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A generalization of Zeeman's family

by

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Abstract. E. C. Zeeman [2] described the behaviour of the iterates of the difference equation $x_{n+1} = R(x_n, x_{n-1}, \ldots, x_{n-k})/Q(x_n, x_{n-1}, \ldots, x_{n-k})$, $n \geq k$, R, Q polynomials in the case $k = 1, Q = x_{n-1}$ and $R = x_n + \alpha$, x_1, x_2 positive, α nonnegative. We generalize his results as well as those of Beukers and Cushman on the existence of an invariant measure in the case when R, Q are affine and $k = 1$. We prove that the totally invariant set remains residual when the coefficients vary.

1. Introduction. Recently E. C. Zeeman [2] described the behaviour of accumulation points of sequences $S = (x_1, x_2, \ldots)$ of positive numbers generated by the difference equation

$$x_{n+1} = \frac{\gamma + x_n}{x_{n-1}}, \quad n \geq 2, \tag{1}$$

where the parameter γ is nonnegative and the initial terms x_1, x_2 are positive. These sequences may be treated as projections of the trajectories of the 2-dimensional system

$$\Phi(u, v) = \left(v, \frac{\gamma + v}{u}\right). \tag{2}$$

The map $\Phi : \mathbb{R}^2_+ \to \mathbb{R}^2_+$ is called the *unfolding* of the difference equation (1). The sequence S is the projection onto the u-axis of the orbit $\mathcal{O} = ((x_1, x_2), (x_2, x_3), (x_3, x_4), \ldots) = (X, \Phi(X), \Phi^2(X), \ldots)$, $X = (x_1, x_2)$.

In this paper we consider the following generalization of (1):

$$x_{n+1} = \frac{\gamma + Bx_n + Ax_{n-1}}{E + Dx_n + Cx_{n-1}}, \quad n \geq 2, \tag{3}$$

or equivalently the generalization of (2):

$$\Phi(u, v) = \left(v, \frac{\gamma + Bv + Au}{E + Dv + Cu}\right) \tag{4}$$

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which is defined on $\mathbb{R}^2$ less the critical line $Cu + Dv + E = 0$. Here all the coefficients γ, A, B, C, D, E and also the initial terms are real numbers. We want to emphasize similarities between the behaviour of sequences of the form (1) and sequences derived from generalized Zeeman's equation (3). Using the 2-dimensional unfolding (2) of equation (1) we show that for every parameter γ the complement of the set of preimages of the critical line is residual in $\mathbb{R}^2$.

In this paper we only deal with maps in the family (4) satisfying $D = 0$. As we want to investigate a family including Zeeman's case we assume $BC \neq 0$. Examples given in the last section show that for $BC = 0$ and $D \neq 0$ there exist maps in the family (4) whose asymptotic behaviour is not similar to Zeeman's maps. Under all these assumptions, if we divide by B and then use the chart $x = Cu$, $y = Cv$ then the map Φ takes the form

$$F(x,y) = \left(y, \frac{\alpha + y + ax}{x + e}\right). \tag{5}$$

For $a = e = 0$ and $\alpha \geq 0$ we obtain the unfolding considered by Zeeman [2].

Remark 1. Recently [3] Zeeman considered another generalization of (1), namely

$$x_{n+1} = \frac{\gamma + x_n + x_{n-1} + \ldots + x_{n-k+2}}{x_{n-k+1}}, \qquad n \geq k.$$

He described the behaviour of the sequences of this form for $k = 3$, where, as in (1), the parameter γ is nonnegative and the initial terms are positive.

2. Definitions and notation. We shall use the following notation:

- $d(X,Y)$ is the standard euclidean metric on $\mathbb{R}^n$,
- $B(X,r) = \{Y \in \mathbb{R}^2 : d(X,Y) < r\}$,
- $\operatorname{dist}(\Omega_1, \Omega_2) = \inf\{d(X,Y) : X \in \Omega_1, Y \in \Omega_2\}$,
- $\mathbb{R}^2_a = \{(x,y) : x > a,\ y > a\}$.

