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Expansions of the real line by open sets: o-minimality and open cores

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Abstract. The open core of a structure $\mathfrak{R} := (\mathbb{R}, <, \ldots)$ is defined to be the reduct (in the sense of definability) of $\mathfrak{R}$ generated by all of its definable open sets. If the open core of $\mathfrak{R}$ is o-minimal, then the topological closure of any definable set has finitely many connected components. We show that if every definable subset of $\mathbb{R}$ is finite or uncountable, or if $\mathfrak{R}$ defines addition and multiplication and every definable open subset of $\mathbb{R}$ has finitely many connected components, then the open core of $\mathfrak{R}$ is o-minimal.

An expansion $\mathfrak{R}$ of the real line $(\mathbb{R}, <)$ is *o-minimal* if every definable subset of $\mathbb{R}$ is a finite union of points and open intervals (that is, has finitely many connected components). Such structures—particularly, o-minimal expansions of the field of real numbers—have many nice properties, and are of interest not only to model theorists, but to analysts and geometers as well. (See e.g. [D2], [DM] for expositions of the subject.)

Conventions. Throughout, given $A \subseteq \mathbb{R}$, "A-definable" means "definable (in the structure under consideration) using parameters from A", and "definable" means "$\mathbb{R}$-definable". We use "reduct" and "expansion" in the sense of definability, that is, given structures $\mathfrak{R}_1$ and $\mathfrak{R}_2$ with underlying set $\mathbb{R}$, we say that $\mathfrak{R}_1$ is a *reduct* of $\mathfrak{R}_2$—equivalently, $\mathfrak{R}_2$ is an *expansion* of $\mathfrak{R}_1$, or $\mathfrak{R}_2$ *expands* $\mathfrak{R}_1$—if every $A \subseteq \mathbb{R}^n$ definable in $\mathfrak{R}_1$ is definable in $\mathfrak{R}_2$, for every $n \in \mathbb{N}$. ($\mathbb{R}^0$ denotes the one-point space $\{0\}$.)

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Here is the main result of this paper:

Theorem. *Let $U_\gamma \subseteq \mathbb{R}^{n(\gamma)}$ be open, γ in some index set Γ. Then*:

(a) *The structure $(\mathbb{R}, <, (U_\gamma)_{\gamma\in\Gamma})$ is o-minimal if and only if every definable subset of $\mathbb{R}$ is finite or uncountable.*

(b) *The structure $(\mathbb{R}, +, \cdot, (U_\gamma)_{\gamma\in\Gamma})$ is o-minimal if and only if every open definable subset of $\mathbb{R}$ has finitely many connected components* (*if and only if every discrete definable subset of $\mathbb{R}$ is finite*).

Given an expansion $\mathfrak{R}$ of $(\mathbb{R}, <)$, we define the *open core* of $\mathfrak{R}$, denoted by $\mathfrak{R}^\circ$, to be the reduct of $\mathfrak{R}$ generated by the collection of all open subsets of $\mathbb{R}^n$ (n ranging over all positive integers) definable in $\mathfrak{R}$. Note that $\mathfrak{R}^\circ$ expands $(\mathbb{R}, <)$.

Corollary. *Let $\mathfrak{R}$ be an expansion of $(\mathbb{R}, <)$.*

(1) *If every definable subset of $\mathbb{R}$ is finite or uncountable*, *then $\mathfrak{R}^\circ$ is o-minimal.*

(2) *If $\mathfrak{R}$ expands $(\mathbb{R}, +, \cdot)$ and every open definable subset of $\mathbb{R}$ has finitely many connected components*, *then $\mathfrak{R}^\circ$ is o-minimal.*

The Theorem and Corollary hold with "$\emptyset$-definable" in place of "definable", provided that the $\emptyset$-definable points of $\mathbb{R}$ are dense in $\mathbb{R}$, in particular, if $\mathfrak{R}$ expands $(\mathbb{R}, <, +, 1)$. We obtain this modification as a corollary of the proof of the Theorem, together with the following:

Proposition 1. *Let $\mathfrak{R}$ be an expansion of $(\mathbb{R}, <)$. Suppose that the $\emptyset$-definable points of $\mathbb{R}$ are dense in $\mathbb{R}$. Then $\mathfrak{R}$ is o-minimal if and only if every $\emptyset$-definable subset of $\mathbb{R}$ has finitely many connected components.*

Given an expansion $\mathfrak{R}$ of $(\mathbb{R}, <)$, every boolean combination of open sets definable in $\mathfrak{R}$ is definable in $\mathfrak{R}^\circ$. Hence, by the cell decomposition theorem for o-minimal structures, every o-minimal reduct of $\mathfrak{R}$ is a reduct of $\mathfrak{R}^\circ$. Given $A \subseteq \mathbb{R}^n$ definable in $\mathfrak{R}$, both the closure and the interior of A are definable in $\mathfrak{R}^\circ$. So if $\mathfrak{R}^\circ$ is o-minimal, then $\mathfrak{R}$ is, loosely speaking, topologically close to being o-minimal. We make this precise:

Proposition 2. *Let $\mathfrak{R}$ be an expansion of $(\mathbb{R}, <)$ and suppose that $\mathfrak{R}^\circ$ is o-minimal. Let $n \in \mathbb{N}$ and $\mathcal{A}$ be a finite collection of subsets of $\mathbb{R}^n$ definable in $\mathfrak{R}$. Then there is a finite partition $\mathcal{C}$ of $\mathbb{R}^n$ into cells*, *definable in $\mathfrak{R}^\circ$*, *such that for each $A \in \mathcal{A}$ and $C \in \mathcal{C}$*, *either A is disjoint from C*, *or A contains C*, *or A is dense and codense in C.*

We prove Theorem (a) in Section 1. The proof of (b) requires minor modifications, which we describe in Section 2. Section 3 contains the proof of Proposition 2, as well as an application. We provide some examples and counterexamples in Section 4.

The proof of the Theorem uses only elementary real topology and the notion of first-order definability. On the other hand, Sections 3 and 4 require a fair amount of familiarity with o-minimality and associated model theory; the expository paper by van den Dries [D2] contains a quick introduction to the necessary material, as well as an extensive bibliography of original sources.

Proposition 1 is elementary and easy to establish, so we do this right away.

Notation. For $A \subseteq \mathbb{R}^{m+n}$ and $x \in \mathbb{R}^m$, let A_x denote the fiber of A over x, that is, the set $\{y \in \mathbb{R}^n : (x,y) \in A\}$.

(It should be clear from context when subscripts indicate taking fibers and when they are used as indices.)

Proof of Proposition 1. Let $\mathfrak{R}$ be an expansion of $(\mathbb{R},<)$. Suppose that the $\emptyset$-definable points of $\mathbb{R}$ are dense, and that every $\emptyset$-definable subset of $\mathbb{R}$ has finitely many connected components. We show that every definable subset of $\mathbb{R}$ has finitely many connected components (and hence $\mathfrak{R}$ is o-minimal).

