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Homotopy orbits of free loop spaces

by

Marcel Bökstedt (Århus) and **Iver Ottosen** (Paris)

Abstract. Let X be a space with free loop space ΛX and mod two cohomology $R = H^*X$. We construct functors $\Omega_\lambda(R)$ and $\ell(R)$ together with algebra homomorphisms $e : \Omega_\lambda(R) \to H^*(\Lambda X)$ and $\psi : \ell(R) \to H^*(ES^1 \times_{S^1} \Lambda X)$. When X is 1-connected and R is a symmetric algebra we show that these are isomorphisms.

1. Introduction. The purpose of this paper is to present a new approach to the cohomology of the S^1 homotopy orbits of the free loop space ΛX.

Let X be a space, and C_n the cyclic group considered as acting on X^n by permuting coordinates. The ($\mathbb{Z}/2$-) cohomology of $EC_n \times_{C_n} X^n$ can be computed from $R = H^*(X;\mathbb{Z}/2)$ by a Serre spectral sequence. It is a fundamental fact, derived from the equivariance of the Eilenberg–Zilber map, that this spectral sequence collapses for all spaces X and all n. In particular, this cohomology is a functor in R, and does not depend in any other way on the topology of X. The second author has investigated this situation closely in [Ottosen].

In §3 and §4 we begin the study of the cohomology of the orbits with some preliminaries and generalities about spaces with an S^1-action.

We continue in §6 by using the S^1-transfer to construct maps connecting the cohomology of X, respectively $EC_2 \times_{C_2} X^2$, with $ES^1 \times_{S^1} \Lambda X$. These maps are not multiplicative, but they do satisfy a version of Frobenius reciprocity. This forces the images of the maps to satisfy universal relations, some of these involving multiplications.

In this way, taking sums of products of the cohomology classes and taking the universal relations into account, we obtain a group $\ell(R)$, which is again a functor of R. This group maps by its definition to $H^*(ES^1 \times_{S^1} \Lambda X;\mathbb{Z}/2)$, and we think of the map as an approximation.

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Via relations to the Steenrod diagonal, the approximation is shown in §7 to be nontrivial in many cases.

Surprisingly, in §8 it turns out that the approximating group has nice algebraic properties. Investigating it by purely algebraic methods we find that it is closely related to a version of de Rham cohomology of the ring R. There is a map from $\mathcal{L}(R)$ (a slight variation on $\ell(R)$) to the de Rham cycles. The de Rham differential factors over this map. Using this, we see that in the (very) special case when R is a polynomial ring, ℓ is precisely given by de Rham cycles and de Rham cohomology.

Finally, in §9 and §10, we discover that in the polynomial case, the approximation is precise, that is, $\ell(R) \to H^*(ES^1 \times_{S^1} \Lambda X; \mathbb{Z}/2)$ is an isomorphism. This is the main result of the paper.

In the two appendices §11 and §12 we show that the approximation functor has other good algebraic properties. In §11 we extend the functor $\ell(R)$ to the case where R is nongraded. In §12, we show that $\mathcal{L}(R)$ is easily computable from R in terms of generators and relations.

The authors wonder if there is some variation on the functor ℓ that computes the cohomology of the S^1 homotopy orbits of the free loop space in greater generality.

They would also like to know about possible connections to the Lannes' functor, which in certain situations computes the cohomology of spaces of homotopy fixed points (cf. [Schwartz]).

2. Notation. Let us fix some notation and conventions. The coefficient ring in all cohomology groups is the field $\mathbb{F}_2 = \mathbb{Z}/2$ when no other ring appears in the notation. $\mathcal{A}$ denotes the mod two Steenrod algebra. We only consider spaces X which are of finite $\mathbb{F}_2$-type, i.e. $H_i(X;\mathbb{F}_2)$ is finite-dimensional for each i.

The cohomology of the circle and of its classifying space is denoted by

$$H^*(S^1) = \Lambda(v), \qquad H^*(BS^1) = \mathbb{F}_2[u],$$

where v has degree one and u has degree two.

For the cohomology of the classifying space of a cyclic group of two power order we use the notation

$$H^*(BC_{2^m}) = \begin{cases} \Lambda(v) \otimes \mathbb{F}_2[u] & \text{when } m \geq 2, \\ \mathbb{F}_2[t] & \text{when } m = 1; \end{cases}$$

here t and v are one-dimensional and u is two-dimensional. The inclusion $j : C_{2^m} \subseteq S^1$ satisfies $(Bj)^*(u) = u$ when $m \geq 2$ and $(Bj)^*(u) = t^2$ when $m = 1$ as seen by the Leray–Serre spectral sequence associated with the fibration $S^1 \to BC_{2^m} \to BS^1$. Also, $i^*(v) = v$ when $i : S^1 \to BC_{2^m}$ denotes the inclusion of the fiber.

For any unstable $\mathcal{A}$-algebra we define the operation $\lambda x = \mathrm{Sq}^{|x|-1} x$. By the Cartan formula this is a derivation over Frobenius, i.e.

$$\lambda(xy) = x^2 \lambda y + y^2 \lambda x.$$

3. Connected S^1-spaces. Let Y be a connected S^1-space with action map $\eta : S^1 \times Y \to Y$. The map η^* is injective since $\eta \circ \gamma = \mathrm{id}$ where γ is the map defined by $\gamma(y) = (1, y)$. We write Y_0 for the space Y with trivial S^1-action.

DEFINITION 3.1. Define a degree -1 map $d : H^*(Y) \to H^{*-1}(Y)$ by

$$\eta^*(y) = 1 \otimes y + v \otimes dy.$$

PROPOSITION 3.2. *The map d satisfies the following equations*:

$$d \circ d = 0, \tag{1}$$

$$d(x + y) = dx + dy, \tag{2}$$

$$d(xy) = xdy + ydx, \tag{3}$$

$$\mathrm{Sq}^i(dx) = d(\mathrm{Sq}^i x), \tag{4}$$

$$(dx)^2 = d(\lambda x). \tag{5}$$

Proof. (1) There is a commutative diagram

$$\begin{tikzcd} S^1 \times S^1 \times Y \arrow[r, "1\times\eta"] \arrow[d, "\mu\times 1"'] & S^1 \times Y \arrow[d, "\eta"] \\ S^1 \times Y \arrow[r, "\eta"] & Y \end{tikzcd}$$

where $\mu : S^1 \times S^1 \to S^1$ is the multiplication map. In cohomology it satisfies $\mu^*(v) = v \otimes 1 + 1 \otimes v$. The result follows by pulling a class $y \in H^*Y$ back along the two different ways in the diagram.

(2) follows from $\eta^*(x + y) = \eta^*(x) + \eta^*(y)$.

(3) follows from $\eta^*(xy) = \eta^*(x)\eta^*(y)$.

(4) follows by the Cartan formula and the fact that η^* is $\mathcal{A}$-linear.

(5) follows from (4). ■

By our next result the differential d also appears in the Leray–Serre spectral sequence.

PROPOSITION 3.3. *The fibration $Y \to ES^1 \times_{S^1} Y \to BS^1$ has the following Leray–Serre spectral sequence*:

$$E_2^{**} = H^*(BS^1) \otimes H^*(Y) \Rightarrow H^*(ES^1 \times_{S^1} Y).$$

The differential in the E_2-term is given by

$$d_2 : H^*(Y) \to uH^*(Y), \quad d_2(y) = ud(y),$$

where d is the differential from Definition 3.1.

Proof. The action map $\eta : S^1 \times Y_0 \to Y$ is an S^1-equivariant map, hence it induces a map of fibrations:

$$\begin{tikzcd}
S^1 \times Y_0 \arrow[r] \arrow[d, "\eta"] & ES^1 \times_{S^1} (S^1 \times Y_0) \arrow[r] \arrow[d, "1\times\eta"] & BS^1 \arrow[d, "\mathrm{id}"] \\
Y \arrow[r] & ES^1 \times_{S^1} Y \arrow[r] & BS^1
\end{tikzcd}$$

We get a map $\eta^* : E \to \widehat{E}$ of the corresponding spectral sequences where $\widehat{E}$ is the spectral sequence associated with the upper fibration. Since the upper fibration has total space homeomorphic to $ES^1 \times Y$ it looks like

$$\widehat{E}_2^{**} = H^*(BS^1) \otimes H^*(S^1) \otimes H^*(Y) \Rightarrow H^*(Y).$$

In $\widehat{E}_2$ the differential on the fiber elements is given by

$$d_2(v \otimes z) = u(1 \otimes z), \quad d_2(1 \otimes z) = 0. \tag{6}$$

To see this, assume that Y is contractible. Then the E_∞-term has a $\mathbb{F}_2$ placed at $(0,0)$ and zero at all other places. It follows that $d_2(v) = u$ and $d_2(1) = 0$. Since the action on Y_0 was trivial this case implies (6). Let $y \in H^*Y$ be a fiber class in E_2. By η^* it maps to $1 \otimes y + v \otimes dy$ and applying the $\widehat{E}_2$-differential to this we get $u(1 \otimes dy)$. We see that $d_2(y) = ux$ where $x \in H^*Y$ is a class with $\eta^*(x) = 1 \otimes dy$. Since $x = dy$ satisfies this equation and η^* is injective the result follows. ■

4. Homotopy orbits of connected S^1-spaces

Definition 4.1. Define the spaces E_nY for $n = 0, 1, \ldots, \infty$ by

$$E_nY = ES^1 \times_{C_{2^n}} Y, \quad n < \infty, \qquad E_\infty Y = ES^1 \times_{S^1} Y.$$

For nonnegative integers n and m with $m > n$ define the maps

$$q_m^n : H^*E_mY \to H^*E_nY, \qquad \tau_n^m : H^*E_nY \to H^*E_mY$$

by letting q_m^n be the map induced by the quotient map and τ_n^m be the transfer map. Also, define

$$q_\infty^n : H^*E_\infty Y \to H^*E_nY$$

as the map induced by the quotient.

