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# On ergodicity of some cylinder flows

by

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**Abstract.** We study ergodicity of cylinder flows of the form

$$T_f : \mathbb{T} \times \mathbb{R} \to \mathbb{T} \times \mathbb{R}, \quad T_f(x, y) = (x + \alpha, y + f(x)),$$

where $f : \mathbb{T} \to \mathbb{R}$ is a measurable cocycle with zero integral. We show a new class of smooth ergodic cocycles. Let $k$ be a natural number and let $f$ be a function such that $D^k f$ is piecewise absolutely continuous (but not continuous) with zero sum of jumps. We show that if the points of discontinuity of $D^k f$ have some good properties, then $T_f$ is ergodic. Moreover, there exists $\varepsilon_f > 0$ such that if $v : \mathbb{T} \to \mathbb{R}$ is a function with zero integral such that $D^k v$ is of bounded variation with $\mathrm{Var}(D^k v) < \varepsilon_f$, then $T_{f+v}$ is ergodic.

**1. Introduction.** Assume that $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ is an ergodic measure-preserving automorphism of a standard Borel space. Each measurable function $f : X \to \mathbb{R}$ is called a *cocycle*. For every $n \in \mathbb{Z}$, let

$$f^{(n)}(x) = \begin{cases} f(x) + f(Tx) + \ldots + f(T^{n-1}x) & \text{if } n > 0, \\ 0 & \text{if } n = 0, \\ -(f(T^n x) + f(T^{n+1}x) + \ldots + f(T^{-1}x)) & \text{if } n < 0. \end{cases}$$

Let $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ be the one-point Aleksandrov compactification of $\mathbb{R}$. Then $r \in \overline{\mathbb{R}}$ is said to be an *extended essential value* of $f$ (see [10]) if for each open neighbourhood $U(r)$ of $r$ and an arbitrary set $C \in \mathcal{B}$ with $\mu(C) > 0$, there exists an integer $n$ such that

$$\mu(C \cap T^{-n}C \cap \{x \in X : f^{(n)} \in U(r)\}) > 0.$$

The set of extended essential values will be denoted by $\overline{E}(f)$. The set $E(f) = \overline{E}(f) \cap \mathbb{R}$ is called the set of *essential values* of $f$. The skew product

$$T_f : (X \times \mathbb{R}, \widetilde{\mathcal{B}}, \widetilde{\mu}) \to (X \times \mathbb{R}, \widetilde{\mathcal{B}}, \widetilde{\mu}), \quad T_f(x, y) = (Tx, y + f(x)),$$

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is said to be the *cylinder flow*. Here $\widetilde{\mu}$ denotes the product measure of $\mu$ and infinite Lebesgue measure on the line. It is shown in [10] that $E(f)$ is a closed subgroup of $\mathbb{R}$ and it is the collection of *periods* of $T_f$-invariant functions, i.e.

$$E(f) = \{r \in \mathbb{R} : \forall_{\phi:X\times\mathbb{R}\to\mathbb{R},\, \phi\circ T_f=\phi}\ \phi(x, y + r) = \phi(x, y)\ \widetilde{\mu}\text{-a.e.}\}.$$

In particular, $T_f$ is ergodic iff $E(f) = \mathbb{R}$.

We say that a strictly increasing sequence $\{q_n\}_{n\in\mathbb{N}}$ is a *rigid time* for $T$ if

$$\lim_{n\to\infty} \mu(T^{q_n}A \triangle A) = 0 \quad \text{for any } A \in \mathcal{B}.$$

In [6], Lemańczyk, Parreau and Volný have proved

Proposition 1. *Suppose that $f : X \to \mathbb{R}$ is an integrable cocycle such that the sequence $\{\|f^{(q_n)}\|_{L^1}\}_{n\in\mathbb{N}}$ is bounded, where $\{q_n\}_{n\in\mathbb{N}}$ is a rigid time for $T$. If*

$$\limsup_{n\to\infty} \Big|\int_X e^{2\pi i l f^{(q_n)}}\, d\mu\Big| \le c < 1$$

*for all $l$ large enough, then $T_f$ is ergodic.*

We denote by $\mathbb{T}$ the group $\mathbb{R}/\mathbb{Z}$ which will be identified with the interval $[0, 1)$ with addition mod 1. Let $\lambda$ denote the Lebesgue measure on $\mathbb{T}$. Let $\widetilde{<} \subset \mathbb{T}\times\mathbb{T}$ be defined by: $x \mathrel{\widetilde{<}} y$ iff $0 < y - x < 1/2$, where $< \subset \mathbb{T}\times\mathbb{T}$ is the usual order on $[0, 1)$. By $\{t\}$ we denote the fractional part of $t$ and $\|t\|$ is the distance of $t$ from the set of integers.

Assume that $\alpha \in [0, 1)$ is an irrational with continued fraction expansion

$$\alpha = [0; a_1, a_2, \ldots].$$

The natural numbers $a_n$ are said to be *the partial quotients* of $\alpha$. Put

$$\begin{aligned} &r_0 = 0, \quad r_1 = 1, \quad r_{n+1} = a_{n+1}r_n + r_{n-1},\\ &s_0 = 1, \quad s_1 = a_1, \quad s_{n+1} = a_{n+1}s_n + s_{n-1}. \end{aligned}$$

The rationals $r_n/s_n$ are called the *convergents*, and $s_n$ is the $n$th *denominator* of $\alpha$. We have the inequality

$$\frac{1}{2s_ns_{n+1}} < \left|\alpha - \frac{r_n}{s_n}\right| < \frac{1}{s_ns_{n+1}}.$$

For every nonnegative integer $k$, let $S_k$ denote the subset of irrational numbers $\alpha$ such that

$$\liminf_{n\to\infty} s_n^{k+1}\|s_n\alpha\| < \infty$$

and let $S_k^0$ denote the subset of irrational numbers $\alpha$ such that

$$\liminf_{n\to\infty} s_n^{k+1}\|s_n\alpha\| = 0.$$

The above sets are residual in $\mathbb{T}$.

