FUNDAMENTA
MATHEMATICAE
163 (2000)

Dimensionsgrad for locally connected Polish spaces

by

Vitaly V. Fedorchuk (Moscow) and **Jan van Mill** (Amsterdam)

Abstract. It is shown that for every $n \geq 2$ there exists an n-dimensional locally connected Polish space with Dimensionsgrad 1.

1. Introduction. *All spaces under discussion are separable and metrizable.* Let A and B be disjoint subsets of a space X. Recall that a closed set $S \subseteq X$ is said to be a *partition* in X between A and B if there are open sets U and V in X such that $A \subseteq U$, $B \subseteq V$, $X \setminus S = U \cup V$, and $U \cap V = \emptyset$. A different but related concept is that of a cut. Let A and B be disjoint subsets of a space X. A closed set $K \subseteq X$ is called a *cut* in X between A and B if $K \cap (A \cup B) = \emptyset$ and $K \cap Y \neq \emptyset$ for every continuum (= compact connected space) $Y \subseteq X$ such that $A \cap Y \neq \emptyset \neq B \cap Y$. In particular, if there is no continuum $Y \subseteq X$ which meets both A and B then $\emptyset$ is a cut in X between A and B. So a partition is a cut; the converse need not be true.

In 1913, Brouwer [2] presented the first definition of a dimensional invariant intended for Polish spaces without isolated points (in the terminology of his days: normal sets in the sense of Fréchet). He called it "Dimensionsgrad". A Polish space X has Dimensionsgrad 0 if it does not contain any continuum of size larger than one, and has Dimensionsgrad less than or equal to $n \geq 1$ if for every pair A, B of disjoint closed subsets there exists a closed set $T \subseteq X$ which is a cut between A and B and has Dimensionsgrad less than or equal to $n - 1$. If there is no positive n such that X has Dimensionsgrad less than or equal to n then we say that it has Dimensionsgrad ∞.

It will be convenient to let $\operatorname{Dg} X$ denote the Dimensionsgrad of X.

2000 *Mathematics Subject Classification*: Primary 54F45, 55M10.
Key words and phrases: Dimensionsgrad, dimension, locally connected space.
The first-named author was partially supported by RFBI (grant N 97-01-357), and INTAS (grant N 94-3763). He also thanks the Division of Mathematics and Computer Science of Vrije Universiteit for generous hospitality and support.

As was pointed out by Urysohn to Brouwer, the notions of a partition and a cut do not always agree, even in the case of compact spaces. For a description of the troubles this caused in the early days of dimension theory, see e.g. Johnson [10]. But it is easy to see that for *locally connected* Polish spaces the notions are the same. This follows easily from the Mazurkiewicz Theorem in [12].

If X is a *compact* space of Dimensionsgrad 0 then it contains no non-trivial continuum, and hence is zero-dimensional in the (now) usual sense. If moreover X is locally connected and $\operatorname{Dg} X \le 1$ then by the above remarks any two disjoint closed sets can be partitioned by a zero-dimensional set, i.e. the large inductive dimension of X is at most 1. The natural approach is to try to continue this reasoning by induction. But we run into troubles if we consider a locally connected compact space X with $\operatorname{Dg} X \le 2$. We can only conclude that any two disjoint closed subsets of X can be partitioned by a closed set of Dimensionsgrad at most 1. But such a partition is not necessarily locally connected, and so this is a dead end.

In the book by Hurewicz and Wallman [9], one can read on page 4 that Dimensionsgrad and dimension agree in the realm of locally connected spaces. No proof of their statement is given. Similar statements were repeated in several variations in several books, all without proofs: Aleksandrov and Pasynkov [1, page 163] (for locally connected Polish spaces), Engelking [3, page 392] (for locally connected Polish spaces), Fedorchuk [5, page 106] (for locally connected Polish spaces), van Mill [13, page 189] (for locally connected compact spaces) and Engelking [4, page 6] (for locally connected Polish spaces).