The map F defined in (5) has two fixed points W_1, W_2 such that $W_i = (\omega_i, \omega_i)$ with

$$\omega_1 = \frac{a + 1 - e - \sqrt{(a+1-e)^2 + 4\alpha}}{2},$$

$$\omega_2 = \frac{a + 1 - e + \sqrt{(a+1-e)^2 + 4\alpha}}{2}.$$

Because F is differentiable in $\mathbb{R}^2 \setminus \{(x,y) : x = -e\}$ we have the formulas

$$DF(W_i) = \begin{pmatrix} 0 & 1 \\ \frac{(ae-\alpha)-\omega_i}{(\omega_i+e)^2} & \frac{1}{\omega_i+e} \end{pmatrix},$$

$$\Delta_i = (\operatorname{tr} DF(W_i))^2 - 4 \det DF(W_i) = \frac{1}{(\omega_i + e)^2}(1 + 4(ae - \alpha) - 4\omega_i).$$

Definition 1. A periodic point X of a map $H : \mathbb{R}^n \to \mathbb{R}^n$ is called *neutral* iff $\det DH(X) \neq 0$ and if all the eigenvalues λ_i of $DH(X)$ lie on the unit circle.

Definition 2. A fixed point Y of a map $H : \mathbb{R}^n \to \mathbb{R}^n$ is called a centre iff there exists $\varepsilon > 0$ and an open set Ω containing Y such that for any $0 < \varepsilon_1, \varepsilon_2 < \varepsilon$ there exist open sets $\Omega_2 \subset \Omega_1 \subset \Omega$ such that for any $X \in \operatorname{int}(\Omega_1 - \Omega_2)$ we have

$$0 < \varepsilon_2 < d(H^k(X), Y) < \varepsilon_1 < \infty$$

for every integer k.

Definition 3. A set Λ is called *forward invariant* for a map F iff $F(\Lambda) \subset \Lambda$; *backward invariant* iff $F^{-1}(\Lambda) \subset \Lambda$; and *totally invariant* iff $\Lambda = F(\Lambda) = F^{-1}(\Lambda)$.

Property 1. It is easy to see that the fixed point W_i of F is neutral iff $\Delta_i \leq 0$ and $(ae - \alpha) - \omega_i = -(\omega_i + e)^2$.

Property 2. For $e = -a$ and $\alpha \geq -a(a+1)$ the unfolding (5) is a diffeomorphism of $\mathbb{R}^2_a$.

3. Statement of results. We begin with a general result on fixed points of maps of the form (5). This theorem yields the existence of nonhyperbolic maps and also maps with fixed points of centre type within this family.

Theorem 1. *The real fixed points W_i of the map F given by* (5) *have the following properties*:

• W_1 *is neutral iff $a = -e$ and $\alpha \geq -\frac{1}{4}(2a+1)^2 + 1$. Moreover, it is a centre iff it is neutral, the above inequality is strict, and $4\alpha \neq -(2a+1)^2 + 9$.*

• W_2 *is neutral iff $a = -e$ and $\alpha \geq -\frac{1}{4}(2a+1)^2$. Moreover, it is a centre iff it is neutral and the above inequality is strict.*

The following theorem is an extension of results showed for Zeeman's maps (2) in [1] by F. Beukers and R. Cushman and also in [2] by E. C. Zeeman.

Theorem 2. *For $e = -a$ and $\alpha \geq -a(a+1)$ the function $V : \mathbb{R}^2_a \to \mathbb{R}$ given by*

$$V(x,y) = \frac{(x-a+1)(y-a+1)(x+y+a^2-a+\alpha)}{(x-a)(y-a)}$$

is invariant under the map $F : \mathbb{R}^2_a \to \mathbb{R}^2_a$ of the form (5), *that is, $V \circ F = V$. Moreover F preserves the 2-form*

$$\sigma = \frac{dx \wedge dy}{(x-a)(y-a)}.$$

Theorem 3 states that under our assumptions there exists a totally invariant set on which forward and backward iterates of F given by (5) are well defined. This theorem is interesting because F is not a homeomorphism on the whole $\mathbb{R}^2$ and also the totally invariant set depends on the coefficients of F.