A function $f:\mathbb{T}\to\mathbb{R}$ is said to be *piecewise absolutely continuous* (PAC for short) if there are $\beta_0,\ldots,\beta_k\in\mathbb{T}$ such that $f|_{(\beta_j,\beta_{j+1})}$ is absolutely continuous ($\beta_{k+1}=\beta_0$). Set

$$f_+(x)=\lim_{y\to x^+}f(y)\quad\text{and}\quad f_-(x)=\lim_{y\to x^-}f(y).$$

Let $a_j=f_+(\beta_j)-f_-(\beta_j)$ for $j=0,\ldots,k$ and

$$S(f)=\sum_{j=0}^{k}a_j=-\sum_{j=0}^{k}f_-(\beta_j)-f_+(\beta_j)=-\int_{\mathbb{T}}Df(x)\,d\lambda(x).$$

Assume that $\alpha\in[0,1)$ is irrational. Denote by $Tx=x+\alpha \bmod 1$ the corresponding ergodic rotation on $\mathbb{T}$. We shall study skew products of the form

$$T_f:\mathbb{T}\times\mathbb{R}\to\mathbb{T}\times\mathbb{R},\qquad T_f(x,y)=(Tx,y+f(x)),$$

where $f:\mathbb{T}\to\mathbb{R}$ is a measurable cocycle with $\int_{\mathbb{T}}f\,d\lambda=0$.

In [8], Pask has given a class of cocycles which are PAC with $S(f)\neq 0$, and has showed ergodicity for all irrationals $\alpha$. Lemańczyk, Parreau and Volný [6] have proved that the class of cocycles considered in [8] is ergodically stable in the space $\mathrm{BV}(\mathbb{T})_0$ of bounded variation functions with zero integral, i.e. if $f\in\mathrm{PAC}$ with $S(f)\neq 0$ and $\mathrm{Var}(f-g)<|S(f)|$, then $T_g$ is still ergodic. It has been proved in [9] that if $f$ is $k-1$ times differentiable a.e. and $D^{k-1}f$ is PAC with $S(D^{k-1}f)\neq 0$, then $T_f$ is ergodic for $\alpha\in S_k$.

The aim of this paper is to study the ergodicity of $T_f$ in the case where a derivative $D^kf$ of $f$ is piecewise absolutely continuous (but not continuous) and $S(D^kf)=0$.

Let $k$ be a natural number. We denote by $C_0^{k+\mathrm{BV}}$ the space of $k-1$ differentiable functions $f:\mathbb{T}\to\mathbb{R}$ with zero integral such that $D^{k-1}f$ is absolutely continuous and $D^kf$ is of bounded variation. Set $C_0^{0+\mathrm{BV}}=\mathrm{BV}_0$.

Observe that if $f:\mathbb{T}\to\mathbb{R}$ is a function of bounded variation with zero integral, then

$$\sup_{x\in\mathbb{T}}|f(x)|\le\mathrm{Var}(f).\tag{1}$$

Notice that if $f\in C_0^{k+\mathrm{BV}}$, then $\mathrm{Var}(D^{j-1}f)\le\mathrm{Var}(D^jf)$ for $j=1,\ldots,k$. Indeed, since $D^{j-1}f$ is absolutely continuous, we have $\mathrm{Var}(D^{j-1}f)=\int_{\mathbb{T}}|D^jf|\,d\lambda$ and $\int_{\mathbb{T}}D^jf\,d\lambda=0$. From (1) we have

$$\mathrm{Var}(D^{j-1}f)=\int_{\mathbb{T}}|D^jf|\,d\lambda\le\sup_{x\in\mathbb{T}}|D^jf(x)|\le\mathrm{Var}(D^jf).$$

In $C_0^{k+\mathrm{BV}}$ we define the norm $\|f\|_{k+\mathrm{BV}}=\mathrm{Var}(D^kf)$. With this norm, $C_0^{k+\mathrm{BV}}$ becomes a Banach space. Let $C_0^{k+\mathrm{PAC}}$ denote the subspace of functions $f\in C_0^{k+\mathrm{BV}}$ such that $D^kf$ is piecewise absolutely continuous and let $C_0^{k+AC}$

denote the space of functions $f \in C_0^{k+\mathrm{PAC}}$ such that $D^k f$ is absolutely continuous. Recall that the subspace of trigonometric polynomials is dense in $C_0^{k+\mathrm{AC}}$ with respect to the $C_0^{k+\mathrm{BV}}$ norm.

Assume that $f \in C_0^{k+\mathrm{PAC}}$ and $S(D^k f) = 0$. Suppose that $\alpha \in S_k^0$ and $0 = \beta_0 < \beta_1 < \ldots < \beta_d < 1$ are all the discontinuity points of $D^k f$. In this paper we will prove the following theorem.

