On ergodicity of some cylinder flows

by

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Abstract. We study ergodicity of cylinder flows of the form

\[ T_f : T \times \mathbb{R} \to T \times \mathbb{R}, \quad T_f(x, y) = (x + \alpha, y + f(x)) \]

where \( f : T \to \mathbb{R} \) is a measurable cocycle with zero integral. We show a new class of smooth ergodic cocycles. Let \( k \) be a natural number and let \( f \) be a function such that \( D^k f \) is piecewise absolutely continuous (but not continuous) with zero sum of jumps. We show that if the points of discontinuity of \( D^k f \) have some good properties, then \( T_f \) is ergodic. Moreover, there exists \( \varepsilon_f > 0 \) such that if \( v : T \to \mathbb{R} \) is a function with zero integral such that \( D^k v \) is of bounded variation with \( \text{Var}(D^k v) < \varepsilon_f \), then \( T_{f+v} \) is ergodic.

1. Introduction. Assume that \( T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu) \) is an ergodic measure-preserving automorphism of a standard Borel space. Each measurable function \( f : X \to \mathbb{R} \) is called a cocycle. For every \( n \in \mathbb{Z} \), let

\[ f^{(n)}(x) = \begin{cases} f(x) + f(Tx) + \ldots + f(T^{n-1}x) & \text{if } n > 0, \\ 0 & \text{if } n = 0, \\ -(f(T^n x) + f(T^{n+1} x) + \ldots + f(T^{-1}x)) & \text{if } n < 0. \end{cases} \]

Let \( \overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\} \) be the one-point Aleksandrov compactification of \( \mathbb{R} \). Then \( r \in \overline{\mathbb{R}} \) is said to be an extended essential value of \( f \) (see [10]) if for each open neighbourhood \( U(r) \) of \( r \) and an arbitrary set \( C \in \mathcal{B} \) with \( \mu(C) > 0 \), there exists an integer \( n \) such that

\[ \mu(C \cap T^{-n} C \cap \{ x \in X : f^{(n)} \in U(r) \}) > 0. \]

The set of extended essential values will be denoted by \( \overline{E}(f) \). The set \( E(f) = \overline{E}(f) \cap \mathbb{R} \) is called the set of essential values of \( f \). The skew product

\[ T_f : (X \times \overline{\mathbb{R}}, \overline{\mathcal{B}}, \overline{\mu}) \to (X \times \overline{\mathbb{R}}, \overline{\mathcal{B}}, \overline{\mu}), \quad T_f(x, y) = (Tx, y + f(x)), \]

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is said to be the cylinder flow. Here \( \tilde{\mu} \) denotes the product measure of \( \mu \) and infinite Lebesgue measure on the line. It is shown in [10] that \( E(f) \) is a closed subgroup of \( \mathbb{R} \) and it is the collection of periods of \( T_f \)-invariant functions, i.e.

\[
E(f) = \{ r \in \mathbb{R} : \forall \phi : X \times \mathbb{R} \to \mathbb{R}, \phi \circ T_f = \phi \cdot (x, y + r) = \phi(x, y) \ \tilde{\mu}\text{-a.e.} \}.
\]

In particular, \( T_f \) is ergodic iff \( E(f) = \mathbb{R} \).

We say that a strictly increasing sequence \( \{q_n\}_{n \in \mathbb{N}} \) is a rigid time for \( T \) if

\[
\lim_{n \to \infty} \mu(T^{q_n} \triangle A) = 0 \quad \text{for any } A \in \mathcal{B}.
\]

In [6], Lemańczyk, Parreau and Volný have proved

**Proposition 1.** Suppose that \( f : X \to \mathbb{R} \) is an integrable cocycle such that the sequence \( \{\|f^{(q_n)}\|_{L^1}\}_{n \in \mathbb{N}} \) is bounded, where \( \{q_n\}_{n \in \mathbb{N}} \) is a rigid time for \( T \). If

\[
\limsup_{n \to \infty} \left| \int_X e^{2\pi i f^{(q_n)}} \, d\mu \right| \leq c < 1
\]

for all \( l \) large enough, then \( T_f \) is ergodic.

We denote by \( \mathbb{T} \) the group \( \mathbb{R}/\mathbb{Z} \) which will be identified with the interval \([0, 1)\) with addition mod 1. Let \( \lambda \) denote the Lebesgue measure on \( \mathbb{T} \). Let \( < \subset \mathbb{T} \times \mathbb{T} \) be defined by: \( x < y \) iff \( 0 < y - x < 1/2 \), where \( < \subset \mathbb{T} \times \mathbb{T} \) is the usual order on \([0, 1)\). By \( \{t\} \) we denote the fractional part of \( t \) and \( \|t\| \) is the distance of \( t \) from the set of integers.

