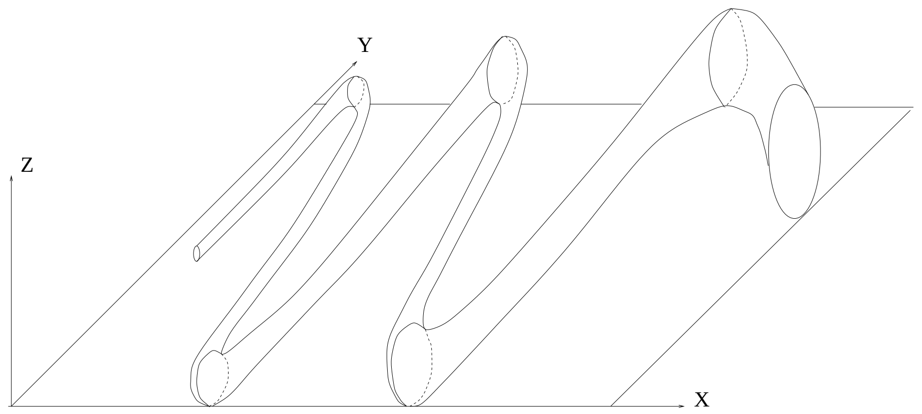

Fig. 5. A $\sin(1/x)$-tube lying on a disk

Let Y_2 (resp. X_2) be the reflection of Y_1 (resp. X_1) about the origin. The spaces $Y_1 \cup Y_2$ and $X_1 \cup X_2$ are not free for essentially the same homotopy reasons as in Example 5.2. We claim Y_1 and X_1 are free. We only prove that Y_1 is free. The proof that X_1 is free uses the fact that Y_1 is free and the argument of Theorem 3.2.

Let $\varepsilon > 0$ and let $f : I \times I \to I$ be as in Lemma 2.1, i.e. f is ε-close to π_1 and the fibers of f are folded relative to π_2. We define a map $h : I^2 \times I \to I^2$.

In the disk I^2 let A_{2i-1} be the line segment from $(1/(2i), 1)$ to $(1/(2i-1), 0)$ and let A_{2i} be the line segment from $(1/(2i), 1)$ to $(1/(2i+1), 0)$ for each positive integer i. Then $(\{0\} \times I) \cup \bigcup_{i=1}^{\infty}(A_{2i-1} \cup A_{2i})$ is a $\sin(1/x)$-curve. Let $(x, y, z) \in I^3$.

(i) Suppose that $(x, y, z) \in A_{2i} \times I$. If $(y, z) \in J_t$ for some $t \in [0, \varepsilon]$, let $h(x, y, z) \in A_{2i+1} \cap (I \times \{f(y, z)\})$. Otherwise, let $h(x, y, z) \in A_{2i} \cap (I \times \{f(y, z)\})$.

(ii) Suppose that $(x, y, z) \in A_{2i-1} \times I$. If $(y, z) \in J_t$ for some $t \in [1-\varepsilon, 1]$, let $h(x, y, z) \in A_{2i} \cap (I \times \{f(y, z)\})$. Otherwise, let $h(x, y, z) \in A_{2i-1} \cap (I \times \{f(y, z)\})$.

(iii) If $x = 0, 1$ define $h(x, y, z) = (x, f(y, z))$.

(iv) If (x, y) is in the component of $I^2 - (A_{2i} \cup A_{2i-1})$ which misses $\{0, 1\} \times I$ let $(x', y) \in A_{2i-1}$ and $(x'', y) \in A_{2i}$. There is a number t so that $x = tx' + (1-t)x''$. Let $h(x, y, z) = th(x', y, z) + (1-t)h(x'', y, z)$, the linear combination of $h(x', y, z)$ and $h(x'', y, z)$.

(v) If (x, y) is in the component of $I^2 - (A_{2i} \cup A_{2i+1})$ which misses $\{0, 1\} \times I^2$ let $(x', y) \in A_{2i}$ and $(x'', y) \in A_{2i+1}$. There is a number t so that $x = tx' + (1-t)x''$. Let $h(x, y, z) = th(x', y, z) + (1-t)h(x'', y, z)$.

(vi) If (x, y) is in the component of $I^2 - A_1$ which meets $\{1\} \times I$ let $(x', y) \in A_1 \cap (I \times \{f(y, z)\})$. There is a number t so that $x = tx' + (1-t)$. Let $h(x, y, z) = th(x', y, z) + (1-t)h(1, y, z)$.

Now I^2 is homeomorphic to $[0, 1] \times [0, 2]$ and $\bigcup_{i=1}^{\infty} A_i$ is homeomorphic to S. We regard h as a mapping of $P_1 \times I$ to P_1. This map extends using the partial product structure on V_1 to a map $\widetilde{h} : Y_1 \times I \to Y_1$ so that $\widetilde{h}$ is ε-close to $\pi_1 : Y_1 \times I \to Y_1$ and each fiber of $\widetilde{h}$ is folded relative to $\pi_2 : Y_1 \times I \to I$.

Example 5.4. Solenoids are free.

A solenoid S is a continuum obtained as the inverse limit

$$S = \varprojlim(S_i,\ z^{p_i})$$

where each S_i is the unit circle S^1, p_i is a positive integer and z^{p_i} represents the p_i-fold covering mapping $z \mapsto z^{p_i}$ of S^1 onto itself.

We prove that for each $\varepsilon > 0$ there is a mapping $\phi : S \times I \to S$ so that $d(\phi, \pi_1) < \varepsilon$ and each fiber of ϕ is folded relative to π_2. We construct an infinite commutative diagram

$$\begin{tikzcd}
S_1 \times I \arrow[d, "\phi_1"] & S_2 \times I \arrow[l, "z^{p_1} \times \mathrm{id}"'] \arrow[d, "\phi_2"] & \arrow[l] \\
S_1 & S_2 \arrow[l, "z^{p_1}"'] & \arrow[l]
\end{tikzcd}$$

The map ϕ_1 is induced by the map $f : I \times I \to I$ defined in the proof of Lemma 2.1 and by the exponential map $\eta : I \to S_1$, $\eta(x) = e^{2\pi ix}$. More precisely, let $(x, y) \in I \times I$. If $(x, y) \in J_t$ for some $t \in [0, \varepsilon]$ let $\phi_1(\eta(x), y) = \eta(f(x, y)) = \eta(t) \in S^1$ since $f(J_t) = \{t\}$. If $(x, y) \in J_t$ for some $t \in [1-\varepsilon, 1]$ then $(1 - x, y) \in J_{1-t}$ so we define $\phi_1(\eta(x), y) = \eta(f(1 - x, y)) = \eta(1 - t)$. Otherwise, let $\phi_1(\eta(x), y) = \eta(f(x, y))$.

Since $z^{p_1} \times \mathrm{id}$ and z^{p_1} are both p_1-fold covering maps there is a unique way to define ϕ_2 to make the diagram commute. The step $i = 2$ is the general step in the inductive definition of ϕ_i. Let $\phi = \varprojlim \phi_i$. If $(x, t) \in S \times I$ then x and $\phi(x)$ are contained in an arc of S of diameter less than ε. Each fiber of ϕ is folded relative to the projection $\pi_2 : S \times I \to I$.

Questions. (1) Is each ANR free?

(2) Is each bundle space over a free space free?

(3) Is each dendroid free?

(4) Is the hyperspace over the pseudo-arc free?

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