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Brown–Peterson cohomology and Morava K-theory of $DI(4)$ and its classifying space

by

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Abstract. $DI(4)$ is the only known example of an exotic 2-compact group, and is conjectured to be the only one. In this work, we study generalized cohomology theories for $DI(4)$ and its classifying space. Specifically, we compute the Morava K-theories, and the $P(n)$-cohomology of $DI(4)$. We use the non-commutativity of the spectrum $P(n)$ at $p = 2$ to prove the non-homotopy nilpotency of $DI(4)$. Concerning the classifying space, we prove that the BP-cohomology and the Morava K-theories of $BDI(4)$ are all concentrated in even degrees.

1. Introduction and statement of results. The concept of a p-compact group was introduced by Dwyer and Wilkerson in [D-W1] as a homotopy-theoretic generalization of a compact Lie group. The first examples of p-compact groups were the p-completions of compact Lie groups. A connected p-compact group is called *exotic* if it is not of the form $G_p^\wedge$ (the Bousfield–Kan p-completion of G) for any connected compact Lie group G. There are many known examples of exotic p-compact groups at odd primes: Sullivan spheres and many of the Clark–Ewing p-compact groups are exotic. However, there is only one example of an exotic 2-compact group: Dwyer and Wilkerson constructed in [D-W2] a 2-complete space $BDI(4)$ whose mod two cohomology is isomorphic to the ring of rank 4 mod 2 Dickson invariants. The loop space $DI(4) = \Omega BDI(4)$ is an exotic 2-compact group. Standard methods show that $H^*(DI(4), \mathbb{Z}/2)$ is multiplicatively generated by elements x_7, y_{11} and z_{13}, with $\mathrm{Sq}^4 x = y$, $\mathrm{Sq}^2 y = z$, $\mathrm{Sq}^1 z = x^2 \neq 0$, and $x^4 = y^2 = z^2 = 0$.

In this paper, we study generalized cohomology theories for $DI(4)$ and its classifying space $BDI(4)$. In particular, we compute the algebra structure of the Morava K-theory and the $P(n)$-cohomology of $DI(4)$:

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Theorem 1.1. (i) *There are* $K(n)^*$*-algebra isomorphisms*:

$$K(n)^*(DI(4)) \cong K(n)^* \otimes \mathbb{Z}/2[x_7]/(x_7^4) \otimes \Lambda(x_{11}, x_{13}) \quad \textit{for all } n \geq 3,$$
$$K(2)^*(DI(4)) \cong K(2)^* \otimes \Lambda(x_{11}, x_{13}, x_{21}),$$
$$K(1)^*(DI(4)) \cong K(1)^* \otimes \Lambda(x_7, x_{13}, x_{25}).$$

(ii) *There are* $P(n)^*$*-algebra isomorphisms*:

$$P(n)^*(DI(4)) \cong P(n)^* \otimes \mathbb{Z}/2[x_7]/(x_7^4) \otimes \Lambda(x_{11}, x_{13}) \quad \textit{for all } n \geq 3,$$
$$P(2)^*(DI(4)) \cong (P(2)^* \otimes \Lambda(x_{21}) \oplus P(3)^* \otimes \mathbb{Z}/2\{x_{14}\}) \otimes \Lambda(x_{11}, x_{13}),$$
$$P(1)^*(DI(4)) \cong ((P(1)^* \otimes \Lambda(x_5, x_{21}, x_{25}, x_{32}) \oplus P(3)^* \otimes \mathbb{Z}/2\{x_{14}\}) \otimes \Lambda(x_{13}))/I,$$

where I *is the ideal generated by* $\{x_{25}x_{32}, x_{21}x_{32}, x_{21}x_{25}, x_5x_{32}, x_5x_{25} + v_1x_{32}, x_5x_{21} + v_2x_{32}, v_1x_{21} + v_2x_{25}\}$.

Using the non-commutativity of the spectrum $P(n)$ at $p = 2$, $DI(4)$ is shown not to be homotopy nilpotent (as was to be expected, by analogy with the behaviour of compact Lie groups):

Theorem 1.2. $DI(4)$ *is not homotopy nilpotent.*

Concerning the classifying space $BDI(4)$, we again use the behaviour of compact Lie groups as a reference, and show the following result:

Theorem 1.3. (i) $K(n)^*(BDI(4))$ *is concentrated in even degrees for all* $n \geq 1$.

(ii) *For all* $n \geq 0$, $P(n)^*(BDI(4))$ *is concentrated in even degrees and has no* v_i*-torsion for* $i \geq n$.

This paper is organized as follows. In Section 2, we recall the basic facts about the generalized cohomology theories associated with the spectra BP, $P(n)$ and $K(n)$. Section 3 is devoted to the proof of Theorem 1.3. In Section 4, we obtain some technical lemmas concerning the Atiyah–Hirzebruch spectral sequence for $P(n)$ and $K(n)$, which will be useful in Section 5 to compute $K(n)^*(DI(4))$ and $P(n)^*(DI(4))$ for all $n \geq 1$. We finish Section 5 with the proof of the non-homotopy nilpotency of $DI(4)$.

Notation. Let X be a space, $n \geq 1$, and $F = P(n)$ or $K(n)$. We write $E_r^{**}(F)$ for the E_r-term in the Atiyah–Hirzebruch spectral sequence of $F^*(X)$. If $x \in E_r^{p,q}(F)$, then $|x|$ denotes the total degree of x, that is, $p+q$. If $x_1, \ldots, x_s \in F^*(X)$, then $F^*\{x_1, \ldots, x_s\}$ denotes the F^*-submodule of $F^*(X)$ generated by $x_1, \ldots, x_s$. When we say that a $P(n)^*$-algebra is of the form

$$(P(n)^* \otimes A_n) \oplus (P(n+1)^* \otimes A_{n+1}) \oplus \ldots \oplus (P(n+s)^* \otimes A_{n+s})$$

the products xy are supposed to be zero if $x \in A_i$ and $y \in A_j$, $i \neq j$.

$\mathbb{Z}/p\{x_1, \ldots, x_s\}$ will denote the free $\mathbb{Z}/p$-module generated by $x_1, \ldots, x_s$, and $F^* \otimes \mathbb{Z}/p\{x_1, \ldots, x_s\}$ the free F^*-module generated by these elements. The symbol Λ will be used to denote the exterior algebra over $\mathbb{Z}/p$.

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2. Preliminary results. Let BP be the Brown–Peterson spectrum at a fixed prime p. It is a ring spectrum which represents the cohomology theory $BP^*(-)$ with coefficient ring $BP^* \cong \mathbb{Z}_{(p)}[v_1, v_2, \ldots]$, where the degree of v_i is $|v_i| = -2(p^i - 1)$.

For all $n \geq 0$, there are BP^*-module spectra $P(n)$ and multiplicative cohomology theories $P(n)^*(-)$ with coefficients $P(n)^* \cong BP^*/I_n$, where $I_n = (p, v_1, \ldots, v_{n-1})$ denotes the nth invariant prime ideal of BP^* (see [J-W] for details). These cohomology theories are related by exact triangles

$$\begin{tikzcd} P(n)^*(-) \arrow[rr, "v_n"] & & P(n)^*(-) \arrow[dl, "i_n"] \\ & P(n+1)^*(-) \arrow[ul, "\delta_n"] & \end{tikzcd}$$

where v_n acts as multiplication by the coefficient v_n, δ_n has degree $2p^n - 1$, and i_n has degree 0. All maps displayed above are morphisms of BP^*-modules.

Note that $P(0)^*(-)$ is the Brown–Peterson cohomology, $P(1)^*(-)$ is the Brown–Peterson cohomology with mod p coefficients, and $P(\infty)^*(-)$ is the ordinary mod p cohomology $H^*(-, \mathbb{Z}/p)$. We have the following tower of cohomology theories:

$$P(0)^*(-) \xrightarrow{i_0} P(1)^*(-) \xrightarrow{i_1} \ldots$$
$$\ldots \to P(n)^*(-) \xrightarrow{i_n} P(n+1)^*(-) \to \ldots \to H^*(-, \mathbb{Z}/p)$$

which can be used to compute $BP^*(-) = P(0)^*(-)$ and $BP^*(-, \mathbb{Z}/p) = P(1)^*(-)$ from $H^*(-, \mathbb{Z}/p)$.

For all $0 \leq i < n$, there are cohomology operations $Q_i : P(n)^*(-) \to P(n)^*(-)$, with degree $2p^i - 1$, that commute with the maps $P(n)^*(-) \to P(n+1)^*(-)$, and that correspond to the Milnor operations in ordinary cohomology theory. In particular, $i_n \delta_n = Q_n$.

Let $K(n)^*(-)$ be the Morava K-theory with coefficients $K(n)^* \cong \mathbb{Z}/p[v_n, v_n^{-1}]$ (see [J-W] for details). By construction of the spectra $P(n)$ and $K(n)$, one has a canonical map $P(n) \xrightarrow{\lambda_n} K(n)$. In [Ya1] it has been proved that there exists a $P(n)^*$-module isomorphism

$$(*) \qquad P(n)^*(-) \otimes_{P(n)^*} K(n)^* \cong K(n)^*(-).$$

The composition of the natural inclusion $P(n)^*(-) \to P(n)^*(-) \otimes_{P(n)^*} K(n)^*$ with the above isomorphism is the map induced on cohomology by the map of spectra λ_n.

In [S-Ya] it has been shown, by using a geometrical approach, that $P(n)$ and $K(n)$ can be given an associative product. Wurgler ([Wu1] and [Wu2]) used a homotopy-theoretic approach that gives more information and showed that, for $p = 2$, the product is not commutative:

PROPOSITION 2.1 ([Wu1]). *Suppose $p = 2$ and $n \geq 1$. There are exactly two products $m_n, m'_n : P(n) \wedge P(n) \to P(n)$ which make $P(n)$ a BP-algebra spectrum compatible with the given BP-module structure. Both are associative and are related by the formula*

$$m'_n = m_n + v_n m_n (Q_{n-1} \wedge Q_{n-1}).$$

Using 2.1 and the isomorphism $(*)$, one easily sees that an analogous statement is also true for $K(n)$. Moreover, $P(n)^*(-) \xrightarrow{i_n} P(n+1)^*(-)$ and $P(n)^*(-) \xrightarrow{\lambda_n} K(n)^*(-)$ are maps of $P(n)^*$-algebras, and the Q_i's are derivations with respect to any product chosen.

As an immediate consequence of Proposition 2.1, we have the following two corollaries. Let $F(n)$ denote one of the spectra $P(n)$ or $K(n)$.