Assume that \( \alpha \in [0, 1) \) is an irrational with continued fraction expansion

\[
\alpha = [0; a_1, a_2, \ldots].
\]

The natural numbers \( a_n \) are said to be the partial quotients of \( \alpha \). Put

\[
\begin{align*}
r_0 &= 0, \quad r_1 = 1, \quad r_{n+1} = a_{n+1}r_n + r_{n-1}, \\
s_0 &= 1, \quad s_1 = a_1, \quad s_{n+1} = a_{n+1}s_n + s_{n-1}.
\end{align*}
\]

The rationals \( r_n/s_n \) are called the convergents, and \( s_n \) is the \( n \)th denominator of \( \alpha \). We have the inequality

\[
\frac{1}{2s_ns_{n+1}} < \left| \alpha - \frac{r_n}{s_n} \right| < \frac{1}{s_ns_{n+1}}.
\]

For every nonnegative integer \( k \), let \( S_k \) denote the subset of irrational numbers \( \alpha \) such that

\[
\liminf_{n \to \infty} s_{n+1}^k \|s_n\alpha\| < \infty
\]

and let \( S^0_k \) denote the subset of irrational numbers \( \alpha \) such that

\[
\liminf_{n \to \infty} s_{n+1}^k \|s_n\alpha\| = 0.
\]

The above sets are residual in \( \mathbb{T} \).
A function $f : \mathbb{T} \to \mathbb{R}$ is said to be piecewise absolutely continuous (PAC for short) if there are $\beta_0, \ldots, \beta_k \in \mathbb{T}$ such that $f|_{(\beta_j, \beta_{j+1})}$ is absolutely continuous ($\beta_{k+1} = \beta_0$). Set

$$f_+(x) = \lim_{y \to x^+} f(y) \quad \text{and} \quad f_-(x) = \lim_{y \to x^-} f(y).$$

Let $a_j = f_+(\beta_j) - f_-(\beta_j)$ for $j = 0, \ldots, k$ and

$$S(f) = \sum_{j=0}^{k} a_j = -\sum_{j=0}^{k} f_-(\beta_j) - f_+(\beta_j) = -\int_{\mathbb{T}} Df(x) \, d\lambda(x).$$

Assume that $\alpha \in [0, 1)$ is irrational. Denote by $Tx = x + \alpha \mod 1$ the corresponding ergodic rotation on $\mathbb{T}$. We shall study skew products of the form

$$T_f : \mathbb{T} \times \mathbb{R} \to \mathbb{T} \times \mathbb{R}, \quad T_f(x, y) = (Tx, y + f(x)),$$

where $f : \mathbb{T} \to \mathbb{R}$ is a measurable cocycle with $\int_{\mathbb{T}} f \, d\lambda = 0$.

In [8], Pask has given a class of cocycles which are PAC with $S(f) \neq 0$, and has showed ergodicity for all irrationals $\alpha$. Lemańczyk, Parreau and Volný [6] have proved that the class of cocycles considered in [8] is ergodically stable in the space $BV(\mathbb{T})_0$ of bounded variation functions with zero integral, i.e. if $f \in$ PAC with $S(f) \neq 0$ and Var$(f - g) < |S(f)|$, then $T_g$ is still ergodic. It has been proved in [9] that if $f$ is $k-1$ times differentiable a.e. and $D^{k-1}f$ is PAC with $S(D^{k-1}f) \neq 0$, then $T_f$ is ergodic for $\alpha \in S_k$.

The aim of this paper is to study the ergodicity of $T_f$ in the case where a derivative $D^k f$ of $f$ is piecewise absolutely continuous (but not continuous) and $S(D^k f) = 0$.

Let $k$ be a natural number. We denote by $C^{k+BV}_0$ the space of $k - 1$ differentiable functions $f : \mathbb{T} \to \mathbb{R}$ with zero integral such that $D^{k-1}f$ is absolutely continuous and $D^k f$ is of bounded variation. Set $C^{0+BV}_0 = BV_0$.

Observe that if $f : \mathbb{T} \to \mathbb{R}$ is a function of bounded variation with zero integral, then

$$\sup_{x \in \mathbb{T}} |f(x)| \leq \text{Var}(f). \quad (1)$$

Notice that if $f \in C^{k+BV}_0$, then Var$(D^j f) \leq \text{Var}(D^j f)$ for $j = 1, \ldots, k$. Indeed, since $D^{j-1}f$ is absolutely continuous, we have Var$(D^{j-1} f) = \int_{\mathbb{T}} |D^j f| \, d\lambda$ and $\int_{\mathbb{T}} D^j f \, d\lambda = 0$. From (1) we have

$$\text{Var}(D^{j-1} f) = \int_{\mathbb{T}} |D^j f| \, d\lambda \leq \sup_{x \in \mathbb{T}} |D^j f(x)| \leq \text{Var}(D^j f).$$

In $C^{k+BV}_0$ we define the norm $\|f\|_{k+BV} = \text{Var}(D^k f)$. With this norm, $C^{k+BV}_0$ becomes a Banach space. Let $C^{k+PAC}_0$ denote the subspace of functions $f \in C^{k+BV}_0$ such that $D^k f$ is piecewise absolutely continuous and let $C^{k+AC}_0$
denote the space of functions $f \in C^{k}_{0} + \text{PAC}$ such that $D^k f$ is absolutely continuous. Recall that the subspace of trigonometric polynomials is dense in $C^{k}_{0} + \text{AC}$ with respect to the $C^{k+BV}_{0}$ norm.

Assume that $f \in C^{k}_{0} + \text{PAC}$ and $S(D^k f) = 0$. Suppose that $\alpha \in S^0_k$ and $0 = \beta_0 < \beta_1 < \ldots < \beta_d < 1$ are all the discontinuity points of $D^k f$. In this paper we will prove the following theorem.