First, note that every nonempty $\emptyset$-definable subset of $\mathbb{R}$ contains a $\emptyset$-definable point of $\mathbb{R}$. An easy induction then shows that for every $n \in \mathbb{N}$, every nonempty $\emptyset$-definable subset of $\mathbb{R}^n$ contains a $\emptyset$-definable point of $\mathbb{R}^n$.

Let $A \subseteq \mathbb{R}$ be definable. It suffices to show that the boundary of A is bounded and discrete (hence finite). Suppose otherwise. Now, $A = B_u$ for some $n \in \mathbb{N}$, $u \in \mathbb{R}^n$ and $\emptyset$-definable $B \subseteq \mathbb{R}^{n+1}$. Hence, the $\emptyset$-definable set C, consisting of all $x \in \mathbb{R}^n$ such that the boundary of B_x is either unbounded or not discrete, is nonempty. By the preceding paragraph, C contains a $\emptyset$-definable $y \in \mathbb{R}^n$. But the fiber B_y is a $\emptyset$-definable subset of $\mathbb{R}$, hence its boundary is finite; contradiction. ∎

Before beginning the remaining proofs, we establish some notation, and review some relevant elementary facts from topology.

Given $A \subseteq \mathbb{R}^n$ we let $\mathrm{int}(A)$ and $\mathrm{cl}(A)$ denote respectively the interior and closure of A. The *frontier* of A, denoted by $\mathrm{fr}(A)$, is the set $\mathrm{cl}(A) \setminus A$. (Note: In general, the frontier of A is not equal to the *boundary* $\mathrm{bd}(A) = \mathrm{cl}(A) \setminus \mathrm{int}(A)$.) We say that A *has interior* if $\mathrm{int}(A) \neq \emptyset$, and that A *has no interior* if $\mathrm{int}(A) = \emptyset$.

A set $A \subseteq \mathbb{R}^n$ is *locally closed* if for each $x \in A$ there is an open neighborhood U of x such that $U \cap A = U \cap \mathrm{cl}(A)$, or equivalently, if there exists an open $U \subseteq \mathbb{R}^n$ such that $A = \mathrm{cl}(A) \cap U$. Any boolean combination of open subsets of $\mathbb{R}^n$ is a finite union of locally closed sets. Given $A \subseteq \mathbb{R}^n$, we put $\mathrm{lc}(A) := A \setminus \mathrm{cl}(\mathrm{fr}(A))$; in other words, $\mathrm{lc}(A)$ is the relative interior of A in $\mathrm{cl}(A)$. Note that $\mathrm{lc}(A)$ is locally closed.

Let $\mathfrak{R}$ be an expansion of $(\mathbb{R},<)$. It is easy to see that given definable sets $A\subseteq B\subseteq\mathbb{R}^n$ with A relatively open in B, there is a definable open $C\subseteq\mathbb{R}^n$ such that $A=B\cap C$, and similarly with "closed" in place of "open". In particular, $A\subseteq\mathbb{R}^n$ is definable and locally closed if and only if there is a definable open $U\subseteq\mathbb{R}^n$ such that $A=\operatorname{cl}(A)\cap U$. Consequently, for each $n\in\mathbb{N}$, the collection of all finite unions of locally closed definable sets in $\mathbb{R}^n$ is a boolean algebra, and every locally closed $A\subseteq\mathbb{R}^n$ definable in $\mathfrak{R}$ is definable in $\mathfrak{R}^\circ$. If $A\subseteq\mathbb{R}^n$ is definable, then so is $\operatorname{lc}(A)$.

Remark. The complement of the locally closed definable set

$$\{0\}\times((-\infty,0)\cup(0,\infty))\subseteq\mathbb{R}^2$$

is not locally closed.

1. Proof of part (a). Let $U_\gamma\subseteq\mathbb{R}^{n(\gamma)}$ be open, γ in some index set Γ. Put $\mathfrak{R}:=(\mathbb{R},<,(U_\gamma)_{\gamma\in\Gamma})$. Assume that every definable subset of $\mathfrak{R}$ is finite or uncountable. We show that every definable subset of $\mathbb{R}$ has finitely many connected components.

1.1. *A definable subset of* $\mathbb{R}$ *is locally closed if and only if it has finitely many connected components.*

Proof. It suffices to show that every definable open $A\subseteq\mathbb{R}$ is a finite union of open intervals, and for this just note that the set of endpoints of the bounded connected components of A is definable and countable, hence finite. ∎

It now suffices to show that every definable subset of $\mathbb{R}$ is a finite union of locally closed definable sets, and for this it suffices to show that for every $n\in\mathbb{N}$, the projection of any locally closed definable set in $\mathbb{R}^{n+1}$ on the first n coordinates is a finite union of locally closed definable sets in $\mathbb{R}^n$ (since then every definable set, in each $\mathbb{R}^n$, is a finite union of locally closed definable sets). In order to do this, we introduce a collection of definable sets that is closed under projection and contains all finite unions of locally closed definable sets.

Definition. A set $A\subseteq\mathbb{R}^n$ is D_σ if it is definable and a countable increasing union of definable compact subsets of $\mathbb{R}^n$. We sometimes write "$A\in\mathrm{D}_\sigma(n)$" or just "$A\in\mathrm{D}_\sigma$".

It is easy to check that finite unions and intersections of D_σ sets are D_σ. If $A\in\mathrm{D}_\sigma(m)$ and if $f:A\to\mathbb{R}^n$ is a continuous definable map, then $f(A)\in\mathrm{D}_\sigma(n)$ and $f^{-1}\{y\}\in\mathrm{D}_\sigma(m)$ for every $y\in\mathbb{R}^n$; in particular, coordinate projections, as well as the associated fibers, of D_σ sets are D_σ. Of course, every D_σ set is F_σ.

1.2. *Every definable locally closed set is* D_σ. *Hence*, *if* $A \in \mathrm{D}_\sigma(n)$ *and* $B \subseteq \mathbb{R}^n$ *is definable and locally closed*, *then* $A \cap B$, $A \cup B$ *and* $A \setminus B$ *are* D_σ.

Proof. Every open set is a countable union of compact boxes. Every closed set $A \subseteq \mathbb{R}^n$ is the union of all sets $A \cap [-m, m]^n$, $m \in \mathbb{N}$. Hence, every definable locally closed set is D_σ. The complement of a locally closed set is a finite union of locally closed sets. ■

1.3. *A subset of* $\mathbb{R}$ *is* D_σ *if and only if it has finitely many connected components.*

Proof. Let $A \subseteq \mathbb{R}$ be D_σ. By 1.1, $\mathrm{int}(A)$ is a finite union of open intervals. Since $A \setminus \mathrm{int}(A)$ is D_σ, we suppose that A has no interior. Write $A = \bigcup_{k\in\mathbb{N}} A_k$, with each A_k compact and definable. Each A_k has no interior, so each A_k is finite (again by 1.1). Then A is countable, hence finite. ■

Given integers n and d with $n \geq d \geq 0$, we let $\Pi(n, d)$ denote the collection of all projection maps $(x_1, \ldots, x_n) \mapsto (x_{\lambda(1)}, \ldots, x_{\lambda(d)}) : \mathbb{R}^n \to \mathbb{R}^d$, where λ is a strictly increasing function from $\{1, \ldots, d\}$ into $\{1, \ldots, n\}$.