COROLLARY 2.2. *Suppose $p = 2$. Let X be a space, and $x, y \in F(n)^*(X)$. Then*

$$xy = yx + v_n (Q_{n-1} y)(Q_{n-1} x).$$

If X is an H-space with $F(n)^*(X)$ free, then $F(n)^*(X)$ is both an algebra and a coalgebra, but not necessarily a Hopf algebra if $p = 2$. The correction is given by:

COROLLARY 2.3 ([R]). *Suppose $p = 2$ and X is an H-space such that $F(n)^*(X)$ is free as an $F(n)^*$-module. Let x and y be elements of $F(n)^*(X)$ and Ψ be the coproduct. Then*

$$\Psi(xy) = \Psi(x)\Psi(y) + v_n((\mathrm{id} \otimes Q_{n-1})\Psi(x))((Q_{n-1} \otimes \mathrm{id})\Psi(y)).$$

The same statement is true in homology.

3. BP-cohomology and Morava K-theory of $BDI(4)$. It is well known that if G is a compact Lie group, p a fixed prime, and $H^*(G)$ is p-torsion free, then the classifying space BG satisfies:

(i) $K(n)^*(BG)$ is concentrated in even degrees for all $n \geq 1$.
(ii) $P(n)^*(BG)$ is concentrated in even degrees for all $n \geq 0$.

In [K-Ya], Kono and Yagita show that the above properties hold in some cases even if G has p-torsion, and conjecture that they hold for all compact Lie groups. Ravenel, Wilson and Yagita ([R-W-Ya]) have recently proved the following result: if $K(n)^*(-)$ is concentrated in even degrees for all $n \geq 1$, then $P(n)^*(-)$ is concentrated in even degrees for all $n \geq 0$. We prove in this section that the BP-cohomology, the Morava K-theory and the $P(n)$-theory of $BDI(4)$ are all concentrated in even degrees.

Recall that $H^*(BDI(4), \mathbb{Z}/2)$ is isomorphic, as an algebra over the Steenrod algebra, to the ring of rank 4 mod 2 Dickson invariants:

$$H^*(BDI(4), \mathbb{Z}/2) \cong D(4) \cong \mathbb{Z}/2[t_1, t_2, t_3, t_4]^{\mathrm{Gl}(4,\mathbb{Z}/2)} \cong \mathbb{Z}/2[c_8, c_{12}, c_{14}, c_{15}]$$

where $|t_i| = 1$, $|c_j| = j$, and the generators c_j are the coefficients of the polynomial

$$p(x) = \prod_{v \in V} (x + v) = x^{16} + c_8 x^8 + c_{12} x^4 + c_{14} x^2 + c_{15} x$$

where V is a 4-dimensional vector space over $\mathbb{Z}/2$.

The action of the Steenrod algebra on the Dickson algebras is well known ([S-S]). In the case of $D(4)$, this action is determined by $\mathrm{Sq}^4 c_8 = c_{12}$, $\mathrm{Sq}^2 c_{12} = c_{14}$, $\mathrm{Sq}^1 c_{14} = c_{15}$, $\mathrm{Sq}^8 c_i = c_8 c_i$, $i = 8, 12, 14, 15$. In particular, we are interested in the action of the Milnor operations Q_i, because of the role they play in the study of the Atiyah–Hirzebruch spectral sequence for $K(n)$- and $P(n)$-theory. Since the Q_i are derivations, $\mathrm{Im}\, Q_i \subset \mathrm{Ker}\, Q_i$ for all $i \geq 0$. What we prove is that the other inclusion is also true, for all elements of odd degree in $H^*(BDI(4), \mathbb{Z}/2)$:

PROPOSITION 3.1. *Let $a \in H^*(BDI(4), \mathbb{Z}/2)$ have odd degree, and $n \geq 0$. If $Q_n a = 0$, then $a \in \mathrm{Im}\, Q_n$.*

Using this result, which will be proved later, we can easily prove the following theorem:

THEOREM 3.2. (i) *$K(n)^*(BDI(4))$ is concentrated in even degrees for all $n \geq 1$.*

(ii) *For all $n \geq 0$, $P(n)^*(BDI(4))$ is concentrated in even degrees and has no v_i-torsion for $i \geq n$.*

Proof. (i) Let $n \geq 1$. We consider the Atiyah–Hirzebruch spectral sequence for $K(n)^*(BDI(4))$:

$$E_2^{**} \cong H^*(BDI(4), K(n)^*) \Rightarrow K(n)^*(BDI(4)).$$

Recall ([Ya4]) that the first non-trivial differential is $d_{2^{n+1}-1} = v_n \otimes Q_n$, where Q_n is the Milnor operation in ordinary mod 2 cohomology. From Proposition 3.1, $E_{2^{n+1}}^{\mathrm{odd}}(K(n)) = 0$, and hence $E_\infty^{**}(K(n)) \cong E_{2^{n+1}}^{**}(K(n))$ is concentrated in even degrees.

(ii) From [R-W-Ya], $P(n)^{\mathrm{odd}}(BDI(4)) = 0$ for all $n \geq 0$. Since the maps δ_n in the exact triangles relating $P(n)$ and $P(n+1)$ for $n \geq 0$ all have odd degrees, $\operatorname{Im}\delta_n = \operatorname{Ker} v_n = \{0\}$. Therefore, for all $n \geq 0$, $P(n)^*(BDI(4))$ has no v_n-torsion.

Recall that, for any complex X, we have an exact sequence

$$0 \to \lim{}^1 P(n)^{q-1}(X^m) \to P(n)^q(X) \to \lim{}^0 P(n)^q(X^m) \to 0.$$

In our case, since $P(n)^{\mathrm{odd}}(X) = 0$, $\lim^1 = 0$.

Let $0 \neq x \in P(n)^*(BDI(4))$, and suppose $v_{n+1}x = 0$. Then $i_n(v_{n+1}x) = v_{n+1}i_n(x) = 0$. Since $P(n+1)^*(BDI(4))$ has no v_{n+1}-torsion, $i_n(x) = 0$, and hence $x = v_n x_1$ for some $x_1 \in P(n)^*(BDI(4))$. Therefore, $v_n v_{n+1} x_1 = 0$ and this implies $v_{n+1}x_1 = 0$. The iteration of this process implies that x is divisible infinitely many times by v_n, and hence x is in the kernel of the maps $P(n)^q(X) \to \lim^0 P(n)^q(X^m)$. This is a contradiction, since $\lim^1 = 0$. The conclusion is that, for all $n \geq 0$, $P(n)^*(BDI(4))$ has no v_{n+1}-torsion. In this way, we can prove by induction that $P(n)^*(BDI(4))$ has no v_i-torsion for all $i \geq n$. ∎

The rest of this section is devoted to the proof of Proposition 3.1.

The action of the Milnor operations Q_0, Q_1, Q_2 on the generators $c_8, c_{12}, c_{14}, c_{15}$ of $H^*(BDI(4), \mathbb{Z}/2)$ is easily computed from the definition of the Q_i (recall that $Q_0 = \mathrm{Sq}^1$ and $Q_i = [\mathrm{Sq}^{2^i}, Q_{i-1}], i \geq 1$). We obtain $Q_0 c_{14} = Q_1 c_{12} = Q_2 c_8 = c_{15}$, and zero in the other cases. Then 3.1 can be proved easily for $n = 0, 1, 2$:

Lemma 3.3. *Let $a \in H^{\mathrm{odd}}(BDI(4), \mathbb{Z}/2)$, and $n = 0, 1, 2$. If $Q_n a = 0$, then $a \in \operatorname{Im} Q_n$.*

Proof. $\operatorname{Ker} Q_0 = \mathbb{Z}/2[c_8, c_{12}, c_{14}^2, c_{15}]$. If $a \in (\operatorname{Ker} Q_0)^{\mathrm{odd}}$, then $a = c_{15}b$ for some $b \in \operatorname{Ker} Q_0$, and hence $a = Q_0(c_{14}b)$. Arguing similarly, we can see that $a = Q_1(c_{12}b)$ if $a \in (\operatorname{Ker} Q_1)^{\mathrm{odd}}$, and $a = Q_2(c_8 b)$ if $a \in (\operatorname{Ker} Q_2)^{\mathrm{odd}}$, for some b in $\operatorname{Ker} Q_1$ or $\operatorname{Ker} Q_2$ respectively. ∎

The action of Q_3 on the generators is still not difficult to calculate: $Q_3 c_8 = c_8 c_{15}$, $Q_3 c_{12} = c_{12} c_{15}$, $Q_3 c_{14} = c_{14} c_{15}$, $Q_3 c_{15} = c_{15}^2$. From $r = 4$ on, it is getting more and more complicated to calculate the action of Q_r on the generators. For all $n \geq 0$, define the following determinants:

$$A_n = \begin{vmatrix} t_1 & t_2 & t_3 & t_4 \\ t_1^2 & t_2^2 & t_3^2 & t_4^2 \\ t_1^4 & t_2^4 & t_3^4 & t_4^4 \\ t_1^{2^n} & t_2^{2^n} & t_3^{2^n} & t_4^{2^n} \end{vmatrix}, \quad B_n = \begin{vmatrix} t_1 & t_2 & t_3 & t_4 \\ t_1^2 & t_2^2 & t_3^2 & t_4^2 \\ t_1^8 & t_2^8 & t_3^8 & t_4^8 \\ t_1^{2^n} & t_2^{2^n} & t_3^{2^n} & t_4^{2^n} \end{vmatrix},$$

$$C_n = \begin{vmatrix} t_1 & t_2 & t_3 & t_4 \\ t_1^4 & t_2^4 & t_3^4 & t_4^4 \\ t_1^8 & t_2^8 & t_3^8 & t_4^8 \\ t_1^{2^n} & t_2^{2^n} & t_3^{2^n} & t_4^{2^n} \end{vmatrix}, \quad D_n = \begin{vmatrix} t_1^2 & t_2^2 & t_3^2 & t_4^2 \\ t_1^4 & t_2^4 & t_3^4 & t_4^4 \\ t_1^8 & t_2^8 & t_3^8 & t_4^8 \\ t_1^{2^n} & t_2^{2^n} & t_3^{2^n} & t_4^{2^n} \end{vmatrix}.$$

Note that $c_{15} = A_3 = B_2 = C_1 = D_0$, and $A_n, B_n, C_n, D_n \neq 0$ for all $n \geq 4$.

The action of the Steenrod algebra on $\mathbb{Z}/2[t_1, t_2, t_3, t_4]$ is well known. In particular,

$$\mathrm{Sq}^i(t_j^{2^n}) = \begin{cases} t_j^{2^{n+1}} & \text{if } i = 2^n, \\ 0 & \text{otherwise.} \end{cases}$$

Using that, it is a trivial computation to check that:

LEMMA 3.4. (i) $\mathrm{Sq}^{2^n} A_n = A_{n+1}$ *and* $\mathrm{Sq}^4 A_n = B_n$ *for all* $n \geq 3$.

(ii) $\mathrm{Sq}^2 B_n = C_n$ *for all* $n \geq 2$.

(iii) $\mathrm{Sq}^1 C_n = D_n$ *for all* $n \geq 1$.

