FUNDAMENTA
MATHEMATICAE
162 (1999)

Topological entropy on zero-dimensional spaces

by

Jozef Bobok and **Ondřej Zindulka** (Praha)

Abstract. Let X be an uncountable compact metrizable space of topological dimension zero. Given any $a \in [0,\infty]$ there is a homeomorphism on X whose topological entropy is a.

1. Introduction. It is well known that each homeomorphism of the closed interval has topological entropy zero. On the other hand, the two-sided shift of the Cantor cube $2^{\mathbb{Z}}$ has positive entropy.

The Cantor cube is metrizable, zero-dimensional, compact and uncountable. We ask if these features of a metric space are enough to ensure the existence of a homeomorphism with positive entropy.

The answer is affirmative. As a matter of fact, more holds for such a space. If X is a zero-dimensional, compact, uncountable metric space, then for each $a \in [0,\infty]$ there is a *homeomorphism* $T : X \to X$ such that $h_{\text{top}}(T) = a$.

We prove this result in Section 4. Section 2 recalls basic concepts of topological and measure-theoretic entropy. In Section 3 we investigate the local structure of zero-dimensional compacta and set a theorem on extending homeomorphisms as a preparation for the main result. Section 5 contains some counterexamples showing that none of the four conditions that are listed above can be dropped.

2. Preliminaries. The following notation of basic sets is adopted. $\mathbb{Z}$ denotes the set of all integers. The set of all positive integers including zero

1991 *Mathematics Subject Classification*: 54H20, 54C70.

Key words and phrases: dynamical system, topological entropy, homeomorphism, zero-dimensional compact space.

The research was partially supported by Czech Technical University, contract no. J04/98/210000010. The first author was partially supported by Grant Agency of Czech Republic, contract no. 201/97/0001. The second author was partially supported by Grant Agency of Czech Technical University, contract no. 309912121.

is denoted by ω, instead of the more common notation $\mathbb{N}$, because it is often considered an ordinal number. A nonnegative integer $n \in \omega$ is sometimes also considered an ordinal number, i.e. $n = \{0, 1, \ldots, n-1\}$. From the topological point of view, n is a discrete space consisting of exactly n points. 2^ω and $2^\mathbb{Z}$ denote, respectively, the sets of all binary sequences and all binary bisequences. Both are topologically identical to the Cantor set.

We will several times appeal to the following fact. Recall that a topological space is *zero-dimensional* if it has a base consisting of clopen sets, and *perfect* if it has no isolated points.

2.1. Proposition ([6, 6.2.A(c)]). *A perfect, compact, metrizable, zero-dimensional space is homeomorphic to* 2^ω.

We now recall the notion of topological entropy and related material. $|A|$ denotes the cardinality of a set A. If $T : X \to X$ is a mapping and $A \subseteq X$, then TA and $T^{-1}A$ denote, respectively, the image and preimage of A. For $n \in \mathbb{Z}$, the set $T^n A$ is defined as follows:

$$T^n A = \begin{cases} A & \text{if } n = 0, \\ TT^{n-1}A & \text{if } n > 0, \\ T^{-1}T^{n+1}A & \text{if } n < 0. \end{cases}$$

If $\mathcal{A}$ is a family of subsets of X and $n \in \mathbb{Z}$, then

$$T^n\mathcal{A} = \{T^n A : A \in \mathcal{A}\}.$$

For two families of sets $\mathcal{A}$ and $\mathcal{B}$ define

$$\mathcal{A} \vee \mathcal{B} = \{A \cap B : A \in \mathcal{A}, B \in \mathcal{B}\}.$$

2.2. Definition. Let X be a compact space and $\mathcal{U}$ an open cover of X. Define

$$H(\mathcal{U}) = \log \min\{|\mathcal{V}| : \mathcal{V} \text{ is a finite subcover of } \mathcal{U}\}.$$

Let $T : X \to X$ be a continuous mapping. Define

$$h(T, \mathcal{U}) = \lim_{n\to\infty} \frac{1}{n} H(\mathcal{U} \vee T^{-1}\mathcal{U} \vee \ldots \vee T^{-(n-1)}\mathcal{U})$$

(the limit exists, see [8, Theorem 7.1]) and

$$h_{\mathrm{top}}(T) = \sup\{h(T, \mathcal{U}) : \mathcal{U} \text{ is a cover of } X\}.$$

The quantity $h_{\mathrm{top}}(T)$ is called the *topological entropy* of T, or, more precisely, the *topological entropy of the dynamical system* (X, T).

It follows directly from its definition that the topological entropy is an invariant of topological conjugacy. We shall need the following. Recall that the *two-sided shift* $\sigma : n^\mathbb{Z} \to n^\mathbb{Z}$ is defined by $\sigma(\langle x_i : i \in \mathbb{Z}\rangle) = \langle x_{i+1} : i \in \mathbb{Z}\rangle$. It is obviously a homeomorphism.

2.3. Proposition ([1, Lemma 4.1.10]). *Let X be a metrizable space and $T : X \to X$ a continuous mapping. If $X = \bigcup_{n\in\omega} X_n$ and the sets X_n are* *closed and T-invariant, then*

$$h_{\mathrm{top}}(T)=\sup_{n\in\omega} h_{\mathrm{top}}(T\restriction X_n).$$

2.4. THEOREM ([8, Theorem 7.13]). *For each $a>0$ there exists $n\in\omega$ and a perfect, shift-invariant subset $\Omega\subseteq n^{\mathbb{Z}}$ such that $h_{\mathrm{top}}(\sigma\restriction\Omega)=a$.*

2.5. COROLLARY. *For each $a\in[0,\infty]$ there exists a homeomorphism $T:2^\omega\to 2^\omega$ such that $h_{\mathrm{top}}(T)=a$.*

Proof. If $a=0$ let T be the identity map on 2^ω. If $0<a<\infty$, consider the space Ω of Theorem 2.4. By Proposition 2.1 it is homeomorphic to 2^ω. Consider $\sigma\restriction\Omega$ and take for T its conjugate by this homeomorphism. We construct T for the case $a=\infty$. Let $\widetilde{0}\in 2^\omega$ denote the sequence that is identically zero. For each $n\in\omega$ let $p_n\in 2^{n+1}$ be defined by

$$p_n(i)=\begin{cases}0 & \text{if } 0\le i<n,\\ 1 & \text{if } i=n,\end{cases}$$

and $X_n=\{f\in 2^\omega : p_n\subseteq f\}$. Each X_n is homeomorphic to 2^ω, therefore there exists a homeomorphism $T_n:X_n\to X_n$ satisfying $h_{\mathrm{top}}(T_n)=n$. The family $\{X_n : n\in\omega\}\cup\{\{\widetilde{0}\}\}$ obviously forms a disjoint cover of 2^ω. Therefore the following formula defines a mapping $T:2^\omega\to 2^\omega$:

$$T(f)=\begin{cases}T_n(f) & \text{for } n\in\omega,\ f\in X_n,\\ \widetilde{0} & \text{for } f=\widetilde{0}.\end{cases}$$