Definition. The *dimension* of a nonempty set $A \subseteq \mathbb{R}^n$, denoted by $\dim A$, is the maximal integer d such that πA has interior for some $\pi \in \Pi(n, d)$. (Equivalently, d is the maximal integer such that, after some permutation of coordinates, the projection of A on the first d coordinates has interior.) Put $\dim \emptyset := -\infty$.

Here is an outline of the rest of the proof: Since every locally closed definable set is D_σ, and every coordinate projection of a D_σ set is D_σ, we are reduced to showing that every D_σ is a finite union of locally closed definable sets. In fact, we will show that given $A \in \mathrm{D}_\sigma$, there exist $m \in \mathbb{N}$ and $B_1, \ldots, B_m, C \in \mathrm{D}_\sigma$ such that: $A = B_1 \cup \ldots \cup B_m \cup C$; $\dim(B_i \setminus \mathrm{lc}(B_i)) < \dim B_i$ for $i = 1, \ldots, m$; and $\dim C < \dim A$. We then finish by induction on $\dim A$.

We resume the proof. An easy induction using 1.3 shows that

1.4. *Every* 0*-dimensional* D_σ *set is finite.*

We note some easy facts about F_σ sets.

1.5. (1) *Every* F_σ *set is meager or has interior.*

(2) *If* $(A_k)_{k\in\mathbb{N}}$ *is a sequence of* F_σ *sets*, *then*

$$\dim \bigcup_{k\in\mathbb{N}} A_k = \max\{\dim A_k : k \in \mathbb{N}\}.$$

(3) *An* F_σ *set* $A \subseteq \mathbb{R}^{m+n}$ *has interior if and only if* $\{x \in \mathbb{R}^m : A_x$ *has interior*$\}$ *is nonmeager.*

(4) *Let* $\emptyset \neq A \subseteq \mathbb{R}^n$ *be* F_σ, $d := \dim A$. *Then* $\{x \in \mathbb{R}^d : \dim A_x > 0\}$ *is meager.*

Proof. Items (1) and (2) are left to the reader.

For (3), let $A \subseteq \mathbb{R}^n$ be F_σ, say A is the countable union of closed sets A_k, $k \in \mathbb{N}$. Suppose that $\{x \in \mathbb{R}^m : A_x \text{ has interior}\}$ is nonmeager. Then there exist $k \in \mathbb{N}$ and an open box $V \subseteq \mathbb{R}^n$ such that the set $\{x \in \mathbb{R}^m : V \subseteq (A_k)_x\}$ is somewhere dense, say in some open box $U \subseteq \mathbb{R}^m$. Then $U \times V \subseteq A_k$.

Now, (4) is obvious if $d = 0$ or $d = n$, so suppose that $0 < d < n$. Let B denote the projection of A on the first d coordinates. If B is meager, then we are done, so suppose otherwise; then B has interior (since it is F_σ). Let $k \in \{d+1, \ldots, n\}$ and π denote the map projecting the last $n-d$ coordinates on the kth coordinate. The image of A under the projection $(v_1, \ldots, v_n) \mapsto (v_1, \ldots, v_d, v_k)$ has no interior (otherwise $\dim A > d$). It follows from (3) that $\{x \in B : \dim \pi(A_x) > 0\}$ is meager. This is true for every $k \in \{d+1, \ldots, n\}$, so $\{x \in B : \dim A_x > 0\}$ is meager. ∎

Next, we have an important technical lemma.

1.6. *For every $n, p \in \mathbb{N}$ and $A \in D_\sigma(n+p)$, the set $\{x \in \mathbb{R}^n : \mathrm{cl}(A)_x \neq \mathrm{cl}(A_x)\}$ is meager.*

We establish this by induction on $n \geq 1$, showing the following in turn:

$(1)_n$ *Every $D_\sigma(n)$ has interior or is nowhere dense.*

$(2)_n$ *For every $p \in \mathbb{N}$ and $A \in D_\sigma(n+p)$, the set $\{x \in \mathbb{R}^n : \mathrm{cl}(A)_x \neq \mathrm{cl}(A_x)\}$ is meager.*

Proof. $(1)_1$ is immediate by 1.3.

For $(2)_1$, let $p \in \mathbb{N}$ and $A \in D_\sigma(1+p)$. It suffices to show that given an open box $U \subseteq \mathbb{R}^p$, the set

$$B := \{x \in \mathbb{R} : U \cap \mathrm{cl}(A)_x \neq \emptyset,\ U \cap \mathrm{cl}(A_x) = \emptyset\}$$

is meager. Let $C \subseteq \mathbb{R}$ denote the projection of $A \cap (\mathbb{R} \times U)$ on the first coordinate; then C is D_σ and $\mathrm{fr}(C)$ is finite (by 1.3). Now note $B \subseteq \mathrm{fr}(C)$.

Suppose now that $n > 1$ and $(1)_m$, $(2)_m$ hold for all $m < n$.

For $(1)_n$, let $A \in D_\sigma(n)$ with $d := \dim A$. The result is clear if $d = 0$ (then A is finite) or $d = n$, so suppose that $0 < d < n$. We must show that A is nowhere dense, that is, $\mathrm{cl}(A)$ has no interior. Without loss of generality, assume that the projection of A on the first d coordinates has interior. By 1.5(3), the set of all $x \in \mathbb{R}^d$ such that A_x has interior is meager. Hence, by $(1)_{n-d}$, the set of all $x \in \mathbb{R}^d$ such that A_x is nowhere dense is comeager. By $(2)_d$, the set of all points $x \in \mathbb{R}^d$ such that $\mathrm{cl}(A)_x$ has interior is meager. Now apply 1.5(3) again.

It follows from $(1)_n$ that the frontier of every $D_\sigma(n)$ is nowhere dense. (For any set X we have $\mathrm{fr}(X) \subseteq \mathrm{fr}(\mathrm{int}(X)) \cup \mathrm{fr}(X \setminus \mathrm{int}(X))$.) Using this, the argument for $(2)_n$ now proceeds similarly to that for $(2)_1$. ∎

Note the following (trivially) equivalent statement of 1.6:

1.7. *For every* $n, p \in \mathbb{N}$ *and* $A \in \mathrm{D}_\sigma(n+p)$, *the set* $\{x \in \mathbb{R}^n : \mathrm{fr}(A)_x \neq \mathrm{fr}(A_x)\}$ *is meager.*

Remark. Another statement equivalent to 1.6 is that $\dim \mathrm{cl}(A) = \dim A$ for every $A \in \mathrm{D}_\sigma$, but we will not need this.