As a consequence of Lemma 3.4, we find that the elements A_n, B_n, C_n, D_n are precisely $Q_{n-1}c_i$ for $i = 8, 12, 14, 15$ respectively:

LEMMA 3.5. *For all* $n \geq 0$, $Q_n c_8 = A_{n+1}$, $Q_n c_{12} = B_{n+1}$, $Q_n c_{14} = C_{n+1}$ *and* $Q_n c_{15} = D_{n+1}$.

Proof. For c_8, the statement follows by induction: $Q_0 c_8 = 0 = A_1$, $Q_1 c_8 = 0 = A_2$, $Q_2 c_8 = c_{15} = A_3$ and, for $n \geq 3$, $Q_n c_8 = [\mathrm{Sq}^{2^n}, Q_{n-1}]c_8 = \mathrm{Sq}^{2^n} A_n = A_{n+1}$.

Recall that $[\mathrm{Sq}^r, Q_i] = Q_{i+1}\mathrm{Sq}^{r-2^{i+1}}$ (it is understood that $\mathrm{Sq}^j = 0$ if $j < 0$). In particular, $[\mathrm{Sq}^4, Q_i] = 0$ for $i \geq 2$, $[\mathrm{Sq}^2, Q_i] = 0$ for $i \geq 1$, and $[\mathrm{Sq}^1, Q_i] = 0$ for $i \geq 0$.

From these relations and 3.4, the lemma follows easily for c_{12}, c_{14} and c_{15}: $Q_0 c_{12} = 0 = B_1, Q_1 c_{12} = c_{15} = B_2$ and, for $n \geq 2, Q_n c_{12} = Q_n \mathrm{Sq}^4 c_8 = \mathrm{Sq}^4 Q_n c_8 = \mathrm{Sq}^4 A_{n+1} = B_{n+1}$. Analogous arguments can be used to prove the result for c_{14} and c_{15}. ∎

Notice now that, since $p(t_i) = 0$ for $i = 1, 2, 3, 4$,

$$t_i^{2^n} = c_8^{2^{n-4}} t_i^{2^{n-1}} + c_{12}^{2^{n-4}} t_i^{2^{n-2}} + c_{14}^{2^{n-4}} t_i^{2^{n-3}} + c_{15}^{2^{n-4}} t_i^{2^{n-4}}$$

for all $n \geq 4$, $i = 1, 2, 3, 4$. Therefore, we get:

LEMMA 3.6. *For all* $n \geq 4$,

$$\begin{aligned}
A_n &= c_8^{2^{n-4}} A_{n-1} + c_{12}^{2^{n-4}} A_{n-2} + c_{14}^{2^{n-4}} A_{n-3} + c_{15}^{2^{n-4}} A_{n-4}, \\
B_n &= c_8^{2^{n-4}} B_{n-1} + c_{12}^{2^{n-4}} B_{n-2} + c_{14}^{2^{n-4}} B_{n-3} + c_{15}^{2^{n-4}} B_{n-4}, \\
C_n &= c_8^{2^{n-4}} C_{n-1} + c_{12}^{2^{n-4}} C_{n-2} + c_{14}^{2^{n-4}} C_{n-3} + c_{15}^{2^{n-4}} C_{n-4}, \\
D_n &= c_8^{2^{n-4}} D_{n-1} + c_{12}^{2^{n-4}} D_{n-2} + c_{14}^{2^{n-4}} D_{n-3} + c_{15}^{2^{n-4}} D_{n-4}.
\end{aligned}$$

It is known ([A-W]) that any five derivations Q_n, Q_{n-1}, Q_{n-2}, Q_{n-3}, Q_{n-4} are linearly dependent on $\mathbb{Z}/2[c_8, c_{12}, c_{14}, c_{15}]$. Using Lemma 3.6, we write explicitly the coefficients:

Lemma 3.7. (i) *For all* $n \geq 4$ *and* $x \in H^*(BDI(4), \mathbb{Z}/2)$,

$$Q_n x = c_8^{2^{n-3}} Q_{n-1}x + c_{12}^{2^{n-3}} Q_{n-2}x + c_{14}^{2^{n-3}} Q_{n-3}x + c_{15}^{2^{n-3}} Q_{n-4}x.$$

(ii) *If* $x \in H^{\mathrm{even}}(BDI(4), \mathbb{Z}/2)$ *and* $n \geq 0$, *then*

$$c_{15} Q_n x = A_{n+1} Q_2 x + B_{n+1} Q_1 x + C_{n+1} Q_0 x.$$

(iii) *If* $a \in H^{\mathrm{odd}}(BDI(4), \mathbb{Z}/2)$ *and* $n \geq 0$, *then*

$$c_{15} Q_n a = A_{n+1} Q_2 a + B_{n+1} Q_1 a + C_{n+1} Q_0 a + D_{n+1} a.$$

Proof. (i) As a consequence of 3.5 and 3.6, the result is true for the generators $c_8, c_{12}, c_{14}, c_{15}$. Since the Q_i are derivations, it is also true for any $x \in H^*(BDI(4), \mathbb{Z}/2)$.

(ii) Let $x \in H^{\mathrm{even}}(BDI(4), \mathbb{Z}/2)$. For $n = 0, 1, 2, 3$, it can be easily checked by a direct computation that the formula holds for c_8, c_{12}, c_{14}, and hence for any element in $H^{\mathrm{even}}(BDI(4), \mathbb{Z}/2)$. Let $n \geq 4$. Using (i) and 3.6, we can prove the result by induction:

$$\begin{aligned} Q_n x &= c_8^{2^{n-3}} Q_{n-1}x + c_{12}^{2^{n-3}} Q_{n-2}x + c_{14}^{2^{n-3}} Q_{n-3}x + c_{15}^{2^{n-3}} Q_{n-4}x, \\ c_{15} Q_n x &= c_8^{2^{n-3}} (A_n Q_2 x + B_n Q_1 x + C_n Q_0 x) \\ &\quad + c_{12}^{2^{n-3}} (A_{n-1} Q_2 x + B_{n-1} Q_1 x + C_{n-1} Q_0 x) \\ &\quad + c_{14}^{2^{n-3}} (A_{n-2} Q_2 x + B_{n-2} Q_1 x + C_{n-2} Q_0 x) \\ &\quad + c_{15}^{2^{n-3}} (A_{n-3} Q_2 x + B_{n-3} Q_1 x + C_{n-3} Q_0 x) \\ &= A_{n+1} Q_2 x + B_{n+1} Q_1 x + C_{n+1} Q_0 x. \end{aligned}$$

(iii) Let $a = xc_{15}$ be an element in $H^{\mathrm{odd}}(BDI(4), \mathbb{Z}/2)$. Then $Q_n a = c_{15} Q_n x + x Q_n c_{15} = A_{n+1} Q_2 x + B_{n+1} Q_1 x + C_{n+1} Q_0 x + D_{n+1} x$. Multiplying by c_{15}, we get the result. ∎

Recall that our hypothesis in Proposition 3.1 is that $Q_n a = 0$, and we want to prove that this implies $a \in \operatorname{Im} Q_n$. From 3.7, if $Q_n a = 0$, then $D_{n+1} a = A_{n+1} Q_2 a + B_{n+1} Q_1 a + C_{n+1} Q_0 a$. We would like to deduce that $a \in (A_{n+1}, B_{n+1}, C_{n+1})$ (the ideal of $D(4)$ generated by these elements). This is what we prove in the following lemma; then the proof of Proposition 3.1 follows easily. Since A_n, B_n, C_n always have odd degree, we can write $A_n = \widetilde{A}_n c_{15}$, $B_n = \widetilde{B}_n c_{15}$, $C_n = \widetilde{C}_n c_{15}$. Moreover, for $n \geq 4$, $D_n = A_{n-1}^2 = \widetilde{A}_{n-1}^2 c_{15}^2$, and we can also write $D_n = \widetilde{D}_n c_{15}$.

Lemma 3.8. *For all* $n \geq 4$, $\{\widetilde{A}_n, \widetilde{B}_n, \widetilde{C}_n, \widetilde{D}_n\}$ *is a regular sequence in* $\mathbb{Z}/2[c_8, c_{12}, c_{14}, c_{15}]$.

Proof. For $n \geq 4$ and $i = 1, 2, 3, 4$,

$$\begin{vmatrix} t_1 & t_2 & t_3 & t_4 & t_i \\ t_1^2 & t_2^2 & t_3^2 & t_4^2 & t_i^2 \\ t_1^4 & t_2^4 & t_3^4 & t_4^4 & t_i^4 \\ t_1^8 & t_2^8 & t_3^8 & t_4^8 & t_i^8 \\ t_1^{2^n} & t_2^{2^n} & t_3^{2^n} & t_4^{2^n} & t_i^{2^n} \end{vmatrix} = 0 = t_i D_n + t_i^2 C_n + t_i^4 B_n + t_i^8 A_n + t_i^{2^n} c_{15}.$$

Therefore, $\mathbb{Z}/2[\widetilde{A}_n, \widetilde{B}_n, \widetilde{C}_n, \widetilde{D}_n] \subset \mathbb{Z}/2[t_1, t_2, t_3, t_4]$ is an integral extension, and hence so is $\mathbb{Z}/2[\widetilde{A}_n, \widetilde{B}_n, \widetilde{C}_n, \widetilde{D}_n] \subset \mathbb{Z}/2[c_8, c_{12}, c_{14}, c_{15}]$. This means that $\mathbb{Z}/2[c_8, c_{12}, c_{14}, c_{15}]$ is a finitely generated $\mathbb{Z}/2[\widetilde{A}_n, \widetilde{B}_n, \widetilde{C}_n, \widetilde{D}_n]$-module. From [B] (Lemma 5.5.5), to prove that $\{\widetilde{A}_n, \widetilde{B}_n, \widetilde{C}_n, \widetilde{D}_n\}$ is regular, it suffices to show that these elements are algebraically independent. By the Derivation Lemma, it suffices to find four derivations $\delta_1, \delta_2, \delta_3, \delta_4$ such that $\det(\delta_i \widetilde{A}_j) \neq 0$. For $n \geq 4$, $Q_3 D_n = Q_3(A_{n-1}^2) = 0 = (Q_3 \widetilde{D}_n) c_{15} + D_n c_{15}$. This implies $Q_3 \widetilde{D}_n \neq 0$. Therefore,

$$\begin{vmatrix} Q_0\widetilde{A}_n & Q_0\widetilde{B}_n & Q_0\widetilde{C}_n & Q_0\widetilde{D}_n \\ Q_1\widetilde{A}_n & Q_1\widetilde{B}_n & Q_1\widetilde{C}_n & Q_1\widetilde{D}_n \\ Q_2\widetilde{A}_n & Q_2\widetilde{B}_n & Q_2\widetilde{C}_n & Q_2\widetilde{D}_n \\ Q_3\widetilde{A}_n & Q_3\widetilde{B}_n & Q_3\widetilde{C}_n & Q_3\widetilde{D}_n \end{vmatrix} = \begin{vmatrix} 0 & 0 & \widetilde{D}_n & 0 \\ 0 & \widetilde{D}_n & 0 & 0 \\ \widetilde{D}_n & 0 & 0 & 0 \\ Q_3\widetilde{A}_n & Q_3\widetilde{B}_n & Q_3\widetilde{C}_n & Q_3\widetilde{D}_n \end{vmatrix} \neq 0. \blacksquare$$