Since each X_n is a clopen subset of 2^ω and $\operatorname{diam} X_n\to 0$, it follows that T is a homeomorphism. Since each X_n is closed and T-invariant, Proposition 2.3 yields

$$h_{\mathrm{top}}(T)\ge \sup_{n\in\omega} h_{\mathrm{top}}(T\restriction X_n)=\sup_{n\in\omega} h_{\mathrm{top}}(T_n)=\sup_{n\in\omega} n=\infty. \ \blacksquare$$

There is also another, metric-dependent definition of topological entropy due to Bowen [2]. For compact metric spaces it is equivalent to the one given above, but it also makes sense for noncompact spaces.

2.6. DEFINITION. Let (X,ϱ) be a (not necessarily compact) metric space and let $T:X\to X$ be a uniformly continuous mapping. For each $n>0$ the function $\varrho_n : X\times X\to\mathbb{R}$ given by $\varrho_n(x,y)=\max_{0\le i<n}\varrho(T^i(x),T^i(y))$ is a metric on X equivalent to ϱ. A set $E\subset X$ is called (n,ε)-*separated* if $\varrho_n(x,y)>\varepsilon$ for all $x,y\in E$, $x\ne y$. For a compact set $K\subset X$ define $s_n(T,\varepsilon,K)$ to be the maximal cardinality of an (n,ε)-separated subset of K. Put

$$h_\varrho(T,K)=\lim_{\varepsilon\to 0^+}\limsup_{n\to\infty}\frac{1}{n}\log s_n(T,\varepsilon,K)$$

and

$$h_\varrho(T)=\sup\{h_\varrho(T,K) : K\subseteq X \text{ compact}\}.$$

The quantity $h_\varrho(T)$ is called the *Bowen entropy* of T.

2.7. Proposition ([8, Theorem 7.4, Corollary 7.5.2]). *Let* (X,ϱ) *be a metric space and* $T : X \to X$ *a uniformly continuous mapping.*

(i) *If* ϱ' *is a metric on* X *uniformly equivalent to* ϱ, *then* $h_\varrho(T) = h_{\varrho'}(T)$.
(ii) *If* X *is compact*, *then* $h_\varrho(T) = h_{\mathrm{top}}(T)$.

2.8. Proposition ([4, Proposition 14.21(b)]). *If* $X_1, X_2 \subset X$ *satisfy* $T(X_1) = X_1$, $T(X_2) = X_2$ *and* $X_1 \cup X_2 = X$, *then*

$$h_\varrho(T) = \max(h_\varrho(T{\restriction}X_1), h_\varrho(T{\restriction}X_2)).$$

Recall the definition of measure-theoretic entropy. If X is a metrizable space and $T : X \to X$ a Borel measurable mapping, a finite Borel measure on X is called T-*invariant* if $\mu T^{-1}E = \mu E$ for each Borel set E.

2.9. Definition. Let X be a metric space, μ a Borel probability measure in X and $\mathcal{P}$ a Borel partition of X. Define

$$H_\mu(\mathcal{P}) = -\sum_{P\in\mathcal{P}} \mu(P)\log\mu(P).$$

Let $T : X \to X$ be a measurable mapping such that μ is T-invariant. Define

$$h_\mu(T,\mathcal{P}) = \lim_{n\to\infty} \frac{1}{n} H_\mu(\mathcal{P} \vee T^{-1}\mathcal{P} \vee \ldots \vee T^{-(n-1)}\mathcal{P})$$

(the limit exists, see [8, Corollary 4.9.1]) and

$$h_\mu(T) = \sup\{h_\mu(T,\mathcal{P}) : \mathcal{P} \text{ is a finite Borel partition}\}.$$

The quantity $h_\mu(T)$ is called the *measure-theoretic entropy* of T.

We will make use of the following consequence of the so-called Variational Principle. A T-invariant probability measure μ is called T-*ergodic* if $T^{-1}E = E$ implies $\mu E = 0$ or $\mu E = 1$ for each Borel set E.

2.10. Theorem. ([8, Corollary 8.6.1(i)]). *If* X *is a compact metric space and* $T : X \to X$ *a continuous mapping*, *then*

$$h_{\mathrm{top}}(T) = \sup\{h_\mu(T) : \mu \textit{ is a } T\textit{-ergodic probability measure}\}.$$

3. Extending homeomorphisms. The goal of the paper is to find, on a zero-dimensional compact space X, for a prescribed value a, a homeomorphism of entropy a. If we supposed that the space X had no isolated points, then the existence of such a homeomorphism would follow directly from Theorem 2.4. When X has isolated points, the situation is a little more complicated. Using the Cantor–Bendixson Theorem, one can split X into a compact subset C without isolated points and a countable subset S.

The plan is to apply Theorem 2.4 to C and then extend carefully the homeomorphism of C to $C \cup S$ preserving entropy.

In order to do that we build up a taxonomy of points in zero-dimensional compacta. The technique used is akin to that used for instance in [7], where similar results are established. While we consider zero-dimensional locally compact metric spaces, [7] deals with first-countable scattered compacta.

The first goal is to describe the local structure of countable, completely metrizable spaces.

3.1. Definition. Let X be a separable metric space.

(i) For $A \subseteq X$ denote by $\iota(A)$ the set of all isolated points of A and by $A' = A \setminus \iota(A)$ the *derived set* of A, i.e. the set of cluster points of A.

(ii) For $A \subseteq X$ define recursively, for each ordinal α,

$$A^{(0)} = A, \quad A^{(\alpha+1)} = (A^{(\alpha)})', \quad A^{(\alpha)} = \bigcap_{\beta<\alpha} A^{(\beta)} \quad \text{for } \alpha \text{ limit.}$$

3.2. Lemma. *If X is a countable, completely metrizable space, then there is a countable ordinal η such that $X^{(\eta)} = \emptyset$.*

Proof. The set $X^{(\alpha)}$ is obviously closed for each ordinal α. Therefore it is completely metrizable. Thus when it is nonempty, the Baire Category Theorem yields an $x \in X^{(\alpha)}$ such that $\{x\}$ is open in $X^{(\alpha)}$, i.e. x is isolated in $X^{(\alpha)}$. Therefore $X^{(\alpha+1)} \subsetneq X^{(\alpha)}$. It follows that the sequence $\langle X^{(\alpha)} : X^{(\alpha)} \neq \emptyset \rangle$ is strictly decreasing and thus countable. ∎

3.3. Remark. If either of the hypotheses is dropped, the lemma can fail. The real line and the set of rationals are, respectively, examples of an uncountable complete space and a countable incomplete space for which $X^{(\alpha)} = X$ for all α.