1.8. *Let* $\emptyset \neq A \in \mathrm{D}_\sigma$. *Then* $\dim \mathrm{fr}(A) < \dim A$.

Proof. Let $A \in \mathrm{D}_\sigma(n)$, $\dim A := d$. The result is easy if $d = 0$ (A is finite) or $d = n$ (the frontier of any set has no interior), so suppose that $0 < d < n$. By 1.4 and 1.5(4), $\{x \in \mathbb{R}^d : A_x \text{ is infinite}\}$ is meager, and hence $\{x \in \mathbb{R}^d : \mathrm{fr}(A)_x \neq \emptyset\}$ is meager by 1.7. Then for every $\pi \in \Pi(n, d)$, $\pi\, \mathrm{fr}(A)$ has no interior (that is, $\dim \mathrm{fr}(A) < d$). ■

Definition. Given $A \subseteq \mathbb{R}^n$ we let $\mathrm{reg}(A)$ denote the set of all $x \in A$ such that $\dim(A \cap U) = \dim A$ for every neighborhood U of x. Note that $\mathrm{reg}(A)$ is closed in A; hence, $\mathrm{reg}(A)$ and $A \setminus \mathrm{reg}(A)$ are F_σ if A is F_σ, and similarly with "D_σ" in place of "F_σ".

1.9. *Let* $\emptyset \neq A \in \mathrm{F}_\sigma$. *Then* $\dim(A \setminus \mathrm{reg}(A)) < \dim A$.

Proof. Note that $A \setminus \mathrm{reg}(A)$ is a countable union of F_σ sets, each having dimension less than that of A, and apply 1.5(2). ■

Let $\emptyset \neq A \in \mathrm{D}_\sigma$. Applying the previous result, and recalling the paragraph immediately preceding 1.4, we now seek $B_1, \ldots, B_m \in \mathrm{D}_\sigma$ such that $\mathrm{reg}(A)$ is the union of the B_is, and $\dim(B_i \setminus \mathrm{lc}(B_i)) < \dim B_i$ for each i.

Definition. Given $\emptyset \neq A \subseteq \mathbb{R}^n$ and $\pi \in \Pi(n, \dim A)$, we let $\mathrm{reg}_\pi(A)$ denote the set of all $x \in A$ such that $\pi(A \cap U)$ has interior for every neighborhood U of x.

Note that $\mathrm{reg}_\pi(A)$ is closed in A, hence both $\mathrm{reg}_\pi(A)$ and $A \setminus \mathrm{reg}_\pi(A)$ are F_σ if A is F_σ, and similarly with "D_σ" in place of "F_σ". We have $\mathrm{reg}(A) = \bigcup_\pi \mathrm{reg}_\pi(A)$ where π ranges over $\Pi(n, \dim A)$. For convenience, we put $\mathrm{reg}_\pi(\emptyset) := \emptyset$ for every $\pi \in \Pi(n, m)$ and $m \in \{0, \ldots, n\}$.

1.10. *Let* $\emptyset \neq A \subseteq \mathbb{R}^n$ *be* F_σ, $\pi \in \Pi(n, \dim A)$. *The following are equivalent*:

(1) $\mathrm{reg}_\pi(A) \neq \emptyset$.
(2) $\mathrm{int}(\pi A) \neq \emptyset$.
(3) $\mathrm{int}(\pi\, \mathrm{reg}_\pi(A)) \neq \emptyset$.
(4) $\dim \mathrm{reg}_\pi(A) = \dim A$.

Proof. (2)⇒(3) is the only nontrivial implication. It follows easily from the definitions (and 1.5(1)) that $\pi(A \setminus \mathrm{reg}_\pi(A))$ is meager. Hence, if πA has interior then $\pi\, \mathrm{reg}_\pi(A)$ is nonmeager and F_σ, and thus has interior. ■

DEFINITION. Let $\pi \in \Pi(n,d)$, $0 \le d \le n$. A set $A \subseteq \mathbb{R}^n$ is *π-regular* if $\dim A = d$ and $A = \mathrm{reg}_\pi(A)$. (We also define $\emptyset$ to be π-regular for every $\pi \in \Pi(n,m)$ and $0 \le m \le n$.)

Noting that $U \cap \mathrm{reg}_\pi(A) = \mathrm{reg}_\pi(U \cap A)$ for every $A \subseteq \mathbb{R}^n$ and open box $U \subseteq \mathbb{R}^n$, the following is immediate from 1.10:

1.11. *Let* $\emptyset \ne A \in \mathrm{F}_\sigma$ *and* $\pi \in \Pi(n, \dim A)$. *Then* $\mathrm{reg}_\pi(A)$ *is* π*-regular.*

1.12. *Let* $\emptyset \ne A \in \mathrm{D}_\sigma(n)$ *and let* $\pi \in \Pi(n, \dim A)$ *be such that* A *is* π*-regular. Then* $\dim(A \setminus \mathrm{lc}(A)) < \dim A$.

Proof. Put $d := \dim A$. The result is clear if $d = 0$ (since then A is finite) or if $d = n$, so suppose that $0 < d < n$. We may assume that π is the projection on the first d coordinates. It now suffices to show that $\mathrm{lc}(A)$ is dense in A, for then $A \setminus \mathrm{lc}(A) \subseteq \mathrm{fr}(\mathrm{lc}(A))$, and $\dim(A \setminus \mathrm{lc}(A)) < \dim A$ by 1.8. Since the intersection of A with any open box is D_σ and π-regular, we are reduced to showing that $\mathrm{lc}(A) \ne \emptyset$. Choose $B \subseteq A$ closed and definable such that πB has interior. (Such a B exists since πA is D_σ and has interior.) By 1.4, 1.5(4) and 1.6 (and the Baire Category Theorem) the set

$$\{x \in \pi\,\mathrm{cl}(A) : B_x \text{ has interior in } \mathrm{cl}(A)_x\} = \bigcup_{j \in \mathbb{N}} \{x \in \mathbb{R}^d : B_x \cap V_j = \mathrm{cl}(A)_x \cap V_j \ne \emptyset\}$$

is nonmeager, where $(V_j)_{j \in \mathbb{N}}$ is an enumeration of all open (in $\mathbb{R}^{n-d}$) boxes with rational vertices. Hence, there exists an open box $V \subseteq \mathbb{R}^{n-d}$ such that the set

$$C := \{x \in \mathbb{R}^d : B_x \cap V = \mathrm{cl}(A)_x \cap V \ne \emptyset\}$$

is somewhere dense, that is, there is an open box $U \subseteq \mathbb{R}^d$ such that C is dense in U. Let $W \subseteq U \times V$ be an open box intersecting $\mathrm{cl}(A)$. Now, $\pi(A \cap W)$ has interior (since A is π-regular), so it contains a point of C; then W intersects B. Since B is closed, we have $B \cap (U \times V) = \mathrm{cl}(A) \cap (U \times V)$. Hence,

$$A \cap (U \times V) = \mathrm{cl}(A) \cap (U \times V) \ne \emptyset$$

and $\mathrm{lc}(A) \ne \emptyset$. ∎

End of proof of Theorem (a). Let $A \in \mathrm{D}_\sigma$, $d := \dim A$. We proceed by induction on $d \ge 0$ to show that A is a finite union of locally closed definable sets. If $d = 0$, then A is finite, so assume $d > 0$ and that the result holds for all $d' < d$. By 1.9 and 1.11, we may reduce to the case that A is π-regular. Apply 1.12 and note that $A \setminus \mathrm{lc}(A)$ is D_σ. ∎