THEOREM 1.1 (Main Theorem). *Let $k \in \mathbb{N}$ and $f \in C_0^{k+\mathrm{PAC}}$ be such that $S(D^k f) = 0$. If there exists a sequence $\{q_n\}_{n\in\mathbb{N}}$ of denominators of $\alpha$ such that*

$$\lim_{n\to\infty} q_n^{k+1}\|q_n\alpha\| = 0 \quad \text{and} \quad \lim_{n\to\infty}\{q_n\beta_i\} = \gamma_i,$$

*where $\gamma_i \neq \gamma_j$ for $i \neq j$, $i, j = 0, \ldots, d$, then $T_f$ is ergodic. Moreover, there exists $\varepsilon > 0$ such that if $v \in C_0^{k+\mathrm{BV}}$ and $\|v\|_{k+\mathrm{BV}} < \varepsilon$, then $T_{f+v}$ is ergodic.*

**2. Some generalizations of the Denjoy–Koksma inequality.** In this section we prove some generalizations of the Denjoy–Koksma inequality which will be needed to prove the main theorem. Let $Q_n$ be a partition of $\mathbb{T}$ into the intervals defined by the points $\{i\alpha\}_{i=0}^{s_n-1}$. Then for all $n$, each interval of $Q_n$ has length $\|s_{n-1}\alpha\| + \|s_n\alpha\|$ or $\|s_{n-1}\alpha\|$.

THEOREM 2.1. *For a given nonnegative integer $k$ there is a positive constant $M_k = M$ such that if $f \in C_0^{k+\mathrm{BV}}$, then*

$$s_n^k|f^{(s_n)}(x)| \le M(1 + s_n^{k+1}\|s_n\alpha\|)\operatorname{Var}(D^k f) \tag{2}$$

*for any natural $n$.*

Proof (by induction on $k$). For $k = 0$ the inequality (2) is the ordinary Denjoy–Koksma inequality (see [5], p. 73).

Assuming (2) to hold for a certain $k$, we will prove that there exists $M_{k+1} > 0$ such that if $f \in C_0^{k+1+\mathrm{BV}}$, then

$$s_n^{k+1}|f^{(s_n)}(x)| \le M_{k+1}(1 + s_n^{k+2}\|s_n\alpha\|)\operatorname{Var}(D^{k+1} f).$$

Let $I$ be an interval of size $\|s_{n-1}\alpha\|$. Then

$$\Big|\int_I f^{(s_n)}(x)\,dx\Big| = \Big|\int_{\bigcup_{i=0}^{s_n-1} T^i I} f(x)\,dx\Big| = \Big|\int_{\mathbb{T}\setminus\bigcup_{i=0}^{s_n-1} T^i I} f(x)\,dx\Big|.$$

Since

$$\mathbb{T} \setminus \bigcup_{i=0}^{s_n-1} T^i I = \bigcup_{j=0}^{s_{n-1}-1} T^j J,$$

where $J$ is an interval of size $\|s_n\alpha\|$, we have

$$\Big|\int_I f^{(s_n)}(x)\,dx\Big| = \Big|\int_J f^{(s_{n-1})}(x)\,dx\Big| \le |J|\operatorname{Var}(f) \le \|s_n\alpha\|\operatorname{Var}(D^{k+1}f).$$

If $|I| = \|s_{n-1}\alpha\| + \|s_n\alpha\|$, then we split this into two chunks, one $I_1$ of size $\|s_{n-1}\alpha\|$, the other $I_2$ of size $\|s_n\alpha\|$. Then

$$\Big|\int_I f^{(s_n)}(x)\,dx\Big| = \Big|\int_{I_1} f^{(s_n)}(x)\,dx\Big| + \Big|\int_{I_2} f^{(s_n)}(x)\,dx\Big| \le 2\|s_n\alpha\|\operatorname{Var}(D^{k+1}f).$$

It follows that for each interval $I$ of $Q_n$ there is $x_I \in I$ with

$$|f^{(s_n)}(x_I)| \le 4s_n\|s_n\alpha\|\operatorname{Var}(D^{k+1}f).$$

Indeed, if $f^{(s_n)}|_I$ changes sign, then we can take $x_I$ such that $f^{(s_n)}(x_I) = 0$. Assume that $f^{(s_n)}|_I$ does not change sign. Suppose that

$$|f^{(s_n)}(x)| \ge 4s_n\|s_n\alpha\|\operatorname{Var}(D^{k+1}f)$$

for any $x \in I$. Then

$$\Big|\int_I f^{(s_n)}(x)\,dx\Big| > |I|4s_n\|s_n\alpha\|\operatorname{Var}(D^{k+1}f) > 2\|s_n\alpha\|\operatorname{Var}(D^{k+1}f),$$

a contradiction. Since $f$ is absolutely continuous and the formula (2) is true for $k$, we have

$$\begin{aligned}|f^{(s_n)}(b) - f^{(s_n)}(a)| &= \Big|\int_a^b Df^{(s_n)}(x)\,dx\Big| \\ &\le M_k(1 + s_n^{k+1}\|s_n\alpha\|)\operatorname{Var}(D^{k+1}f)\frac{|b-a|}{s_n^k}\end{aligned}$$

for all $a, b \in \mathbb{T}$. If $x \in I \in Q_n$, then

$$\begin{aligned}|f^{(s_n)}(x) - f^{(s_n)}(x_I)| &\le 2\frac{\|s_{n-1}\alpha\|}{s_n^k}M_k(1 + s_n^{k+1}\|s_n\alpha\|)\operatorname{Var}(D^{k+1}f) \\ &\le \frac{2M_k}{s_n^{k+1}}(1 + s_n^{k+1}\|s_n\alpha\|)\operatorname{Var}(D^{k+1}f)\end{aligned}$$

and finally

$$\begin{aligned}s_n^{k+1}|f^{(s_n)}(x)| &\le s_n^{k+1}|f^{(s_n)}(x) - f^{(s_n)}(x_I)| + s_n^{k+1}|f^{(s_n)}(x_I)| \\ &\le (2M_k(1 + s_n^{k+1}\|s_n\alpha\|) + 4s_n^{k+2}\|s_n\alpha\|)\operatorname{Var}(D^{k+1}f) \\ &\le (2M_k + 4)(1 + s_n^{k+2}\|s_n\alpha\|)\operatorname{Var}(D^{k+1}f). \ \blacksquare\end{aligned}$$