Recently, in Fedorchuk, Levin and Shchepin [7] it was shown that for an arbitrary compact space X we have $\operatorname{Dg} X = \dim X$. So for compact spaces there are no troubles and local connectivity plays no role. But for complete spaces the above claims are false. The aim of this note is among other things to show that for locally connected Polish spaces the gap between Dimensionsgrad and dimension can be arbitrarily large.

We are indebted to the referee for some helpful comments.

2. The examples. Now we present our main result. Recall that a space X is called *hereditarily disconnected* if X does not contain any connected subspace of size larger than 1. A space X is said to be *totally disconnected* if for any two distinct points $x, y \in X$ there is a clopen (both open and closed) set $E \subseteq X$ such that $x \in E \subseteq X \setminus \{y\}$. It is clear that a totally disconnected space is hereditarily disconnected (the converse need not be true).

Our construction consists of two steps:

A. The Hilbert cube Q can be embedded in a continuuum Z such that $\dim(Z \setminus Q) = 1$ and for any $G \subseteq Q$ the set $X(G) = G \cup (Z \setminus Q)$ is locally connected.
B. If $G \subseteq Q$ is hereditarily disconnected then in $X(G)$ each pair of disjoint closed sets can be separated by a partition which is hereditarily disconnected (and thus has Dimensiongrad 0).

For the proof of Statement A we use a continuum due to Kuratowski [11, §50, IV]. Let C be the continuum in the plane consisting of the segment $(0 \leq x \leq 1, y = 0)$, which we denote by C_0, of the vertical segments $(x = m/2^{n+1}, 0 \leq y \leq 1/2^n)$ with $0 \leq m \leq 2^{n+1}$ and of the level segments $(0 \leq x \leq 1, y = 1/2^n)$, where $n = 0, 1, \ldots$ Kuratowski used C as an example of a locally connected continuum which is not hereditarily locally connected while it is the union of two hereditarily locally connected subcontinua.

Let C_1 denote $C \setminus C_0$.

It is clear that every point of C_0 has arbitrarily small neighborhoods U in C such that $U \cap C_1$ is connected. Let $\mathcal{U}$ be the collection of all those neighborhoods of points of C_0.

Lemma 2.1. *If* $G \subseteq C_0$ *then* $G \cup C_1$ *is locally connected.*

Proof. In fact, a point $c \in G \subseteq G \cup C_1$ has arbitrarily small neighborhoods of the form

$$(U \cap G) \cup (U \cap C_1),$$

where $U \in \mathcal{U}$. This set is connected, since it contains the dense connected subset $U \cap C_1$. ∎

Let $f : C_0 \to Q$ be a continuous surjection from C onto the Hilbert cube Q. The map f defines a decomposition of C whose members are the singletons from C_1 and the sets $f^{-1}(q)$, $q \in Q$. It is well known and easy to show that this decomposition is upper semicontinuous and that its quotient space Z is compact and metrizable. Let $g : C \to Z$ be the quotient map. So Z is the disjoint union of C_1 and the Hilbert cube Q.

For $G \subseteq Q$ let $X(G) = G \cup C_1$.

Lemma 2.2. $X(G)$ *is locally connected for every* $G \subseteq Q$.

Proof. By Lemma 2.1, $g^{-1}[X(G)]$ is locally connected. Since the map g is closed due to the compactness of C this implies that $X(G)$ is locally connected as well ([8, Lemma 3-21]). ∎

Since the dimension of $Z \setminus Q$ is clearly 1, this completes the proof of Statement A. We proceed to proving Statement B.

Lemma 2.3. *If* $G \subseteq Q$ *is hereditarily disconnected then in* $X(G)$ *each pair of disjoint closed sets can be separated by a partition which is hereditarily disconnected* (*and thus* $X(G)$ *has Dimensiongrad at most* 1).

P r o o f. Let A and B be disjoint closed subsets of $X(G)$. By Kuratowski [11, §27.II] there is a partition S between A and B in $X(G)$ such that $\dim(S \cap C_1) \leq 0$. We claim that S is hereditarily disconnected. To show this, let K be a connected subset of S. If $K \cap C_1 \neq \emptyset$ then K is a singleton since $S \cap C_1$ is zero-dimensional. So we may assume without loss of generality that $K \subseteq G$. But then K is a singleton since G is hereditarily disconnected. ■

Now that we proved Statements A and B, it remains to pick a suitable G to complete the proof of our main result.