Theorem 4.2. *Let $m \geq 1$ be an integer. Consider the diagram*

$$\begin{tikzcd}
E_mY \arrow[r, "Q"] \arrow[d, "\mathrm{pr}_1"] & E_\infty Y \arrow[d, "\mathrm{pr}_1"] \\
BC_{2^m} \arrow[r, "Bj"] & BS^1
\end{tikzcd} \tag{7}$$

where Q denotes the quotient map. There is an isomorphism

$$\theta : H^*(BC_{2^m}) \otimes_{H^*(BS^1)} H^*(E_\infty Y) \cong H^*(E_m Y)$$

defined by $x \otimes y \mapsto \mathrm{pr}_1^*(x) q_\infty^m(y)$. The transfer map $\tau_m^{m+1} : H^* E_m Y \to H^* E_{m+1} Y$ is zero on elements of the form $\theta(1 \otimes y)$ and the identity on elements of the form $\theta(v \otimes y)$ (here $v = t$ when $m = 1$). We get an isomorphism

$$\operatorname{colim} H^* E_m Y = v H^* E_\infty Y \cong \widetilde{H}^*(\Sigma(E_\infty Y)_+).$$

Proof. Filling in the fibers of diagram (7) we get

$$\begin{tikzcd}
* \arrow[r] \arrow[d] & Y \arrow[r, "\mathrm{id}"] \arrow[d] & Y \arrow[d] \\
S^1 \arrow[r] \arrow[d] & E_m Y \arrow[r, "Q"] \arrow[d, "\mathrm{pr}_1"'] & E_\infty Y \arrow[d, "\mathrm{pr}_1"'] \\
S^1 \arrow[r] & BC_{2^m} \arrow[r, "Bj"] & BS^1
\end{tikzcd}$$

Since the fundamental group of BS^1 is zero, the Leray–Serre spectral sequence $\widehat{E}$ of the lowest horizontal fibration has trivial coefficients, and since Q is the pull back of this, the Leray–Serre spectral sequence E associated with Q also has trivial coefficients. We get

$$E_2^{**} = H^*(E_\infty Y) \otimes H^*(S^1) \Rightarrow H^*(E_m Y),$$
$$\widehat{E}_2^{**} = H^*(BS^1) \otimes H^*(S^1) \Rightarrow H^*(BC_{2^m}),$$

and pr_1 gives a map of these two spectral sequences. We know that $d_2(v) = 0$ in $\widehat{E}$. Since $\mathrm{pr}_1^*(v) = v$ we get $d_2 v = 0$ in E, hence $E_2 = E_\infty$. It is now obvious that the map θ is an isomorphism. Using pr_1^* we get $\tau_m^{m+1}(v) = v$. By Frobenius reciprocity the description of the transfer map follows. ∎

The above theorem is inspired by the following result of Tom Goodwillie, which can be found on page 279 of [Madsen].

THEOREM 4.3 (Goodwillie). *For any based S^1-space Z, there is a map*

$$\tau : \widetilde{Q}(\Sigma ES^1_+ \wedge_{S^1} Z) \to \operatorname{holim} \widetilde{Q}(ES^1_+ \wedge_{C_{p^n}} Z)$$

which induces an isomorphism on homotopy groups with $\mathbb{F}_p$ coefficients.

We may now give our own definition of the S^1-transfer.

DEFINITION 4.4. For any nonnegative integer n the S^1-*transfer* $\tau_n^\infty : H^* E_n Y \to H^* E_\infty Y$ is defined as the composite

$$H^* E_n Y \to \operatorname{colim} H^* E_m Y \xrightarrow{v^{-1}} H^* E_\infty Y$$

where the direct limit is over the transfer maps τ_n^m. Note that the degree of τ_n^∞ is -1.

Definition 4.5. Let θ_0 denote the S^1-equivariant map

$$\theta_0 : S^1 \times Y_0 \to ES^1 \times Y, \quad (z, y) \mapsto (ze, zy),$$

and let θ_n for $n = 1, 2, \ldots, \infty$ be the maps one obtains by passing to the quotients:

$$\theta_n : S^1/C_{2^n} \times Y \to E_m Y, \quad m < \infty, \quad \theta_\infty : * \times Y \to E_\infty Y$$

Proposition 4.6. *For nonnegative integers m and n with $n < m$ the following squares commute*:

$$\begin{tikzcd} H^* E_n Y \arrow[r, "\theta_n^*"] & H^*(S^1 \times Y) \\ H^* E_m Y \arrow[u, "q_m^n"] \arrow[r, "\theta_m^*"] & H^*(S^1 \times Y) \arrow[u, "q_m^n \otimes 1"'] \end{tikzcd} \qquad \begin{tikzcd} H^* E_n Y \arrow[r, "\theta_n^*"] \arrow[d, "\tau_n^m"'] & H^*(S^1 \times Y) \arrow[d, "\tau_n^m \otimes 1"] \\ H^* E_m Y \arrow[r, "\theta_m^*"] & H^*(S^1 \times Y) \end{tikzcd}$$

There are also commutative squares

$$\begin{tikzcd} H^* E_n Y \arrow[r, "\theta_n^*"] & H^*(S^1 \times Y) \\ H^* E_\infty Y \arrow[u, "q_\infty^n"] \arrow[r, "\theta_\infty^*"] & H^* Y \arrow[u, "\mathrm{pr}_2^*"'] \end{tikzcd} \qquad \begin{tikzcd} H^* E_n Y \arrow[r, "\theta_n^*"] \arrow[d, "\tau_n^\infty"'] & H^*(S^1 \times Y) \arrow[d, "\tau_n^\infty"] \\ H^* E_\infty Y \arrow[r, "\theta_\infty^*"] & H^* Y \end{tikzcd}$$

where $\theta_\infty^* = q_\infty^0$ is the map induced by the inclusion of the fiber, *and the transfer on the right hand side is given by $1 \otimes y \mapsto 0$ and $v \otimes y \mapsto y$.*

Proof. The first three diagrams are obvious. The last diagram follows by Theorem 4.2 when passing to the direct limit over m in the diagram with the finite transfers. ∎

Proposition 4.7. *For any nonnegative integer n Frobenius reciprocity holds*, *i.e.*

$$\tau_n^\infty(q_\infty^n(x)y) = x\tau_n^\infty(y).$$

Proof. When $n \geq 1$ this is an easy consequence of the description of the transfer in Theorem 4.2. For $n = 0$ we use Frobenius reciprocity of τ_0^2 as follows:

$$\tau_0^\infty(q_\infty(x)y) = \tau_2^\infty \circ \tau_0^2(q_2^0(q_\infty^2(x))y) = \tau_2^\infty(q_\infty^2(x)\tau_0^2(y))$$

and use the $n = 2$ case. ∎

Proposition 4.8. *We have*

$$\tau_0^\infty \circ q_\infty^0 = 0, \tag{8}$$

$$q_\infty^0 \circ \tau_0^\infty = d. \tag{9}$$

Proof. (8) We can factor the composite as $\tau_0^\infty \circ q_\infty^0 = \tau_1^\infty \circ \tau_0^1 \circ q_1^0 \circ q_\infty^1$, and $\tau_0^1 \circ q_1^0$ is multiplication by 2, thus the zero map.

(9) We can factor θ_0 as

$$\theta : S^1 \times Y_0 \xrightarrow{\Delta\times 1} S^1 \times S^1 \times Y \xrightarrow{1\times\eta} S^1 \times Y \to ES^1 \times Y$$

and hence $\theta_0^*(1\otimes y) = 1\otimes y + v\otimes dy$. The result follows from the last diagram of Proposition 4.6. ■

DEFINITION 4.9. Define the map

$$f'_Y : S^1 \times Y \to ES^1 \times Y^2, \qquad (z,y) \mapsto (ze, zy, -zy).$$

We let C_2 act on the left space by $(-1)\cdot(z,y) = (-z,y)$ and on the right space by $(-1)\cdot(e,y,y') = ((-1)e, y', y)$. Then the above map is C_2-equivariant. Passing to the quotients we get a map

$$f_Y : S^1/C_2 \times Y \to ES^1 \times_{C_2} Y^2.$$

Note that this map is natural in Y with respect to C_2-equivariant maps. Recall that there is a natural isomorphism

$$H^*(ES^1 \times_{C_2} Y^2) \cong H^*(C_2; H^*(Y)^{\otimes 2}).$$

For a homogeneous element $y \in H^*Y$ the C_2 invariant $y\otimes y$ defines an element $1\otimes y^{\otimes 2}$ in the zeroth cohomology group of C_2. If $x \in H^*Y$ is another homogeneous element with $x \neq y$ then the invariant $x\otimes y + y\otimes x$ also defines an element $1\otimes(1+T)x\otimes y$ in the zeroth cohomology group of C_2. Using this notation we have the following results.

LEMMA 4.10. $f_{Y_0}^*(1\otimes y^{\otimes 2}) = 1\otimes y^2 + v\otimes \lambda y$.

Proof. We have a factorization

$$f_{Y_0} : S^1/C_2 \times Y_0 \xrightarrow{i\times 1} ES^1/C_2 \times Y_0 \xrightarrow{1\times\Delta} ES^1 \times_{C_2} Y_0^2.$$

This together with Steenrod's definition of the squares gives the result. ■

LEMMA 4.11. $f_Y^*(1\otimes(1+T)x\otimes y) = v\otimes d(xy)$.