Proof of Proposition 3.1. We first suppose $a \in H^{\mathrm{odd}}(BDI(4), \mathbb{Z}/2)$ is such that $Q_n a = 0$ and $a \in (A_{n+1}, B_{n+1})$. Applying Q_n we get $0 = (Q_n\alpha)A_{n+1} + (Q_n\beta)B_{n+1}$ for some α, β in even degrees. It follows from Lemma 3.8 that $Q_n\alpha \in (B_{n+1})$ and $Q_n\beta \in (A_{n+1})$. That is, $Q_n\alpha = Q_n(c_{12}t)$ and $Q_n\beta = Q_n(c_8 t)$ for some $t \in \operatorname{Ker} Q_n$. Therefore, $a = \alpha A_{n+1} + \beta B_{n+1} = (c_{12}t + t_1)A_{n+1} + (c_8 t + t_2)B_{n+1} = Q_n(c_8 c_{12} t + c_8 t_1 + c_{12} t_2)$ for some $t, t_1, t_2 \in \operatorname{Ker} Q_n$. Analogous results can be proved if $a \in (B_{n+1}, C_{n+1})$ or $a \in (A_{n+1}, C_{n+1})$.

Now, let $a \in H^{\mathrm{odd}}(BDI(4), \mathbb{Z}/2)$ be such that $Q_n a = 0$ for some $n \geq 3$. Then $D_{n+1} a = A_{n+1} Q_2 a + B_{n+1} Q_1 a + C_{n+1} Q_0 a$. From Lemma 3.8, $a \in (\widetilde{A}_{n+1}, \widetilde{B}_{n+1}, \widetilde{C}_{n+1})$. Since a has odd degree, $a = A_{n+1}x + B_{n+1}y + C_{n+1}z$. Applying Q_n, we obtain

$$(*) \qquad 0 = A_{n+1}(Q_n x) + B_{n+1}(Q_n y) + C_{n+1}(Q_n z).$$

Again by 3.8, this implies $Q_n x \in (B_{n+1}, C_{n+1})$, $Q_n y \in (A_{n+1}, C_{n+1})$, $Q_n z \in (A_{n+1}, B_{n+1})$. Using the fact that $Q_n x$, $Q_n y$, $Q_n z$ satisfy $(*)$, it is easy to check that

$$\begin{cases} Q_n x = \alpha B_{n+1} + \beta C_{n+1}, \\ Q_n y = \alpha A_{n+1} + \delta C_{n+1}, \\ Q_n z = \beta A_{n+1} + \delta B_{n+1}, \end{cases}$$

for some elements $\alpha, \beta, \delta \in H^*(BDI(4), \mathbb{Z}/2)$. As we have seen before, this implies

$$\begin{cases} Q_n x = Q_n(c_{12}c_{14}t + c_{12}t_1 + c_{14}t_2), \\ Q_n y = Q_n(c_8c_{14}t + c_8t_1 + c_{14}t_3), \\ Q_n z = Q_n(c_8c_{12}t + c_8t_2 + c_{12}t_3), \end{cases}$$

where $t, t_i \in \operatorname{Ker} Q_n$. It follows that $a \in \operatorname{Im} Q_n$. ∎

NOTE. The result of Proposition 3.1 can be proved in the same way for any Dickson algebra $D(n), n \geq 1$.

4. The Atiyah–Hirzebruch spectral sequence for $P(n)$ and $K(n)$. Our main tool to compute $K(n)^*(DI(4))$ and $P(n)^*(DI(4))$ is the Atiyah–Hirzebruch spectral sequence (in the sequel abbreviated to AHss):

$$E_2^{**}(K(n)) = H^*(-, K(n)^*) \Rightarrow K(n)^*(-),$$
$$E_2^{**}(P(n)) = H^*(-, P(n)^*) \Rightarrow P(n)^*(-).$$

Recall ([Ya4]) that the first non-trivial differential in the AHss for both $P(n)$-theory and $K(n)$-theory is $d_{2p^n-1} = v_n \otimes Q_n$, where Q_n is the Milnor operation in ordinary mod p cohomology.

In the case of $K(n)$-theory, the AHss has some properties that make the computation easier. The possible non-trivial differentials are $d_{2(p^n-1)s+1}$ with $s \geq 1$. We set $\delta_s = v_n^{-s} d_r$, where $r = 2(p^n-1)s+1$. Then

$$E_{r+1}^{**}(K(n)) \cong K(n)^* \otimes H(E_r^{*0}, \delta_s).$$

This means that each term in the AHss is a free $K(n)^*$-module, and the spectral sequence for $K(n)^*(X)$ is a spectral sequence of $K(n)^*$-Hopf algebras if X is an associative H-space. As a consequence, we get the following lemma, which will be useful to compute the AHss for $K(n)^*(DI(4))$:

LEMMA 4.1. *Let X be an H-space. Suppose that there exists $r \geq 2$ such that $E_r^{**}(K(n)) \cong K(n)^* \otimes A$, where $A \subset H^*(X, \mathbb{Z}/p)$ is a biprimitive Hopf algebra on odd degree generators. Then $E_\infty^{**}(K(n)) \cong E_r^{**}(K(n))$.*

Proof. Recall that a Hopf algebra A is said to be *biprimitive* if it is primitively generated, and all the primitive elements of A are indecomposable. The differential $\delta_s = v_n^{-s} d_r$ commutes with the coproduct and sends primitive elements to primitive elements. But δ_s has odd degree, and there are no primitive elements x, y in A such that $|x| - |y|$ is odd. Therefore, we have $\delta_s = 0$, and $d_r = 0$. ∎

Define $P(n) \xrightarrow{j_{n,s}} P(n+s)$ to be the composition

$$P(n) \xrightarrow{i_n} P(n+1) \xrightarrow{i_{n+1}} \ldots \xrightarrow{i_{n+s-1}} P(n+s)$$

for $s \geq 0$ ($j_{n,0}$ is the identity on $P(n)$, and $j_{n,1} = i_n$).

The maps of spectral sequences induced by the canonical maps of spectra $P(n) \xrightarrow{i_n} P(n+1)$, $P(n) \xrightarrow{\lambda_n} K(n)$, $P(n) \xrightarrow{j_{n,s}} P(n+s)$ will also be denoted by $i_n, \lambda_n, j_{n,s}$.

LEMMA 4.2. *Let X be a space, $n \geq 1$, $r \geq 2$, and $\alpha \in E_r^{p,q}(P(n))$ a permanent cycle in the AHss for $P(n)^*(X)$. Suppose that the following conditions hold*:

(i) *$E_r^{i,j}(P(n)) = 0$ if $i+j > p+q$ and $i \leq p$.*

(ii) *$\lambda_n(\alpha) \neq 0$ in $E_\infty^{**}(K(n))$.*

Then the $P(n)^*$-module generated by α in $E_\infty^{**}(P(n))$ is $P(n)^*$-free.

Proof. The argument used in the proof is analogous to that of Lemma 2.1 in [Ya2]. Assume that there is a relation $v\alpha = 0$ in $E_\infty^{**}(P(n))$ for some $v \in P(n)^*$. Then $\lambda_n(v\alpha) = \lambda_n(v)\lambda_n(\alpha) = 0$ in $E_\infty^{**}(K(n))$. Assumption (ii) implies that $v \in (v_{n+1}, v_{n+2}, \ldots) \subset P(n)^*$ (the ideal generated in $P(n)^*$ by these coefficients).

Let $v\alpha = v_{n+s}^{i_s} \ldots v_{n+1}^{i_1} v_n^{i_0} \alpha + \sum_j \omega_j \alpha$, where $\{i_s, \ldots, i_1, i_0\}$ is the largest sequence under the lexicographical order. There exists ([J-W]) a cohomology operation $r \in P(n)^*P(n)$ such that $r(v_{n+s}^{i_s} \ldots v_{n+1}^{i_1} v_n^{i_0}) = v_n^{i_0+i_1+\ldots+i_s}$ and $r(\omega_j) = 0$ for all j.

Recall that the associated filtration for a complex X with skeleta $\{X^q\}$ is defined as

$$F_m = \operatorname{Ker}(P(n)^*(X) \to P(n)^*(X^{m-1}))$$

and $E_\infty^{m,*} \cong F_m/F_{m+1}$.

That $v\alpha = 0$ in $E_\infty^{**}(P(n))$ means that $v\alpha \in F_{p+1}$, and the naturality of the operations implies that $r(v\alpha) \in F_{p+1}$. From assumption (i), each element $\beta \in E_\infty^{**}(P(n))$ with total degree $|\beta| > |\alpha|$ belongs to F_{p+1}. This implies $v_n^{i_0+\ldots+i_s}\alpha \in F_{p+1}$, and hence $v_n^{i_0+\ldots+i_s}\alpha = 0$ in $E_\infty^{**}(P(n))$ and $v_n^{i_0+\ldots+i_s}\lambda_n(\alpha) = 0$ in $E_\infty^{**}(K(n))$. This is a contradiction, since $E_\infty^{**}(K(n))$ is $K(n)^*$-free. ∎

In order to simplify the notation, we introduce the following definition:

DEFINITION 4.3. (i) An element $\alpha \in E_r^{p,q}(P(n))$ is said to be *maximal* if it is the only non-trivial element in $E_r^{i,j}(P(n))$ for $i+j \geq p+q$ and $i \leq p$.

(ii) We will say that $E_r^{**}(P(n))$ is *maximally generated as $P(n)^*$-module* if there exists a set $\{x_1, \ldots, x_s\}$ of generators where all x_i's are maximal.

As a consequence of 4.2, we prove the following result, which will be useful in the next section to compute the AHss of $P(n)^*(DI(4))$ from those of $K(m)^*(DI(4))$ for $m \geq n$, and $P(m)^*(DI(4))$ for $m > n$.