**Theorem 1.1 (Main Theorem).** Let $k \in \mathbb{N}$ and $f \in C^{k}_{0} + \text{PAC}$ be such that $S(D^k f) = 0$. If there exists a sequence $\{q_n\}_{n \in \mathbb{N}}$ of denominators of $\alpha$ such that
\[
\lim_{n \to \infty} q_n^{k+1} \|q_n \alpha\| = 0 \quad \text{and} \quad \lim_{n \to \infty} \{q_n \beta_i\} = \gamma_i,
\]
where $\gamma_i \neq \gamma_j$ for $i \neq j$, $i, j = 0, \ldots, d$, then $T_f$ is ergodic. Moreover, there exists $\varepsilon > 0$ such that if $v \in C^{k+BV}_{0}$ and $\|v\|_{k+BV} < \varepsilon$, then $T_f + v$ is ergodic.

2. Some generalizations of the Denjoy–Koksma inequality. In this section we prove some generalizations of the Denjoy–Koksma inequality which will be needed to prove the main theorem. Let $Q_n$ be a partition of $T$ into the intervals defined by the points $\{i \alpha\}^{s_n - 1}_{i = 0}$. Then for all $n$, each interval of $Q_n$ has length $\|s_n - 1 \alpha\|$ or $\|s_n \alpha\|$.

**Theorem 2.1.** For a given nonnegative integer $k$ there is a positive constant $M_k = M$ such that if $f \in C^{k+BV}_{0}$, then
\[
s_k^n |f(s_n)(x)| \leq M(1 + s_{n+1}^k \|s_n \alpha\|) \text{Var}(D^k f)
\]
for any natural $n$.

**Proof (by induction on $k$).** For $k = 0$ the inequality (2) is the ordinary Denjoy–Koksma inequality (see [5], p. 73).

Assuming (2) to hold for a certain $k$, we will prove that there exists $M_{k+1} > 0$ such that if $f \in C^{k+1+BV}_{0}$, then
\[
s_{n+1}^k |f(s_n)(x)| \leq M_{k+1}(1 + s_{n+2}^{k+2} \|s_n \alpha\|) \text{Var}(D^{k+1} f).
\]
Let $I$ be an interval of size $\|s_n - 1 \alpha\|$. Then
\[
\left| \int_I f(s_n)(x) \, dx \right| = \left| \int_{\bigcup_{i=0}^{s_n - 1} T^i I} f(x) \, dx \right| = \left| \int_{T \setminus \bigcup_{i=0}^{s_n - 1} T^i I} f(x) \, dx \right|.
\]
Since
\[
T \setminus \bigcup_{i=0}^{s_n - 1} T^i I = \bigcup_{j=0}^{s_n - 1 - 1} T^j J,
\]
where $J$ is an interval of size $\|s_n \alpha\|$, we have
If \( |I| = \|s_{n-1}\alpha\| + \|s_n\alpha\| \), then we split this into two chunks, one \( I_1 \) of size \( \|s_{n-1}\alpha\| \), the other \( I_2 \) of size \( \|s_n\alpha\| \). Then

\[
\left| \int_I f^{(s_n)}(x) \, dx \right| = \left| \int_{I_1} f^{(s_n)}(x) \, dx \right| + \left| \int_{I_2} f^{(s_n)}(x) \, dx \right| \leq 2\|s_n\alpha\| \text{Var}(D^{k+1}f).
\]

It follows that for each interval \( I \) of \( Q_n \) there is \( x_I \in I \) with

\[
|f^{(s_n)}(x_I)| \leq 4s_n\|s_n\alpha\| \text{Var}(D^{k+1}f).
\]

Indeed, if \( f^{(s_n)}|_I \) changes sign, then we can take \( x_I \) such that \( f^{(s_n)}(x_I) = 0 \). Assume that \( f^{(s_n)}|_I \) does not change sign. Suppose that

\[
|f^{(s_n)}(x)| \geq 4s_n\|s_n\alpha\| \text{Var}(D^{k+1}f)
\]

for any \( x \in I \). Then

\[
\left| \int_I f^{(s_n)}(x) \, dx \right| > |I|4s_n\|s_n\alpha\| \text{Var}(D^{k+1}f) > 2\|s_n\alpha\| \text{Var}(D^{k+1}f),
\]

a contradiction. Since \( f \) is absolutely continuous and the formula (2) is true for \( k \), we have

\[
|f^{(s_n)}(b) - f^{(s_n)}(a)| = \left| \int_a^b Df^{(s_n)}(x) \, dx \right|
\]

\[
\leq M_k(1 + s_n^{k+1}\|s_n\alpha\|) \text{Var}(D^{k+1}f) \frac{|b - a|}{s_n^k}
\]

for all \( a, b \in \mathbb{T} \). If \( x \in I \in Q_n \), then

\[
|f^{(s_n)}(x) - f^{(s_n)}(x_I)| \leq 2\|s_{n-1}\alpha\| \frac{M_k(1 + s_n^{k+1}\|s_n\alpha\|)}{s_n^k} \text{Var}(D^{k+1}f)
\]