THEOREM 3. *For $e = -a$ the map F has a totally invariant set Λ which is residual.*

4. Proofs

4.1. *Proof of Theorem 1*

LEMMA 1. *The fixed point $W_i = (\omega_i, \omega_i)$ of F is neutral iff the following conditions are satisfied*:

$$(a+1-e)^2 + 4\alpha \geq 0, \tag{6}$$

$$1 + 4(ae - \alpha) - 4\omega_i \leq 0, \tag{7}$$

$$(ae - \alpha) - \omega_i = -(\omega_i + e)^2. \tag{8}$$

Proof. Condition (6) ensures the existence of real fixed points of F. Furthermore by Property 1 conditions (7) and (8) are equivalent to the neutrality of W_i. ∎

Condition (8) can be reformulated as $\omega_i = t_j$, $i, j = 1, 2$, where

$$t_1 = \frac{1-2e-\sqrt{1-4(e(a+1)-\alpha)}}{2}, \qquad t_2 = \frac{1-2e+\sqrt{1-4(e(a+1)-\alpha)}}{2}$$

are solutions of the equation $(ae - \alpha) - t = -(t+e)^2$. Because we are interested only in real fixed points of F, we want the equation (8) to have real solutions. Hence we assume

$$\alpha \geq e(a+1) - 1/4. \tag{9}$$

Suppose $t_1 = \omega_1$ or $t_1 = \omega_2$. After elementary calculations we obtain

$$2(a+e)\sqrt{1-4(e(a+1)-\alpha)} = 2(a+e),$$

so $a = -e$ or $\alpha = e(a+1)$. On the other hand if $t_2 = \omega_1$ or $t_2 = \omega_2$ then

$$-2(a+e)\sqrt{1-4(e(a+1)-\alpha)} = 2(a+e)$$

and so $a = -e$.

For $\alpha = e(a+1)$ condition (6) takes the form

$$(a+1-e)^2 + 4\alpha = (a+1+e)^2 \geq 0$$

and is trivial, and (7) does not hold because $\alpha = e(a+1)$ implies $t_1 = -e$.

If $a = -e$, then $(a+1-e)^2 + 4\alpha = -4e(a+1) + 1 + 4\alpha$ and (6) holds as a consequence of (9). Because in this case $1 + 4(ae - \alpha) = 1 - 4(a^2 + \alpha)$, at W_1 for $\alpha \ge -a(a+1) + 3/4$ we have

$$(2a+1)^2 + 4\alpha \ge 2((2a+1)^2 + 4\alpha)^{1/2},$$
$$-4a(a+1) - 4\alpha - 1 \le -2((2a+1)^2 + 4\alpha)^{1/2},$$
$$1 + 4(ae - \alpha) \le 2((a+1-e) - ((a+1-e)^2 + 4\alpha)^{1/2}) = 4\omega_1.$$

Hence (7) holds. We have thus proved the neutrality of the real fixed point W_1. The proof for W_2 is analogous.

We now proceed to determine when W_1 is a centre. Differentiating V gives

$$\frac{\partial V}{\partial x} = \frac{(y-a+1)(x^2 - 2ax - y - \alpha)}{(x-a)^2(y-a)},$$
$$\frac{\partial V}{\partial y} = \frac{(x-a+1)(y^2 - 2ay - x - \alpha)}{(x-a)(y-a)^2},$$
$$\frac{\partial^2 V}{\partial x^2} = \frac{2(y-a+1)(y + a^2 + \alpha)}{(x-a)^3(y-a)},$$
$$\frac{\partial^2 V}{\partial y^2} = \frac{2(x-a+1)(x + a^2 + \alpha)}{(x-a)(y-a)^3},$$
$$\frac{\partial^2 V}{\partial x \partial y} = \frac{\alpha - (x-a)^2 - (y-a)^2 + a^2 + a}{(x-a)^2(y-a)^2}.$$

Because $\omega_i + a^2 + \alpha = (\omega_i - a)^2$ at W_i, we have

$$\frac{\partial V}{\partial x} = \frac{\partial V}{\partial y} = 0, \qquad \frac{\partial^2 V}{\partial x^2} = \frac{2(\omega_i - a + 1)}{(\omega_i - a)^2}.$$