LEMMA 4.4. *Let $n \geq 1$ and $r \geq 2$. Suppose that $E_r^{**}(P(n))$ is a maximally generated $P(n)^*$-module of the form*

$$E_r^{**}(P(n)) \cong (P(n)^* \otimes A_n) \oplus (P(n+1)^* \otimes A_{n+1}) \oplus \ldots \oplus (P(n+s)^* \otimes A_{n+s})$$

for some $s \geq 0$, where the A_i are finitely generated free $\mathbb{Z}/2$-modules. Moreover, suppose that the following conditions hold:

(i) *$E_r^{**}(K(m)) \cong E_\infty^{**}(K(m))$ for all $m \geq n$.*
(ii) *$E_r^{**}(P(m)) \cong E_\infty^{**}(P(m))$ for all $m > n$.*
(iii) *For all $0 \leq t \leq s$ and $0 \neq \alpha \in A_{n+t}$, $j_{n,t}(\alpha)$ is maximal in $E_r^{**}(P(n+t))$ and $\lambda_{n+t} j_{n,t}(\alpha) \neq 0$ in $E_r^{**}(K(n+t))$.*

Then $E_\infty^{**}(P(n)) \cong E_r^{**}(P(n))$.

Proof. Let $\{x_1, \ldots, x_k\}$ be a minimal set of $P(n)^*$-generators of $E_r^{**}(P(n))$, where all x_i's are maximal. Then $x_i \in A_{n+t}$ for some $0 \leq t \leq s$, and we can suppose that $x_i \in E_r^{a_i,*}$ with $a_1 < \ldots < a_k$. Since $\lambda_{n+t} j_{n,t}(x_i) \neq 0$ in $E_r^{**}(K(n+t)) \cong E_\infty^{**}(K(n+t))$ there is no differential in the AHss that kills x_i. Therefore, if we want to prove that x_i is permanent, it suffices to show that x_i is in the kernel of all differentials in the spectral sequence.

Suppose that the following assertion is true:

(∗) If x_i is permanent, then so is the $P(n)^*$-module generated by x_i in $E_r^{**}(P(n))$.

Then the lemma follows by induction: it is clear that x_k is a permanent cycle and, by (∗), the $P(n)^*$-module generated by x_k in $E_r^{**}(P(n))$ is also permanent. Assume that the $P(n)^*$-module generated by x_i in $E_r^{**}(P(n))$ is permanent for all $i > m$. This implies $dx_m = 0$ for any differential d in the AHss, and hence x_m is permanent.

Finally, we have to prove (∗). If $x_i \in A_n$, it follows from assumption (iii) that $\lambda_n(x_i) \neq 0$ in $E_r^{**}(K(n)) \cong E_\infty^{**}(K(n))$. Hence, we are under the conditions of Lemma 4.2, and (∗) holds. If $x_i \in A_{n+t}$ with $t \geq 1$, our hypotheses imply that $j_{n,t}(x_i)$ satisfies the conditions of Lemma 4.2: $j_{n,t}(x_i)$ is maximal in $E_r^{**}(P(n+t)) \cong E_\infty^{**}(P(n+t))$, and $\lambda_{n+t} j_{n,t}(x_i) \neq 0$ in $E_r^{**}(K(n+t)) \cong E_\infty^{**}(K(n+t))$. Therefore, the $P(n+t)^*$-module generated by $j_{n,t}(x_i)$ in $E_\infty^{**}(P(n+t))$ is $P(n+t)^*$-free, and this implies that the $P(n+t)^*$-module generated by x_i in $E_\infty^{**}(P(n))$ is $P(n+t)^*$-free. ∎

5. $K(n)$- and $P(n)$-cohomology of $DI(4)$ for $n \geq 1$. This section is devoted to the proof of Theorem 1.1.

Recall that the ordinary mod 2 cohomology of $DI(4)$ has the following algebra structure:

$$H^*(DI(4), \mathbb{Z}/2) \cong \mathbb{Z}/2[x_7]/(x_7^4) \otimes \Lambda(y_{11}, z_{13}),$$

and the Steenrod algebra acts by $\mathrm{Sq}^4 x = y$, $\mathrm{Sq}^2 y = z$, $\mathrm{Sq}^1 z = x^2$. In particular, the action of the Milnor operations on the generators is trivial except for $Q_2 x = Q_1 y = Q_0 z = x^2$. The primitive elements in $H^*(DI(4), \mathbb{Z}/2)$ are $\{x, y, z, x^2\}$.

We first compute the E_∞-term in the AHss for $K(n)^*(DI(4))$ and $P(n)^*(DI(4))$, by using the results of Section 4.

LEMMA 5.1. *For all $n \geq 3$, there are $K(n)^*$-algebra, respectively $P(n)^*$-algebra isomorphisms*:

(a) $E_\infty^{**}(K(n)) \cong K(n)^* \otimes \mathbb{Z}/2[x_7]/(x_7^4) \otimes \Lambda(y_{11}, z_{13})$,
(b) $E_\infty^{**}(P(n)) \cong P(n)^* \otimes \mathbb{Z}/2[x_7]/(x_7^4) \otimes \Lambda(y_{11}, z_{13})$.

Proof. The first potentially non-trivial differential in the AHss for both $P(n)$- and $K(n)$-theory is $d_{2^{n+1}-1} = v_n \otimes Q_n$. But, for $n \geq 3$, $Q_n x = Q_n y = Q_n z = 0$.

For dimensional reasons, the differentials d_r with $r > 2^{n+1} - 1$ are forced to be zero, in the $K(n)$-AHss for all $n \geq 3$, and in the $P(n)$-AHss for all $n \geq 4$. Hence,

$$E_\infty^{**}(K(n)) \cong E_2^{**}(K(n)) \cong K(n)^* \otimes H^*(DI(4), \mathbb{Z}/2) \quad \text{for all } n \geq 3,$$
$$E_\infty^{**}(P(n)) \cong E_2^{**}(P(n)) \cong P(n)^* \otimes H^*(DI(4), \mathbb{Z}/2) \quad \text{for all } n \geq 4.$$

Finally, if we set $A_3 = H^*(DI(4), \mathbb{Z}/2)$, then $E_2^{**}(P(3)) \cong P(3)^* \otimes A_3$ is a maximally generated $P(3)^*$-module, and $\lambda_3 : E_2^{**}(P(3)) \to E_2^{**}(K(3))$ acts as the identity on A_3. Hence, we are under the conditions of Lemma 4.4, and $E_\infty^{**}(P(3)) \cong E_2^{**}(P(3))$. ∎

LEMMA 5.2. *There exist $K(2)^*$-algebra, respectively $P(2)^*$-algebra isomorphisms*:

(a) $E_\infty^{**}(K(2)) \cong K(2)^* \otimes \Lambda(y_{11}, z_{13}, t_{21})$,
(b) $E_\infty^{**}(P(2)) \cong (P(2)^* \otimes \Lambda(t_{21}) \oplus P(3)^* \otimes \mathbb{Z}/2\{\omega_{14}\}) \otimes \Lambda(y_{11}, z_{13})$.

Proof. The first non-trivial differential in the AHss for both $K(2)$- and $P(2)$-theory is $d_7 = v_2 \otimes Q_2$, acting as $d_7 x = v_2 x^2, d_7 y = d_7 z = 0$.

We start with the AHss for $K(2)$. The E_8-term is

$$E_8^{**}(K(2)) \cong K(2)^* \otimes \Lambda(y, z, x^3).$$

Now, we can apply Lemma 4.1 to conclude that $E_8^{**}(K(2)) \cong E_\infty^{**}(K(2))$. Rename $t = x^3$, and the isomorphism (a) is proved.

For the $P(2)$-AHss, we can compute the E_8-term as

$$E_8^{**}(P(2)) \cong (P(2)^* \otimes \Lambda(y, z, x^3)) \oplus (P(3)^* \otimes \mathbb{Z}/2\{x^2\} \otimes \Lambda(y, z)).$$

If we set $B_2 = \Lambda(y, z, x^3)$ and $B_3 = \mathbb{Z}/2\{x^2\} \otimes \Lambda(y, z)$, then $E_8^{**}(P(2)) \cong (P(2)^* \otimes B_2) \oplus (P(3)^* \otimes B_3)$ is a maximally generated $P(2)^*$-module and we are again under the conditions of Lemma 4.4:

(i) $E_\infty^{**}(K(m)) \cong E_8^{**}(K(m))$ for all $m \ge 2$,
(ii) $E_\infty^{**}(P(m)) \cong E_8^{**}(P(m))$ for all $m \ge 3$,
(iii) λ_2 is the identity on B_2, and $i_2(B_3) \subset A_3$.

Therefore, $E_\infty^{**}(P(2)) \cong E_8^{**}(P(2))$. Rename $t = x^3$, $\omega = x^2$, and the lemma is proved. ∎

LEMMA 5.3. *There exist $K(1)^*$-algebra, respectively $P(1)^*$-algebra isomorphisms*:

(a) $E_\infty^{**}(K(1)) \cong K(1)^* \otimes \Lambda(x_7, z_{13}, r_{25})$,
(b) $E_\infty^{**}(P(1)) \cong ((P(1)^* \otimes \Lambda(u_5, r_{25}, s_{32}) \oplus P(2)^* \otimes \mathbb{Z}/2\{t_{21}\} \oplus P(3)^* \otimes \mathbb{Z}/2\{\omega_{14}\}) \otimes \Lambda(z_{13}))/I$, *where I is the ideal generated by* $\{ur + v_1 s, us, rs\}$.

Proof. The first non-trivial differential is $d_3 = v_1 \otimes Q_1$, acting as $d_3 y = v_1 x^2$, $d_3 x = d_3 z = 0$. In the AHss for $K(1)$, the E_4-term is

$$E_4^{**}(K(1)) \cong K(1)^* \otimes \Lambda(x, z, x^2 y).$$

It follows from Lemma 4.1 that $E_\infty^{**}(K(1)) \cong E_4^{**}(K(1))$. Rename $r = x^2 y$, and we have the isomorphism (a). In the AHss for $P(1)$, the E_4-term is

$$E_4^{**}(P(1)) \cong (P(1)^* \otimes \Lambda(x, z, x^2 y)) \oplus (P(2)^* \otimes \mathbb{Z}/2\{x^2\} \otimes \Lambda(x, z)).$$

The differential d_5 is trivial for dimensional reasons. We next consider the differential d_7, and recall that it commutes with the map $E_7^{**}(P(1)) \xrightarrow{i_1} E_7^{**}P(2))$. Therefore,

$$\begin{cases} d_7 x = v_2 x^2 \bmod (v_1), \\ d_7 z = 0 \bmod (v_1), \\ d_7(x^2 y) = 0 \bmod (v_1). \end{cases}$$

Since x^2 is v_1-torsion,

$$\begin{cases} d_7 x = v_2 x^2, \\ d_7 z = \alpha v_1^3 x z, \\ d_7(x^2 y) = \beta v_1^3 x^3 y, \end{cases}$$

with $\alpha, \beta = 0, 1$. On the other hand, d_7 also has to commute with $E_7^{**}(P(1)) \xrightarrow{\lambda_1} E_7^{**}(K(1))$. But d_7 acts trivially on $E_7^{**}(K(1))$, and λ_1 is the identity over $\Lambda(x, z, x^2 y)$. This implies $\alpha = \beta = 0$. We can now compute the E_8-term in the AHss for $P(1)$ as

$$\begin{aligned} E_8^{**}(P(1)) \cong P(1)^* &\otimes \mathbb{Z}/2\{v_1 x, z, v_1 x z, x^2 y, x^3 y, x^2 y z, x^3 y z\} \\ &\oplus P(2)^* \otimes \mathbb{Z}/2\{x^3, x^3 z\} \oplus P(3)^* \otimes \mathbb{Z}/2\{x^2, x^2 z\}. \end{aligned}$$