3.4. Definition. Let X be a countable, completely metrizable space.

(i) Define the *depth* of X to be the ordinal $\eta_X = \min\{\alpha : X^{(\alpha)} = \emptyset\}$.

(ii) For $x \in X$ define the *type* of x in X to be the ordinal

$$t_X(x) = \min\{\alpha : x \notin X^{(\alpha+1)}\}.$$

Lemma 3.2 obviously implies that $t_X(x)$ is defined for each $x \in X$ and that $t_X(x) < \eta_X$.

3.5. Remark. The depth and type, as well as the sets S_X and C_X defined by (3.2) below, obviously depend on the underlying space X. If X is given and $A \subset X$ is a completely metrizable subspace, then $t_A(x)$, η_A etc. make sense.

3.6. Lemma. *Let X be a countable, completely metrizable space. For each $x \in X$ and each neighborhood V of x there is a neighborhood $U \subseteq V$ such that*

(i) *$t_X(y) < t_X(x)$ for all $y \in U$, $y \neq x$,*

(ii) *for each $\beta < t_X(x)$ there is $y \in U$, $y \neq x$ such that $t_X(y) \geq \beta$.*

Proof. Let $\alpha = t_X(x)$.

(i) By definition, x is an isolated point of $X^{(\alpha)}$. Hence there is an open subset $W \subseteq X$ such that $W \cap X^{(\alpha)} = \{x\}$. Thus if $y \in W$ and $y \neq x$, then $y \notin X^{(\alpha)}$, hence there is $\beta < \alpha$ such that $y \in X^{(\beta)}$, and $t_X(y) < \alpha$ follows. Put $V = W \cap U$.

(ii) Assume on the contrary that there is $\beta < \alpha$ such that $t_X(y) < \beta$ for all $y \in V$, $y \neq x$. Since $t_X(y) < \beta$ yields $y \notin X^{(\beta)}$, it follows that $V \setminus \{x\} \subseteq X \setminus X^{(\beta)}$, which in turn implies that $V \cap X^{(\beta)} \subseteq \{x\}$. If $V \cap X^{(\beta)} = \emptyset$, then $x \notin X^{(\beta)}$ and thus obviously $t_X(x) < \beta$. If $V \cap X^{(\beta)} = \{x\}$, then x is an isolated point of $X^{(\beta)}$. It follows that $x \notin X^{(\beta+1)}$, whence $t_X(x) \leq \beta$. In either case $t_X(x) \leq \beta$, which is the desired contradiction. ∎

Arithmetic of ordinals. We now need a little ordinal arithmetic and topology. Let α and β be ordinals. Recall:

• The sum $\alpha + \beta$ is the unique ordinal whose order type is that of the set $\{0\} \times \alpha \cup \{1\} \times \beta$ provided with the lexicographic order. Informally, $\alpha + \beta$ is the ordinal that one obtains by putting β next to α. Addition of ordinals is associative but not commutative.

• When $\langle \alpha_n : n \in \omega \rangle$ is a sequence of ordinals, then

$$\sum_{n \in \omega} \alpha_n = \sup_{n \in \omega} (\alpha_0 + \alpha_1 + \ldots + \alpha_n).$$

• The product $\alpha \cdot \beta$ of two ordinals is the unique ordinal whose order type is that of the cartesian product $\beta \times \alpha$ provided with the lexicographic order.

• The power ω^α is defined by recursion as follows:

$$\omega^0 = 1, \qquad \omega^{\alpha+1} = \omega^\alpha \cdot \omega, \qquad \omega^\alpha = \sup_{\beta < \alpha} \omega^\beta \quad \text{for } \alpha \text{ limit.}$$

We also define $(\omega^\alpha)^*$ to be the ordinal

$$(\omega^\alpha)^* = \begin{cases} 1 & \text{if } \alpha = 0, \\ \omega^\alpha + 1 & \text{otherwise.} \end{cases} \tag{3.1}$$

We list some basic properties of ω^α. Though most of them hold for any value of α, we only consider countable ordinals. The proofs are elementary.

3.7. PROPOSITION. *Let $\alpha > 0$ be a countable ordinal.*

(i) *ω^α is a limit countable ordinal.*

(ii) $(\omega^\alpha)^* \cdot \omega = \omega^{\alpha+1}$.

Let $\langle \alpha_n : n \in \omega \rangle$ be a sequence of ordinals such that $\alpha = \sup_{n \in \omega} \alpha_n$.

(iii) *If $\beta < \alpha$, then $(\omega^\beta)^* + (\omega^\alpha)^* = (\omega^\alpha)^*$ and $(\omega^\beta)^* + \omega^\alpha = \omega^\alpha$.*

(iv) *If $\alpha = \alpha_n$ for infinitely many $n \in \omega$, then $\sum_{n \in \omega} (\omega^{\alpha_n})^* = \omega^{\alpha+1}$.*

(v) *If $\alpha > \alpha_n$ for all $n \in \omega$, then $\sum_{n \in \omega} (\omega^{\alpha_n})^* = \omega^\alpha$.*

Proof. We only prove (iv), for (v) is proved in a similar manner and the rest is trivial. Let $\langle n(i) : i \in \omega\rangle$ be the increasing enumeration of $\{n \in \omega : \alpha_n = \alpha\}$. Put

$$I_0 = \{0,1,\ldots,n(0)\},\ I_1 = \{n(0)+1,\ldots,n(1)\},\ I_2 = \{n(1)+1,\ldots,n(2)\},\ldots$$

By assumption and (iii),

$$\sum_{n\in I_i}(\omega^{\alpha_n})^* = (\omega^{\alpha_{n(i)}})^* = (\omega^{\alpha})^*.$$

Therefore (ii) yields

$$\sum_{n\in\omega}(\omega^{\alpha_n})^* = \sum_{i\in\omega}\sum_{n\in I_i}(\omega^{\alpha_n})^* = \sum_{i\in\omega}(\omega^{\alpha})^* = (\omega^{\alpha})^*\cdot\omega = \omega^{\alpha+1}. \ \blacksquare$$

Topology of ordinals. Ordinals have a natural topology induced by the well-ordering. If α is an ordinal and $\beta < \alpha$, then β is isolated in the topology if it is a nonlimit ordinal, and if β is a limit, then the sets of the form $(\gamma,\beta]$, where $\gamma < \beta$, form a base of the neighborhood system of β. In other words, the base for the topology is formed by open intervals.