For $i = 1$ using the condition $4\alpha \ne -(2a+1)^2 + 9$ we have $\partial^2 V/\partial x^2 \ne 0$. Since at W_1,

$$\det \operatorname{Hess} V = \frac{\partial^2 V}{\partial x^2}\frac{\partial^2 V}{\partial y^2} - \left(\frac{\partial^2 V}{\partial x \partial y}\right)^2 > 0,$$

and the principal minor $\partial^2 V/\partial x^2$ of the second derivative of V at W_1 is not 0, it follows that the second derivative of V at W_1 is definite. In other words, W_1 is a nondegenerate extreme point of V. Thus W_1 is a fixed point of centre type. The proof for W_2 is analogous. ∎

Remark 2. A neutral fixed point may not be a centre; e.g. for $H(x,y) = (y, (y+2)/x)$, the point $W_1 = (-1,-1)$ is fixed and neutral, but for $S = (-1, t)$, where $-1 < t < 0$, we obtain $\lim_{n\to\infty} H^{3n}(S) = (-1,-1)$.

Remark 3. It is easy to check that for the map F given by (5) the fixed point W_2 lies in $\mathbb{R}^2_a$.

Remark 4. For $a = 0$ and $\alpha \geq 0$ the fixed point W_2 lies in $\mathbb{R}^2_+$ and it is a centre. This is Zeeman's case [2].

4.2. *Proof of Theorem 2.* Although, as we show below, the function V is a formal integral for the map F, that is, V is constant along the orbits of F, some problems appear because F is not a homeomorphism and its image contains points of the critical line $x + e = 0$. In order to define all forward iterates of F we shall determine a forward invariant set of F, i.e., a set Λ with $F(\Lambda) \subset \Lambda$.

First we study a restriction of F which is a diffeomorphism of an open set. Our next task is to find global properties of invariant sets of F.

By property (2), we have $F : \mathbb{R}^2_a \to \mathbb{R}^2_a$. We have to show that V is invariant under $F_{|\mathbb{R}^2_a}$. Now $e = -a$. Therefore

$$\begin{aligned}
V \circ F(x,y) &= V\left(y, \frac{ax+y+\alpha}{x-a}\right) \\
&= \frac{(y-a+1)(\frac{ax+y+\alpha}{x-a} - a + 1)(y + \frac{ax+y+\alpha}{x-a} + a^2 - a + \alpha)}{(y-a)(\frac{ax+y+\alpha}{x-a} - a)} \\
&= \frac{(y-a+1)(x+y+a^2-a+\alpha)}{(x-a)(y-a)(y+\alpha+a^2)} \\
&\quad \times (xy - ay + y + \alpha + a^2x - a^3 + a^2 + \alpha x - a\alpha) \\
&= \frac{(y-a+1)(x+y+a^2-a+\alpha)(x-a+1)(y+\alpha+a^2)}{(x-a)(y-a)(y+\alpha+a^2)} = V(x,y).
\end{aligned}$$

Therefore $V \circ F = V$, as required. The proof of the fact that F preserves the form σ is analogous. ∎

4.3. *Proof of Theorem 3*

Remark 5. *For $e = -a$ the maps F are topologically conjugate to Zeeman's maps G of the form* (2) (*that is*, *for each F there exists a map G and a homeomorphism H with $F \circ H = H \circ G$*).

Proof. Given F let $H_a : \mathbb{R}^2 \to \mathbb{R}^2$ denote the translation given by $H_a(x,y) = (x-a, y-a)$. Then for

$$G(x,y) = \left(y, \frac{y + a(a+1) + \alpha}{x}\right)$$

we have

$$\begin{aligned}
H \circ F(x,y) &= H\left(y, \frac{ax+y+\alpha}{x-a}\right) = \left(y-a, \frac{y+a^2+\alpha}{x-a}\right) \\
&= \left(y-a, \frac{y-a+a(a+1)+\alpha}{x-a}\right) = G(x-a, y-a) = G \circ H(x,y). \ \blacksquare
\end{aligned}$$

REMARK 6. The generalized family given by (5) contains maps which are not topologically conjugate to any Zeeman map. For example in this family there exist maps with two hyperbolic fixed points (see Section 5) while Zeeman maps with two fixed points have at least one which is not hyperbolic.