\[
\leq \frac{2M_k}{s_n^{k+1}}(1 + s_n^{k+1}\|s_n\alpha\|) \text{Var}(D^{k+1}f)
\]

and finally

\[
s_n^{k+1}|f^{(s_n)}(x)| \leq s_n^{k+1}|f^{(s_n)}(x) - f^{(s_n)}(x_I)| + s_n^{k+1}|f^{(s_n)}(x_I)|
\]

\[
\leq (2M_k(1 + s_n^{k+1}\|s_n\alpha\|) + 4s_n^{k+2}\|s_n\alpha\|) \text{Var}(D^{k+1}f)
\]

\[
\leq (2M_k + 4)(1 + s_n^{k+2}\|s_n\alpha\|) \text{Var}(D^{k+1}f). \quad \blacksquare
\]

**Corollary 2.1.** Assume that \( \alpha \in S_k \) and \( \{q_n\}_{n \in \mathbb{N}} \) is a sequence of denominators of \( \alpha \) such that the sequence \( \{q_n^{k+1}\|q_n\alpha\|\}_{n \in \mathbb{N}} \) is bounded. Then there is a constant \( K \geq 1 \) such that

\[
q_n^{k}|f^{(s_n)}(x)| \leq K\|f\|_{k+BV}
\]
for any \( f \in C_0^{k+BV} \) and \( n \in \mathbb{N} \). Moreover, if \( f \in C_0^{k+AC} \), then the sequence \( \{q_n^k f(q_n)\}_{n \in \mathbb{N}} \) uniformly converges to zero.

Proof. Notice that Theorem 2.1 implies the first part of the corollary. Since for every \( f \in C_0^{k+AC} \) there exists a sequence \( \{P_m \}_{m \in \mathbb{N}} \) of trigonometric polynomials with zero integral such that

\[
\lim_{m \to \infty} \|P_m - f\|_{k+BV} = 0,
\]

it suffices to show that for every trigonometric polynomial \( f \) with zero integral the sequence \( \{q_n^k f(q_n)\}_{n \in \mathbb{N}} \) uniformly converges to zero. Let

\[
f(x) = \sum_{m=-M}^{M} a_m e^{2\pi i mx}
\]

where \( a_0 = 0 \). Then

\[
|q_n^k f(q_n)(x)| = \left| q_n^k \sum_{m=-M}^{M} a_m \frac{e^{2\pi i q_n x} - 1}{e^{2\pi i \alpha} - 1} \right|
\]

\[
\leq 2q_n^k \sum_{m=-M}^{M} |a_m| \frac{m\|q_n \alpha\|}{\|m \alpha\|} = q_n^k \|q_n \alpha\| \sum_{m=-M}^{M} 2|m|\frac{|a_m|}{\|m \alpha\|}.
\]

It follows that \( q_n^k f(q_n) \) uniformly converges to zero, which completes the proof.

3. Ergodicity of differentiable cocycles. We need auxiliary lemmas.

Lemma 3.1. Let \( 0 = \beta_0 < \beta_1 < \ldots < \beta_d < \beta_{d+1} = 1 \) and let \( a_1, \ldots, a_{d+1} \) be real numbers with zero sum. Consider a function \( h : \mathbb{T} \to \mathbb{R} \) with zero integral given by

\[
h = h(0) + \sum_{i=1}^{d+1} a_i 1_{[\beta_i, 1]}.
\]

Then \( h(0) = \sum_{i=1}^{d+1} a_i \beta_i \) and

\[
h^{(q)} = h^{(q)}(0) + \sum_{s=0}^{q-1} \sum_{i=1}^{d+1} a_i 1_{[\beta_i - s\alpha, 1]}
\]

for any natural \( q \), where \( T : \mathbb{T} \to \mathbb{T} \) is the rotation through \( \alpha \).

Proof. Since \( \int_T h \, d\lambda = 0 \) and \( a_1 + \ldots + a_{d+1} = 0 \), we have

\[
0 = h(0) + \sum_{i=1}^{d+1} a_i (1 - \beta_i) = h(0) - \sum_{i=1}^{d+1} a_i \beta_i.
\]
For all $a, b, x \in \mathbb{T}$, we have

$$1_{[b, 1)}(x + a) - 1_{[b, 1)}(a) = 1_{[b-a, 1)}(x) - 1_{[1-a, 1)}(x).$$

It follows that

$$h(x + a) - h(a) = \sum_{i=1}^{d+1} a_i(1_{[\beta_i, 1)}(x + a) - 1_{[\beta_i, 1)}(x))$$

$$= \sum_{i=1}^{d+1} a_i(1_{[\beta_i-a, 1)}(x) - 1_{[1-a, 1)}(x)) = \sum_{i=1}^{d+1} a_i(1_{[\beta_i-a, 1)}(x).$$