$E_8^{**}(P(1))$ is a maximally generated $P(1)^*$-module, and it is easy to check that we are again under the conditions of Lemma 4.4. Thus,

$$\begin{aligned} E_\infty^{**}(P(1)) &\cong E_8^{**}(P(1)) \\ &\cong (P(1)^* \otimes \Lambda(v_1x, z, x^2y, x^3y) \\ &\qquad \oplus P(2)^* \otimes \mathbb{Z}/2\{x^3\} \otimes \Lambda(z) \oplus P(3)^* \otimes \mathbb{Z}/2\{x^2\} \otimes \Lambda(z))/R, \end{aligned}$$

where $R \equiv \{(v_1x)(x^2y) = v_1(x^3y), (v_1x)(x^3y) = (x^2y)(x^3y) = 0\}$. Rename $u = v_1x$, $r = x^2y$, $s = x^3y$, $t = x^3$, $\omega = x^2$, and the lemma is proved. ∎

We still have to solve some extension problems to obtain the algebra structure of $P(n)^*(DI(4))$ and $K(n)^*(DI(4))$. However, the algebra structure of the E_∞-term gives us quite a lot of information about the algebra structure of the corresponding cohomology theory. In this sense, once we have obtained the $F(n)^*$-module structure of $F(n)^*(-)$ (where $F(n)$ denotes one of the spectra $P(n)$ or $K(n)$), we will say that we know $F(n)^*(-)$ as an $F(n)^*$-*module*, meaning by this that we not only know the module structure, but we also have some information about the algebra structure.

In the following computations it will be useful to know explicitly how the map $P(n)^*(DI(4)) \xrightarrow{i_n} P(n+1)^*(DI(4))$ acts, for all $n \geq 1$.

A minimal set of $P(n)^*$-*module* generators of $P(n)^*(DI(4))$, for $n \geq 3$, is the one represented in the following table, where the number above each element is its degree:

7	11	13	14	18	20	21	24	25	27	31	32	34	38	45
x	y	z	x^2	xy	xz	x^3	yz	x^2y	x^2z	xyz	x^3y	x^3z	x^2yz	x^3yz

For dimensional reasons, it is easy to check that i_n acts as the identity over these generators, for all $n \geq 3$. (Note: we are abusing notation by denoting the generators in cohomology the same as those in the E_∞-term).

Lemma 5.2 yields that a minimal set of $P(2)^*$-*module* generators of $P(2)^*(DI(4))$ is:

11	13	14	21	24	25	27	32	34	38	45
y	z	ω	t	yz	ωy	ωz	ty	tz	ωyz	tyz

We first study the dimensional possibilities for the map i_2 acting over these generators, and then check that we can rechoose generators in such a way that the map i_2 acts as follows (denoting the new generators the same as the old ones): $i_2(y) = y$, $i_2(z) = z$, $i_2(\omega) = x^2$, $i_2(t) = x^3$, and extending multiplicatively over the other generators.

Lemma 5.3 shows that a minimal set of $P(1)^*$-*module* generators of $P(1)^*(DI(4))$ is:

5	13	14	18	21	25	27	32	34	38	45
u	z	ω	uz	t	r	ωz	s	tz	rz	sz

We study the dimensional possibilities and rechoose generators such that the map i_1 acts by: $i_1(u) = v_2 y$, $i_1(z) = z$, $i_1(\omega) = \omega$, $i_1(t) = t$, $i_1(r) = \omega y$, $i_1(s) = ty$, and we extend it multiplicatively over the other generators.

Using i_n and the exact triangle relating $P(n)$- and $P(n+1)$-theory, we will be able to compute the $P(n)^*$-*module* structure of $P(n)^*(DI(4))$ from that of $P(n+1)^*(DI(4))$. The $K(n)^*$-*module* structure of $K(n)^*(DI(4))$ is given directly by the E_∞-term, as $E_\infty^{**}(K(n))$ is $K(n)^*$-free.

To determine the algebra structure of $P(n)^*(DI(4))$ and $K(n)^*(DI(4))$, it will be useful to apply the map $P(n)^*(-) \xrightarrow{\lambda_n} K(n)^*(-)$. Recall that λ_n is the composition

$$P(n)^*(-) \xrightarrow{j_n} P(n)^*(-) \otimes_{P(n)^*} K(n)^* \cong K(n)^*(-)$$

where j_n is the natural inclusion. It is easy to check that $\operatorname{Ker} \lambda_n = \operatorname{Ker} j_n = \{x \in P(n)^*(-) : v_n^s x \in (v_{n+1}, v_{n+2}, \ldots) \text{ for some } s \geq 0\}$. (Here $(v_{n+1}, v_{n+2}, \ldots)$ is the ideal of $P(n)^*(-)$ generated by these elements.)

As noted before, $K(n)$-theory has the advantage that we can use arguments involving the coproduct. Such arguments are used to prove the following lemma, which will allow us to determine relations of the form $x^2 = 0$ in $K(n)^*(DI(4))$, and use it to study these relations in $P(n)^*(DI(4))$, through λ_n.

LEMMA 5.4. *Let X be an H-space, $p = 2$ and $n \geq 1$. Suppose that $K(n)^*(X)$ is a commutative $K(n)^*$-algebra, and the E_∞-term in the AHss is*

$$E_\infty^{**}(K(n)) \cong K(n)^* \otimes \Lambda(x, y, z)$$

where $x, y, z \in H^*(X, \mathbb{Z}/2)$ have odd degree, and $|x| < |y| < |z|$. Then $K(n)^*(X) \cong E_\infty^{**}(K(n))$ as $K(n)^*$-algebras.

Proof. By dimensional reasons, $z^2 = 0$, $y^2 \in K(n)^*\{yz, xz\}$ and $x^2 \in K(n)^*\{xz, xy, yz\}$ in $K(n)^*(X)$. This implies that $x^2z, y^2z, y^2x, x^2y, y^3, x^3 \in K(n)^*\{xyz\}$. Therefore, each decomposable element with odd degree in the $\mathbb{Z}/2$-algebra generated by x, y, z belongs to $K(n)^*\{xyz\}$.

Denote by Ψ the coproduct in $K(n)^*(X)$. Then

$$\Psi(x) = 1 \otimes x + x \otimes 1 + \sum_i (x_i \otimes x'_i),$$

$$\Psi(y) = 1 \otimes y + y \otimes 1 + \sum_j (y_j \otimes y'_j),$$

$$\Psi(z) = 1 \otimes z + z \otimes 1 + \sum_k (z_k \otimes z'_k),$$

for some $x_i, x'_i, y_j, y'_j, z_k, z'_k \in K(n)^*(X)$. Corollary 2.3 yields

$$\begin{aligned}
\Psi(x^2) &= \Big(1\otimes x + x\otimes 1 + \sum_i x_i\otimes x'_i\Big)^2 \\
&\quad + v_n\Big(1\otimes Q_{n-1}x + \sum_i x_i\otimes Q_{n-1}x'_i\Big)\Big(Q_{n-1}x\otimes 1 + \sum_i Q_{n-1}x_i\otimes x'_i\Big),\\
\Psi(xy) &= \Big(1\otimes x + x\otimes 1 + \sum_i x_i\otimes x'_i\Big)\Big(1\otimes y + y\otimes 1 + \sum_j y_j\otimes y'_j\Big)\\
&\quad + v_n\Big(1\otimes Q_{n-1}x + \sum_i x_i\otimes Q_{n-1}x'_i\Big)\Big(Q_{n-1}y\otimes 1 + \sum_j Q_{n-1}y_j\otimes y'_j\Big)\\
&= x\otimes y + y\otimes x + \{\text{other terms}\},
\end{aligned}$$

and analogous expressions for $\Psi(yz)$ and $\Psi(xz)$.

The only possibility for $x^2 \neq 0$ is that there exists a relation of the form $x = vd$, or $y = vd$, or $z = vd$, where $v \in K(n)^*$ and d is a decomposable element in the $\mathbb{Z}/2$-algebra generated by x, y, z. This implies $x \in K(n)^*\{xyz\}$, or $y \in K(n)^*\{xyz\}$, or $z \in K(n)^*\{xyz\}$, which is a contradiction with the module structure of $K(n)^*(X)$. Thus, $x^2 = 0$. Arguing similarly, we can prove that $y^2 = 0$. ∎

In the following three propositions, we compute the algebra structure of $K(n)^*(DI(4))$ and $P(n)^*(DI(4))$ for all $n \geq 1$.

Proposition 5.5. *For all $n \geq 3$, there are $K(n)^*$-algebra, respectively $P(n)^*$-algebra isomorphisms*

$$K(n)^*(DI(4)) \cong E_\infty^{**}(K(n)), \qquad P(n)^*(DI(4)) \cong E_\infty^{**}(P(n)).$$

Proof. In Lemma 5.1 we proved that

$$\begin{aligned}
E_\infty^{**}(K(n)) &\cong K(n)^* \otimes \mathbb{Z}/2[x_7]/(x_7^4) \otimes \Lambda(y_{11}, z_{13}),\\
E_\infty^{**}(P(n)) &\cong P(n)^* \otimes \mathbb{Z}/2[x_7]/(x_7^4) \otimes \Lambda(y_{11}, z_{13}),
\end{aligned}$$

for all $n \geq 3$. The E_∞-term gives directly the $P(n)^*$-*module* structure of $P(n)^*(DI(4))$, and the $K(n)^*$-*module* structure of $K(n)^*(DI(4))$. For the $P(n)^*$-algebra isomorphism, we only need to prove that $x^4 = y^2 = z^2 = 0$, and the commutativity.

The relations $x^4 = y^2 = z^2 = 0$ in $P(n)^*(DI(4))$ follow for dimensional reasons. The correction to commutativity is given in terms of $Q_{n-1}x$, $Q_{n-1}y$, and $Q_{n-1}z$ (see Corollary 2.2). Since i_n acts as the identity for $n \geq 3$, the action of the Q_i's on $P(n)$-cohomology is the same (up to coefficients) as the action of the Milnor operations on ordinary mod 2 cohomology. Looking at the dimensional possibilities, we find that

$$Q_{n-1}x = Q_{n-1}y = Q_{n-1}z = 0 \quad \text{in } P(n)^*(DI(4)), \text{ for all } n \geq 4,$$
$$Q_2x = x^2,\ Q_2y = \alpha v_3x^3y,\ Q_2z = \beta v_3x^3z \text{ with } \alpha, \beta = 0, 1 \text{ in } P(3)^*(DI(4)).$$

Therefore, $(Q_2x)(Q_2y) = (Q_2x)(Q_2z) = (Q_2y)(Q_2z) = 0$, and $P(n)^*(DI(4))$ is a commutative algebra for all $n \geq 3$.