Because topological conjugacy preserves the topological properties of trajectories it is enough to prove Theorem 3 for Zeeman's maps. Given a Zeeman map G, which is defined on $\mathbb{R}^2$ less the line $x = 0$, and its invariant function V, we observe that each orbit of G lies on a level curve $V = \text{const.}$ If we denote by C a level curve of V then C is the cubic curve in $\mathbb{R}^2$ given by

$$(x+1)(y+1)(x+y+\alpha) - vxy = 0. \tag{10}$$

If we think of the variables x, y as being complex, then (10) defines a family of elliptic curves $\mathcal{C}$ in the space $\mathbb{C}^2$. The closure $\overline{\mathcal{C}}$ of $\mathcal{C}$ in the complex projective space $\mathbb{C}P^2$ is defined by

$$(x+z)(y+z)(x+y+\alpha z) - vxyz = 0. \tag{11}$$

When we wish to emphasize the parameter v we shall add it as a subscript in the following. For any V let $\overline{\mathcal{C}}_v$ be the closure of the level curve $V = v$ defined by (11). We see that $\overline{\mathcal{C}}$ is obtained from $\mathcal{C}$ by adding three points at infinity, namely $(1,0,0)$, $(0,1,0)$, $(1,-1,0)$. One of the most interesting properties of G is that $G = I \circ J$, where I and J are involutions defined on $\mathbb{R}^2$ less the line $x = 0$ given by

$$I(x,y) = (y,x), \qquad J(x,y) = \left(\frac{y+\alpha}{x}, y\right).$$

As Zeeman shows [2], I, J, G extend to maps

$$\begin{aligned}
&\bar{I} : \mathbb{C}P^2 \to \mathbb{C}P^2, \quad (x,y,z) \mapsto (y,x,z),\\
&\bar{J} : \mathbb{C}P^2 \to \mathbb{C}P^2, \quad (x,y,z) \mapsto (z(y+az), xy, xz),\\
&\overline{G} : \mathbb{C}P^2 \to \mathbb{C}P^2, \quad (x,y,z) \mapsto (xy, (y+\alpha z)z, xz),
\end{aligned}$$

which preserve the curve $\overline{\mathcal{C}}_v$ for each v. When $\overline{\mathcal{C}}_v$ is a nonsingular elliptic curve, diffeomorphic to the 2-torus, its intersection with the real projective plane $\mathbb{R}P^2$ is either the union of two disjoint closed curves C_v' and C_v'' or one closed curve C_v'. The curve C_v'' corresponds to the convex component of the intersection $\overline{\mathcal{C}}_v \cap \mathbb{R}^2$ and C_v' corresponds to another component which is not convex. To describe the invariant set of G we need to know the evolution on each level curve. This is achieved by means of the following lemma.

LEMMA 2. *For each nonsingular elliptic curve $\overline{\mathcal{C}}_v$ the maps $\overline{G}_{|C_v'}$ and $\overline{G}_{|C_v''}$ are smoothly conjugate to a rotation.*

Proof. Although in his Theorem 3 ([2], pp. 12–16) Zeeman proves the assertion of Lemma 2 for $v > v_2$ only, the same proof applies for any real value of v assuming that $\overline{\mathcal{C}}_v$ is nonsingular. ∎

In the pencil of cubics given by varying the parameter v in the extended complex plane there are five values (namely, $0, \infty, \alpha - 1, v_1$ and v_2, the last two being the values of the integral V at the fixed points of G) where the curve $\overline{\mathcal{C}}_v$ is not elliptic (see [2], p. 13). It is worth noticing that at each of these five exceptional values, the intersection of $\overline{\mathcal{C}}_v$ with $\mathbb{R}^2$ is a closed set whose complement is dense in $\mathbb{R}^2$. Let $\mathcal{D}$ be the intersection of all these five complements, and $\mathcal{R}$ the set of parameters v for which $\overline{\mathcal{C}}_v$ is nonsingular, i.e. $\mathcal{R} = \mathbb{R} \setminus \{0, \infty, \alpha - 1, v_1, v_2\}$.