Therefore

$$h^{(q)} = h(0) + \sum_{s=0}^{q-1} \sum_{i=1}^{d+1} a_i(1_{[\beta_i-s, 1)}$$

for any natural $q$. ■

**Lemma 3.2.** Let $I \subset \mathbb{R}$ be an interval and $k$ be a natural number. If $P$ is a real polynomial of the form $P(x) = c_k x^k + \ldots + c_0$, $c_k \neq 0$, then there exists a closed subinterval $J \subset I$ with $|J| \geq |I|/4^k$ such that

$$x \in J \Rightarrow |P(x)| \geq k! |c_k|(|I|/4)^k.$$

**Proof.** Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function with continuous derivative. Suppose that there exists a closed interval $I \subset \mathbb{R}$ such that $|Df(x)| \geq a > 0$ for any $x \in I$. We first show that there exists an interval $J \subset I$ with $|J| \geq |I|/4$ and $|f(x)| \geq a|I|/4$ for any $x \in J$. Without loss of generality we can assume that $Df(x) \geq a > 0$ for any $x \in I$. Suppose that for every interval $J \subset I$ with $|J| \geq |I|/4$ there exists $x \in J$ such that $|f(x)| < a|I|/4$. Since $f$ increases on $I$, we can find $x, y \in I$ such that $x - y \geq |I|/2$ and $|f(x)|, |f(y)| < a|I|/4$. It follows that

$$a|I|/2 \leq a|x - y| \leq |f(x) - f(y)| < a|I|/2,$$

a contradiction. Applying the above fact to derivatives of $P$ we obtain our assertion. ■

Let $f \in C_0^{k+\text{PAC}}$ be such that $S(D^k f) = 0$. Let $\alpha \in S_0^k$ and let $0 = \beta_0 < \beta_1 < \ldots < \beta_d < 1$ be all the discontinuities of $D^k f$. Suppose that there exists a sequence $\{q_n\}_{n \in \mathbb{N}}$ of denominators of $\alpha$ such that

$$\lim_{n \to \infty} q_n^{k+1} \|q_n \alpha\| = 0 \quad \text{and} \quad \lim_{n \to \infty} \{q_n \beta_i\} = \gamma_i,$$

where $\gamma_i \neq \gamma_j$ for $i \neq j$, $i, j = 0, \ldots, d$. It is clear that the function $f$ can
be represented as $f = g + h$, where $g \in C_0^{k+AC}$, $h \in C_0^{k+FAC}$ and $D^kh$ is constant on each interval $(\beta_i, \beta_{i+1})$. Then

$$D^kh_+(\beta_i) - D^kh_-(\beta_i) = D^kf_+(\beta_i) - D^kf_-(\beta_i) = a_i \neq 0$$

for $i = 0, \ldots, d$ and

$$D^kh_+ = D^kh_+(0) + \sum_{i=1}^{d+1} a_i 1_{[\beta_i, 1)}$$

with $D^kh_+(0) = \sum_{i=1}^{d+1} a_i \beta_i$. By Lemma 3.1,

$$(4) \quad D^k h_+^{(q)} = D^k h_+^{(q)}(0) + \sum_{s=0}^{q-1} \sum_{i=1}^{d+1} a_i 1_{[\beta_i - s\alpha, 1)}$$

for any natural $q$. Let $\sigma$ be a permutation of the set $\{0, 1, \ldots, d\}$ such that

$$0 = \gamma_{\sigma(0)} < \gamma_{\sigma(1)} < \ldots < \gamma_{\sigma(d)} < \gamma_{\sigma(d+1)} = 1,$$

where $\sigma(0) = \sigma(d + 1)$. For given $1 \leq i \leq d + 1$ and $0 \leq j < q_n$, let $t^{(j)}_i$ be the unique integer satisfying $0 \leq t^{(j)}_i < q_n$ and

$$t^{(j)}_i p_n + j = [q_n \beta_i] \mod q_n,$$

where $\{p_n/q_n\}_{n \in \mathbb{N}}$ is the sequence of convergents of $\alpha$. Then

$$\beta_i - t^{(j)}_i \alpha = \frac{[q_n \beta_i]}{q_n} + \frac{\{q_n \beta_i\}}{q_n} - t^{(j)}_i \frac{p_n}{q_n} - t^{(j)}_i \frac{\delta_n}{q_n}$$

$$= \frac{j}{q_n} + \frac{1}{q_n} \left( \{q_n \beta_i\} - t^{(j)}_i \delta_n \right) \mod 1,$$

where $|\delta_n| = \|q_n \alpha\|$. It follows that

$$\beta_{\sigma(0)} - t^{(j)}_{\sigma(0)} \alpha \lesssim \beta_{\sigma(1)} - t^{(j)}_{\sigma(1)} \alpha \lesssim \ldots \lesssim \beta_{\sigma(d)} - t^{(j)}_{\sigma(d)} \alpha \lesssim \beta_{\sigma(0)} - t^{(j + 1)}_{\sigma(0)} \alpha$$

for $j = 0, \ldots, q_n - 1$. Let $0 \leq j \leq q_n - 1$ and $0 \leq i \leq d$. Set

$$t^{(j)}_i = \begin{cases} (\beta_{\sigma(i)} - t^{(j)}_{\sigma(i)} \alpha, \beta_{\sigma(i + 1)} - t^{(j)}_{\sigma(i + 1)} \alpha) & \text{if } 0 \leq i < d, \\ (\beta_{\sigma(d)} - t^{(j)}_{\sigma(d)} \alpha, \beta_{\sigma(0)} - t^{(j + 1)}_{\sigma(0)} \alpha) & \text{if } i = d. \end{cases}$$