Proof. There is a commutative diagram

$$\begin{tikzcd} H^*(S^1/C_2 \times Y) & H^*(ES^1 \times_{C_2} Y^2) \arrow[l, "f_Y^*"'] \\ H^*(S^1 \times Y) \arrow[u, "\tau_0^1\otimes 1"] & H^*(ES^1 \times Y^2) \arrow[l, "f_Y'^*"'] \arrow[u, "\tau_0^1"'] \end{tikzcd}$$

The lower horizontal map is given by $f_Y'^*(1\otimes x\otimes y) = 1\otimes xy + v\otimes d(xy)$ as seen by the factorization

$$f'_Y : S^1 \times Y \xrightarrow{\Delta} (S^1\times Y)^2 \xrightarrow{\mathrm{pr}_1\times\Delta} S^1 \times (S^1\times Y)^2 \xrightarrow{i\times\eta\times\eta} ES^1 \times Y^2 \xrightarrow{1\times 1\times(-1)} ES^1 \times Y^2.$$

Since the norm class is hit by the transfer $\tau_0^1(1 \otimes x \otimes y) = 1 \otimes (1+T)x \otimes y$ the result follows. ∎

THEOREM 4.12. $f_Y^*(1 \otimes y^{\otimes 2}) = 1 \otimes y^2 + v \otimes (ydy + \lambda y)$.

Proof. Because of the degrees there is a constant $k \in \mathbb{F}_2$ such that

$$f_{S^1}^*(1 \otimes v^{\otimes 2}) = kv \otimes v.$$

The two projection maps $\mathrm{pr}_1 : S^1 \times Y_0 \to S^1$ and $\mathrm{pr}_2 : S^1 \times Y_0 \to Y_0$ are S^1-equivariant. Thus we can use naturality together with Lemma 4.10 and the above equation respectively to find the first two of the three equations

$$\begin{aligned} f_{S^1\times Y_0}^*(1 \otimes (1 \otimes y)^{\otimes 2}) &= 1 \otimes 1 \otimes y^2 + v \otimes 1 \otimes \lambda y, \\ f_{S^1\times Y_0}^*(1 \otimes (v \otimes 1)^{\otimes 2}) &= kv \otimes v \otimes 1, \\ f_{S^1\times Y_0}^*(1 \otimes (v \otimes dy)^{\otimes 2}) &= kv \otimes v \otimes (dy)^2. \end{aligned}$$

The third equation follows from the other two.

The action map $\eta : S^1 \times Y_0 \to Y$ is also an S^1-equivariant map, hence by naturality we have a commutative diagram

$$\begin{array}{ccc} S^1/C_2 \times (S^1 \times Y_0) & \xrightarrow{f_{S^1\times Y_0}} & ES^1 \times_{C_2} (S^1 \times Y_0)^2 \\ \Big\downarrow{\scriptstyle 1\times\eta} & & \Big\downarrow{\scriptstyle 1\times\eta^2} \\ S^1/C_2 \times Y & \xrightarrow{f_Y} & ES^1 \times_{C_2} Y^2 \end{array}$$

We compute the pull back of the class $1 \otimes y^{\otimes 2}$ to the cohomology of the upper left corner. First we find

$$\begin{aligned} (1 \times \eta^2)^*(1 \otimes y^{\otimes 2}) &= 1 \otimes (1 \otimes y + v \otimes dy)^{\otimes 2} \\ &= 1 \otimes (1 \otimes y)^{\otimes 2} + 1 \otimes (v \otimes dy)^{\otimes 2} \\ &\quad + 1 \otimes (1+T)(1 \otimes y) \otimes (v \otimes dy). \end{aligned}$$

By Lemma 4.11 we can compute $f_{S^1\times Y_0}^*$ applied to the last term:

$$\begin{aligned} &f_{S^1\times Y_0}^*(1 \otimes (1+T)(1 \otimes y) \otimes (v \otimes dy)) \\ &\qquad = v \otimes d_{S^1\times Y_0}(v \otimes ydy) \\ &\qquad = v \otimes (d_{S^1}(v) \otimes ydy + v \otimes d_{Y_0}(ydy)) = v \otimes 1 \otimes ydy. \end{aligned}$$

Altogether we have

$$\begin{aligned} (1 \otimes \eta^*) \circ f_Y^*(1 \otimes y^{\otimes 2}) &= f_{S^1\times Y_0}^* \circ (1 \times \eta^2)^*(1 \otimes y^{\otimes 2}) \\ &= 1 \otimes 1 \otimes y^2 + v \otimes 1 \otimes (ydy + \lambda y) \\ &\quad + kv \otimes v \otimes (dy)^2. \end{aligned}$$

We now apply $1 \otimes \gamma^*$ on both sides (for the map γ, see the beginning of §3):

$$\begin{aligned} f_Y^*(1\otimes y^{\otimes 2}) &= (1\otimes\gamma^*)\circ(1\otimes\eta^*)\circ f_Y^*(1\otimes y^{\otimes 2})\\ &= 1\otimes\gamma^*(1\otimes 1\otimes y^2+v\otimes 1\otimes(ydy+\lambda y)+kv\otimes v\otimes(dy)^2)\\ &= 1\otimes y^2+v\otimes(ydy+\lambda y). \ \blacksquare \end{aligned}$$

5. The approximation functors. Let $\mathbf{A}$ be a graded ring over $\mathbb{F}_2$. Let $\lambda:\mathbf{A}\to\mathbf{A}$ be a linear map with $|\lambda x|=2|x|-1$ which is a derivation over Frobenius, i.e. $\lambda(ab)=\lambda(a)b^2+a^2\lambda(b)$. We consider a version of the de Rham complex, relative to the derivation λ.

DEFINITION 5.1. The algebra $\Omega_\lambda(\mathbf{A})$ is generated by a, da for $a\in\mathbf{A}$ where $d(a)$ is given the degree $|a|-1$. The relations are

$$d(a+b)=da+db, \tag{10}$$

$$d(ab)=(da)b+a(db), \tag{11}$$

$$(da)^2=d(\lambda(a)). \tag{12}$$

Of course, there is also a differential on $\Omega_\lambda(\mathbf{A})$, defined as a derivation over $\mathbf{A}$ by the formula $d(a)=da$. This is compatible with the relations, since $d((da)^2)=0=d(d(\lambda a))$. Note that this differential depends on λ. When λ is the trivial derivation $\lambda(a)=0$ we get the ordinary de Rham complex.

Clearly, Ω_λ is a functor from the category of graded rings with derivations over Frobenius to the category of differential graded algebras.

DEFINITION 5.2. We define $\ell(\mathbf{A})$ to be the graded algebra over $\mathbb{F}_2$ generated by u of degree two and the following symbols, defined for each homogeneous element $a\in\mathbf{A}^n$:

$$\begin{aligned} &\phi(a), \quad \deg(\phi(a))=2n,\\ &\delta(a), \quad \deg(\delta(a))=n-1,\\ &q(a), \quad \deg(q(a))=2n-1, \end{aligned}$$

and satisfying the following relations:

$$\phi(a+b)=\phi(a)+\phi(b), \tag{13}$$

$$\delta(a+b)=\delta(a)+\delta(b), \tag{14}$$

$$q(a+b)=q(a)+q(b)+\delta(ab), \tag{15}$$

$$0=\delta(xy)\delta(z)+\delta(yz)\delta(x)+\delta(zx)\delta(y), \tag{16}$$

$$\phi(ab)=\phi(a)\phi(b)+uq(a)q(b), \tag{17}$$

$$q(ab)=q(a)\phi(b)+\phi(a)q(b), \tag{18}$$

$$\delta(a)^2=\delta(\lambda(a)), \tag{19}$$

$$q(a)^2=\phi(\lambda(a))+\delta(a^2\lambda(a)), \tag{20}$$

$$\delta(a)\phi(b)=\delta(ab^2), \tag{21}$$

(22) $$\delta(a)q(b) = \delta(a\lambda(b)) + \delta(ab)\delta(b),$$

(23) $$u\delta(a) = 0.$$

We note some immediate consequences of these relations:

- By (13), (14) and (15), $\phi(0) = \delta(0) = q(0) = 0$.
- By (18), $q(1) = q(1)\phi(1) + \phi(1)q(1) = 0$.
- Using this and (15), we see that $\delta(1) = 0$.
- Using (17), (18) and (21) we now see that $\phi(1)$ acts as a one element. So we can put $1 = \phi(1)$, to make $\ell(\mathbf{A})$ into a unital algebra. Finally, $q(a^2) = q(a)\phi(a) + \phi(a)q(a) = 0$, and also $\delta(a^2) = \delta(1)\phi(a) = 0$.

Definition 5.3. Define $\mathcal{L}(\mathbf{A})$ to be the quotient algebra $\ell(\mathbf{A})/(u)$.

Note that the map $\phi : \mathbf{A} \to \mathcal{L}(\mathbf{A})$ is a unital ring homomorphism, and doubles the degree. ℓ and $\mathcal{L}$ are functors from graded rings with derivation over Frobenius to graded rings.

6. The free loop space. Let X be a connected space and let ΛX denote its free loop space. We also assume that ΛX is connected (otherwise we may apply the theory given below on each component of ΛX). Then we may take $Y = \Lambda X$ in §3 and §4.

Definition 6.1. Define the evaluation maps

$$\mathrm{ev}_0 : \Lambda X \to X, \quad \mathrm{ev}_1' : \Lambda X \to X^2, \quad \mathrm{ev}_1 : ES^1 \times_{C_2} \Lambda X \to ES^1 \times_{C_2} X^2$$

by $\mathrm{ev}_0(\omega) = \omega(1)$, $\mathrm{ev}_1'(\omega) = (\omega(1), \omega(-1))$ and $\mathrm{ev}_1 = 1 \times \mathrm{ev}_1'$.