If α is countable, then the topology of ω^α is second countable and thus metrizable, and locally compact; moreover, it is noncompact unless $\alpha = 0$. (Warning: ω^α is *not* a topological product of α many countable discrete spaces!) The topological space $(\omega^\alpha)^*$ defined by (3.1) above is its Aleksandrov one-point compactification. If $\alpha = 0$, then the space $\omega^\alpha = 1$ is obviously compact. Overall, $(\omega^\alpha)^*$ is, for each countable α, the smallest compact space containing ω^α, and due to Proposition 3.7(i) it is a countable ordinal.

In the proof of Lemma 3.10 we shall use the following property of compact spaces. If X is compact, $x \in X$ and $Y = X \setminus \{x\}$ is not compact, then X is the Aleksandrov compactification of Y.

We refer the reader to [6, 3.5.11] for more information on the Aleksandrov one-point compactification.

The following are topological counterparts of Proposition 3.7(iii), (iv) and (v).

3.8. Lemma. *Let α and β be countable ordinals, $\alpha < \beta$. Let K_α and K_β be spaces homeomorphic to $(\omega^\alpha)^*$ and ω^β respectively. Then the topological sum $K_\alpha \oplus K_\beta$ is homeomorphic to ω^β.*

3.9. Lemma. *Let $\langle\alpha_n : n \in \omega\rangle$ be a sequence of countable ordinals and $\alpha = \sup_{n\in\omega}\alpha_n$. Let $\{K_n : n \in \omega\}$ be a family of topological spaces such that K_n is homeomorphic to $(\omega^{\alpha_n})^*$ for all $n \in \omega$ and let $K = \bigoplus_{n\in\omega} K_n$ be the topological sum of K_n's.*

(i) *If $\alpha = \alpha_n$ for infinitely many $n \in \omega$, then K is homeomorphic to $\omega^{\alpha+1}$.*

(ii) *If* $\alpha > \alpha_n$ *for all* $n \in \omega$, *then* K *is homeomorphic to* ω^α.

Proof. For each $n \in \omega$, let $\phi_n : (\omega^{\alpha_n})^* \to K_n$ be the homeomorphism. Let $\phi : \sum_{n\in\omega}(\omega^{\alpha_n})^* \to K$ be the unique common extension of the ϕ_n's. It is clear that ϕ is a homeomorphism. Proposition 3.7(iv) and (v) concludes the proof. ∎

We have enough background to state and prove the main lemma on the local structure.

3.10. Lemma. *Let* X *be a countable*, *locally compact metrizable space and* $x \in X$. *If* $t_X(x) = \alpha$, *then for each neighborhood* V *of* x *there is a clopen neighborhood* $U \subseteq V$ *of* x *that is homeomorphic to* $(\omega^\alpha)^*$.

Proof. If $\alpha = 0$, then x is isolated. Therefore there is nothing to prove. We proceed by induction up to η_X.

Assume that $\alpha > 0$ and that the assertion is true for each $\beta < \alpha$. By Lemma 3.6 there is a neighborhood $U \subseteq V$ of x such that $t_X(y) < \alpha$ for all $y \in U \setminus \{x\}$. As X is countable and locally compact, *mutatis mutandis* U can be assumed to be clopen and compact, and it is of course countable. Let $\langle x_n : n \in \omega\rangle$ be an enumeration of $U \setminus \{x\}$. For each $n \in \omega$ put $\alpha_n = t_X(x_n) < \alpha$.

We inductively construct a countable disjoint clopen cover $\mathcal{W}$ of $U\setminus\{x\}$. By the induction hypothesis there is a clopen neighborhood $W_0 \subseteq U \setminus \{x\}$ of x_0 that is homeomorphic to $(\omega^{\alpha_0})^*$. Put $\mathcal{W}_0 = \{W_0\}$.

Now assume that $n \in \omega$ and that $\mathcal{W}_n$ is already defined. If $x_{n+1} \in \bigcup \mathcal{W}_n$, put $\mathcal{W}_{n+1} = \mathcal{W}_n$. Otherwise, as $U \setminus \{x\}$ is open and $\bigcup \mathcal{W}_n$ is closed, the induction hypothesis yields a neighborhood $W_{n+1} \subseteq U \setminus \{x\} \setminus \bigcup \mathcal{W}_n$ of x_{n+1} that is homeomorphic to $(\omega^{\alpha_{n+1}})^*$. Put $\mathcal{W}_{n+1} = \mathcal{W}_n \cup \{W_{n+1}\}$.

When $\mathcal{W}_n$ is constructed for each $n \in \omega$, put $\mathcal{W} = \bigcup_{n\in\omega} \mathcal{W}_n$. The family $\mathcal{W}$ is obviously disjoint and covers $U\setminus\{x\}$. Each $W \in \mathcal{W}$ is homeomorphic to some $(\omega^{\alpha_n})^*$. Lemma 3.6 ensures that one can apply Lemma 3.9 to conclude that $U \setminus \{x\}$ is homeomorphic to ω^α. Since $\alpha > 0$, it follows in particular that $U \setminus \{x\}$ is a locally compact, noncompact space. As U is compact, it is the Aleksandrov compactification of $U \setminus \{x\}$, which is homeomorphic to $(\omega^\alpha)^*$. The induction step is finished, and so is the proof. ∎

We now attempt to extend the definition of type to an uncountable compact metric space. For a space X put

$$S_X = \bigcup\{U \subseteq X : U \text{ open}, |U| \le \omega\}, \quad C_X = X \setminus S_X. \tag{3.2}$$

We list some properties of S_X and C_X. (i) below, the first part of (ii) and (iii) are trivial, and the last part of (ii) is Proposition 2.1. Recall that a topological space is *locally countable* if each of its points has a countable neighborhood.

3.11. LEMMA. *Let X be an uncountable metrizable space.*

(i) *S_X is open and locally countable.*

(ii) *C_X is closed and perfect. If X is compact and zero-dimensional, then C_X is homeomorphic to 2^ω.*

(iii) *If $U \subseteq X$ is open, then $S_U = U \cap S_X$ and $C_U = U \cap C_X$.*

If X is locally compact, separable and metrizable, then it has a countable base and therefore S_X is countable. The set $S_X \cup \{x\}$, being a union of a closed set and an open set, is a G_δ-set in the locally compact space X. It follows that $S_X \cup \{x\}$ is a countable, completely metrizable space. Therefore the following definition makes sense in view of Definition 3.4.