**Lemma 3.3.** If $x \in t^{(j)}_i$, then

$$D^k h_+(q_n)(x) = \sum_{m=1}^{d} a_m \{q_n \beta_m\} + \sum_{n=0}^{i} a_m.$$
Proof. Let $x \in I_i^{(j)}$. From (4), we have

$$D^{k}h_{+}^{(q_{n})}(x) = D^{k}h_{+}^{(q_{n})}(0) + \sum_{l=0}^{q_{n}-1} \sum_{m=1}^{d+1} a_{m} 1_{[\beta_{\sigma_{m}}^{l} - \ell^{(l)}_{\sigma_{m}}\alpha_{1}]}(x)$$

$$= D^{k}h_{+}^{(q_{n})}(0) + \sum_{l=0}^{q_{n}-1} \sum_{m=1}^{d+1} a_{m} + \sum_{m=1}^{d+1} a_{m} 1_{[\beta_{\sigma_{m}}^{l} - \ell^{(l)}_{\sigma_{m}}\alpha_{1}]}(x)$$

$$= D^{k}h_{+}^{(q_{n})}(0) + \sum_{m=1}^{d} a_{m}.$$ 

Moreover

$$D^{k}h_{+}^{(q_{n})}(0) = \sum_{j=0}^{q_{n}-1} D^{k}h_{+}(j\alpha) = \sum_{j=0}^{q_{n}-1} \left( D^{k}h_{+}(0) + \sum_{i=1}^{d} a_{i} 1_{[\beta_{i}]}(j\alpha) \right)$$

$$= q_{n} D^{k}h_{+}(0) + \sum_{i=1}^{d} q_{n}^{-1} \sum_{j=0}^{q_{n}-1} 1_{[\beta_{i}]}(j\alpha).$$

On the other hand,

$$\sum_{j=0}^{q_{n}-1} 1_{[\beta_{i}]}(j\alpha)$$

$$= \text{card}\{0 \leq j < q_{n} : \{j\alpha\} > \beta_{i}\}$$

$$= \text{card}\{0 \leq j < q_{n} : j\delta_{n}/q_{n} > [q_{n}\beta_{i}]/q_{n} + \{q_{n}\beta_{i}\}/q_{n}\}$$

$$= \text{card}\{0 \leq j < q_{n} : j\delta_{n}/q_{n} > [q_{n}\beta_{i}]/q_{n}\}$$

$$= q_{n} - [q_{n}\beta_{i}] - 1.$$

Therefore

$$D^{k}h_{+}^{(q_{n})}(0) = q_{n} \sum_{i=1}^{d} a_{i} \beta_{i} + \sum_{i=1}^{d} a_{i} (q_{n} - [q_{n}\beta_{i}] - 1)$$

$$= q_{n} \sum_{i=1}^{d} a_{i} \beta_{i} + \sum_{i=1}^{d} a_{i} ([q_{n}\beta_{i}] - q_{n}\beta_{i}) + a_{0}$$

$$= \sum_{i=1}^{d} a_{i} [q_{n}\beta_{i}] + a_{0}$$

and consequently

$$D^{k}h_{+}^{(q_{n})}(x) = \sum_{m=1}^{d} a_{m} [q_{n}\beta_{m}] + \sum_{m=0}^{q_{n}^{-1}} a_{m}. \quad \blackbox$$
Let $0 \leq j \leq q_n - 1$ and $0 \leq i \leq d$. Let $\tilde{I}^{(j)}_i$ denote the interval 

$$(\beta_{\sigma(i)} - \ell_{\sigma(i)}^{(j)} \alpha + q_i^k\|q_n\alpha\|, \beta_{\sigma(i+1)} - \ell_{\sigma(i+1)}^{(j)} \alpha - q_i^k\|q_n\alpha\|)$$

if $0 < i < d$, and the interval 

$$(\beta_{\sigma(d)} - \ell_{\sigma(d)}^{(j)} \alpha + q_i^k\|q_n\alpha\|, \beta_{\sigma(0)} - \ell_{\sigma(0)}^{(j+1)} \alpha - q_i^k\|q_n\alpha\|)$$

if $i = d$. Since $q_i^k \|q_n\alpha\| \to 0$ as $n \to \infty$, we have 

$$|\tilde{I}^{(j)}_i| = \frac{1}{q_i^k} |\{q_i^k \beta_{\sigma(i+1)} - \{q_i^k \beta_{\sigma(i)}\} - \delta_n (\ell_{\sigma(i+1)}^{(j)} - \ell_{\sigma(i)}^{(j)}) - 2q_i^k \|q_n\alpha\|)|$$

for all $n$ large enough.