The integral V is defined on $\mathbb{R}^2$ less the lines $\{(x, y) : xy = 0\}$. Observe that for (x, y) in the domain of V its G-image also lies in the domain of V iff $y \neq -\alpha$. Moreover for any x, $G(x, -\alpha) = (-\alpha, 0)$. Denote the point $(-\alpha, 0)$ by Y. At Y we have

$$(-\alpha + 1) \cdot (0 + 1) \cdot (-\alpha + 0 + \alpha) - v \cdot (-\alpha) \cdot 0 \cdot 1 = 0,$$

so $Y \in \overline{\mathcal{C}}_v \cap \mathbb{R}P^2$ for each v. Furthermore $Y \in C'_v$ (see [2], p. 23).

Let $\Lambda_+ \in \mathbb{R}^2$ be the forward invariant set of the map (2) with $\gamma \in \mathbb{R}$. The complement of Λ_+ in $\mathcal{D}$ is the set of all preimages of Y under $\overline{G}_{|C'_v}$ for all $v \in \mathcal{R}$. To simplify notation let $\overline{G}_v^{-i}(Y)$ denote the ith preimage of Y under $\overline{G}_{|C'_v}$.

Setting $\mathcal{A} = \bigcup_{i \in \mathbb{N},\, v \in \mathcal{R}} \overline{G}_v^{-i}(Y)$ we have $\Lambda_+ = \mathcal{D} \setminus \mathcal{A}$. Moreover, define

$$\mathcal{A}_i = \bigcup_{v \in \mathcal{R}} \overline{G}_v^{-i}(Y) \quad \text{and} \quad \Lambda_i = \mathcal{D} \setminus \mathcal{A}_i.$$

Lemma 3. *Λ_i is dense in $\mathcal{D}$.*

Proof. Let $X \in \mathcal{D}$. For $v_X = V(X)$ we have $X \in C'_{v_X} \cup C''_{v_X}$. Because $X \in \mathcal{D}$, X is not a critical point of the integral V, so by the implicit function theorem we find that for any $\varepsilon > 0$, $B(X, \varepsilon) = \{Y : d(X, Y) < \varepsilon\}$ contains points other than X, lying on the level curve C'_{v_X} passing through X. By Lemma 2 the map $\overline{G}$ restricted to C'_{v_X} and C''_{v_X} is smoothly conjugate to a rotation, so $\sharp\{\mathcal{A}_i \cap (C'_{v_X} \cup C''_{v_X})\} = 1$. Thus there also exists in $B(X, \varepsilon)$ a point $W \in \mathcal{D}$ lying in C'_{v_X} which is not in $\mathcal{A}_i$ (the intersection of $\mathcal{A}_i$ with $B(X, \varepsilon)$ consists of at most one point). We have $V(W) = V(X)$. From this it follows that $W \notin \mathcal{A}_i$. ∎

Lemma 4. *Λ_i is an open set.*

Proof. Let ϱ_v denote the rotation number of $\overline{G}_{C'_v}$ and $\overline{G}_{C''_v}$. Fix $X \in \Lambda_i$. By Lemma 2 there exists δ_0 such that $B(X, r) \cap \overline{G}_{v_X}^{-i}(Y) = \emptyset$ for any $r < \delta_0$. Fix $r_0 < \delta_0$. Choose $\varepsilon_X > 0$ such that $I_{\varepsilon_X} = (v_X - \varepsilon_X, v_X + \varepsilon_X) \subset \mathcal{R}$.

Since the rotation number of a continuous family of homeomorphisms is continuous (see [2] and references given there), ϱ_v is a continuous function of v in I_{ε_X}. It follows that there exists δ_1 such that $d(X, \overline{G}_v^{-i}(Y)) > r_0$ for $|v - v_X| < \delta_1$. Because V is continuous at X there exists δ_2 such that $|V(W) - V(X)| < \delta_1$ for W satisfying $d(W, X) < \delta_2$. Because $\mathcal{D}^c$ is closed and $X \in \mathcal{D}$, it follows that the distance of X to $\mathcal{D}^c$ is nonzero. Let δ_3 be smaller than $\operatorname{dist}(X, \mathcal{D}^c)$. For $\delta = \min(r_0, \delta_2, \delta_3)$ we have $B(X, \delta) \cap \mathcal{A}_i = \emptyset$. Thus $B(X, \delta) \subset \Lambda_i$, which proves that Λ_i is open. ∎