3.12. DEFINITION. Let X be a locally compact separable metric space. For each $x \in X$ put

$$t_X(x) = t_{S_X \cup \{x\}}(x).$$

3.13. LEMMA. *Let X be a locally compact separable metric space.*

(i) *$t_X(x) \le \eta_{S_X}$ for each $x \in X$.*

(ii) *The set $\{x \in X : t_X(x) \le \alpha\}$ is open for each ordinal α.*

(iii) *If $U \subseteq X$ is open and $x \in U$, then $t_U(x) = t_X(x)$.*

Proof. (i) If $x \in S_X$, then $t_X(x) = t_{S_X}(x)$, so $t_X(x) < \eta_{S_X}$ by the remark preceding Lemma 3.6. If $x \in C_X$, then obviously $(S_X \cup \{x\})^{(\alpha)} \subseteq S_X^{(\alpha)} \cup \{x\}$ for each α, whence $(S_X \cup \{x\})^{(\eta_{S_X})} \subseteq \{x\}$. Therefore $(S_X \cup \{x\})^{(\eta_{S_X}+1)} \subseteq \{x\}' = \emptyset$, and $t_X(x) \le \eta_{S_X}$ follows.

(ii) We prove that if $t_X(x) = \alpha$, then there is a neighborhood U of x such that $t_X(y) \le \alpha$ for each $y \in U$. If $x \in S_X$, then the assertion follows directly from Lemma 3.6. If $x \in C_X$, then there is an open set $U \subseteq X$ such that $U \cap S_X$ satisfies the conditions (i) and (ii) of Lemma 3.6. Let $y \in U$. If $y \in S_X$, then $t_X(y) < \alpha$ by Lemma 3.6(i). Let $y \in C_X$. Assume $t_X(y) > \alpha$. As $U \cap (S_X \cup \{y\})$ is a neighborhood of y in $S_X \cup \{y\}$, Lemma 3.6(ii) yields a $z \in U \cap S_X$ such that $t_X(z) \ge \alpha$, a contradiction.

(iii) Obviously $\iota(U) = \iota(X) \cap U$. Thus $U' = X' \cap U$. A straightforward application of the latter fact yields $U^{(\alpha)} = X^{(\alpha)} \cap U$ for each α, which in turn implies $t_U(x) = t_X(x)$. ■

3.14. LEMMA. *Let X be a locally compact separable metric space and $x \in C_X$.*

(i) *If $t_X(x) = 0$, then $x \notin \overline{S}_X$.*

(ii) *If $t_X(x) = \alpha > 0$, then for each neighborhood $V \subseteq X$ of x there exists an open set $U \subseteq V \cap S_X$ homeomorphic to ω^α such that $\overline{U} = U \cup \{x\}$. In particular, $\overline{U}$ is homeomorphic to $(\omega^\alpha)^*$.*

Proof. (i) If $t_X(x)=0$, then $x\notin(S_X\cup\{x\})'$. *A fortiori* $x\notin(S_X)'$. As $\overline{S}_X=S_X\cup S_X'$, it follows that $x\notin\overline{S}_X$.

(ii) Lemma 3.6 implies that there is a sequence $\langle x_n : n\in\omega\rangle$ in $V\cap S_X$ such that $x_n\to x$, $t_X(x_n)<\alpha$ for all $n\in\omega$ and

(a) if $\alpha=\beta+1$, then $t_X(x_n)=\beta$,
(b) if α is a limit, then $t_X(x_n)\to\alpha$.

In either case, the set $\{x_n : n\in\omega\}$ is closed and discrete in S_X. As metrizable spaces are collectionwise Hausdorff, Lemma 3.10 yields a sequence $\langle U_n : n\in\omega\rangle$ of disjoint clopen compact subsets of S_X such that

(c) U_n is a neighborhood of x_n for each $n\in\omega$,
(d) $\operatorname{diam}U_n\to 0$,
(e) U_n is homeomorphic to $(\omega^{t_X(x_n)})^*$.

Thus (a), (b) and Lemma 3.9 imply that $U=\bigcup_{n\in\omega}U_n\subseteq V\cap S_X$ is an open set that is homeomorphic to ω^α.

Obviously $x\in\overline{U}$. Let $z\in\overline{U}$. There is a sequence $\langle z_j : j\in\omega\rangle$ in U that converges to z. Passing to a subsequence if necessary, we can assume that either the z_j's pass through infinitely many U_n's, and then (d) yields $z_j\to x$, i.e. $z=x$, or else the z_j's stay within one U_n, and then $z\in U_n$, because U_n is closed. We have proved that $\overline{U}=U\cup\{x\}$. ∎

3.15. Lemma. *Let X be a locally compact, zero-dimensional, metrizable space that is not locally countable. Then there is a clopen compact subspace $K\subseteq X$ such that*

(i) *C_K is homeomorphic to 2^ω,*
(ii) *$t_K(x)=\eta_K$ for all $x\in C_K$,*
(iii) *$t_K(x)<\eta_K$ for all $x\in S_K$.*

Proof. By assumption, X has a base consisting of clopen compact sets. At least one of them is uncountable. Thus we may assume without loss of generality that X itself is compact and uncountable.

Consider the ordinal $\eta=\min\{t_X(x) : x\in C_X\}$ and the set $Y=\{x\in X : t_X(x)\le\eta\}$. According to Lemma 3.13(ii) the set Y is open. Choose any $y\in C_Y=Y\cap C_X$. By Lemma 3.6(i) there is a clopen neighborhood $K\subseteq Y$ of y such that $t_Y(x)<\eta$ whenever $x\in K\setminus C_Y$. We prove that K is the required set. By Lemma 3.11(iii), $C_K=K\cap C_Y$. By Lemma 3.13(iii), $t_K(x)=t_Y(x)$ for $x\in K$. Since C_K is nonempty, by Lemma 3.11(ii) it is homeomorphic to 2^ω. ∎

The following is the core result on extending homeomorphisms from C_X to X. It is a crucial ingredient of the proof of Theorem 4.3.

3.16. Lemma. *Let K be a compact, zero-dimensional, uncountable metrizable space such that*

(i) $t_K(x) = \eta_K$ *for all* $x \in C_K$,
(ii) $t_K(x) < \eta_K$ *for all* $x \in S_K$.

Then for each homeomorphism $T : C_K \to C_K$ *there is a homeomorphism* $\widehat{T} : K \to K$ *that extends* T, *i.e.* $\widehat{T}\restriction C_K = T$.