Definition 6.2. Define an algebra homomorphism

$$e : \Omega^*_\lambda(H^*X) \to H^*(\Lambda X)$$

by $e(x) = \mathrm{ev}_0^*(x)$ and $e(dx) = d\,\mathrm{ev}_0^*(x)$.

Note that e is well defined by Proposition 3.2.

Definition 6.3. For any homogeneous class $x \in H^*X$ define the classes $\delta(x)$, $\phi(x)$, $q(x)$ in $H^*E_\infty\Lambda X$ of degrees $|x|-1$, $2|x|$, $2|x|-1$ respectively by

$$\delta(x) = \tau_0^\infty \circ \mathrm{ev}_0^*(x), \quad \phi(x) = \tau_1^\infty \circ \mathrm{ev}_1^*(t \otimes x^{\otimes 2}), \quad q(x) = \tau_1^\infty \circ \mathrm{ev}_1^*(1 \otimes x^{\otimes 2}).$$

Any constant loop is an S^1-fixpoint in the free loop space; it defines a section to the fibration $\mathrm{pr}_1 : E_\infty \Lambda X \to BS^1$. Thus pr_1^* is injective and we define u in $H^*E_\infty\Lambda X$ by $u = \mathrm{pr}_1^*(u)$.

LEMMA 6.4. *There is a commutative diagram*

$$\begin{tikzcd}
H^*(ES^1 \times_{C_2} X^2) \arrow[r, "\mathrm{ev}_1^*"] & H^*(ES^1 \times_{C_2} \Lambda X) \arrow[r, "\tau_1^\infty"] & H^*(ES^1 \times_{S^1} \Lambda X) \\
H^*(X^2) \arrow[u, "\mathrm{Tr}_0^1"] \arrow[r, "\mathrm{ev}_1'^*"] \arrow[d, "\Delta^*"'] & H^*(\Lambda X) \arrow[u, "\tau_0^1"'] \arrow[d, equal] & \\
H^*(X) \arrow[r, "\mathrm{ev}_0^*"] & H^*(\Lambda X) &
\end{tikzcd}$$

and hence $\delta(xy) = \tau_1^\infty \circ \mathrm{ev}_1^*(1 \otimes (1+T)x \otimes y)$ *and* $\delta(x^2) = 0$.

Proof. The upper square is obviously commutative. The lower square is induced by a homotopy commutative diagram. The homotopy

$$H : I \times \Lambda X \to X^2, \qquad (s, \omega) \mapsto (\omega(1), \omega(e^{\pi i s})),$$

has the evaluation map ev_1' at one end and $\Delta \circ \mathrm{ev}_0$ at the other. The last statement follows since $\mathrm{Tr}_0^1(x \otimes y) = 1 \otimes (1+T)x \otimes y$. ∎

PROPOSITION 6.5. *We have*

$$\mathrm{ev}_1^*(1 \otimes x^{\otimes 2}) = \phi(x) + tq(x).$$

Proof. This follows directly from Theorem 4.2. ∎

THEOREM 6.6. *The map* $q_\infty^0 : H^* E_\infty \Lambda X \to H^* \Lambda X$ *maps the above classes as follows*:

$$q(x) \mapsto e(xdx + \lambda x), \qquad \phi(x) \mapsto e(x^2), \qquad \delta(x) \mapsto e(dx).$$

Proof. It follows directly from Proposition 4.8 that $\delta(x)$ is mapped as stated. For the other two classes we use Proposition 4.6:

$$q_\infty^0 \circ \tau_1^\infty \circ \mathrm{ev}_1^* = \tau_1^\infty \circ \theta_1^* \circ \mathrm{ev}_1^* = \tau_1^\infty \circ (\mathrm{ev}_1 \circ \theta_1)^*.$$

Let $f = f_{\Lambda X}$ be the map from Definition 4.9. We see that $\mathrm{ev}_1 \circ \theta_1$ equals the composite

$$S^1/C_2 \times \Lambda X \xrightarrow{f} ES^1 \times_{C_2} (\Lambda X)^2 \xrightarrow{1 \times \mathrm{ev}_0^2} ES^1 \times_{C_2} X^2.$$

Thus we have

$$q_\infty^0 \circ \tau_1^\infty \circ \mathrm{ev}_1^* = \tau_1^\infty \circ f^* \circ (1 \times \mathrm{ev}_0^2)^*.$$

Applying this and Theorem 4.12 we get

$$q_\infty^0(q(a)) = \tau_1^\infty \circ f^*(1 \otimes \mathrm{ev}_0^*(a)^{\otimes 2}) = e(ada + \lambda a),$$
$$q_\infty^0(\phi(a)) = \tau_1^\infty \circ f^*(t \otimes \mathrm{ev}_0^*(a)^{\otimes 2}) = e(a^2). \ ∎$$

PROPOSITION 6.7. *The subalgebra of* $H^* E_\infty \Lambda X$ *generated by the classes* $\delta(x)$, $\phi(x)$, $q(x)$ *for* $x \in H^* X$ *and* u *is closed under the* $\mathcal{A}$*-action. The action* *is explicitly given by*

$$\mathrm{Sq}^i\,\delta(x) = \delta(\mathrm{Sq}^i\,x), \tag{24}$$

$$\mathrm{Sq}^i\,\phi(x) = \sum_{j\ge 0}\binom{|x|+1-j}{i-2j} \times u^{[(i+1)/2]-j}((i+1)\phi(\mathrm{Sq}^j\,x) + iq(\mathrm{Sq}^j\,x)), \tag{25}$$

$$\mathrm{Sq}^i\,q(x) = \sum_{j\ge 0}\binom{|x|-j}{i-2j} \times u^{[i/2]-j}(i\phi(\mathrm{Sq}^j\,x) + (i+1)q(\mathrm{Sq}^j\,x)) + \delta(Q^i(x)). \tag{26}$$

By convention a binomial coefficient is zero when its lower parameter is negative. The Q^i operation in the last formula is defined by

$$Q^i(x) = \sum_{r=0}^{[i/2]} \mathrm{Sq}^r(x)\,\mathrm{Sq}^{i-r}(x).$$

P r o o f. The first formula (24) is obvious. For the other two we use the formula for the $\mathcal{A}$-action on $H^*(ES^1 \times_{C_2} X^2)$ which can be found e.g. in [Milgram]:

$$\mathrm{Sq}^i(t^k \otimes x^{\otimes 2}) = \sum_{j\ge 0}\binom{k+|x|-j}{i-2j} t^{k+i-2j} \otimes (\mathrm{Sq}^j\,x)^{\otimes 2} + \delta_{k,0}\sum_{r=0}^{[(i-1)/2]} 1 \otimes (1+T)\,\mathrm{Sq}^r\,x \otimes \mathrm{Sq}^{i-r}\,x.$$

Here $\delta_{i,j}$ is the Kronecker delta. The second formula (25) follows directly and the last formula (26) follows by applying Lemma 6.4. ∎

THEOREM 6.8. *Let X be a connected space with connected free loop space ΛX. Then there are natural algebra homomorphisms*

$$e : \Omega_\lambda(H^*X) \to H^*(\Lambda X), \qquad \psi : \ell(H^*X) \to H^*(ES^1 \times_{S^1} \Lambda X).$$

The first is defined by $e(x) = \mathrm{ev}_0^*(x)$ and $e(dx) = d\,\mathrm{ev}_0^*(x)$. The second is defined by $\psi(u) = u$ and

$$\psi(\delta(x)) = \tau_0^\infty \circ \mathrm{ev}_0^*(x),$$
$$\psi(\phi(x)) = \tau_1^\infty \circ \mathrm{ev}_1^*(t \otimes x^{\otimes 2}),$$
$$\psi(q(x)) = \tau_1^\infty \circ \mathrm{ev}_1^*(1 \otimes x^{\otimes 2}).$$

P r o o f. We have already defined the map e. To establish the map ψ we must show that the classes $\delta(x)$, $\phi(x)$, $q(x)$ from Definition 6.3 satisfy the relations (13)–(23) from the definition of the ℓ-functor.

(13) We have the expansion

$$1\otimes(a+b)^{\otimes 2}=1\otimes a^{\otimes 2}+1\otimes b^{\otimes 2}+1\otimes(1+T)a\otimes b;$$

if we multiply this equation by t the norm element goes to zero and the relation follows.

(14) is obvious.

(15) We apply $\tau_1^\infty\circ \mathrm{ev}_1^*$ to the expansion and identify the last term by Lemma 6.4.

(16) By Frobenius reciprocity we get the first equality in the formula below, and by Proposition 4.8 we get the second equality:

$$\begin{aligned}\tau_0^\infty\circ \mathrm{ev}_0^*(xy)\tau_0^\infty\circ \mathrm{ev}_0^*(z)&=\tau_0^\infty(q_\infty^0\circ\tau_0^\infty\circ \mathrm{ev}_0^*(xy)\,\mathrm{ev}_0^*(z))\\&=\tau_0^\infty(d(\mathrm{ev}_0^*(xy))\,\mathrm{ev}_0^*(z)).\end{aligned}$$

From this we see that the right hand side of (16) equals $\tau_0^\infty\circ e$ applied to

$$d(xy)z+d(yz)x+d(zx)y.$$

Since this element is zero the relation follows.

(17) & (18) The map ev_1^* is a ring homomorphism. Thus we have

$$\mathrm{ev}_1^*(1\otimes(ab)^{\otimes 2})=\mathrm{ev}_1^*(1\otimes a^{\otimes 2})\,\mathrm{ev}_1^*(1\otimes b^{\otimes 2}).$$

This and Proposition 6.5 give the result (like $e^{i\theta}=\cos\theta+i\sin\theta$).