COROLLARY 2.1. *Assume that $\alpha \in S_k$ and $\{q_n\}_{n\in\mathbb{N}}$ is a sequence of denominators of $\alpha$ such that the sequence $\{q_n^{k+1}\|q_n\alpha\|\}_{n\in\mathbb{N}}$ is bounded. Then there is a constant $K \ge 1$ such that*

$$q_n^k|f^{(q_n)}(x)| \le K\|f\|_{k+\mathrm{BV}}$$

*for any* $f \in C_0^{k+\mathrm{BV}}$ *and* $n \in \mathbb{N}$. *Moreover*, *if* $f \in C_0^{k+\mathrm{AC}}$, *then the sequence* $\{q_n^k f^{(q_n)}\}_{n\in\mathbb{N}}$ *uniformly converges to zero.*

Proof. Notice that Theorem 2.1 implies the first part of the corollary. Since for every $f \in C_0^{k+\mathrm{AC}}$ there exists a sequence $\{P_m\}_{m\in\mathbb{N}}$ of trigonometric polynomials with zero integral such that

$$\lim_{m\to\infty} \|P_m - f\|_{k+\mathrm{BV}} = 0,$$

it suffices to show that for every trigonometric polynomial $f$ with zero integral the sequence $\{q_n^k f^{(q_n)}\}_{n\in\mathbb{N}}$ uniformly converges to zero. Let

$$f(x) = \sum_{m=-M}^{M} a_m e^{2\pi imx}$$

where $a_0 = 0$. Then

$$\begin{aligned} |q_n^k f^{(q_n)}(x)| &= \left| q_n^k \sum_{m=-M}^{M} a_m \frac{e^{2\pi imq_n\alpha} - 1}{e^{2\pi im\alpha} - 1} e^{2\pi imx} \right| \\ &\le 2q_n^k \sum_{m=-M}^{M} |a_m| \frac{m\|q_n\alpha\|}{\|m\alpha\|} = q_n^k \|q_n\alpha\| \sum_{m=-M}^{M} \frac{2|a_m|m}{\|m\alpha\|}. \end{aligned}$$

It follows that $q_n^k f^{(q_n)}$ uniformly converges to zero, which completes the proof. ■

**3. Ergodicity of differentiable cocycles.** We need auxiliary lemmas.

LEMMA 3.1. *Let* $0 = \beta_0 < \beta_1 < \ldots < \beta_d < \beta_{d+1} = 1$ *and let* $a_1, \ldots, a_{d+1}$ *be real numbers with zero sum. Consider a function* $h : \mathbb{T} \to \mathbb{R}$ *with zero integral given by*

$$h = h(0) + \sum_{i=1}^{d+1} a_i \mathbf{1}_{[\beta_i,1)}.$$

*Then* $h(0) = \sum_{i=1}^{d+1} a_i\beta_i$ *and*

$$h^{(q)} = h^{(q)}(0) + \sum_{s=0}^{q-1} \sum_{i=1}^{d+1} a_i \mathbf{1}_{[\beta_i - s\alpha,1)} \tag{3}$$

*for any natural* $q$, *where* $T : \mathbb{T} \to \mathbb{T}$ *is the rotation through* $\alpha$.

Proof. Since $\int_{\mathbb{T}} h\,d\lambda = 0$ and $a_1 + \ldots + a_{d+1} = 0$, we have

$$0 = h(0) + \sum_{i=1}^{d+1} a_i(1 - \beta_i) = h(0) - \sum_{i=1}^{d+1} a_i\beta_i.$$

For all $a, b, x \in \mathbb{T}$, we have

$$\mathbf{1}_{[b,1)}(x+a) - \mathbf{1}_{[b,1)}(a) = \mathbf{1}_{[b-a,1)}(x) - \mathbf{1}_{[1-a,1)}(x).$$

It follows that

$$\begin{aligned} h(x+a) - h(a) &= \sum_{i=1}^{d+1} a_i(\mathbf{1}_{[\beta_i,1)}(x+a) - \mathbf{1}_{[\beta_i,1)}(x)) \\ &= \sum_{i=1}^{d+1} a_i(\mathbf{1}_{[\beta_i-a,1)}(x) - \mathbf{1}_{[1-a,1)}(x)) = \sum_{i=1}^{d+1} a_i \mathbf{1}_{[\beta_i-a,1)}(x). \end{aligned}$$