**Corollary 3.1.** If $x \in \tilde{I}^{(j)}_i$, then 

$$D^k h(q_i^{k+1})(x) = q_i^k \left( \sum_{m=1}^d a_m \{q_i^k \beta_m\} + \sum_{m=0}^i a_m \right).$$

**Proof.** For every $x \in T$, we have 

$$D^k h(q_i^{k+1})(x) = D^k h(q_n)(x)$$

$$+ D^k h(q_n)(x + q_n \alpha) + \ldots + D^k h(q_n)(x + (q_i^k - 1)q_n \alpha).$$

If $x \in \tilde{I}^{(j)}_i$, then $x + lq_n \alpha \in I^{(j)}_i$ for $l = 0, 1, \ldots, q_i^k - 1$. It follows that 

$$D^k h(q_i^{k+1})(x) = q_i^k \left( \sum_{m=1}^d a_m \{q_i^k \beta_m\} + \sum_{m=0}^i a_m \right).$$

**Corollary 3.2.** There exists a collection $\{J_j\}_{j=0}^{q_n-1}$ of pairwise disjoint closed intervals and there exist constants $0 < C < 1$, $M > 0$ such that 

$$|J_j| \geq \frac{C}{q_n}$$

and $x \in J_j \Rightarrow |D h(q_i^{k+1})(x)| \geq M q_n$ for $j = 0, \ldots, q_n - 1$.

**Proof.** Fix 

$$c_i = \sum_{m=1}^d a_m \gamma_m + \sum_{m=0}^i a_m.$$

At least one of the numbers $c_i$ is not zero. Indeed, if we suppose that $c_i = 0$ for $i = 0, \ldots, d$, then $a_i = c_i - c_{i-1} = 0$ for $i = 0, \ldots, d$, which is impossible. Take $i_0$ such that $c_{i_0} \neq 0$. Set 

$$b^{(i_0)} = \sum_{m=1}^d a_m \{q_i \beta_m\} + \sum_{m=0}^{i_0} a_m.$$
Since $D^k h(q_n^{k+1}) = q_n^k b^{(n)}$ on $\hat{T}_{i_o}^{(j)}$, we have

$$D h(q_n^{k+1}) (x) = q_n^k b^{(n)} x^{k-1} + P_j (x)$$
on $\hat{T}_{i_o}^{(j)}$, where $P_j$ is a polynomial with $\deg(P_j) < k - 1$ ($j = 0, \ldots, q_n - 1$).

By Lemma 3.2, there exist closed subintervals $J_j \subset \hat{T}_{i_o}^{(j)}$ such that

$$|J_j| \geq \frac{1}{4^{k-1}} |\hat{T}_{i_o}^{(j)}| \geq \frac{\gamma_{\sigma(i_o+1)} - \gamma_{\sigma(i_o)}}{4^k q_n}$$
and if $x \in J_j$, then

$$|D h(q_n^{k+1}) (x)| \geq q_n^k |b^{(n)}| \left( \frac{|\hat{T}_{i_o}^{(j)}|}{4} \right)^{k-1} \geq \frac{1}{2} q_n^k \left( \frac{\gamma_{\sigma(i_o+1)} - \gamma_{\sigma(i_o)}}{4^k q_n} \right)^{k-1}$$
for $j = 0, \ldots, q_n - 1$. It follows that we can set

$$C = \frac{\gamma_{\sigma(i_o+1)} - \gamma_{\sigma(i_o)}}{4^k} \quad \text{and} \quad M = \frac{|c_{i_o}| (\gamma_{\sigma(i_o+1)} - \gamma_{\sigma(i_o)})^{k-1}}{4^k}.$$  

Proof of Theorem 1.1. Notice that $\{q_n^{k+1}\}_{n \in \mathbb{N}}$ is a rigid time for the rotation $T x = x + \alpha$. By Corollary 2.1, the sequence $\{\|f + v\| (q_n^{k+1})\}_{n \in \mathbb{N}}$ is bounded, because $\|g (q_n^{k+1})\|_{\infty} \leq q_n^k \|g (q_n)\|_{\infty}$ and $f + v \in C_0^{k+BV}$.

By Proposition 1, it suffices to find $\varepsilon > 0$ such that $\text{Var}(D^k v) < \varepsilon$ implies

$$\lim_{n \to \infty} \left| \int_T e^{2\pi i l (f + v) (q_n^{k+1}) (x)} dx \right| \leq c < 1$$
for all $l$ large enough.

Represent $f$ as the sum of functions $g \in C_0^{k+AC}$ and $h \in C_0^{k+PAC}$, where $D^k h$ is constant on intervals $[\beta_j, \beta_{j+1})$. Since $\|g (q_n^{k+1})\|_{\infty} \leq q_n^k \|g (q_n)\|_{\infty}$, the sequence $\{g (q_n^{k+1})\}_{n \in \mathbb{N}}$ uniformly converges to zero, by Corollary 2.1. Therefore

$$\lim_{n \to \infty} \left| \int_T e^{2\pi i l (f + v) (q_n^{k+1}) (x)} dx - \int_{\frac{\beta_j}{q_n}} e^{2\pi i l (h + v) (q_n^{k+1}) (x)} dx \right| = 0.$$ 

It follows that it suffices to compute

$$\lim_{n \to \infty} \left| \int_T e^{2\pi i l (h + v) (q_n^{k+1}) (x)} dx \right|.$$ 