Proof. Assume that $\eta_K > 0$, otherwise there is nothing to prove. In order to simplify notation write $\eta = \eta_K$, $C = C_K$, $S = S_K$ and $t(x) = t_K(x)$. Let d denote a fixed metric on K. Let D be a countable dense subset of C that is invariant with respect to both T and T^{-1}. Such a set exists: it is enough to take any countable set E that is dense in C and put $D = \bigcup_{j\in\mathbb{Z}} T^j E$. Let $\langle c_n : n \in \omega\rangle$ be an enumeration of D. To enumerate the countable set S, assign to each $s \in S$ some $c \in D$ so that $d(s, c) < 2d(s, C)$ and the assignment $s \mapsto c$ is one-to-one. This is possible as D is dense and C has no isolated points. Thus there is a set $I \subseteq \omega$ such that $\langle s_n : n \in I\rangle$ enumerates all points of S and

$$d(s_n, c_n) < 2d(s_n, C) \quad \text{for all } n \in I. \tag{3.3}$$

We now inductively construct sequences $\langle V_n : n \in \omega\rangle$, $\langle W_n : n \in \omega\rangle$ and $\langle U_n : n \in \omega\rangle$ of sets satisfying the following conditions.

(a) $V_n \cap W_n = \emptyset$ and $U_n = V_n \cup W_n$ for all $n \in \omega$,
(b) $W_n = \emptyset$ whenever $n \in \omega \setminus I$,
(c) the family $\{U_n : n \in \omega\}$ is a disjoint cover of S,
(d) U_n is a clopen subset of S homeomorphic to ω^η and $\overline{U}_n^K = U_n \cup \{c_n\}$,
(e) $V_n \subseteq B(c_n, 2^{-n})$ for all $n \in \omega$,
(f) $W_n \subseteq B\big(s_n, \frac{1}{2}d(s_n, C)\big)$ for all $n \in I$.

During the construction we define for each $n \in \omega$ a set

$$A_n = \begin{cases} \emptyset & \text{when } n = 0,\\ \bigcup_{i<n} U_n & \text{when } n > 0.\end{cases}$$

According to (d), $c_n \notin \overline{A}_n^K$. Thus $B\big(c_n, 2^{-n}\big) \setminus A_n$ is a neighborhood of c_n. Let V_n be a subset of this neighborhood that is homeomorphic to ω^η, clopen in S and satisfies $\overline{V}_n = V_n \cup \{c_n\}$. Its existence is ensured by Lemma 3.14(ii).

The set W_n is defined as follows. If $n \notin I$ or $s_n \in A_n$, then put $W_n = \emptyset$. If $n \in I$ and $s_n \notin A_n$, then $B\big(s_n, \frac{1}{2}d(s_n, C)\big)\setminus(A_n \cup V_n)$ is a neighborhood of s_n. Let W_n be a subset of this neighborhood that is a clopen neighborhood of s_n and is homeomorphic to $(\omega^{t(s_n)})^*$. Its existence is ensured by Lemma 3.10.

Finally put $U_n = V_n \cup W_n$. If $W_n = \emptyset$, then (d) obviously holds. If $W_n \neq \emptyset$, then (d) follows from Lemma 3.8. Properties (a), (b), (e) and (f) are obviously satisfied. As to (c), it is clear that the collection $\{U_n : n \in \omega\}$ is disjoint, and as $s_n \in W_n \subseteq U_n$ whenever $n \in I$, it is also a cover.

It follows from (c) and (d) that the set S can be identified with a set $D \times \omega^\eta$. This identification is topological in the sense that S is homeomorphic to the topological product of a discrete space D and the ordinal topological space ω^η. We shall thus assume that $S = D \times \omega^\eta$. Note that as $\{c\} \times \omega^\eta$ is a clopen subset of S for each $c \in D$, the mappings $\{c\} \times \omega^\eta \to \{Tc\} \times \omega^\eta$, $\langle c, \alpha\rangle \mapsto \langle Tc, \alpha\rangle$, and $\{c\} \times \omega^\eta \to \{T^{-1}c\} \times \omega^\eta$, $\langle c, \alpha\rangle \mapsto \langle T^{-1}c, \alpha\rangle$, are continuous.

Define the extension $\widehat{T}$ of T by

$$\widehat{T}(x) = \begin{cases} \langle T(c), \alpha\rangle & \text{when } x = \langle c, \alpha\rangle \in S, \\ T(x) & \text{when } x \in C. \end{cases} \tag{3.4}$$

$\widehat{T}$ obviously extends T. As T is one-to-one and onto, so is $\widehat{T}$, because $T{\restriction}D : D \to D$ is bijective. So to prove that $\widehat{T} : K \to K$ is a homeomorphism, it suffices to show that $\widehat{T}$ is continuous. A simple argument shows that it is actually enough to prove that

$$\lim \widehat{T}(x_j) = \widehat{T}(\lim x_j) \tag{3.5}$$

for any convergent sequence $\langle x_j : j \in \omega\rangle$ in K such that $\langle \widehat{T}(x_j) : j \in \omega\rangle$ converges as well. Consider such a sequence. We can clearly assume that $x_j \in S$ for each $j \in \omega$. Put $x = \lim x_j$ and $y = \lim \widehat{T}(x_j)$.

First assume that $x \in S$. Then there is $c \in D$ such that $x_j \in \{c\} \times \omega^\eta$ for all but finitely many j's. Therefore $\widehat{T}(x) = \lim \widehat{T}(x_j)$ because of the continuity of the mapping $\{c\} \times \omega^\eta \to \{Tc\} \times \omega^\eta$, $\langle c, \alpha\rangle \mapsto \langle Tc, \alpha\rangle$. So in this case (3.5) is proved.

Now assume that $x \in C$. For each $j \in \omega$ there is $c_{n_j} \in D$ and an ordinal $\alpha_j < \omega^\eta$ such that $x_j = \langle c_{n_j}, \alpha_j\rangle$. We show that

$$\lim_{j\to\infty} c_{n_j} = x. \tag{3.6}$$

For each $n \in \omega$ consider the set $I_n = \{j \in \omega : n_j = n\}$. There is at most one n such that I_n is infinite. Indeed, if I_n is infinite, then (d) yields $x = \lim_{j\in I_n} x_j = c_n$, so if there were two distinct infinite sets I_n, I_m, we would have $x = c_n$ and $x = c_m$. Put

$$\begin{aligned} J_0 &= \begin{cases} \{j \in \omega : n_j = n\} & \text{if there is } n \text{ such that } I_n \text{ is infinite}, \\ \emptyset & \text{otherwise}, \end{cases} \\ J_1 &= \{j \in \omega : x_j \in V_{n_j}\} \setminus J_0, \\ J_2 &= \{j \in \omega : x_j \in W_{n_j}\} \setminus J_0. \end{aligned}$$

If J_0 is infinite then, as mentioned above, (d) yields

$$\lim_{j\in J_0} c_{n_j} = x. \tag{3.7}$$

If $j \in J_1$, then the triangle inequality and (e) imply

$$d(c_{n_j}, x) \le d(c_{n_j}, x_j) + d(x_j, x) \le 2^{-n_j} + d(x_j, x). \tag{3.8}$$