Therefore

$$h^{(q)} = h^{(q)}(0) + \sum_{s=0}^{q-1} \sum_{i=1}^{d+1} a_i \mathbf{1}_{[\beta_i - s\alpha,1)}$$

for any natural $q$. ∎

LEMMA 3.2. *Let $I \subset \mathbb{R}$ be an interval and $k$ be a natural number. If $P$ is a real polynomial of the form $P(x) = c_k x^k + \ldots + c_0$, $c_k \neq 0$, then there exists a closed subinterval $J \subset I$ with $|J| \geq |I|/4^k$ such that*

$$x \in J \Rightarrow |P(x)| \geq k!|c_k|(|I|/4)^k.$$

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function with continuous derivative. Suppose that there exists a closed interval $I \subset \mathbb{R}$ such that $|Df(x)| \geq a > 0$ for any $x \in I$. We first show that there exists an interval $J \subset I$ with $|J| \geq |I|/4$ and $|f(x)| \geq a|I|/4$ for any $x \in J$. Without loss of generality we can assume that $Df(x) \geq a > 0$ for any $x \in I$. Suppose that for every interval $J \subset I$ with $|J| \geq |I|/4$ there exists $x \in J$ such that $|f(x)| < a|I|/4$. Since $f$ increases on $I$, we can find $x, y \in I$ such that $x - y \geq |I|/2$ and $|f(x)|, |f(y)| < a|I|/4$. It follows that

$$a|I|/2 \leq a|x-y| \leq |f(x) - f(y)| < a|I|/2,$$

a contradiction. Applying the above fact to derivatives of $P$ we obtain our assertion. ∎

Let $f \in C_0^{k+\mathrm{PAC}}$ be such that $S(D^k f) = 0$. Let $\alpha \in S_k^0$ and let $0 = \beta_0 < \beta_1 < \ldots < \beta_d < 1$ be all the discontinuities of $D^k f$. Suppose that there exists a sequence $\{q_n\}_{n\in\mathbb{N}}$ of denominators of $\alpha$ such that

$$\lim_{n\to\infty} q_n^{k+1} \|q_n\alpha\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \{q_n\beta_i\} = \gamma_i,$$

where $\gamma_i \neq \gamma_j$ for $i \neq j$, $i, j = 0, \ldots, d$. It is clear that the function $f$ can be represented as $f = g + h$, where $g \in C_0^{k+\mathrm{AC}}$, $h \in C_0^{k+\mathrm{PAC}}$ and $D^k h$ is constant on each interval $(\beta_i, \beta_{i+1})$. Then

$$D^k h_+(\beta_i) - D^k h_-(\beta_i) = D^k f_+(\beta_i) - D^k f_-(\beta_i) = a_i \neq 0$$

for $i = 0, \ldots, d$ and

$$D^k h_+ = D^k h_+(0) + \sum_{i=1}^{d+1} a_i \mathbf{1}_{[\beta_i, 1)}$$

with $D^k h_+(0) = \sum_{i=1}^{d+1} a_i \beta_i$. By Lemma 3.1,

$$D^k h_+^{(q)} = D^k h_+^{(q)}(0) + \sum_{s=0}^{q-1} \sum_{i=1}^{d+1} a_i \mathbf{1}_{[\beta_i - s\alpha, 1)} \tag{4}$$

for any natural $q$. Let $\sigma$ be a permutation of the set $\{0, 1, \ldots, d\}$ such that

$$0 = \gamma_{\sigma(0)} < \gamma_{\sigma(1)} < \ldots < \gamma_{\sigma(d)} < \gamma_{\sigma(d+1)} = 1,$$

where $\sigma(0) = \sigma(d+1)$. For given $1 \le i \le d+1$ and $0 \le j < q_n$, let $t_i^{(j)}$ be the unique integer satisfying $0 \le t_i^{(j)} < q_n$ and

$$t_i^{(j)} p_n + j = [q_n \beta_i] \bmod q_n,$$

where $\{p_n/q_n\}_{n \in \mathbb{N}}$ is the sequence of convergents of $\alpha$. Then

$$\begin{aligned}\beta_i - t_i^{(j)} \alpha &= \frac{[q_n \beta_i]}{q_n} + \frac{\{q_n \beta_i\}}{q_n} - t_i^{(j)} \frac{p_n}{q_n} - t_i^{(j)} \frac{\delta_n}{q_n} \\ &= \frac{j}{q_n} + \frac{1}{q_n}(\{q_n \beta_i\} - t_i^{(j)} \delta_n) \bmod 1,\end{aligned}$$

where $|\delta_n| = \|q_n \alpha\|$. It follows that

$$\beta_{\sigma(0)} - t_{\sigma(0)}^{(j)} \alpha \mathrel{\widetilde{<}} \beta_{\sigma(1)} - t_{\sigma(1)}^{(j)} \alpha \mathrel{\widetilde{<}} \ldots \mathrel{\widetilde{<}} \beta_{\sigma(d)} - t_{\sigma(d)}^{(j)} \alpha \mathrel{\widetilde{<}} \beta_{\sigma(0)} - t_{\sigma(0)}^{(j+1)} \alpha$$

for $j = 0, \ldots, q_n - 1$. Let $0 \le j \le q_n - 1$ and $0 \le i \le d$. Set

$$I_i^{(j)} = \begin{cases} (\beta_{\sigma(i)} - t_{\sigma(i)}^{(j)} \alpha, \beta_{\sigma(i+1)} - t_{\sigma(i+1)}^{(j)} \alpha) & \text{if } 0 \le i < d, \\ (\beta_{\sigma(d)} - t_{\sigma(d)}^{(j)} \alpha, \beta_{\sigma(0)} - t_{\sigma(0)}^{(j+1)} \alpha) & \text{if } i = d. \end{cases}$$