By Corollary 3.2, there exists a collection $\{J_j : j = 0, \ldots, q_n - 1\}$ of pairwise disjoint closed intervals and there exist $0 < C < 1$, $M > 0$ such that

$$|J_j| \geq \frac{C}{q_n} \quad \text{and} \quad x \in J_j \Rightarrow |D h(q_n^{k+1}) (x)| \geq M q_n$$
for \( j = 0, \ldots, q_n - 1 \). Let \( J_j = [a_j, b_j] \) for \( j = 0, \ldots, q_n - 1 \). Applying integration by parts we get

\[
\left| \int_\mathbb{T} e^{2\pi il(h+v)(q_n^{k+1})(x)} \, dx \right| \\
\leq 1 - \sum_{j=0}^{q_n-1} |J_j| + \left| \sum_{j=0}^{q_n-1} b_j \int_\mathbb{T} e^{2\pi il(h+v)(q_n^{k+1})(x)} \, dx \right| \\
\leq 1 - C + \left| \sum_{j=0}^{q_n-1} b_j \int_\mathbb{T} e^{2\pi il(h+v)(q_n^{k+1})(x)} \, dx \right| \\
= 1 - C + \left| \sum_{j=0}^{q_n-1} b_j \int_\mathbb{T} e^{2\pi il(h+v)(q_n^{k+1})(x)} \, dx \right| \\
\leq 1 - C \\
+ \left| \sum_{j=0}^{q_n-1} \left( \frac{e^{2\pi il(h+v)(q_n^{k+1})(b_j)}}{2\pi ilDh(q_n^{k+1})(b_j)} - \frac{e^{2\pi il(h+v)(q_n^{k+1})(a_j)}}{2\pi ilDh(q_n^{k+1})(a_j)} \right) \\
- \frac{b_j}{a_j} \int_\mathbb{T} e^{2\pi il(h+v)(q_n^{k+1})(x)} \, dx \right|.
\]

Since \( |Dh(q_n^{k+1})(x)| \geq Mq_n \) for every \( x \in J_j \), we obtain

\[
\left| \sum_{j=0}^{q_n-1} \left( \frac{e^{2\pi il(h+v)(q_n^{k+1})(b_j)}}{2\pi ilDh(q_n^{k+1})(b_j)} - \frac{e^{2\pi il(h+v)(q_n^{k+1})(a_j)}}{2\pi ilDh(q_n^{k+1})(a_j)} \right) \right| \leq \frac{1}{lM\pi}
\]

and

\[
\left| \frac{b_j}{a_j} \int_\mathbb{T} e^{2\pi il(h+v)(q_n^{k+1})(x)} \, dx \right| \leq \text{Var}_{a_j}^b \left( \frac{e^{2\pi il(h+v)(q_n^{k+1})}}{Dh(q_n^{k+1})} \right) \\
\leq \frac{2\pi l \text{Var}_{a_j}^b \left( v(q_n^{k+1}) \right)}{\inf_{(a_j, b_j)} |Dh(q_n^{k+1})|} + \text{Var}_{a_j}^b \left( \frac{1}{Dh(q_n^{k+1})} \right) \\
\leq \frac{2\pi l}{Mq_n} \int_\mathbb{T} |Dv(q_n^{k+1})| \, d\lambda + \frac{\text{Var}_{a_j}^b (Dh(q_n^{k+1}))}{M^2q_n^2}
\]

for \( j = 0, \ldots, q_n - 1 \). It follows that

\[
\left| \int_\mathbb{T} e^{2\pi il(h+v)(q_n^{k+1})(x)} \, dx \right| \leq 1 - C + \frac{1}{lM\pi} \\
+ \frac{1}{Mq_n} \int_\mathbb{T} |Dv(q_n^{k+1})| \, d\lambda + \frac{\text{Var}(Dh(q_n^{k+1}))}{2\pi lM^2q_n^2}.
\]
By Corollary 2.1, we have
\[ \int_{\mathbb{T}} |Dv(q_n^{k+1})| \, d\lambda \leq q_n^k \int_{\mathbb{T}} |Dv(q_n)| \, d\lambda \leq Kq_n \|v\|_{k+BV}. \]

Moreover,
\[ \text{Var}(Dh(q_n^{k+1})) \leq Kq_n^2 \|h\|_{k+BV}. \]

Indeed, for \( k = 1 \), we have
\[ \text{Var}(Dh(q_n^{k+1})) \leq q_n \text{Var}(Dh) \]
and
\[ \text{Var}(Dh(q_n^{k+1})) = \int_{\mathbb{T}} |D^2 h(q_n^{k+1})| \, d\lambda \leq q_n^2 \int_{\mathbb{T}} |D^2 h(q_n)| \, d\lambda \leq Kq_n^2 \text{Var}(D^k h) \]
for \( k > 1 \), by Corollary 2.1. It follows that
\[ \limsup_{n \to \infty} \left| \int_{\mathbb{T}} e^{2\pi i l (h+v)(q_n^{k+1})} \, dx \right| \leq 1 - C + \frac{1}{LM} + \frac{K}{M} \|v\|_{k+BV} + \frac{K}{LM^2} \|h\|_{k+BV}. \]

Let \( v \in C_0^{k+BV} \). Suppose that \( \|v\|_{k+BV} < MC/K \). Then
\[ \limsup_{n \to \infty} \left| \int_{\mathbb{T}} e^{2\pi i l (h+v)(q_n^{k+1})} \, dx \right| \leq 1 - \frac{1}{2} \left( C - \frac{K}{M} \|v\|_{k+BV} \right) < 1 \]
for all \( l \) large enough, which completes the proof. ■

References


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