COROLLARY. *Let* $U_\gamma \subseteq \mathbb{R}^{n(\gamma)}$ *be open,* γ *in some index set* Γ. *Suppose that the points of* $\mathbb{R}$ *that are* $\emptyset$*-definable in the structure* $(\mathbb{R}, <, (U_\gamma)_{\gamma \in \Gamma})$ *are dense in* $\mathbb{R}$. *Then* $(\mathbb{R}, <, (U_\gamma)_{\gamma \in \Gamma})$ *is o-minimal if and only if every* $\emptyset$*-definable subset of* $\mathbb{R}$ *is finite or uncountable.*

Proof. Throughout the proof of part (a), replace "definable" by "$\emptyset$-definable", and use $\emptyset$-definable points instead of rational points as necessary. Then apply Proposition 1. ∎

2. Proof of part (b). First, note that in any expansion of $(\mathbb{R},<,+)$, the assumption that every definable open subset of $\mathbb{R}$ has finitely many connected components is equivalent to the assumption that every definable discrete subset of $\mathbb{R}$ is finite. (The midpoints of the bounded connected components of a definable open subset of $\mathbb{R}$ form a definable discrete set.)

Again, let $\mathfrak{R}$ be an expansion of $(\mathbb{R},<)$ by open sets $U_\gamma \subseteq \mathbb{R}^{n(\gamma)}$, γ in some index set Γ. Suppose that $\mathfrak{R}$ defines addition and multiplication and that every open definable subset of $\mathbb{R}$ has finitely many connected components. We show that $\mathfrak{R}$ is o-minimal.

It is immediate that:

2.1. *A subset of $\mathbb{R}$ is locally closed and definable if and only if it has finitely many connected components.*

Definition. A set $A\subseteq\mathbb{R}^n$ is D_Σ if there is a definable set $X\subseteq(0,\infty)\times\mathbb{R}^n$ such that $A=\bigcup_{r>0}X_r$, each fiber X_r is compact, and $X_r\subseteq X_s$ for all $0<r\le s$.

2.2. *Every locally closed definable $A\subseteq\mathbb{R}^n$ is D_Σ.*

Proof. If A is closed, we have $A=\bigcup_{r>0}\{x\in A:|x|\le r\}$; if A is not closed, then $\mathrm{fr}(A)$ is closed and nonempty, so

$$A=\bigcup_{r>0}\{x\in \mathrm{cl}(A):|x|\le r \ \&\ \mathrm{d}(x,\mathrm{fr}(A))\ge 1/r\}.$$

(Here, $|\cdot|$ denotes the sup norm on $\mathbb{R}^n$ and $\mathrm{d}(\cdot,\mathrm{fr}(A))$ denotes the associated distance function.) ∎

Note. The above statement can be strengthened: Given any locally closed definable $A\subseteq\mathbb{R}^n$ there is a closed definable $B\subseteq\mathbb{R}^{n+1}$ such that B projects homeomorphically onto A. If A is closed, this is clear; otherwise, put

$$B:=\{(x,t)\in\mathrm{cl}(A)\times(0,\infty):\mathrm{d}(x,\mathrm{fr}(A))=1/t\}.$$

2.3. *A subset of $\mathbb{R}$ is D_Σ if and only if it has finitely many connected components.*

Proof. Let $A\subseteq\mathbb{R}$ be D_Σ. Since $\mathrm{int}(A)$ is a finite union of open intervals, we may assume that A has no interior. Write $A=\bigcup X_r$ (as in the definition of D_Σ). Each X_r has no interior and thus is finite. Let B be the set of all $r>0$ such that $X_s=X_r$ for all s in some open interval containing r. Then B is open and definable, hence a finite union of open intervals. Now, $(0,\infty)\setminus B$

has no interior—since (X_r) is an increasing family of finite sets—so it is finite. Thus, there exists $r > 0$ such that $X_r = X_s$ for all $s \geq r$, that is, $A = X_r$. Hence, A is finite. ∎

End of proof of Theorem (b). Replace "D_σ" by "D_Σ" in 1.4 through 1.12. ∎

Similarly to Section 1, we obtain:

Corollary. *Let $U_\gamma \subseteq \mathbb{R}^{n(\gamma)}$ be open, γ in some index set Γ. Then the structure $(\mathbb{R}, +, \cdot, (U_\gamma)_{\gamma\in\Gamma})$ is o-minimal if and only if every $\emptyset$-definable open subset of $\mathbb{R}$ has finitely many connected components.*

Remark. The only use of the assumption that $\mathfrak{R}$ defines multiplication was in the proof of 2.2, and there all we actually needed was a definable decreasing homeomorphism from some interval $(0, a)$ onto an interval (b, ∞) with $a, b > 0$. Hence, part (b) of the Theorem holds under the weaker assumption that $\mathfrak{R}$ defines addition and a homeomorphism between a bounded interval and an unbounded interval. (We leave it to the interested reader to apply this generalization appropriately throughout the rest of this paper.)

Every F_σ set $A \subseteq \mathbb{R}^n$ is the projection of a closed subset of $\mathbb{R}^{n+1}$. (Write $A = \bigcup_{k\in\mathbb{N}} A_k$ with each A_k closed; then A is the projection of the closed set $\{(x, k) : k \in \mathbb{N},\ x \in A_k\}$.) As a corollary of Theorem (b), we obtain a relativized version of this fact for expansions of the real field.

Corollary. *Let $\mathfrak{R}$ be an expansion of $(\mathbb{R}, +, \cdot)$ and let $A \subseteq \mathbb{R}^n$ be definable. Then the following are equivalent*:

(1) *There exist $p \in \mathbb{N}$ and a closed definable $B \subseteq \mathbb{R}^{n+p}$ such that A is the projection of B on the first n coordinates.*

(2) *A is* D_Σ.

(3) *There exists a locally closed definable $B \subseteq \mathbb{R}^{n+1}$ such that A is the projection of B on the first n coordinates.*

(4) *There exists a closed definable $C \subseteq \mathbb{R}^{n+2}$ such that A is the projection of C on the first n coordinates.*

Proof. We only prove that (2) implies (3).