(19) & (20) These are special cases of Proposition 6.7.

(21) We first use Frobenius reciprocity and then Theorem 6.6:

$$\phi(b)\delta(a)=\tau_0^\infty(q_\infty^0(\phi(b))\,\mathrm{ev}_0^*(a))=\tau_0^\infty(\mathrm{ev}_0^*(b^2a))=\delta(ab^2).$$

(22) Again we first use Frobenius reciprocity and then Theorem 6.6:

$$q(a)\delta(b)=\tau_0^\infty(q_\infty^0(q(b))\,\mathrm{ev}_0^*(a))=\tau_0^\infty(\mathrm{ev}_0^*(ab)d\,\mathrm{ev}_0^*(b)+\mathrm{ev}_0^*(a\lambda(b))).$$

The second term gives $\delta(a\lambda b)$ and if we use 4.8 to replace the d by $q_\infty^0\circ\tau_0^\infty$ in the first term we see by Frobenius reciprocity that it gives $\delta(ab)\delta(b)$.

(23) We use Lemma 6.4. When we multiply a norm element by t^2 we get zero. ■

The functors $\Omega_\lambda(H^*X)$ and $\ell(H^*x)$ are viewed as approximations to $H^*(\Lambda X)$ and $H^*(ES^1\times_{S^1}\Lambda X)$ respectively via these maps.

We can motivate the definition of the functor $\mathcal{L}$ from 5.3 as follows. Since the composite $q_0^\infty\circ\psi$ vanishes on the ideal $(u)\subseteq\ell(H^*X)$, it induces a map

$$\mathcal{L}(H^*X)\to \mathrm{Im}(q_0^\infty: H^*(ES^1\times_{S^1}\Lambda X)\to H^*(\Lambda X)).$$

Hence $\mathcal{L}(H^*X)$ may be viewed as an approximation to the image of q_0^∞ via this map.

7. Relations with the Steenrod diagonal. Assume that X is a connected space with connected free loop space. Let $i: X\to\Lambda X$ denote the map which sends a point x in X to the constant loop with value x. This is an S^1-equivariant map when X is given the trivial S^1-action. Thus $1 \times i$ defines maps

$$i_n : E_n \Lambda X \to BC_{2^n} \times X, \quad n \geq 0, \qquad i_\infty : E_\infty \Lambda X \to BS^1 \times X.$$

PROPOSITION 7.1. *For each integer $m \geq 1$ there is a commutative diagram*

$$\begin{array}{ccc} H^*(E_\infty \Lambda X) & \xrightarrow{i_\infty^*} & H^*(BS^1) \otimes H^*(X) \\ \uparrow{\scriptstyle \tau_m^\infty} & & \uparrow{\scriptstyle \tau_m^\infty \otimes 1} \\ H^*(E_m \Lambda X) & \xrightarrow{i_m^*} & H^*(BC_{2^m}) \otimes H^*(X) \end{array}$$

P r o o f. There is a commutative square

$$\begin{array}{ccc} BC_{2^m} \times X & \xrightarrow{Bj \times 1} & BS^1 \times X \\ \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle \mathrm{pr}_1} \\ BC_{2^m} & \xrightarrow{Bj} & BS^1 \end{array}$$

If we take $Y = \Lambda X$ in Theorem 4.2, it maps into the square (7) by the maps i_m and i_∞ which sit over the respective identity maps. The resulting cube is commutative. From the cube we get a commutative diagram

$$\begin{array}{ccc} H^* E_m \Lambda X & \xrightarrow{i_m^*} & H^*(BC_{2^m} \times X) \\ \uparrow{\scriptstyle \theta} & & \uparrow \\ H^*(BC_{2^m}) \otimes_{H^*(BS^1)} H^*(E_\infty \Lambda X) & \xrightarrow{1 \otimes i_\infty^*} & H^*(BC_{2^m}) \otimes_{H^*(BS^1)} H^*(BS^1 \times X) \end{array}$$

Here θ is an isomorphism and it is easy to see that the right vertical map is also an isomorphism. The result follows by passing to the direct limit over transfer maps. ∎

Note that $\mathrm{ev}_1 \circ i_1$ is the diagonal map. Hence

$$i_1^* \circ \mathrm{ev}_1^*(t^k \otimes x^{\otimes 2}) = \sum_{j \geq 0} t^{k+j} \otimes \mathrm{Sq}^{|x|-j} x \tag{27}$$

according to the definition of the Steenrod squares.

PROPOSITION 7.2. *Let X be a connected space with connected free loop space. Then the composite map*

$$\ell(H^* X) \xrightarrow{\psi} H^*(ES^1 \times_{S^1} \Lambda X) \xrightarrow{i_\infty^*} \mathbb{F}_2[u] \otimes H^* X$$

is given by $\delta(x) \mapsto 0$ *and*

$$\phi(x) \mapsto \sum_{i\ge 0} u^i \otimes \mathrm{Sq}^{|x|-2i}\, x, \qquad q(x) \mapsto \sum_{i\ge 0} u^i \otimes \mathrm{Sq}^{|x|-2i-1}\, x.$$

Proof. The classes $\phi(x)$ and $q(x)$ are mapped as stated by (27) and Proposition 7.1. By Lemma 6.4 we find

$$\begin{aligned} i_\infty^*(\delta(x)) &= i_\infty^* \circ \tau_1^\infty \circ \mathrm{ev}_1^*(1 \otimes (1+T)x \otimes 1) \\ &= (\tau_1^\infty \otimes 1) \circ i_1^* \circ \mathrm{ev}_1^*(1 \otimes (1+T)x \otimes 1). \end{aligned}$$

This is zero since the Steenrod diagonal maps norm elements to zero (see [St-Ep]). ∎

As a consequence the approximation is nontrivial in many cases. The additive map $\phi : H^*X \to H^*(ES^1 \times_{S^1} \Lambda X)$ is e.g. always injective when restricted to the subalgebra of H^*X consisting of all even-dimensional classes.

8. De Rham cohomology. In this section we relate the $\mathcal{L}$-functor to the de Rham cycles. This will be used later in the proof of our main result, Theorem 10.1.

There is a filtration on $\Omega_\lambda(\mathbf{A})$ induced by giving each element in $\mathbf{A}$ itself the filtration 0, and the symbol $a_0(da_1)\ldots(da_n)$ the filtration n. The multiplication decreases filtration, even if it does not necessarily preserve it.

The corresponding graded algebra can be identified with the de Rham complex corresponding to the trivial derivation λ, $\lambda(a) = 0$.

The differential increases the filtration above by 1, but it does not preserve the corresponding grading.

As usual, the composite of the de Rham differential with itself is zero, so we can consider the cohomology groups. Sometimes these can be computed explicitly.

We consider another version of the de Rham complex. Let $\widetilde{\Omega}_\lambda(\mathbf{A})$ be the algebra generated by the same generators as $\mathbf{A}$, satisfying (10) and (11) but instead of (12) the relation

$$(da)^2 = \lambda(a). \tag{28}$$

Note that this is not a homogeneous relation!

Let us now define a Frobenius map. It is a multiplicative map, defined on generators by

$$\Phi : \widetilde{\Omega}_\lambda(\mathbf{A}) \to H^*(\Omega_\lambda(\mathbf{A})), \quad \Phi(a) = a^2, \quad \Phi(da) = ada + \lambda(a).$$

Proposition 8.1. *The above formula defines a ring map. In case* $\mathbf{A}$ *is a polynomial algebra* (*no matter what* λ *is*), $\Phi : \widetilde{\Omega}_\lambda(\mathbf{A}) \to H^*(\Omega_\lambda(\mathbf{A}))$ *is an isomorphism of rings. This also holds when the polynomial algebra has infinitely many generators.*

Proof. The map Φ is a linear map, compatible with the relations in $\widetilde{\Omega}_\lambda$ modulo boundaries, since

$$\begin{aligned}
\Phi(d(a+b)) &= (a+b)d(a+b)+\lambda(a+b)\\
&= d(ab)+ada+bdb+\lambda(a)+\lambda(b)\\
&= d(ab)+\Phi(da)+\Phi(db),\\
\Phi(d(ab)) &= abd(ab)+\lambda(ab) = a^2bdb+b^2ada+a^2\lambda(b)+b^2\lambda(a)\\
&= \Phi(a)\Phi(db)+\Phi(da)\Phi(b),\\
\Phi(d(a)^2) &= (ada+\lambda(a))^2 = a^2d\lambda(a)+\lambda(a)^2\\
&= d(a^2\lambda(a))+\Phi(\lambda(a)).
\end{aligned}$$

The map goes into the homology, since d is a derivation, $d\Phi(a)=0$ and $d\Phi(da)=d(ada+\lambda(a))=(da)^2+d\lambda(a)=0$. Let $\mathbf{A}$ be a polynomial algebra on generators $a_1,\ldots,a_n$. Assume first that $\lambda=0$. Then $\Omega^*_\lambda(\mathbf{A})$ and $\widetilde{\Omega}^*_\lambda(\mathbf{A})$ are both isomorphic to the tensor product of the differential graded algebras $\mathbb{F}_2[a_i,da_i]/(da_i)^2=0$. By the Künneth formula it suffices to check the $n=1$ case, which is easily done by inspection.

Let $\mathbf{A}=\mathbb{F}_2[a_1,\ldots,a_n]$, and λ be an arbitrary derivation over Frobenius. In this general case, we compare the chain complex Ω_λ with the corresponding graded complex

$$\ldots \to F^i/F^{i-1} \xrightarrow{d} F^{i+1}/F^i \to \ldots$$

This graded complex is the complex of the case $\lambda=0$. We can define a corresponding filtration on $\widetilde{\Omega}_\lambda$ and the map Φ is filtration preserving. By the above argument, Φ induces an isomorphism between graded rings.