Proof of Theorem 3. Due to the definition of Λ_+ we have $\Lambda_+ = \mathcal{D} \setminus \mathcal{A} = \mathcal{D} \setminus \bigcup_{i=0}^{\infty} \mathcal{A}_i = \bigcap_{i=0}^{\infty} (\mathcal{D} \setminus \mathcal{A}_i) = \bigcap_{i=0}^{\infty} \Lambda_i$. By Lemmas 3 and 4 and since $\mathcal{D}$ is a dense subset of $\mathbb{R}^2$ we infer that Λ_+ is a residual set. If we denote by Λ_- the backward invariant set for the map (2), now considering images of the point $Z = (0, -\alpha)$ we prove in an analogous way that Λ_- is residual. One can check that in this case the set $\Lambda = \Lambda_+ \cap \Lambda_-$ is totally invariant. This ends the proof of Theorem 3. ∎

COROLLARY 1. *Λ is dense* (*by Baire's theorem*).

REMARK 7. Although Theorem 3 states that the totally invariant set Λ is residual, it may contain subsets which are homeomorphic to $\mathbb{R}^2$. In Zeeman's case for $\alpha \geq 0$ we have $\mathbb{R}^2_+ \subset \Lambda$ (see Property 2) and also for $\alpha \geq 1, \alpha \neq 2$ we find that the triangle $S_1S_2S_3$ with $S_1 = (-1,-1), S_2 = (-1, 1-\alpha), S_3 = (1-\alpha, -1)$ is contained in Λ.

5. Remarks. As we show in Theorem 1 and in Remark 5 each map with a neutral fixed point contained in the family (5) is topologically conjugate to a Zeeman map. Below we give examples of maps of the form (4) which are not topologically conjugate to any map (2). The principal significance of these examples is that the family of unfoldings given by (4) with at least one neutral fixed point is not contained in the conjugacy class of Zeeman's family. This means that Zeeman's family does not contain all nonhyperbolic maps of the form (4). Furthermore, as mentioned in Remark 6, even the family (5) contains maps not conjugate to Zeeman's family.

EXAMPLE 1. *The map*

$$F_A(x, y) = \left(y, \frac{3 + y + x}{x} \right)$$

contained in the family (5) *has two hyperbolic fixed points.*

Proof. The fixed points of F_A are $A_1 = (-1,-1)$ and $A_2 = (3,3)$. Both are hyperbolic, because in absolute value both eigenvalues are equal to $\sqrt{2}$ at A_1 and $\sqrt{2/3}$ at A_2. ∎

Example 2. *The maps*

$$F_B(x,y)=(y,x),\qquad F_C(x,y)=(y,2y-x),\qquad F_D(x,y)=\left(y,\frac{y}{x+y-1}\right)$$

are not topologically conjugate to any map in the family (2).

Proof. For F_B, F_C the line $\{(x,y) : x = y\}$ is fixed, while Zeeman's maps have at most two real fixed points.

Suppose that F_D is conjugate to a map of the form (2). Let G be a Zeeman map with any real γ. At the fixed points W_i of this map, for $\gamma \neq 0$ we have $G^{-1}(W_i) = W_i$. The origin $(0,0)$ is a neutral fixed point of F_D and $F_D^{-1}(0,0) = \{(x,y) : y = 0\}$. From these facts it follows that if G and F_D are topologically conjugate then γ has to be equal to 0. Recall that $(y, y/x)$ is a periodic map with period 6, but F_D is not periodic with period 6. The first 7 iterations of the point $(0,-1)$ under F_D are the following:

$$\begin{aligned}\overline{F} : (0,-1) &\mapsto (-1,1/2) \mapsto (1/2,-1/3)\\ &\mapsto (-1/3,2/5) \mapsto (2/5,-3/7) \mapsto (-3/7,5/12) \mapsto (5/12,-7/17),\end{aligned}$$

which ends the proof. ∎

The above example gives information about sequences of the form (3) whose limit behaviour differs significantly from those given by (1), in the sense that the sets of accumulation points of these sequences may be very different. It is an interesting question whether in the family (4) there exists a map with a neutral fixed point of centre type which is not topologically conjugate to any member of Zeeman's family.

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