Suppose that A is D_Σ. If $\mathfrak{R}$ has o-minimal open core, then A is a finite union of locally closed definable sets, and hence the projection of a closed definable subset of $\mathbb{R}^{n+1}$ (see the note following 2.2). If the open core of $\mathfrak{R}$ is not o-minimal, then $\mathfrak{R}$ defines an infinite discrete set, and thus an unbounded discrete set $S \subseteq (0, \infty)$. Write A as the increasing union $\bigcup_{r>0} X_r$ of a definable family of compact sets. Then A is the projection of the locally closed (since S is discrete) definable set $\{(x, s) : s \in S,\ x \in X_s\}$. ∎

Remark. Of course, in the above, if $\mathfrak{R}$ defines an infinite discrete *closed* subset of $\mathbb{R}$, then every $\mathrm{D}_\Sigma(n)$ is the projection of a *closed* definable subset of $\mathbb{R}^{n+1}$, but we do not know if an expansion of the real field which defines an infinite discrete subset of $\mathbb{R}$ necessarily defines an infinite discrete closed subset of $\mathbb{R}$.

In the remainder of this paper, we assume that the reader is familiar with the basic topological and model-theoretic properties of o-minimal structures.

3. Proof of Proposition 2. Let $\mathfrak{R}$ be an expansion of $(\mathbb{R}, <)$ and suppose that $\mathfrak{R}^\circ$ is o-minimal. Let $\mathcal{A}$ be a finite collection of $\mathfrak{R}$-definable subsets of $\mathbb{R}^n$. We show that there is a finite partition $\mathcal{C}$ of $\mathbb{R}^n$ into cells such that for each $A \in \mathcal{A}$ and $C \in \mathcal{C}$, either A is disjoint from C, or A contains C, or A is dense and codense in C.

We proceed by induction on $n \geq 0$. If $n = 0$ there is nothing to do, so we assume that $n > 0$ and that the result holds for lower values of n. Since $\mathfrak{R}^\circ$ is o-minimal, there is a finite partition $\mathcal{D}$ of $\mathbb{R}^n$ into cells compatible with the collection $\{\mathrm{cl}(A), \mathrm{int}(A) : A \in \mathcal{A}\}$. It now suffices to find, for each $D \in \mathcal{D}$, a partition $\mathcal{C}_D$ of D into cells such that for each $C \in \mathcal{C}_D$ and $A \in \mathcal{A}$, either A is disjoint from C, or A contains C, or A is dense and codense in C.

Fix $D \in \mathcal{D}$. First, suppose that D is an open cell. Let $A \in \mathcal{A}$ be such that $D \cap A \neq \emptyset$; then either $D \subseteq \mathrm{int}(A)$ or $D \subseteq \mathrm{int}(\mathrm{cl}(A)) \setminus \mathrm{int}(A)$, and in the latter case A is dense and codense in D.

Now, suppose that D is a nonopen cell, say of dimension $d < n$. Then there exists $\pi \in \Pi(n, d)$ such that the restriction $\pi|D$ maps D homeomorphically onto an open cell $U \subseteq \mathbb{R}^d$. Permuting coordinates if necessary, we may assume that π is the projection on the first d coordinates. Let $f : U \to \mathbb{R}^{n-d}$ denote the inverse of $\pi|D$, so $D = \mathrm{gr}(f)$ (the graph of f). Inductively, there is a finite partition $\mathcal{S}$ of $\mathbb{R}^d$ into cells such that for each $S \in \mathcal{S}$ and each $B \in \{U\} \cup \{\pi(A \cap D) : A \in \mathcal{A}\}$, either S is disjoint from B, or S is contained in B, or B is dense and codense in S. Hence, for each $A \in \mathcal{A}$, either $\mathrm{gr}(f|S) \cap A = \emptyset$, or $\mathrm{gr}(f|S) \subseteq A$, or A is dense and codense in $\mathrm{gr}(f|S)$. ∎

Remarks. (i) In the above, note that if $C \in \mathcal{C}$ is open and $C' \subseteq C$ is an open cell intersecting some $A \in \mathcal{A}$, then either $C' \subseteq A$ or A is dense and codense in C'. Thus, by routine o-minimal arguments, it is easy to strengthen the conclusion so that it holds with "decomposition" in place of "finite partition".

(ii) Suppose moreover in the above that $\mathfrak{R}$ defines addition and multiplication. Then the conclusion and the preceding remark hold with "C^p cell" and "C^p decomposition" in place of "cell" and "decomposition", for each fixed $p \in \mathbb{N}$.

(iii) We have confined our attention in this paper to expansions of the real line. However, the definition of open core makes sense for expansions of arbitrary dense linear orders without endpoints. Proposition 2 and the preceding two remarks hold in this more general setting.

The o-minimality of expansions of $(\mathbb{R},+,\cdot)$ is closely linked with good asymptotic behaviour of the definable unary functions. In particular, an expansion of $(\mathbb{R},+,\cdot)$ is o-minimal if and only if every definable unary function is ultimately either constant, or strictly monotone and C^p, for any fixed $p\in\mathbb{N}$. (The forward implication is true for o-minimal expansions of arbitrary ordered fields, but the reverse implication uses the fact that we are working over $\mathbb{R}$.) As an application of our work so far, we obtain a result that is similar in spirit (but not nearly so tight).

Corollary. *Let* $\mathfrak{R}$ *be an expansion of* $(\mathbb{R},+,\cdot)$. *Suppose that for each definable unary function* f *there exists a continuous function* $g:(c,\infty)\to\mathbb{R}$ *such that* $|f(t)|\le g(t)$ *for all* $t>c$. *Then*:

(1) *Every infinite definable subset of* $\mathbb{R}$ *has the cardinality of the continuum.*

(2) *Let* $f:\mathbb{R}\to\mathbb{R}$ *be definable and* p *be a positive integer. Then there exist* $k\in\mathbb{N}$, $c\in\mathbb{R}$ *and definable* C^p *functions* $g_0<\ldots<g_k:(c,\infty)\to\mathbb{R}$ *such that each* g_i *is either constant or strictly monotone*, *and the closure of the graph of* $f|(c,\infty)$ *is equal to the union of the closures of the graphs of* $g_0,\ldots,g_k$.

Proof. First, note that the supposition is equivalent to: For each definable unary function f there are only finitely many $r\in\mathbb{R}$ with $\overline{\lim}_{t\to r}|f(t)| = +\infty$.

(1) Let $A\subset\mathbb{R}$ be definable and infinite. Suppose that A is unbounded. It suffices to show that A contains a transcendence base for $\mathbb{R}$. Suppose otherwise, and let $\alpha\in\mathbb{R}$ be transcendental over A. Then $f:\mathbb{R}\to\mathbb{R}$ given by

$$f(t):=\begin{cases} y & \text{if there exist } x,y\in A \text{ such that } y(t-x)=\alpha,\\ 0 & \text{otherwise}\end{cases}$$

is a well defined, definable unary function which is unbounded at every point of A; contradiction. Now suppose that A is bounded; then it has a limit point $a\in\mathbb{R}$. By the previous case, the set $\{1/(x-a): x\in A\setminus\{a\}\}$ has cardinality that of the continuum, hence so does A.