For the $K(n)^*$-algebra isomorphism, consider the map $\lambda_n : P(n)^*(DI(4)) \to K(n)^*(DI(4))$. Since $(\operatorname{Ker}\lambda_n) \cap (\mathbb{Z}/2[x]/(x^4) \otimes \Lambda(y, z)) = \{0\}$,

$$K(n)^*(DI(4)) \cong K(n)^* \otimes \mathbb{Z}/2[\lambda_n(x)]/(\lambda_n(x)^4) \otimes \Lambda(\lambda_n(y), \lambda_n(z))$$

as $K(n)^*$-*modules*. The fact that λ_n is an algebra morphism implies $\lambda_n(x)^4 = \lambda_n(y)^2 = \lambda_n(z)^2 = 0$, and the commutativity of $K(n)^*(DI(4))$. ∎

PROPOSITION 5.6. *There exist $K(2)^*$-algebra, respectively $P(2)^*$-algebra isomorphisms*

$$K(2)^*(DI(4)) \cong E_\infty^{**}(K(2)), \qquad P(2)^*(DI(4)) \cong E_\infty^{**}(P(2)).$$

Proof. From Lemma 5.2,

$$E_\infty^{**}(K(2)) \cong K(2)^* \otimes \Lambda(y_{11}, z_{13}, t_{21}),$$
$$E_\infty^{**}(P(2)) \cong (P(2)^* \otimes \Lambda(y_{11}, z_{13}, t_{21})) \oplus (P(3)^* \otimes \mathbb{Z}/2\{\omega_{14}\} \otimes \Lambda(y, z)).$$

To prove the $P(2)^*$-*module* isomorphism, consider the exact triangle

$$\begin{tikzcd} P(2)^*(-) \arrow[rr, "v_2"] & & P(2)^*(-) \arrow[dl, "i_2"] \\ & P(3)^*(-) \arrow[ul, "\delta_2"] & \end{tikzcd}$$

Let $x_7 \in P(3)^*(DI(4))$. Then $(i_2\delta_2)(x) = Q_2(x) = x^2 \neq 0$. If we set $\omega = \delta_2 x$, we find that ω is an element in $P(2)^*(DI(4))$ such that $i_2(\omega) = x^2$ and $v_2\omega = 0$. This shows the $P(2)^*$-*module* isomorphism.

For the $P(2)^*$-algebra isomorphism, we need to prove that $y^2 = z^2 = t^2 = \omega^2 = t\omega = 0$, and the commutativity. Dimensional reasons imply that $t^2 = \omega t = 0$, $\omega^2 = \alpha v_2tz$, $y^2 = \beta v_2^2tz$, $z^2 = \gamma v_2ty$, with $\alpha, \beta, \gamma = 0, 1$. Using the module structure gives $v_2\omega^2 = 0 = \alpha v_2^2tz$, and this implies $\alpha = 0$.

For the commutativity, we use the fact that the maps i_n commute with the Q_i to obtain

$$i_2(Q_1\omega) = Q_1i_2(\omega) = Q_1x^2 = 0 \bmod (v_3, v_4, \ldots) \Rightarrow Q_1\omega = 0 \bmod (v_2, v_3, \ldots),$$
$$i_2(Q_1y) = Q_1i_2(y) = Q_1y = x^2 \bmod (v_3, v_4, \ldots) \Rightarrow Q_1y = \omega \bmod (v_2, v_3, \ldots),$$
$$i_2(Q_1z) = Q_1i_2(z) = 0 \bmod (v_3, v_4, \ldots) \Rightarrow Q_1z = 0 \bmod (v_2, v_3, \ldots),$$
$$i_2(Q_1t) = Q_1i_2(t) = Q_1x^3 = 0 \bmod (v_3, v_4, \ldots) \Rightarrow Q_1t = 0 \bmod (v_2, v_3, \ldots).$$

Examining the dimensional possibilities shows that $Q_1\omega \in P(2)^*\{tyz\}$, $Q_1 y \in P(2)^*\{\omega, ty, tz\}$, $Q_1 z \in P(2)^*\{tz\}$, $Q_1 t \in P(2)^*\{\omega yz\}$, and any product in the correction to commutativity is forced to be zero.

Regarding the relations $y^2 = \beta v_2^2 tz$, $z^2 = \gamma v_2 ty$, these cannot be determined by using arguments involving $P(n)$-theory only. Consider the map $P(2)^*(DI(4)) \xrightarrow{\lambda_2} K(2)^*(DI(4))$. Since $P(2)^*(DI(4))$ is commutative, so is $K(2)^*(DI(4))$. Hence, we are under the conditions of Lemma 5.4, and $K(2)^*(DI(4)) \cong K(2)^* \otimes \Lambda(y,z,t)$. Now, $\lambda_2(y^2) = \lambda_2(y)^2 = 0 = \beta v_2^2 \lambda_2(tz)$. But $tz \notin \operatorname{Ker}\lambda_2$, and this implies $\beta = 0$. Arguing similarly, we can prove that $\gamma = 0$. ∎

PROPOSITION 5.7. *There exist $K(1)^*$-algebra, respectively $P(1)^*$-algebra isomorphisms*

$$\begin{aligned} K(1)^*(DI(4)) &\cong E_\infty^{**}(K(1)),\\ P(1)^*(DI(4)) &\cong ((P(1)^* \otimes \Lambda(u_5, z_{13}, t_{21}, r_{25}, s_{32}))\\ &\qquad \oplus (P(3)^* \otimes \mathbb{Z}/2\{\omega_{14}\} \otimes \Lambda(z)))/R. \end{aligned}$$

where $R \equiv \{rs = ts = rt = us = 0,\ ur = v_1 s,\ ut = v_2 s,\ v_1 t = v_2 r\}$.

Proof. In Lemma 5.3 we proved

$$\begin{aligned} &E_\infty^{**}(K(1)) \cong K(1)^* \cong \Lambda(x_7, z_{13}, r_{25}),\\ &E_\infty^{**}(P(1)) \cong (P(1)^* \otimes \Lambda(u_5, z_{13}, r_{25}, s_{32}) \oplus P(2)^* \otimes \mathbb{Z}/2\{t_{21}\} \otimes \Lambda(z)\\ &\qquad\qquad \oplus P(3)^* \otimes \mathbb{Z}/2\{\omega_{14}\} \otimes \Lambda(z))/R, \end{aligned}$$

where $R \equiv \{ur = v_1 s, us = rs = 0\}$.

The relations in the module structure of $E_\infty^{**}(P(1))$ are $v_1 t = v_1\omega = v_2\omega = 0$. Let us study these relations in $P(1)^*(DI(4))$. Consider the exact triangle

$$\begin{tikzcd} P(1)^*(-) \arrow[rr, "v_1"] & & P(1)^*(-) \arrow[dl, "i_1"] \\ & P(2)^*(-) \arrow[ul, "\delta_1"] & \end{tikzcd}$$

Let $y_{11} \in P(2)^*(DI(4))$. Then $(i_1\delta_1)(y) = Q_1 y = \omega \bmod (v_2, v_3, \ldots)$. Moreover, $i_1(u) = v_2 y$ implies $v_2 y \in \operatorname{Im} i_1 = \operatorname{Ker}\delta_1$, and hence $\delta_1(v_2 y) = v_2(\delta_1 y) = 0$. Set $\omega' = \delta_1 y$; we see that ω' is an element in $P(1)^*(DI(4))$ such that $i_1(\omega') = \omega \bmod (v_2, v_3, \ldots)$, and

$$\boxed{v_1\omega' = v_2\omega' = 0}$$

That $i_1(r) = \omega y$ implies $i_1(v_2 r) = 0$, and hence $v_2 r \in \operatorname{Ker} i_1 = \operatorname{Im} v_1$. For dimensional reasons, $v_2 r = \alpha v_1 t + a v_1 r + b v_1 sz$, where $\alpha = 0, 1$, and $a, b \in P(1)^*$. But the $P(1)^*$-module generated in $P(1)^*(DI(4))$ by $\{r, sz\}$ is

$P(1)^*$-free, and this forces $\alpha = 1$, and $v_1 t \neq 0$. Setting $t' = t + ar + bsz$, we get

$$\boxed{v_2 r = v_1 t'}$$

This completes the study of the $P(1)^*$-*module* structure of $P(1)^*(DI(4))$.

For the algebra structure, the relations that we have to study are the following: $u^2, r^2, z^2, s^2, (t')^2, (\omega')^2, ut', rt', st', u\omega', r\omega', t'\omega', s\omega', us, rs, ur + v_1 s$, and the commutativity.

Just by dimensional reasons,

$$\boxed{(t')^2 = s^2 = t's = t'r = sr = \omega' s = r^2 = 0}$$

The elements $(\omega')^2, t'\omega', r\omega', u\omega'$ are in $(\operatorname{Ker} v_1) \cap (\operatorname{Ker} v_2)$, and this implies that they are not in $(\operatorname{Im} v_1) \cup (\operatorname{Im} v_2)$. Again for dimensional reasons, this implies

$$\boxed{(\omega')^2 = t'\omega' = r\omega' = u\omega' = 0}$$

The fact that $ur = v_1 s$ in $E_\infty^{**}(P(1))$ implies $ur = v_1 s + \alpha v_1^4 rz + \beta v_1 v_2 rz$ in $P(1)^*(DI(4))$, where $\alpha, \beta = 0, 1$. Setting $s' = s + \alpha v_1^3 rz + \beta v_2 rz$, we get

$$\boxed{ur = v_1 s'}$$

Combining the two relations $ur = v_1 s'$ and $v_1 t' = v_2 r$, we obtain $v_1(v_2 s' + ut') = 0$. But $i_1(v_2 s') = v_2 ty = i_1(ut')$, and hence $v_2 s' + ut' \in \operatorname{Im} v_1$. This implies

$$\boxed{v_2 s' = ut'}$$

Since we are not going to make any other change of generators, we rename again $\omega = \omega'$, $t = t'$, $s = s'$.