LEMMA 3.3. *If* $x \in I_i^{(j)}$, *then*

$$D^k h^{(q_n)}(x) = \sum_{m=1}^{d} a_m \{q_n \beta_m\} + \sum_{m=0}^{i} a_m.$$

Proof. Let $x \in I_i^{(j)}$. From (4), we have

$$\begin{aligned} D^k h^{(q_n)}(x) &= D^k h_+^{(q_n)}(0) + \sum_{l=0}^{q_n-1} \sum_{m=1}^{d+1} a_m \mathbf{1}_{[\beta_{\sigma(m)} - t_{\sigma(m)}^{(l)} \alpha, 1)}(x) \\ &= D^k h_+^{(q_n)}(0) + \sum_{l=0}^{j-1} \sum_{m=1}^{d+1} a_m + \sum_{m=1}^{d+1} a_m \mathbf{1}_{[\beta_{\sigma(m)} - t_{\sigma(m)}^{(j)} \alpha, 1)}(x) \\ &= D^k h_+^{(q_n)}(0) + \sum_{m=1}^{i} a_m. \end{aligned}$$

Moreover

$$\begin{aligned} D^k h_+^{(q_n)}(0) &= \sum_{j=0}^{q_n-1} D^k h_+(j\alpha) = \sum_{j=0}^{q_n-1} \Big( D^k h_+(0) + \sum_{i=1}^{d} a_i \mathbf{1}_{[\beta_i, 1)}(j\alpha) \Big) \\ &= q_n D^k h_+(0) + \sum_{i=1}^{d} a_i \sum_{j=0}^{q_n-1} \mathbf{1}_{[\beta_i, 1)}(j\alpha). \end{aligned}$$

On the other hand,

$$\begin{aligned} &\sum_{j=0}^{q_n-1} \mathbf{1}_{[\beta_i, 1)}(j\alpha) \\ &\quad = \operatorname{card}\{0 \le j < q_n : \{j\alpha\} > \beta_i\} \\ &\quad = \operatorname{card}\{0 \le j < q_n : \{jp_n/q_n\} + j\delta_n/q_n > [q_n\beta_i]/q_n + \{q_n\beta_i\}/q_n\} \\ &\quad = \operatorname{card}\{0 \le j < q_n : \{jp_n/q_n\} > [q_n\beta_i]/q_n\} \\ &\quad = q_n - [q_n\beta_i] - 1. \end{aligned}$$

Therefore

$$\begin{aligned} D^k h_+^{(q_n)}(0) &= q_n \sum_{i=1}^{d+1} a_i \beta_i + \sum_{i=1}^{d} a_i (q_n - [q_n\beta_i] - 1) \\ &= q_n \sum_{i=1}^{d} a_i \beta_i + \sum_{i=1}^{d} a_i (\{q_n\beta_i\} - q_n\beta_i) + a_0 \\ &= \sum_{i=1}^{d} a_i \{q_n\beta_i\} + a_0 \end{aligned}$$

and consequently

$$D^k h^{(q_n)}(x) = \sum_{m=1}^{d} a_m \{q_n\beta_m\} + \sum_{m=0}^{i} a_m. \quad \blacksquare$$

Let $0 \le j \le q_n - 1$ and $0 \le i \le d$. Let $\widehat{I}_i^{(j)}$ denote the interval

$$(\beta_{\sigma(i)} - t_{\sigma(i)}^{(j)}\alpha + q_n^k\|q_n\alpha\|, \beta_{\sigma(i+1)} - t_{\sigma(i+1)}^{(j)}\alpha - q_n^k\|q_n\alpha\|)$$

if $0 \le i < d$, and the interval

$$(\beta_{\sigma(d)} - t_{\sigma(d)}^{(j)}\alpha + q_n^k\|q_n\alpha\|, \beta_{\sigma(0)} - t_{\sigma(0)}^{(j+1)}\alpha - q_n^k\|q_n\alpha\|)$$

if $i = d$. Since $q_n^{k+1}\|q_n\alpha\| \to 0$ as $n \to \infty$, we have

$$\begin{aligned}|\widehat{I}_i^{(j)}| &= \frac{1}{q_n}|\{q_n\beta_{\sigma(i+1)}\} - \{q_n\beta_{\sigma(i)}\} - \delta_n(t_{\sigma(i+1)}^{(j)} - t_{\sigma(i)}^{(j)}) - 2q_n^{k+1}\|q_n\alpha\|| \\ &\ge \frac{\gamma_{\sigma(i+1)} - \gamma_{\sigma(i)}}{2q_n}\end{aligned}$$

for all $n$ large enough.

COROLLARY 3.1. *If* $x \in \widehat{I}_i^{(j)}$, *then*

$$D^k h^{(q_n^{k+1})}(x) = q_n^k\Big(\sum_{m=1}^{d} a_m\{q_n\beta_m\} + \sum_{m=0}^{i} a_m\Big).$$

Proof. For every $x \in \mathbb{T}$, we have

$$\begin{aligned}D^k h^{(q_n^{k+1})}(x) &= D^k h^{(q_n)}(x) \\ &\quad + D^k h^{(q_n)}(x + q_n\alpha) + \ldots + D^k h^{(q_n)}(x + (q_n^k - 1)q_n\alpha).\end{aligned}$$

If $x \in \widehat{I}_i^{(j)}$, then $x + lq_n\alpha \in I_i^{(j)}$ for $l = 0, 1, \ldots, q_n^k - 1$. It follows that

$$D^k h^{(q_n^{k+1})}(x) = q_n^k\Big(\sum_{m=1}^{d} a_m\{q_n\beta_m\} + \sum_{m=0}^{i} a_m\Big). \ \blacksquare$$

COROLLARY 3.2. *There exists a collection* $\{J_j\}_{j=0}^{q_n-1}$ *of pairwise disjoint closed intervals and there exist constants* $0 < C < 1$, $M > 0$ *such that*

$$|J_j| \ge \frac{C}{q_n} \text{ and } x \in J_j \Rightarrow |Dh^{(q_n^{k+1})}(x)| \ge Mq_n$$

*for* $j = 0, \ldots, q_n - 1$.