(2) By (1), Theorem (b), and the previous Remark (ii), it suffices to show that the graph of f is nowhere dense. Suppose otherwise; then the closure of the graph of f contains an open box $(a,b)\times(c,d)$, with $d\in\mathbb{R}$. Define $h:(a,b)\to\mathbb{R}$ by $h(t)=1/(d-f(t))$ if $f(t)<d$ and $h(t)=0$ otherwise. Then h is definable and unbounded at every $x\in(a,b)$; contradiction. ■

4. Some examples and counterexamples. Let $\mathfrak{R}$ be an expansion of $(\mathbb{R},+,\cdot)$. Consider the following conditions:

(1) $\mathfrak{R}$ is o-minimal.

(2) Every definable unary function is ultimately bounded by a continuous function.

(3) Every infinite definable subset of $\mathbb{R}$ has the cardinality of the continuum.

(4) $\mathfrak{R}^\circ$ is o-minimal.

(5) For every $n \in \mathbb{N}$, every definable subset of $\mathbb{R}^n$ is a finite union of locally closed definable subsets of $\mathbb{R}^n$.

The monotonicity and cell decomposition theorems for o-minimal structures yield (1)⇒(2) and (1)⇒(5) respectively; (2)⇒(3) is by the Corollary in Section 3; and (3)⇒(4) is by Theorem (a). All of the converses fail.

PROPOSITION 3. *Let $\mathfrak{R}$ be an o-minimal expansion of $(\mathbb{R}, +, \cdot)$ in a countable language. Then there exist $Y_1, Y_2, Y_3 \subseteq \mathbb{R}$ such that $(\mathfrak{R}, Y_i)$ satisfies condition $(i+1)$, but not condition (i), for $i = 1, 2, 3$.*

The above is a byproduct of results in the next two subsections.

In the third subsection, we prove the following, showing that $(5)\not\Rightarrow(1)$.

PROPOSITION 4. *Every set definable in $(\mathbb{R}, +, \cdot, \alpha^{\mathbb{Z}})$ is a finite union of locally closed definable sets, where $\alpha > 1$ and $\alpha^{\mathbb{Z}} := \{\alpha^k : k \in \mathbb{Z}\}$.*

4.1. *Dense pairs.* We draw heavily in this subsection on results from [D3]; all citations [...] will be to that paper. We show that $(4)\not\Rightarrow(3)$ and $(3)\not\Rightarrow(2)$.

Let $\mathfrak{R}$ be an o-minimal expansion of $(\mathbb{R}, <, +, -, \cdot, 0, 1)$ in a language extending $\{<, +, -, \cdot, 0, 1\}$. Let M be (the underlying set of) a proper elementary substructure of $\mathfrak{R}$. Then M is dense in $\mathbb{R}$, and the structure $(\mathfrak{R}, M)$ is an example of what is called a *dense pair*. The simplest example is the expansion of the ordered field of real numbers by a predicate for the set of real algebraic numbers.

By [Thm. 4], every open subset of $\mathbb{R}$ definable in $(\mathfrak{R}, M)$ has finitely many connected components, hence, by Theorem (b), the open core of $(\mathfrak{R}, M)$ is o-minimal. (Indeed, the open core of $(\mathfrak{R}, M)$ is just $\mathfrak{R}$; see [Thm. 5].)

Proof of $(4)\not\Rightarrow(3)$. If the language of $\mathfrak{R}$ is countable, then $\mathrm{dcl}(\emptyset)$ is a countably infinite elementary substructure of $\mathfrak{R}$, and the open core of $(\mathfrak{R}, \mathrm{dcl}(\emptyset))$ is o-minimal. ∎

We will need the following easy facts:

Let K be a proper subfield of $(\mathbb{R}, +, \cdot)$ and $I \subseteq \mathbb{R}$ be an open interval. Then:

(a) $\mathrm{card}(I \cap K) = \mathrm{card}(K)$.

(b) $\mathrm{card}(I \setminus K) = \mathrm{card}(\mathbb{R})$.

(c) *The structure $(\mathbb{R}, +, \cdot, K)$ defines a unary function whose graph is dense in the plane.*

We leave the proofs of (a) and (b) to the reader. For (c), let $\alpha \in \mathbb{R} \setminus K$, and define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(t) := \begin{cases} x & \text{if } t = x + \alpha y \in K + \alpha.K, \\ 0 & \text{otherwise.} \end{cases}$$

Note that by (c), the structure $(\mathfrak{R}, M)$ certainly defines unary functions that are not bounded at $+\infty$ by any continuous function.

A set $X \subseteq \mathbb{R}$ is said to be *M-small* (with respect to $\mathfrak{R}$) if X is definable in $(\mathfrak{R}, M)$ and $X \subseteq f(M^n)$ for some $n \in \mathbb{N}$ and $f : \mathbb{R}^n \to \mathbb{R}$ definable in $\mathfrak{R}$.

LEMMA. *Every infinite M-small subset of $\mathbb{R}$ has the cardinality of M.*

Proof. Let $X \subseteq \mathbb{R}$ be infinite and M-small. Clearly, $\operatorname{card}(X) \leq \operatorname{card}(M)$. By C^1 cell decomposition and [Lemma 4.3], X contains a set of the form $f(U \cap M^n)$, where $U \subseteq \mathbb{R}^n$ is an open box, and $f : U \to \mathbb{R}$ is C^1 with no critical points in U. For the case $n = 1$, this implies that f is injective on the open interval U; so $\operatorname{card}(X) \geq \operatorname{card}(f(U \cap M)) = \operatorname{card}(M)$. If $n > 1$, then there is an open box $V \subseteq U$ such that some first partial derivative, say $\partial f/\partial x_n$, has no zeros in V. Write $V := W \times I$ with $W \subseteq \mathbb{R}^{n-1}$ an open box and $I \subseteq \mathbb{R}$ an open interval. Since M^{n-1} is dense in $\mathbb{R}^{n-1}$, there exists a point $Q \in W \cap M^{n-1}$ such that $t \mapsto f(Q, t) : I \to \mathbb{R}$ is injective. Note that $\{Q\} \times (I \cap M) \subseteq S$. ■

Proof of (3)$\not\Rightarrow$(2). Suppose that the language of $\mathfrak{R}$ is countable. We show that M can be chosen so that every infinite subset of $\mathbb{R}$ definable in $(\mathfrak{R}, M)$ has the cardinality of the continuum.