It follows that if $z_i\in F^i(\Omega_\lambda)$ is a cycle, we can find an $a\in F^i\widetilde{\Omega}_\lambda$ and an $x\in F^{i-1}\Omega_\lambda$ so that $z_{i-1}=z_i-\Phi(a)+dx$ is contained in $F^{i-1}\Omega_\lambda$. Moreover, z_{i-1} has to be a cycle. So by induction, we conclude that any cycle in $F^i\Omega_\lambda$ is homologous to an element in the image of Φ. A similar argument proves injectivity.

Since both functors in $\mathbf{A}$ commute with direct limits, the statement also holds in the case of infinitely many polynomial generators. ∎

Theorem 8.2. *There is a well-defined map $\Psi:\Omega_\lambda(\mathbf{A})\to\mathcal{L}\mathbf{A}$ defined by*

$$\Psi(a_0da_1\ldots da_n)=\delta(a_0)\delta(a_1)\ldots\delta(a_n).$$

There is a ring map $R:\mathcal{L}\mathbf{A}\to\Omega_\lambda(\mathbf{A})$ given on generators by

$$R(\phi(a))=a^2,\qquad R(q(a))=ada+\lambda a,\qquad R(\delta(a))=da.$$

Note that $d\circ R=0$.

Proof. We only have to check relations. For the map Ψ is suffices to check that

$$\begin{aligned}&\Psi(a_0da_1\dots da_{i-1}d(a_ia_{i+1})da_{i+2}\dots da_n)\\&\quad=\Psi(a_0a_ida_1\dots da_{i-1}da_{i+1}\dots da_n+a_0a_{i+1}da_1\dots da_ida_{i+2}\dots da_n).\end{aligned}$$

But this is a consequence of (16):

$$\delta(a_0)\delta(a_ia_{i+1})+\delta(a_0a_i)\delta(a_{i+1})+\delta(a_0a_{i+1})\delta(a_i)=0.$$

For the map R, there are more relations to check. Again, (13), (14), (17), (21) and (23) are obvious from linearity. The rest needs small computations:

$$\begin{aligned}R(q(a+b)+q(a)+q(b)+\delta(ab))&=(a+b)d(a+b)+\lambda(a+b)\\&\quad+ada+\lambda(a)+bdb+\lambda(b)+d(ab)\\&=adb+bda+d(ab)=0,\\R(\delta(ab)\delta(c)+\delta(bc)\delta(a)+\delta(ca)\delta(b))&\\&=d(ab)dc+d(bc)da+d(ca)db=0,\\R(q(ab)+q(a)\phi(b)+\phi(a)q(b))&=abd(ab)+\lambda(ab)+(ada+\lambda(a))b^2\\&\quad+a^2(bdb+\lambda(b))\\&=a^2bdb+b^2ada+b^2\lambda(a)+a^2\lambda(b)\\&\quad+ab^2da+b^2\lambda(a)+a^2bdb\\&\quad+a^2\lambda(b)=0,\\R(\delta(a)^2+\delta(\lambda(a)))&=(dx)^2+d(\lambda x)=0,\\R(q(a)^2+\phi(\lambda(a))+\delta(a^2\lambda(a)))&=(ada+\lambda(a))^2+(\lambda(a))^2+d(a^2\lambda(a))\\&=a^2(da)^2+\lambda(a)^2+\lambda(a)^2+a^2d\lambda(a)=0,\\R(\delta(a)\phi(b)+\delta(ab^2))&=(da)b^2+d(ab^2)=0\\R(\delta(a)q(b)+\delta(a\lambda(b))+\delta(ab)\delta(b))&=d(a)(bdb+\lambda(b))+d(a\lambda(b))+d(ab)d(b)\\&=bdadb+da\lambda(b)+da\lambda(b)+ad(\lambda(b))\\&\quad+a(db)^2+bdadb=0.\ \blacksquare\end{aligned}$$

The composition of these maps is not so hard to compute. Let $z=a_0da_1\dots da_n$. Then

$$R\circ\Psi(z)=R\circ\Psi(a_0da_1\dots da_n)=R(\delta(a_0)\dots\delta(a_n))=da_0\dots da_n=dz.$$

Moreover, we have

THEOREM 8.3. *Let Φ be as above. Then the image of Ψ is an ideal in $\mathcal{L}(\mathbf{A})$, and the quotient ring is isomorphic to $\Omega_\lambda(\mathbf{A})$. The isomorphism is realized by a surjective ring map $P:\mathcal{L}(\mathbf{A})\to\widetilde{\Omega}_\lambda(\mathbf{A})$ satisfying*

$$P(\delta(a))=0,\qquad P(\phi(a))=a,\qquad P(q(a))=da.$$

In particular, the map $R : \mathcal{L}(\mathbf{A}) \to \Omega_\lambda(\mathbf{A})$ discussed above factors after projection to $\Omega_\lambda(\mathbf{A})/d\Omega_\lambda(\mathbf{A})$ as $\Phi \circ P$.

Proof. The image of Ψ clearly consists of all sums of products of elements of the form $\delta(a)$. By using relations (21) and (22), we see that these classes do indeed form an ideal, and the quotient $\mathcal{L}(\mathbf{A})/\mathrm{Im}(\Psi)$ is generated by the classes $\phi(a)$ and $q(a)$. The relations these symbols satisfy are given by setting $\delta(a)$ equal to 0 in the list of relations above. We obtain

$$\phi(a+b) = \phi(a) + \phi(b), \quad q(a+b) = q(a) + q(b), \quad \phi(ab) = \phi(a)\phi(b),$$
$$q(ab) = \phi(a)q(b) + q(a)\phi(b), \quad q(a)^2 = \phi(\lambda(a)).$$

It is immediately clear that the formulas above for P define a ring homomorphism and that $\mathrm{Im}\,\Psi \subseteq \ker P$. On the other hand, it is easy to define an inverse homomorphism $P^{-1} : \widetilde{\Omega}_\lambda(\mathbf{A}) \to \mathcal{L}(\mathbf{A})/\mathrm{Im}(\Psi)$ by $P^{-1}(a) = \phi(a)$, $P^{-1}(da) = q(a)$. The claimed factorization of R can be checked on generators:

$$\begin{aligned}
&\Phi \circ P(\phi(a)) = \Phi(a) = a^2 = R(\phi(a)),\\
&\Phi \circ P(\delta(a)) = \Phi(0) = 0 = da + da = R(\delta(a)) + da,\\
&\Phi \circ P(q(a)) = \Phi(da) = ada + \lambda(a) = R(q(a)). \ \blacksquare
\end{aligned}$$

THEOREM 8.4. *The composite $\Psi \circ d : \Omega_\lambda(\mathbf{A}) \to \mathcal{L}(\mathbf{A})$ is trivial, so we can define Ψ as a map on $\Omega_\lambda(\mathbf{A})/\mathrm{Im}(d)$. This allows us to consider the composite $\Psi \circ \Phi : \widetilde{\Omega}_\lambda(\mathbf{A}) \to \mathcal{L}(\mathbf{A})$. This composite is zero.*

Proof. The first claim is easy, since the image of d is the classes of the form $da_1da_2\ldots da_n$. And by definition, Ψ vanishes on those classes.

To show the second claim, we need the following rules: Let $b \in \mathbf{A}$, and $z = a_0da_1\ldots da_n \in \widetilde{\Omega}_\lambda$. Then

$$\Psi(\Phi(b)z) = \phi(b)\Psi(z), \quad \Psi(\Phi(db)z) = q(b)\Psi(z).$$

The first follows from (21), the second is a calculation, using (22) at an essential point:

$$\begin{aligned}
\Psi(\Phi(db)a_0da_1\ldots da_n) &= \Psi((bdb + \lambda(b))a_0da_1\ldots da_n))\\
&= \Psi((ba_0)dbda_1\ldots da_n + \lambda(b)a_0da_1\ldots da_n)\\
&= (\delta(ba_0)\delta(b) + \delta(\lambda(b)a_0))\delta(a_1)\ldots\delta(a_n)\\
&= \delta(a_0)q(b)\delta(a_1)\ldots\delta(a_n)\\
&= q(b)\Phi(z).
\end{aligned}$$

Now the theorem follows from these two formulas and the observation that $\Psi(1) = 0$. $\blacksquare$

We can collect the information we have gathered so far in a diagram:

$$\begin{tikzcd}
 & \widetilde{\Omega}_\lambda(\mathbf{A}) \arrow[d, "\Phi"'] & & \\
\Omega_\lambda(\mathbf{A}) \arrow[r] \arrow[rrd, "d"'] & \Omega_\lambda(\mathbf{A})/d\Omega_\lambda(\mathbf{A}) \arrow[r, "\Psi"] & \mathcal{L}(\mathbf{A}) \arrow[r, "P"] \arrow[d, "R"] & \widetilde{\Omega}_\lambda(\mathbf{A}) \arrow[d, "\Phi"] \\
 & & \Omega_\lambda(\mathbf{A}) \arrow[r] & \Omega_\lambda(\mathbf{A})/d\Omega_\lambda(\mathbf{A})
\end{tikzcd}$$

The results above says that the right hand square commutes, the composite $\Psi \circ \Phi$ vanishes, $\ker P = \operatorname{Im} \Psi$ and finally the composite map from $\Omega_\lambda(\mathbf{A})$ to itself is the de Rham differential.