Proof. Fix

$$c_i = \sum_{m=1}^{d} a_m\gamma_m + \sum_{m=0}^{i} a_m.$$

At least one of the numbers $c_i$ is not zero. Indeed, if we suppose that $c_i = 0$ for $i = 0, \ldots, d$, then $a_i = c_i - c_{i-1} = 0$ for $i = 0, \ldots, d$, which is impossible. Take $i_0$ such that $c_{i_0} \ne 0$. Set

$$b^{(n)} = \sum_{m=1}^{d} a_m\{q_n\beta_m\} + \sum_{m=0}^{i_0} a_m.$$

Since $D^k h^{(q_n^{k+1})} = q_n^k b^{(n)}$ on $\widehat{I}_{i_0}^{(j)}$, we have

$$Dh^{(q_n^{k+1})}(x) = q_n^k b^{(n)} x^{k-1} + P_j(x)$$

on $\widehat{I}_{i_0}^{(j)}$, where $P_j$ is a polynomial with $\deg(P_j) < k-1$ ($j = 0, \ldots, q_n - 1$). By Lemma 3.2, there exist closed subintervals $J_j \subset \widehat{I}_{i_0}^{(j)}$ such that

$$|J_j| \geq \frac{1}{4^{k-1}} |\widehat{I}_{i_0}^{(j)}| \geq \frac{\gamma_{\sigma(i_0+1)} - \gamma_{\sigma(i_0)}}{4^k q_n}$$

and if $x \in J_j$, then

$$|Dh^{(q_n^{k+1})}(x))| \geq q_n^k |b^{(n)}| \bigg( \frac{|\widehat{I}_{i_0}^{(j)}|}{4} \bigg)^{k-1} \geq \frac{1}{2} q_n^k |c_{i_0}| \bigg( \frac{\gamma_{\sigma(i_0+1)} - \gamma_{\sigma(i_0)}}{4^k q_n} \bigg)^{k-1}$$
$$\geq q_n \frac{|c_{i_0}| (\gamma_{\sigma(i_0+1)} - \gamma_{\sigma(i_0+1)})^{k-1}}{4^{k^2}}$$

for $j = 0, \ldots, q_n - 1$. It follows that we can set

$$C = \frac{\gamma_{\sigma(i_0+1)} - \gamma_{\sigma(i_0)}}{4^k} \quad \text{and} \quad M = \frac{|c_{i_0}| (\gamma_{\sigma(i_0+1)} - \gamma_{\sigma(i_0)})^{k-1}}{4^{k^2}}. \ \blacksquare$$

*Proof of Theorem 1.1.* Notice that $\{q_n^{k+1}\}_{n \in \mathbb{N}}$ is a rigid time for the rotation $Tx = x + \alpha$. By Corollary 2.1, the sequence $\{\|(f+v)^{(q_n^{k+1})}\|_\infty\}_{n \in \mathbb{N}}$ is bounded, because $\|(f+v)^{(q_n^{k+1})}\|_\infty \leq q_n^k \|(f+v)^{(q_n)}\|_\infty$ and $f+v \in C_0^{k+\mathrm{BV}}$. By Proposition 1, it suffices to find $\varepsilon > 0$ such that $\mathrm{Var}(D^k v) < \varepsilon$ implies

$$\limsup_{n \to \infty} \Big| \int_{\mathbb{T}} e^{2\pi i l (f+v)^{(q_n^{k+1})}(x)} \, dx \Big| \leq c < 1$$

for all $l$ large enough.

Represent $f$ as the sum of functions $g \in C_0^{k+\mathrm{AC}}$ and $h \in C_0^{k+\mathrm{PAC}}$, where $D^k h$ is constant on intervals $(\beta_i, \beta_{i+1})$. Since $\|g^{(q_n^{k+1})}\|_\infty \leq q_n^k \|g^{(q_n)}\|_\infty$, the sequence $\{g^{(q_n^{k+1})}\}_{n \in \mathbb{N}}$ uniformly converges to zero, by Corollary 2.1. Therefore

$$\lim_{n \to \infty} \Big| \int_{\mathbb{T}} e^{2\pi i l (f+v)^{(q_n^{k+1})}(x)} \, dx - \int_{\mathbb{T}} e^{2\pi i l (h+v)^{(q_n^{k+1})}(x)} \, dx \Big| = 0.$$

It follows that it suffices to compute

$$\limsup_{n \to \infty} \Big| \int_{\mathbb{T}} e^{2\pi i l (h+v)^{(q_n^{k+1})}(x)} \, dx \Big|.$$

By Corollary 3.2, there exists a collection $\{J_j : j = 0, \ldots, q_n - 1\}$ of pairwise disjoint closed intervals and there exist $0 < C < 1$, $M > 0$ such that

$$|J_j| \geq \frac{C}{q_n} \text{ and } x \in J_j \Rightarrow |Dh^{(q_n^{k+1})}(x)| \geq M q_n$$

for $j=0,\ldots,q_n-1$. Let $J_j=[a_j,b_j]$ for $j=0,\ldots,q_n-1$. Applying integration by parts we get