Since the language of $\mathfrak{R}$ is countable, there exist continuum-many proper elementary substructures M of $\mathfrak{R}$ such that the dcl-rank of $\mathfrak{R}$ over M is countably infinite. Choose such an M; then $\operatorname{card}(M) = \operatorname{card}(\mathbb{R})$. Let $S \subseteq \mathbb{R}$ be infinite and definable in $(\mathfrak{R}, M)$. By [Thm. 3(2)], either S is M-small or S contains a set of the form $I \setminus X$, where X is M-small and $I \subseteq \mathbb{R}$ is an open interval. If the former, then $\operatorname{card}(S) = \operatorname{card}(\mathbb{R})$ by the preceding lemma. Assume the latter. By the definition of M-small, there exists a finite set $A \subseteq \mathbb{R}$ such that $X \subseteq \operatorname{dcl}(M \cup A)$. Since the rank of $\mathfrak{R}$ over M is infinite, $\operatorname{dcl}(M \cup A)$ is a proper subfield of $(\mathbb{R}, +, \cdot)$. Now note that S contains $I \setminus \operatorname{dcl}(M \cup A)$. ■

REMARK. We suspect that the infinite co-rank condition used above is unnecessary, leading us to conjecture that every infinite subset of $\mathbb{R}$ definable in a dense pair $(\mathfrak{R}, M)$ has either the cardinality of M or that of the continuum.

4.2. *Generic predicates.* The result below (more precisely, we describe only a special case) is due to H. Friedman [unpublished] and was produced essentially upon demand. It provides examples for (2)$\not\Rightarrow$(1). We omit proofs.

Let $\mathfrak{R}$ be an expansion of $(\mathbb{R}, <)$. A set $P \subseteq \mathbb{R}$ is a *generic predicate* for $\mathfrak{R}$ if the following holds for each $p \in \mathbb{N}$, open interval $I \subseteq \mathbb{R}$, and $\varepsilon \in \{0,1\}^p$: If $f_1, \ldots, f_p : I \to \mathbb{R}$ are definable, strictly monotone, and $f_i(x) \neq f_j(x)$ for every $x \in I$ and $1 \le i < j \le p$, then there is an injective function $\phi : \mathbb{R} \to I$ such that for every $x \in \mathbb{R}$ and $i = 1, \ldots, p$, we have $f_i(\phi(x)) \in P \Leftrightarrow \varepsilon_i = 1$. (The use of the phrase "generic predicate" here is in analogy with a concept from model-theoretic stability theory; see e.g. [CP].) Obviously, any generic predicate P for $\mathfrak{R}$ must be dense and codense in $\mathbb{R}$, so $(\mathfrak{R}, P)$ is not o-minimal.

FACT [Friedman]. *Let $\mathfrak{R}$ be an o-minimal expansion of $(\mathbb{R}, +, \cdot)$ in a countable language. Then*:

(1) *There exist continuum-many F_σ generic predicates for $\mathfrak{R}$.*

(2) *Let $P \subseteq \mathbb{R}$ be a generic predicate for $\mathfrak{R}$ and let $f : \mathbb{R} \to \mathbb{R}$ be definable in $(\mathfrak{R}, P)$. Then there exists a continuous function $g : (c, \infty) \to \mathbb{R}$ definable in $\mathfrak{R}$ such that $|f(t)| \le g(t)$ for all $t > c$.*

Actually, the field structure is not needed: The result holds for any o-minimal expansion $\mathfrak{R}$ of the real line, in a countable language, that defines a unary function F with $F(x) > x$ for all $x \in \mathbb{R}$. It can be shown directly, even in this more general setting, that every infinite subset of $\mathfrak{R}$ definable in $(\mathfrak{R}, P)$ has the cardinality of the continuum, and thus that the open core of $(\mathfrak{R}, P)$ is o-minimal.

4.3. *Proof of Proposition 4.* First, we need an easy result, the proof of which we leave to the reader.

LEMMA. *Let X and Y be topological spaces and $f : A \to Y$ be continuous with A locally closed in X. Then $f^{-1}(B)$ is locally closed in X, for every locally closed $B \subseteq Y$.*

Fix a real number $\alpha > 1$ and put $\alpha^{\mathbb{Z}} := \{\alpha^k : k \in \mathbb{Z}\}$. We show that every set definable in $(\mathbb{R}, +, \cdot, \alpha^{\mathbb{Z}})$ is a finite union of locally closed definable sets.

For each $n \ge 1$ put $P_n := \{\alpha^{nk} : k \in \mathbb{Z}\}$. Each P_n is discrete, hence locally closed. Define the function $\lambda : \mathbb{R} \to \mathbb{R}$ by $\lambda(t) := \max(\alpha^{\mathbb{Z}} \cap (0, t])$ for $t > 0$, and $\lambda(t) := 0$ otherwise. Put

$$\mathfrak{R} := (\mathbb{R}, <, +, -, \cdot, 0, 1, \alpha, \lambda, (P_n)_{n \ge 1}).$$

Clearly, $\mathfrak{R}$ is interdefinable with $(\mathbb{R}, +, \cdot, \alpha^{\mathbb{Z}})$, so we show that every definable (in $\mathfrak{R}$) set is a finite union of locally closed definable (in $\mathfrak{R}$) sets.

Let $A \subseteq \mathbb{R}^n$ be definable. By [D1], $\mathfrak{R}$ admits elimination of quantifiers, so there exist $m \in \mathbb{N}$, $x \in \mathbb{R}^m$ and quantifier-free $\emptyset$-definable $B \subseteq \mathbb{R}^{m+n}$ such that A is equal to the fiber B_x. Since fibers of locally closed sets are locally closed, we are reduced to showing that every quantifier-free $\emptyset$-definable set is a finite union of locally closed definable sets. By the lemma, this reduces to showing that for each $n \in \mathbb{N}$ and n-ary term τ (in the language of $\mathfrak{R}$) there is a finite partition $\mathcal{S}$ of $\mathbb{R}^n$ into locally closed definable sets such that $\tau|S : S \to \mathbb{R}$ is continuous for each $S \in \mathcal{S}$. This is easily established by induction on complexity, using the lemma and noting that the restriction of λ to each of the sets $(-\infty, 0)$, $\{0\}$, $\alpha^{\mathbb{Z}}$ and $(0,\infty) \setminus \alpha^{\mathbb{Z}}$ is continuous. ∎

The techniques used above fail for the structures $(\mathbb{R}, +, \cdot, 2^{\mathbb{Z}}, 3^{\mathbb{Z}})$ and $(\mathbb{R}, +, \cdot, G)$, where $G := \{2^j 3^k : j, k \in \mathbb{Z}\}$. (Since G is dense and codense in $(0,\infty)$, it cannot be a finite union of locally closed subsets of $\mathbb{R}$. Note also that $(0,\infty) \setminus G$ is not F_σ.) Our understanding of these structures is quite limited at present, and we close with the following questions:

(1) Is the open core of $(\mathbb{R}, +, \cdot, G)$ o-minimal?
(2) Is $\mathbb{Z}$ definable in $(\mathbb{R}, +, \cdot, 2^{\mathbb{Z}}, 3^{\mathbb{Z}})$?

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