THEOREM 8.5. *Assume that the map $\Phi : \widetilde{\Omega}_\lambda(A) \to H^*(\Omega_\lambda(\mathbf{A}))$ is an isomorphism. Then so is*

$$R : \mathcal{L}(\mathbf{A}) \to \ker\{d : \Omega_\lambda(\mathbf{A}) \to \Omega_\lambda(\mathbf{A})\}.$$

Proof. We first prove that R is surjective. By assumption, every cycle in $\Omega_\lambda(\mathbf{A})$ is the sum of a boundary and an element in the image of Φ. But since P is surjective, this says that every element in the kernel of d is the sum of a boundary and an element of the image of R. On the other hand, d factors over R, so R is surjective itself.

Now we prove injectivity. Assume that $z \in \mathcal{L}(\mathbf{A})$, and that $R(z) = 0$. Since Φ is injective, $P(z) = 0$, and $z = \Psi(y)$ for some $y \in \Omega_\lambda(\mathbf{A})$. But then $dy = 0$, so that y is a cycle. Using again the fact that Φ is surjective onto the de Rham cohomology, we can write y as the sum of a boundary and an element $\Phi(x)$, where $x \in \widetilde{\Omega}_\lambda$. But both summands map to zero in $\mathcal{L}(\mathbf{A})$, so $z = 0$. ∎

9. Cohomology of free loop spaces. The Eilenberg–Moore spectral sequence can sometimes be used to compute the cohomology of free loop spaces. According to [Smith81] there is a fiber square $\mathcal{F}(X)$ of the following form:

$$\begin{tikzcd}
\Lambda X \arrow[r, "\mathrm{ev}"] \arrow[d, "\mathrm{ev}"'] & X \arrow[d, "\Delta"] \\
X \arrow[r, "\Delta"] & X \times X
\end{tikzcd}$$

with common fiber ΩX. Assume that X is a 1-connected space and let $R^* = H^*X$. The Eilenberg–Moore spectral sequence associated with $\mathcal{F}(X)$ is a second quadrant spectral sequence of cohomology type. It has the form

$$E_2^{-p,q} = \operatorname{Tor}_p^{R^* \otimes R^*}(R^*, R^*)^q \Rightarrow H^*(\Lambda X)$$

where the action of $R^*\otimes R^*$ on R^* is via the multiplication map $\mu : R^*\otimes R^*\to R^*$. Since R^* is a commutative ring the E_2-term can be expressed by Hochschild homology:

$$E_2^{-p,q}=HH_p(R^*)^q.$$

If we assume that R^* is a *smooth* $\mathbb{F}_2$-algebra we get the following isomorphism by the Hochschild–Konstant–Rosenberg theorem ([Loday], p. 102):

$$HH_*(R^*)\cong \Omega^*(R^*).$$

Here $\Omega^*(R^*)$ is the graded exterior differential module of the $\mathbb{F}_2$-algebra R^*; see [Loday], p. 26. Since all algebra generators sit in $E_2^{0,*}$ or $E_2^{-1,*}$ the spectral sequence collapses, i.e. $E_2=E_\infty$.

Since X is 1-connected we have zeros below the line $q=-2p$ in the E_2-term. Thus the decreasing filtration F^iH^* of $H^*:=H^*(\Lambda X)$ is bounded:

$$H^n=F^{-n}H^n\supseteq F^{-n+1}H^n\supseteq\ldots\supseteq F^0H^n\supseteq 0,$$
$$E_\infty^{-p,n+p}=F^{-p}H^n/F^{-p+1}H^n.$$

Note also that $F^0H^n=R^n$.

10. Spaces with polynomial cohomology

Theorem 10.1 *Let X be a 1-connected space. Assume that $H^*X=S(V)$ is a symmetric algebra on the graded vector space V, where V^i is finite-dimensional for each i. Then the two ring homomorphisms*

$$e:\Omega_\lambda(H^*X)\to H^*(\Lambda X),\qquad \psi:\ell(H^*X)\to H^*(ES^1\times_{S^1}\Lambda X)$$

are isomorphisms of unstable $\mathcal{A}$-algebras, where the $\mathcal{A}$-action on $\Omega_\lambda(H^*X)$ is given by $\mathrm{Sq}^i(dx)=d(\mathrm{Sq}^i x)$ and the $\mathcal{A}$-action on $\ell(H^*X)$ is given by the formulas of Proposition 6.7.

Proof. We first show that e is an isomorphism. Since any polynomial algebra is smooth we have an isomorphism of graded $\mathbb{F}_2$-vector spaces

$$H^*(\Lambda X)\cong E_\infty^{**}\cong \Omega^*(R^*).$$

Let $x\in V$. We first observe that $e(x)$ represents x in $E_\infty^{0,*}$. This is a consequence of the construction of the Eilenberg–Moore spectral sequence. The composite

$$R^*\otimes_{R^*\otimes R^*}R^*=\mathrm{Tor}_0^{R^*\otimes R^*}(R^*,R^*)=E_2^{0,*}\to E_\infty^{0,*}\subseteq H^*(\Lambda X)$$

is given by $a\otimes b\mapsto \mathrm{ev}_0^*(a)\mathrm{ev}_0^*(b)$ since we consider the spectral sequence associated with the fiber square $\mathcal{F}(X)$. From this we see that $\mathrm{ev}_0^*(x)$ represents x in $E_\infty^{0,*}$.

Next we show that $e(dx)$ represents dx in $E_\infty^{-1,*}$. Put $n=|x|$ and choose a map $f:X\to K(\mathbb{F}_2,n)$ such that $f^*(\iota_n)=x$. Clearly, $\mathrm{ev}_0\circ\Lambda f=f\circ\mathrm{ev}_0$ where Λf is the map between the corresponding free loop spaces. By

Proposition 4.8 we see that d commutes with the cohomology of such maps and thus

$$d \circ \mathrm{ev}_0^*(x) = d \circ \mathrm{ev}_0^* \circ f^*(\iota_n) = d \circ (\Lambda f)^* \circ \mathrm{ev}_0^*(\iota_n) = (\Lambda f)^* \circ d \circ \mathrm{ev}_0^*(\iota_n).$$

By naturality of the Eilenberg–Moore spectral sequence it is enough to show that $d\,\mathrm{ev}_0^*(\iota_n)$ represents $d\iota_n$ in $E_\infty^{-1,n}$. But $H^{n-1}(\Lambda K(\mathbb{F}_2, n))$, which additively equals $(\Omega^* H^*(K(\mathbb{F}_2, n)))^{n-1}$, is one-dimensional. Thus it is enough to show that $d\,\mathrm{ev}_0^*(\iota_n) \neq 0$. This is done in Lemma 10.2 below.

Note that $\Omega_\lambda^*(R^*)$ is generated by

$$\{dx_1 \ldots dx_r \mid x_s \in S,\ x_i \neq x_j \text{ for } i \neq j,\ r \geq 0\}$$

as an R^*-module. This follows since each time we replace a square $(dx)^2$ in a monomial by $d(\lambda x)$ the number of d's in any of the monomials in the resulting sum is lowered by one. Thus the replacement process terminates. Define $\mathcal{F}^{-p} \subseteq \Omega_\lambda^*(R^*)$ as the subspace generated by

$$\{dx_1 \ldots dx_r \mid x_s \in S,\ x_i \neq x_j \text{ for } i \neq j,\ r \leq p\}$$

as an R^*-module. Then we have a decreasing filtration $\mathcal{F}^i$, $i \leq 0$, of $\Omega_\lambda^*(R^*)$ satisfying $e(\mathcal{F}^i) \subseteq F^i H^*$. The map induced by e between the corresponding graded objects is clearly an isomorphism, hence so is e. The statement about the $\mathcal{A}$-action follows directly by Proposition 3.2.

We now prove that ψ is an isomorphism. By Proposition 3.3 we can write the E_3-term of the Leray–Serre spectral sequence of the fibration $\Lambda X \to E_\infty X \to BS^1$ as

$$E_3^{**} = \ker(d) \oplus uH(d) \oplus u^2 H(d) \oplus \ldots$$

By Proposition 8.1 and Theorem 8.5 the map R is an isomorphism. This together with Theorem 6.6 makes the spectral sequence collapse, i.e. $E_2 = E_\infty$. It is then easy to check that ψ is an isomorphism. The statement about the $\mathcal{A}$-action is immediate by Proposition 6.7. ■

LEMMA 10.2. *Let $n \geq 2$ and put $K = K(\mathbb{F}_2, n)$ with fundamental cohomology class ι_n. Then the class $d\,\mathrm{ev}_0^*(\iota_n)$ is nonzero in $H^*(\Lambda K)$.*

Proof. By Lemma 3 of [Smith84] we have $K_n \times K_{n-1} \simeq K_n \times \Omega K_n \simeq \Lambda K_n$. We can write $d(\iota_n) = k\iota_{n-1}$ where $k \in \mathbb{F}_2$ and we must show $k = 1$. Consider the composite

$$g : S^1 \times \Lambda K \xrightarrow{\eta} \Lambda K \xrightarrow{\mathrm{ev}_0} K$$

which is simply given by the evaluation $g(z, \omega) = \omega(z)$. The pull back of the fundamental class is $g^*(\iota_n) = 1 \otimes \iota_n + kv \otimes \iota_{n-1}$. Assume that $k = 0$. Since K classifies mod two cohomology in degree n we see that g is homotopic to

$$S^1 \times \Lambda K \xrightarrow{\mathrm{pr}_2} \Lambda K \xrightarrow{\mathrm{ev}_0} K.$$

But then the adjoint of g, which is $\mathrm{id}_{\Lambda K}$, is homotopic to the adjoint of $\mathrm{ev}_0 \circ \mathrm{pr}_2$, which is the map from ΛK to itself sending a loop ω to the constant loop $\omega(1)$. Since this is not the case we have $k = 1$. ∎

11. Appendix: The ℓ functor for nonhomogeneous classes. Actually, the ℓ functor is not really limited to homogeneous elements. Let a be a not necessarily homogeneous element. Write it as the sum of homogeneous elements $a_i \in \mathbf{A}_i$. We define the following elements in $\ell(\mathbf{A})$:

$$\phi\Big(\sum_i a_i\Big) = \sum_i \phi(a_i), \qquad \delta\Big(\sum_i a_i\Big) = \sum_i \delta(a_i),$$
$$q\Big(\sum_i a_i\Big) = \sum_i q(a_i) + \sum_{i<j} \delta(a_i a_j).$$

THEOREM 11.1 *The above classes satisfy all the relations* (13)–(23).