If J_1 is infinite, then the set $\{j \in J_1 : n_j = n\}$ is finite for each $n \in \omega$. Therefore $\lim_{j\in J_1} n_j = \infty$. Thus both terms on the right hand side in (3.8) converge to zero, whence

$$\lim_{j\in J_1} c_{n_j} = x. \tag{3.9}$$

If $j \in J_2$, then

$$\begin{aligned} d(s_{n_j}, x) &\le d(s_{n_j}, x_j) + d(x_j, x) \le \tfrac{1}{2} d(s_{n_j}, C) + d(x_j, x) \\ &\le \tfrac{1}{2} d(s_{n_j}, x) + d(x_j, x) \end{aligned}$$

by (f). Therefore

$$d(s_{n_j}, x) \le 2d(x_j, x). \tag{3.10}$$

The inequalities (3.3), (3.10) and again (3.10) thus imply

$$\begin{aligned} d(c_{n_j}, x) &\le d(c_{n_j}, s_{n_j}) + d(s_{n_j}, x) \le 2d(s_{n_j}, C) + 2d(x_j, x) \\ &\le 2d(s_{n_j}, x) + 2d(x_j, x) \le 4d(x_j, x) + 2d(x_j, x) = 6d(x_j, x). \end{aligned}$$

So if J_2 is infinite, then

$$\lim_{j\in J_2} c_{n_j} = x. \tag{3.11}$$

At least one of the sets J_0, J_1, J_2 is obviously infinite. Combining (3.7), (3.9) and (3.11) thus proves (3.6).

As T is continuous on C, it follows that

$$\lim_{j\to\infty} Tc_{n_j} = Tx. \tag{3.12}$$

For each $j \in \omega$ put $y_j = \widehat{T}(x_j)$ and consider the sequence $\langle y_j : j \in \omega\rangle$. By assumption, $y = \lim_{j\to\infty} y_j$ exists. We have $y \in C$. Indeed, if not, then there is $c \in D$ such that $y \in \{c\} \times \omega^\eta$. Therefore all but finitely many y_j's belong to U_n and thus

$$x = \lim x_j = \lim \widehat{T}^{-1} y_j = \widehat{T}^{-1}(y) \in \widehat{T}^{-1}(\{c\} \times \omega^\eta) = \{T^{-1}c\} \times \omega^\eta,$$

because the mapping $\{c\} \times \omega^\eta \to \{T^{-1}c\} \times \omega^\eta$, $\langle c, \alpha\rangle \mapsto \langle T^{-1}c, \alpha\rangle$, is continuous. As $\{T^{-1}c\} \times \omega^\eta$ is disjoint from C, we arrived at a contradiction proving that $y \in C$. Also, as the x_j's belong to S, so do the y_j's.

Therefore we can apply (3.6) to the sequence $\langle y_j : j \in \omega\rangle$. Thus

$$\lim Tc_{n_j} = y.$$

Comparison with (3.12) yields $y = Tx$, so (3.5) is proved. Thus $\widehat{T}$ is a homeomorphism. The proof is complete. ∎

4. The main result. The following is the main result of the paper. It follows at once from a slightly more general Theorem 4.3 below. For its proof we prepare Lemma 4.2.

4.1. Theorem. *Let X be a zero-dimensional*, *uncountable*, *compact metric space. Then for each $a \in [0,\infty]$ there is a homeomorphism $T : X \to X$ such that $h_{\mathrm{top}}(T) = a$.*

4.2. Lemma. *Let X be a compact metric space and $T : X \to X$ a continuous mapping. If $C \subseteq X$ is a closed T-invariant subset of X and $X \setminus C$ is countable and T-invariant*, *then $h_{\mathrm{top}}(T) = h_{\mathrm{top}}(T\restriction C)$.*

Proof. Put $S = X \setminus C$. Clearly $h_{\mathrm{top}}(T) \geq h_{\mathrm{top}}(T\restriction C)$. To prove the opposite inequality we use Theorem 2.10. Assume that $h_{\mathrm{top}}(T) > h_{\mathrm{top}}(T\restriction C)$. Then there is an ergodic probability measure μ in X such that $h_\mu(T) > h_{\mathrm{top}}(T\restriction C)$. The set C is T-invariant, therefore either $\mu(C) = 1$ or $\mu(S) = 1$. The former case would lead to $h_\mu(T) = h_\mu(T\restriction C) \leq h_{\mathrm{top}}(T\restriction C) < h_\mu(T)$. Thus $\mu(S) = 1$, whence $h_\mu(T) = h_\mu(T\restriction S)$.

If $x \in S$, then its two-sided orbit $\mathcal{O}(x) = \{T^j(x) : j \in \mathbb{Z}\}$ is T-invariant and thus either $\mu(\mathcal{O}(x)) = 0$ or $\mu(\mathcal{O}(x)) = 1$. If x is not periodic, then $\mathcal{O}(x)$ is infinite and therefore $\mu(\{x\}) = 0$. So the only points which can have positive measure are periodic. Since distinct orbits are disjoint, the ergodicity of μ implies that there is a unique cycle $\mathcal{O}(x_0) = \{x_0, x_1, \ldots, x_n\}$ such that $\mu(\mathcal{O}(x_0)) = 1$. Therefore $h_\mu(T) = h_\mu(T\restriction S) = h_\mu(T\restriction \mathcal{O}(x_0))$. Since $\mathcal{O}(x_0)$ is a finite space, we conclude that $h_\mu(T) = 0$, a contradiction. ∎

4.3. Theorem. *Let X be a zero-dimensional*, *locally compact*, *metrizable space that is not locally countable. Then for each $a \in [0,\infty]$ there is a homeomorphism $T : X \to X$ such that $h_\varrho(T) = a$ for each metric ϱ in X. Moreover*, *both T and T^{-1} are uniformly continuous.*

Proof. There is an open set $U \subseteq X$ and a point $x \in U$ such that each closed neighborhood of x contained in U is compact and uncountable. As X is zero-dimensional, there is a clopen set F separating x and $X \setminus U$, i.e. $x \in F \subseteq U$. This set is a compact, uncountable, zero-dimensional space. By Lemma 3.15 there is a clopen set $K \subseteq F$ such that

(i) C_K is homeomorphic to 2^ω,
(ii) $t_K(x) = \eta_K$ for all $x \in C_K$,
(iii) $t_K(x) < \eta_K$ for all $x \in S_K$.