$$\begin{aligned}
\Big|\int_{\mathbb{T}} e^{2\pi i l(h+v)^{(q_n^{k+1})}(x)}\,dx\Big|
&\le 1-\sum_{j=0}^{q_n-1}|J_j|+\Big|\sum_{j=0}^{q_n-1}\int_{a_j}^{b_j} e^{2\pi i l(h+v)^{(q_n^{k+1})}(x)}\,dx\Big|\\
&\le 1-C+\Big|\sum_{j=0}^{q_n-1}\int_{a_j}^{b_j}\frac{e^{2\pi i l v^{(q_n^{k+1})}(x)}}{2\pi i l Dh^{(q_n^{k+1})}(x)}\,de^{2\pi i l h^{(q_n^{k+1})}(x)}\Big|\\
&=1-C\\
&\quad+\Big|\sum_{j=0}^{q_n-1}\Big(\frac{e^{2\pi i l(h+v)^{(q_n^{k+1})}(b_j)}}{2\pi i l Dh^{(q_n^{k+1})}(b_j)}-\frac{e^{2\pi i l(h+v)^{(q_n^{k+1})}(a_j)}}{2\pi i l Dh^{(q_n^{k+1})}(a_j)}\\
&\quad-\int_{a_j}^{b_j} e^{2\pi i l h^{(q_n^{k+1})}(x)}\,d\frac{e^{2\pi i l v^{(q_n^{k+1})}(x)}}{2\pi i l Dh^{(q_n^{k+1})}(x)}\Big)\Big|.
\end{aligned}$$

Since $|Dh^{(q_n^{k+1})}(x)|\ge Mq_n$ for every $x\in J_j$, we obtain

$$\Big|\sum_{j=0}^{q_n-1}\Big(\frac{e^{2\pi i l(h+v)^{(q_n^{k+1})}(b_j)}}{2\pi i l Dh^{(q_n^{k+1})}(b_j)}-\frac{e^{2\pi i l(h+v)^{(q_n^{k+1})}(a_j)}}{2\pi i l Dh^{(q_n^{k+1})}(a_j)}\Big)\Big|\le\frac{1}{lM\pi}$$

and

$$\begin{aligned}
\Big|\int_{a_j}^{b_j} e^{2\pi i l h^{(q_n^{k+1})}(x)}\,d\frac{e^{2\pi i l v^{(q_n^{k+1})}(x)}}{Dh^{(q_n^{k+1})}(x)}\Big|
&\le \operatorname{Var}_{a_j}^{b_j}\Big(\frac{e^{2\pi i l v^{(q_n^{k+1})}}}{Dh^{(q_n^{k+1})}}\Big)\\
&\le \frac{2\pi l\operatorname{Var}_{a_j}^{b_j}(v^{(q_n^{k+1})})}{\inf_{(a_j,b_j)}|Dh^{(q_n^{k+1})}|}+\operatorname{Var}_{a_j}^{b_j}\Big(\frac{1}{Dh^{(q_n^{k+1})}}\Big)\\
&\le \frac{2\pi l}{Mq_n}\int_{a_j}^{b_j}|Dv^{(q_n^{k+1})}|\,d\lambda+\frac{\operatorname{Var}_{a_j}^{b_j}(Dh^{(q_n^{k+1})})}{M^2q_n^2}
\end{aligned}$$

for $j=0,\ldots,q_n-1$. It follows that

$$\begin{aligned}
\Big|\int_{\mathbb{T}} e^{2\pi i l(h+v)^{(q_n^{k+1})}(x)}\,dx\Big| &\le 1-C+\frac{1}{lM\pi}\\
&\quad+\frac{1}{Mq_n}\int_{\mathbb{T}}|Dv^{(q_n^{k+1})}|\,d\lambda+\frac{\operatorname{Var}(Dh^{(q_n^{k+1})})}{2\pi l M^2 q_n^2}.
\end{aligned}$$

By Corollary 2.1, we have

$$\int_{\mathbb{T}} |Dv^{(q_n^{k+1})}|\,d\lambda \le q_n^k \int_{\mathbb{T}} |Dv^{(q_n)}|\,d\lambda \le Kq_n\|v\|_{k+\mathrm{BV}}.$$

Moreover,

$$\mathrm{Var}(Dh^{(q_n^{k+1})}) \le Kq_n^2\|h\|_{k+\mathrm{BV}}.$$

Indeed, for $k=1$, we have

$$\mathrm{Var}(Dh^{(q_n^{k+1})}) \le q_n^2\,\mathrm{Var}(Dh)$$

and

$$\mathrm{Var}(Dh^{(q_n^{k+1})}) = \int_{\mathbb{T}} |D^2h^{(q_n^{k+1})}|\,d\lambda \le q_n^k \int_{\mathbb{T}} |D^2h^{(q_n)}|\,d\lambda \le Kq_n^2\,\mathrm{Var}(D^kh)$$

for $k>1$, by Corollary 2.1. It follows that

$$\limsup_{n\to\infty}\Big|\int_{\mathbb{T}} e^{2\pi il(h+v)^{(q_n^{k+1})}(x)}\,dx\Big| \le 1-C+\frac{1}{lM\pi}+\frac{K}{M}\|v\|_{k+\mathrm{BV}}+\frac{K}{lM^2}\|h\|_{k+\mathrm{BV}}.$$

Let $v\in C_0^{k+\mathrm{BV}}$. Suppose that $\|v\|_{k+\mathrm{BV}} < MC/K$. Then

$$\limsup_{n\to\infty}\Big|\int_{\mathbb{T}} e^{2\pi il(h+v)^{(q_n^{k+1})}(x)}\,dx\Big| \le 1-\frac{1}{2}\Big(C-\frac{K}{M}\|v\|_{k+\mathrm{BV}}\Big) < 1$$

for all $l$ large enough, which completes the proof. ∎

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