To complete the proof of the $P(1)^*$-algebra isomorphism in the lemma, we only have to check that $u^2 = z^2 = us = 0$, and the commutativity. Since $Q_0\omega \notin (\operatorname{Im} v_1) \cup (\operatorname{Im} v_2)$, by dimensional reasons we have $Q_0\omega = 0$. Regarding the other generators,

$$\begin{cases} Q_0 u \in P(1)^*\{uz, s, tz, rz\}, \\ Q_0 z \in P(1)^*\{\omega, uz, s, tz, rz\}, \\ Q_0 t \in P(1)^*\{s, tz, rz\}, \\ Q_0 r \in P(1)^*\{s, tz, rz\}, \\ Q_0 s \in P(1)^*\{sz\}. \end{cases}$$

By considering dimensions, any product in the correction to commutativity is forced to be zero, except for the case $(uz)(uz)$. But $(uz)(uz) = uz(zu + v_1 Q_0 z Q_0 u) = uz(zu + v_1 a (uz)^2) = uz^2 u + a v_1 (uz)^3$ for some $a \in P(1)^*$. For dimensional reasons, $(uz)^3 = 0$, and the fact that $z^2 \in P(1)^*\{s, rz\}$ implies $uz^2 u = 0$.

To prove the relations $u^2=z^2=0$, we consider the map $\lambda_1 : P(1)^*(DI(4)) \to K(1)^*(DI(4))$. We are again under the conditions of Lemma 5.4 and hence $K(1)^*(DI(4)) \cong K(1)^* \otimes \Lambda(x,z,r)$ as $K(1)^*$-algebras. We finish the proof of the $P(1)^*$-algebra isomorphism by using an argument completely analogous to the one used in Lemma 4.2: we know that $z^2 = as+brz$, where $a,b \in P(1)^*$. That $\lambda_1(z)^2 = 0$ and $s, rz \notin \operatorname{Ker}\lambda_1$ implies $a,b \in (v_2, v_3, \ldots)$. Now, we order the coefficients lexicographically:

$$z^2 = \sum_{j=1}^{n_1} \omega_{1,j} s + \sum_{j=1}^{n_2} \omega_{2,j} rz \quad \text{ with } \omega_{i,1} > \omega_{i,2} > \ldots > \omega_{i,n_i},\ i = 1,2.$$

There exists an operation $r \in P(1)^*P(1)$ such that $r(\omega_{1,1}) = v_1^k$ for some $k > 0$, and $r(\omega_{1,j}) = 0$ for all $j > 1$. Since $x^2 = 0$ for all $x \in P(1)^*(DI(4))$ with $|x| > |z|$, the Cartan formula implies that $r(z^2) = 0$. Hence, $r(z^2) = 0 = v_1^k s + \sum_i \omega_i x_i$, where $\omega_i \in P(1)^*$ and $|x_i| > |s|$ for all i. This contradicts the module structure of $P(1)^*(DI(4))$. Therefore, $z^2 = 0$. Arguing similarly, we can prove that $u^2 = 0$. Finally, $us \in P(1)^*\{sz\}$, and $v_1(us) = u^2r = 0$. This implies $us = 0$. ∎

We finish this section with the proof of the non-homotopy nilpotency of $DI(4)$, by using the non-commutativity of the spectra $P(n)$ at $p = 2$.

Let X be a finite homotopy associative H-space, and let λ and σ be the multiplication and inverse maps of X. Define c_2 (the *commutator*) to be the composition

$$X \times X \xrightarrow{\Delta_{X\times X}} X \times X \times X \times X \xrightarrow{f} X \times X \times X \times X \xrightarrow{g} X$$

where $\Delta_{X\times X}$ is the diagonal map, $f = \mathrm{id} \times \mathrm{id} \times \sigma \times \sigma$ and $g = \lambda(\lambda \times \lambda)$. Define the iterated commutators $c_n : X^n \to X$ inductively by $c_n = c_2(c_{n-1} \times \mathrm{id})$. Zabrodsky defined X to be *homotopy nilpotent* if the functor $[-, X]$ takes values in the category of nilpotent groups, and proved the following criterion for the homotopy nilpotency of an associative H-space, in terms of the commutators:

PROPOSITION 5.8 ([Z]). *A finite homotopy associative H-space is homotopy nilpotent if and only if c_n is null homotopic for sufficiently large n.*

In [Ho], Hopkins found cohomological criteria for a finite H-space to be homotopy nilpotent, and used it to prove that H-spaces with no torsion in homology are homotopy nilpotent. Hopkins also conjectured that all finite connected homotopy associative H-spaces are homotopy nilpotent. However, Rao [R] found that $\mathrm{Spin}(n), \mathrm{SO}(n), n \geq 7$, and $\mathrm{SO}(3), \mathrm{SO}(4)$ are not homotopy nilpotent. About the same time, Yagita proved the following result:

THEOREM 5.9 ([Ya3]). *Let G be a simply connected Lie group. Then, for each prime p, the p-localization $G_{(p)}$ is homotopy nilpotent if and only if $H^*(G)$ has no p-torsion.*

Since $DI(4)$ has 2-torsion, it is natural to expect that it is not homotopy nilpotent. We prove the non-homotopy nilpotency of $DI(4)$ by showing that the maps induced by the commutators c_n in $P(3)_*(DI(4))$ are non-trivial. Consider the AHss for $P(3)_*(DI(4))$:

$$E^2_{**}(P(3)) \cong P(3)_* \otimes H_*(DI(4), \mathbb{Z}/2) \to P(3)_*(DI(4)).$$

For dimensional reasons, all the differentials are forced to be zero, and $P(3)_*(DI(4)) \cong P(3)_* \otimes H_*(DI(4), \mathbb{Z}/2)$ as $P(3)_*$*-modules.*

By duality, one easily sees that the algebra structure of $H_*(DI(4), \mathbb{Z}/2)$ is

$$H_*(DI(4), \mathbb{Z}/2) \cong \Lambda(z_7, z_{11}, z_{13}, z_{14}).$$

If we denote by Ψ the coproduct, the generators z_7, z_{11}, z_{13} are primitive, and $\Psi(z_{14}) = 1 \otimes z_{14} + z_{14} \otimes 1 + z_7 \otimes z_7$. The action of the Steenrod algebra is given by $\mathrm{Sq}^4 z_{11} = z_7$, $\mathrm{Sq}^2 z_{13} = z_{11}$, $\mathrm{Sq}^1 z_{14} = z_{13}$. In particular, $Q_2 z_{14} = z_7$.

Lemma 5.10. *In* $P(3)_*(DI(4))$, $c_{2*}(z_7 \otimes z_{14}) = [z_7, z_{14}]$.

Proof. As a consequence of Proposition 2.1, we have (see [R])

$$\begin{aligned}
&(\Delta_{X\times X})_*(z_7 \otimes z_{14}) \\
&\quad = \Psi(z_7) \otimes \Psi(z_{14}) + v_3(\mathrm{id} \otimes Q_2 \otimes Q_2 \otimes \mathrm{id})(\Psi(z_7) \otimes \Psi(z_{14})) \\
&\quad = (1 \otimes 1 \otimes z_7 \otimes z_{14}) + (1 \otimes z_{14} \otimes z_7 \otimes 1) + (1 \otimes z_7 \otimes z_7 \otimes z_7) \\
&\quad\quad + (z_7 \otimes 1 \otimes 1 \otimes z_{14}) + (z_7 \otimes z_{14} \otimes 1 \otimes 1) + (z_7 \otimes z_7 \otimes 1 \otimes z_7).
\end{aligned}$$

As for σ_*, note that if $x \in P(3)_*(DI(4))$, and $\Psi(x) = \sum_i x'_i \otimes x''_i$, then $\sum_i x'_i \sigma_*(x''_i) = 0$. Thus, $\sigma_*(z_7) = z_7, \sigma_*(z_{14}) = z_{14} + z_7^2 = z_{14}$ (since $z_7^2 = 0$).

Finally, we obtain $c_{2*}(z_7 \otimes z_{14}) = g_* f_* (\Delta_{X\times X})_*(z_7 \otimes z_{14}) = z_7 z_{14} + z_{14} z_7 + z_7^3 + z_7 z_{14} + z_7 z_{14} + z_7^3 = [z_7, z_{14}]$. ∎

We know that $[z_7, z_{14}] = 0 \bmod (v_3, v_4, \ldots)$ and, by dimensional reasons, we must have $[z_7, z_{14}] = \alpha v_3 z_7$ with $\alpha = 0$ or 1. In the following lemma we prove that $\alpha = 1$.

Lemma 5.11. *In* $P(3)_*(DI(4))$, $[z_7, z_{14}] = v_3 z_7$.

Proof. Corollary 2.3 yields

$$\begin{aligned}
\Psi(z_{14}^2) &= (\Psi(z_{14}))^2 + v_3((Q_2 \otimes \mathrm{id})\Psi(z_{14}))((\mathrm{id} \otimes Q_2)(\Psi(z_{14})) \\
&= \Psi(z_{14})^2 + v_3(z_7 \otimes z_7).
\end{aligned}$$

Since $(\Psi(z_{14}))^2 = 1 \otimes z_{14}^2 + z_{14}^2 \otimes 1 + z_7 \otimes [z_7, z_{14}] + [z_7, z_{14}] \otimes z_7$, we obtain

$$\begin{aligned}
\Psi(z_{14}^2 + v_3 z_{14}) &= 1 \otimes (z_{14}^2 + v_3 z_{14}) + (z_{14}^2 + v_3 z_{14}) \otimes 1 \\
&\quad + [z_7, z_{14}] \otimes z_7 + z_7 \otimes [z_7, z_{14}] \\
&= 1 \otimes (z_{14}^2 + v_3 z_{14}) + (z_{14}^2 + v_3 z_{14}) \otimes 1.
\end{aligned}$$

Then $z_{14}^2+v_3z_{14}$ is a primitive element in even degree. But the only primitive elements in $P(3)_*(DI(4))$ are z_7, z_{11}, z_{13}, up to coefficients. This implies $z_{14}^2+v_3z_{14}=0$, and hence $[z_7, z_{14}]=Q_2(z_{14}^2)=Q_2(v_3z_{14})=v_3Q_2(z_{14})=v_3z_7$. ∎

The proof of the non-homotopy nilpotency of $DI(4)$ is now immediate:

THEOREM 5.12. *$DI(4)$ is not homotopy nilpotent.*

Proof. We show that, for all $n \geq 2$, c_{n*} is not trivial in $P(3)_*(DI(4))$. The proof is by induction. For $n = 2$, $c_{2*}(z_7 \otimes z_{14}) = [z_7, z_{14}] = v_3z_7 \neq 0$. Suppose

$$(c_{n-1})_*(z_7 \otimes \overbrace{z_{14} \otimes \ldots \otimes z_{14}}^{n-2}) = v_3^{n-2}z_7.$$

Then

$$\begin{aligned} c_{n_*}(z_7 \otimes \overbrace{z_{14} \otimes \ldots \otimes z_{14}}^{n-1}) &= c_{2_*}(c_{n-1} \times \mathrm{id})_*(z_7 \otimes z_{14} \otimes \ldots \otimes z_{14}) \\ &= c_{2_*}(v_3^{n-2}z_7 \otimes z_{14}) = v_3^{n-1}z_7 \neq 0. \end{aligned}$$ ∎

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