Let $a \in [0,\infty]$ be given. According to Theorem 2.4 there is a homeomorphism $T_0 : C_K \to C_K$ such that $h_{\mathrm{top}}(T_0) = a$. By Lemmas 3.16 and 4.2 there is a homeomorphism $\widehat{T}_0 : K \to K$ such that $h_{\mathrm{top}}(\widehat{T}_0) = h_{\mathrm{top}}(T_0) = a$. Define a mapping $T : X \to X$ by

$$T(x) = \begin{cases} \widehat{T}_0(x) & \text{for } x \in K, \\ x & \text{for } x \in X \setminus K. \end{cases}$$

As K is a clopen set, T is obviously a homeomorphism. T is an aggregate of an identity and a homeomorphism of a compact space, and thus both T

and T^{-1} are uniformly continuous with respect to any metric in X. Since K and $X \setminus K$ are both T-invariant, it follows from Proposition 2.8 that for each metric ϱ in X,

$$h_\varrho(T) = \max(h_{\mathrm{top}}(\widehat{T}_0), h_\varrho(\mathrm{id}_{X\setminus K})) = a. \ \blacksquare$$

5. Counterexamples. Theorem 4.1 lists four conditions the space X has to satisfy in order to possess homeomorphisms of arbitrary entropies:

- X is uncountable,
- X is compact,
- X is zero-dimensional,
- X is metrizable.

We show that when any of these conditions is dropped, the conclusion of Theorem 4.1 fails.

5.1. Proposition. *If X is a countable, compact metric space and $T : X \to X$ a continuous mapping, then $h_{\mathrm{top}}(T) = 0$.*

Proof. Apply Lemma 4.2 with $C = \emptyset$. $\blacksquare$

So the first condition cannot be dropped.

5.2. Example. Let X be an uncountable set provided with the discrete topology. Then X is zero-dimensional and metrizable. As each compact subset of X is finite, $h_{\mathrm{top}}(T) = 0$ for each continuous mapping $T : X \to X$.

So the second condition cannot be dropped.

5.3. Example. Recall that a *Cook continuum* is a metric continuum X that admits only the identity mapping onto nondegenerate subcontinua. In particular, any continuous mapping $T : X \to X$ is either constant or the identity. Cook continua exist, see e.g. [3]. So a Cook continuum is an example of an uncountable compact metric space of positive dimension such that $h_{\mathrm{top}}(T) = 0$ for each continuous mapping $T : X \to X$.

So the third condition cannot be dropped. Another example of a compact metric space of positive dimension possessing no homeomorphisms of positive entropy is the unit interval. However, Example 5.3 is better, because on the unit interval there are continuous mappings of positive entropy.

5.4. Proposition. *Let ω_1 be the first uncountable ordinal and $X = \omega_1 + 1$ its successor provided with the interval topology. Then X is a zero-dimensional compact space and $h_{\mathrm{top}}(T) = 0$ for any continuous mapping $T : X \to X$.*

Proof. It is well known that X is zero-dimensional and compact. For any family $\mathcal{A}$ of subsets of X and each $n \in \omega$ define

$$\mathcal{A}_n = \mathcal{A} \vee T^{-1}\mathcal{A} \vee \ldots \vee T^{-n}\mathcal{A}, \quad \mathcal{A}_\infty = \bigcup_{n\in\omega} \mathcal{A}_n.$$

Let $\mathcal{V}$ be a finite open cover of X. As X is zero-dimensional, $\mathcal{V}$ has an open disjoint refinement $\mathcal{U}$ (cf. [5, 7.1.7]). Using the notation of Definition 2.2 it is easy to check that

$$H(\mathcal{V}_n) \le H(\mathcal{U}_n) = \log|\mathcal{U}_n|, \tag{5.1}$$

$$H(T,\mathcal{V}) \le H(T,\mathcal{U}). \tag{5.2}$$

For each set $U \in \mathcal{U}_\infty$ pick a point $x_U \in U$ and set $D = \{x_U : U \in \mathcal{U}_\infty\}$. As $\mathcal{U}_\infty$ is countable, so is D. Consider the sets $E = \bigcup_{i\in\omega} T^i D$ and $F = \overline{E}$. The set F is T-invariant and closed. As $D \subseteq F$, it follows that for the family $\mathcal{U}' = \{U \cap F : U \in \mathcal{U}\}$ we have

$$H(T,\mathcal{U}) = H(T{\restriction}F, \mathcal{U}').$$

Combining with (5.2) and Definition 2.2 we get

$$H(T,\mathcal{V}) \le H(T{\restriction}F, \mathcal{U}') \le h_{\mathrm{top}}(T{\restriction}F). \tag{5.3}$$

Since D is countable, so is E. A countable subset of ω_1 is bounded, therefore $E \setminus \{\omega_1\}$ is bounded. The closure of a bounded subset of ω_1 is bounded, therefore $F \setminus \{\omega_1\}$ is bounded. A bounded subset of ω_1 is countable, therefore $F \setminus \{\omega_1\}$, and *a fortiori* F, is countable. As it is also closed, it is a countable compact space. Thus it is also metrizable. Therefore Proposition 5.1 and (5.3) yield $H(T,\mathcal{V}) = 0$. As $\mathcal{V}$ was an arbitrary open cover of X, it follows that $h_{\mathrm{top}}(T) = 0$. ∎

As $\omega_1 + 1$ is zero-dimensional, compact and uncountable, the fourth condition cannot be dropped.

References

[1] L. Alsedà, J. Llibre, and M. Misiurewicz, *Combinatorial Dynamics and Entropy in Dimension One*, Adv. Ser. Nonlinear Dynam. 5, World Sci., Singapore, 1993.

[2] R. Bowen, *Entropy for group endomorphisms and homogeneous spaces*, Trans. Amer. Math. Soc. 153 (1971), 401–414.

[3] H. Cook, *Continua which admit only the identity mapping onto nondegenerate subcontinua*, Fund. Math. 60 (1967), 241–249.

[4] M. Denker, C. Grillenberger, and K. Sigmund, *Ergodic Theory on Compact Spaces*, Lecture Notes in Math. 527, Springer, 1976.

[5] R. Engelking, *Dimension Theory*, North-Holland, Amsterdam, 1978.

[6] —, *General Topology*, Heldermann, Berlin, 1989.

[7] S. Mazurkiewicz et W. Sierpiński, *Contribution à la topologie des ensembles dénombrables*, Fund. Math. 1 (1920), 17–27.

[8] P. Walters, *An Introduction to Ergodic Theory*, Grad. Texts in Math. 79, Springer, New York, 1981.

Department of Mathematics
Faculty of Civil Engineering
Czech Technical University
Thákurova 7
166 29 Praha 6, Czech Republic
E-mail: erastus@mbox.cesnet.cz
zindulka@mat.fsv.cvut.cz

Received 31 August 1998;
in revised form 6 September 1999