Proof. The relations (13), (14), (16), (17), (19), (21), (23) only involve the additive operations λ, ϕ, $uq(-)$ and δ, and follow immediately from linearity. The relations (15), (18), (20) and (22) involve the quadratic operation q, so they each require a computation.

(15) We compute:

$$\begin{aligned} q\Big(\sum_i a_i + \sum_i b_i\Big) &= q\Big(\sum_i (a_i + b_i)\Big) \\ &= \sum_i q(a_i + b_i) + \sum_{i<j} \delta((a_i + b_i)(a_j + b_j)) \\ &= \sum_i (q(a_i) + q(b_i) + \delta(a_i b_i)) \\ &\quad + \sum_{i<j} (\delta(a_i a_j) + \delta(b_i b_j) + \delta(a_i b_j) + \delta(b_i a_j)) \\ &= q(a) + q(b) + \sum_i \delta(a_i b_i) + \sum_{i<j} (\delta(a_i b_j) + \delta(b_i a_j)) \\ &= q(a) + q(b) + \delta\Big(\Big(\sum_i a_i\Big) \sum_j b_j\Big) \\ &= q(a) + q(b) + \delta(ab). \end{aligned}$$

(18) This is the most complicated computation. We do it in two steps, first assuming that b is a homogeneous element, but $a = \sum_i a_i$. Under this assumption,

$$q(ab) = q\Big(\sum_i a_i b\Big) = \sum_i q(a_i b) + \sum_{i<j} \delta(a_i b a_j b)$$

$$
\begin{aligned}
&= \sum_i (\phi(a_i)q(b) + q(a_i)\phi(b)) + \sum_{i<j} \phi(b)\delta(a_ia_j) \\
&= \Big(\sum_i \phi(a_i)\Big)q(b) + \phi(b)\Big(\sum_i q(a_i) + \sum_{i<j} \delta(a_ia_j)\Big) \\
&= \phi(a)q(b) + q(a)\phi(b).
\end{aligned}
$$

In the general case, we do a very similar calculation. Taking into account that in addition to the special case above, we have also proved relation (15), in the general case we compute

$$
\begin{aligned}
q(ab) = q\Big(\sum_i ab_i\Big) &= \sum_i q(ab_i) + \sum_{i<j} \delta(ab_iab_j) \\
&= \sum_i (q(a)\phi(b_i) + \phi(a)q(b_i)) + \sum_{i<j} \phi(a)\delta(b_ib_j) \\
&= q(a)\phi(b) + \phi(a)q(b).
\end{aligned}
$$

(20) The first step is to notice that

$$
\delta(a^2\lambda(a)) = \delta\Big(\Big(\sum_i a_i\Big)^2 \sum_j \lambda(a_j)\Big) = \delta\Big(\sum_{i,j} a_i^2\lambda(a_j)\Big).
$$

The rest is calculation:

$$
\begin{aligned}
q(a)^2 &= \Big(\sum_i q(a_i) + \sum_{i<j} \delta(a_ia_j)\Big)^2 = \sum_i q(a_i)^2 + \sum_{i<j} \delta(a_ia_j)^2 \\
&= \sum_i (\phi(\lambda(a_i)) + \delta(a_i^2\lambda(a_i))) + \sum_{i<j} \delta(\lambda(a_ia_j)) \\
&= \phi(\lambda(a)) + \delta\Big(\sum_i a_i^2\lambda(a_i) + \sum_{i<j}(a_i^2\lambda(a_j) + a_j^2\lambda(a_i))\Big) \\
&= \phi(\lambda(a)) + \delta\Big(\sum_{i,j} a_i^2\lambda(a_j)\Big) \\
&= \phi(\lambda(a)) + \delta(a^2\lambda(a)).
\end{aligned}
$$

(22) We first observe that the relation is additive in a, so we can as well assume that a is homogeneous. Next we compute

$$
\begin{aligned}
\delta(a)q(b) &= \delta(a)\Big(\sum_i q(b_i) + \sum_{i<j} \delta(b_ib_j)\Big) = \sum_i \delta(a)q(b_i) + \sum_{i<j} \delta(a)\delta(b_ib_j) \\
&= \sum_i \delta(a\lambda(b_i)) + \sum_i \delta(ab_i)\delta(b_i) + \sum_{i<j} \delta(a)\delta(b_ib_j).
\end{aligned}
$$

Now we use relation (16) on the last term! We get

$$\delta(a)q(b) = \sum_i \delta(a\lambda(b_i)) + \sum_i \delta(ab_i)\delta(b_i) + \sum_{i<j}(\delta(b_i)\delta(ab_j) + \delta(b_j)\delta(ab_i))$$
$$= \delta(a\lambda(b)) + \sum_{i,j} \delta(b_i)\delta(ab_j) = \delta(a\lambda(b)) + \delta(ab)\delta(b). \ \blacksquare$$

Given a not necessarily graded algebra $\mathbf{A}$, we can define a new functor $\ell'(\mathbf{A})$ to be the ring generated by the symbols $\delta(a), \phi(a), q(a)$, where we now allow any $a \in \mathbf{A}$, and satisfying relations (13)–(23).

If $\mathbf{A}$ is indeed graded there is a forgetful map

$$F : \ell(\mathbf{A}) \to \ell'(\mathbf{A}).$$

The previous computations can be interpreted as saying that there is a ring map $G : \ell'(\mathbf{A}) \to \ell(\mathbf{A})$, mapping $q(a)$ etc. to the sums defined above. It is easy to check that both $F \circ G$ and $G \circ F$ are the identity on generators, so F and G identify the two constructions.

We therefore see that in this way we can extend ℓ to a functor on all $\mathbb{F}_2$-algebras with derivation over Frobenius.

12. Appendix: Generators and relations for the algebra $\mathcal{L}(A)$. Suppose that we have somehow computed $\mathcal{L}(A)$ for a ring A. Let $B = A/I$. The purpose of this section is to give a description of $\mathcal{L}(B)$. In conjunction with the computation in the previous section of $\mathcal{L}(A)$ for a polynomial ring A this gives a method of computing $\mathcal{L}(B)$ in general.

Let $B = A/I$ where I is an ideal, $p : A \to B$ the natural map sending an element to its equivalence class. Let $\{x_i\}$ be a set of generators for I. Let $\{y_i\}$ be a set of generators of I as a module over A with the Frobenius action, that is, so that any element $m \in I$ can be written as a sum

$$m = \sum b_i^2 y_i.$$

If A is a Noetherian ring, we can always choose the sets $\{x_i\}$ and $\{y_i\}$ to be finite.

THEOREM 12.1. *$\mathcal{L}(B) = \mathcal{L}(A)/J$ where J is the ideal generated by the classes $\phi(x_i)$, $q(x_i)$ and $\delta(y_i)$.*

Proof. From the definition it follows that $\mathcal{L}(A) \to \mathcal{L}(B)$ is surjective. More specifically, $\mathcal{L}(B) \cong \mathcal{L}(A)/J'$ where J' is the ideal generated by $\phi(a) - \phi(b)$, $\delta(a) - \delta(b)$, $q(a) - q(a)$ for any pair of classes a, b which satisfy $p(a) = p(b)$. It is clear that $J \subset J'$.

The idea of the proof is to show that the ideal J' is also generated by the classes listed in the theorem, so that $J = J'$. That is, we have to show that any of the generators for J' that we just listed is also an element of J.

First we note that $\phi(x) \in J$ and $\delta(x) \in J$ for all $x \in I$. This follows from the formulas

$$\phi\Big(\sum a_i x_i\Big) = \sum \phi(a_i)\phi(x_i), \qquad \delta\Big(\sum b_i^2 y_i\Big) = \sum \phi(b_i)\delta(y_i).$$

Assume that $a - b \in I$. By linearity and the above remark we see that $\phi(a) - \phi(b)$ and $\delta(a) - \delta(b)$ lies in J. Hence we only need to verify that $q(a) - q(b) \in J$. We have

$$q(a) - q(b) = q(a-b) + \delta(ab).$$

We check that each of the two terms lies in J. Write $a - b = \sum a_i x_i$. Then

$$q(a-b) = \sum_i q(a_i x_i) + \sum_{i<j} \delta(a_i a_j x_i x_j).$$

Since $q(a_i x_i) = \phi(a_i) q(x_i) + q(a_i)\phi(x_i)$ the first sum is in J. Since $a_i a_j x_i x_j \in I$ the last sum also lies in J by the remark above. Finally, we can use the fact that $\delta(a^2) = 0$ to get $\delta(ab) = -\delta(a(a-b))$. This lies in J since $a(a-b) \in I$. ∎

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Department of Mathematics
University of Aarhus
Ny Munkegade
DK-8000 Århus C, Denmark
E-mail: marcel@mi.aau.dk

Département de Mathématiques
Université de Paris Nord, URA 742 du CNRS
Avenue Jean-Baptiste Clément
F-93430 Villetaneuse, France
E-mail: ottosen@math.univ-paris13.fr

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