THEOREM 2.4. *For each* $n = 2, 3, \ldots, \infty$ *there exists a locally connected Polish space* X_n *such that* $\operatorname{Dg} X_n = 1 < \dim X_n = n$.

P r o o f. For any $n \in \mathbb{N}$ there exists a totally disconnected G_δ-set Y_n in Q (cf. [4, 6.2.A]) such that $\dim Y_n = n$. (This interesting result is due to Mazurkiewicz and has proved useful in the construction of various counterexamples in dimension theory.) For every $n \geq 2$ let $X_n = X(Y_n)$. Then X_n is locally connected by Lemma 2.2. In addition, since totally disconnected spaces are hereditarily disconnected, $\operatorname{Dg} X_n \leq 1$ by Lemma 2.3. That $\operatorname{Dg} X_n \geq 1$ is clear since X_n contains an interval.

CLAIM 1. X_n *is a Polish space.*

P r o o f. This is clear since the complement of X_n in the compact space Z is an F_σ-subset of Z. ■

CLAIM 2. $\dim X_n = n$.

P r o o f. Observe that Y_n is a closed n-dimensional subset of X_n. So $\dim X_n \geq n$. On the other hand, $X_n \setminus Y_n$ is a countable union of 1-dimensional compacta. Hence $\dim X_n \leq n$ by the Countable Closed Sum Theorem [4, 1.5.3]. ■

This concludes the proof of the theorem. ■

Let us remark that there also is an infinite-dimensional locally connected Polish space with Dimensionsgrad 1. It suffices to take the topological sum of the spaces X_n in Theorem 2.4. This space is countable-dimensional. There is also a strongly infinite-dimensional example since there is a strongly infinite-dimensional totally disconnected G_δ-subset in Q (cf. [4, 6.2.4]). We do not know whether there is a weakly infinite-dimensional such example which is not countable-dimensional. It seems to us that Pol's method from [14] cannot be applied for answering this question.

3. Remarks. If in the definition of Dimensionsgrad we only consider pairs A and B such that A consists of a single point then we get the definition of $\operatorname{dg} X$, the (small) dimensionsgrad. In Fedorchuk [6] it was shown that

$\operatorname{dg} X = 1$ for an arbitrary hereditarily indecomposable continuum X. This result increased the interest in Brouwer's dimension function, and it was shown subsequently in Fedorchuk, Levin and Shchepin [7] that the following statements are true for an arbitrary (metrizable) compactum X:

(1) $\operatorname{Dg} X = \dim X$.
(2) $\operatorname{dg} X \leq 2$.

REMARK 3.1. The proof of (1) consists of a few lines only. But it is based on three deep results in dimension theory which were not known in 1913: (a) the Countable Closed Sum Theorem, (b) the Addition Theorem, and (c) the G_δ-Enlargement Theorem. It should also be mentioned that the equality (1) holds for an arbitrary σ-compact space (the proof is the same as the one in [7]).

REMARK 3.2. As for the inequality (2), we remark that it holds for arbitrary spaces. The argument is the same as the one in [7]. The essential tool for proving this is Proposition 1.1 from [6]: if K is a hereditarily indecomposable continuum then for any continuum (or singleton) $A \subseteq K$ and closed set $B \subseteq K$ missing A there is a cut in K between A and B consisting of one point only.

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Chair of General Topology and Geometry
Mech. Math. Faculty
Moscow State University
119-899 Moscow, Russia
E-mail: vitaly@gtopol.math.msu.su

Faculteit der Exacte Wetenschappen
Divisie Wiskunde en Informatica
Vrije Universiteit
De Boelelaan 1081a
1081 HV Amsterdam, the Netherlands
E-mail: vanmill@cs.vu.nl

Received 12 May 1999;
in revised form 5 October 1999