FUNDAMENTA
MATHEMATICAE
162 (1999)

Continuous images and other topological properties of Valdivia compacta

by

Ondřej Kalenda (Praha)

Abstract. We study topological properties of Valdivia compact spaces. We prove in particular that a compact Hausdorff space K is Corson provided each continuous image of K is a Valdivia compactum. This answers a question of M. Valdivia (1997). We also prove that the class of Valdivia compacta is stable with respect to arbitrary products and we give a generalization of the fact that Corson compacta are angelic.

1. Introduction. The class of Valdivia compacta was first studied in [AMN] as a generalization of Corson compact spaces. Functional analytic properties of Valdivia compacta and of spaces of continuous functions on them were studied for example in [AMN], [V1], [V2], [DG] and [FGZ]. In the present paper we investigate the topological properties of the class of Valdivia compacta.

An important role in our investigations is played by the notion of Σ-*subset* (see Definition 1.1 below) which makes formulations easier. In Section 2 we study several abstract properties of Σ-subsets which turn out to be a powerful technical tool not only in the present paper. Section 3 contains our main result. We prove that a compact space K is Corson provided each continuous image of K is Valdivia. This answers a question of [V3], where the first example of a non-Valdivia continuous image of a Valdivia compact space is constructed. Another example in this direction is given in [K1]. In Section 4 we study other topological properties of Valdivia compacta, namely permanence properties under products, subsets and finite unions.

Let us start with basic definitions.

1991 *Mathematics Subject Classification*: Primary 54C05, 54D30.

Key words and phrases: Corson compact space, Valdivia compact space, countably compact space, Fréchet–Urysohn space, continuous image.

Supported by Research grant GAUK 190/96 and GAČR 201/97/0216.

Research project J13/98113200007.

1.1. Definition. (1) If Γ is a set and $x \in \mathbb{R}^\Gamma$, we put

$$\operatorname{supp} x = \{\gamma \in \Gamma \mid x(\gamma) \neq 0\}.$$

(2) If Γ is a set, we put

$$\Sigma(\Gamma) = \{x \in \mathbb{R}^\Gamma \mid \operatorname{supp} x \text{ is countable}\}.$$

Let K be a compact Hausdorff space.

(3) We say that $A \subset K$ is a *Σ-subset* of K if there is a homeomorphic injection h of K into some $\mathbb{R}^\Gamma$ such that $h(A) = h(K) \cap \Sigma(\Gamma)$.

(4) K is called a *Corson compact space* if K is a Σ-subset of itself.

(5) K is called a *Valdivia compact space* if K has a dense Σ-subset.

(6) We say that K is a *super-Valdivia compact space* if each $x \in K$ is contained in some dense Σ-subset of K.

1.2. Remarks. (1) It is easy to observe that in the definition of a Σ-subset one can consider $[0,1]^\Gamma$ instead of $\mathbb{R}^\Gamma$.

(2) The standard definition of a Valdivia compact space does not use the concept of Σ-subset. But this notion turns out to be convenient. Using it we are not forced to consider one concrete embedding; it suffices to use the existence of a subset with several important properties.

(3) We introduce the non-standard notion of super-Valdivia compacta. It seems that this concept properly expresses the difference between two canonical examples of Valdivia non-Corson compacta. While $[0,\omega_1]$ has exactly one dense Σ-subset $[0,\omega_1)$, the space $[0,1]^\Gamma$ is easily seen to be super-Valdivia.

2. Properties of Σ-subsets. In this section we collect several important properties of Σ-subsets. They are, in general, not hard to prove but turn out to be powerful tools in the study of the structure of Valdivia compacta. They are used in the proof of our main result, as well as in other results ([K2], [K3], cf. also [FGZ]). We start by defining two topological notions.

2.1. Definition. (1) A topological space X is called a *Fréchet–Urysohn* space if, whenever $A \subset X$ and $x \in \overline{A}$, there exists a sequence $x_n \in A$ with $x_n \to x$.

(2) A subset A of a topological space X is said to be *countably closed* in X if $\overline{C} \subset A$ whenever $C \subset A$ is countable.

2.2. Proposition. *Let K be a compact Hausdorff space and $A \subset K$ be a dense Σ-subset. Then*:

(1) *A is a Fréchet–Urysohn space.*

(2) *A is countably closed in K.*

(3) *$G \cap A$ is dense in G whenever $G \subset K$ is G_δ.*

Proof. (1) We will prove that $\Sigma(\Gamma)$ is a Fréchet–Urysohn space for any Γ. For every $x \in \Sigma(\Gamma)$ we can fix an enumeration $\operatorname{supp} x = \{\gamma_1(x), \gamma_2(x), \ldots\}$. If $\operatorname{supp} x$ is finite, we fill up the sequence $\gamma_k(x)$ with some element of Γ. Now let $A \subset \Sigma(\Gamma)$, $x \in \Sigma(\Gamma)$, $x \in \overline{A}$. By induction we can construct $x_n \in A$ such that $|x_n(\gamma_k(x_l)) - x(\gamma_k(x_l))| < 1/n$ for $0 \le l < n$ and $1 \le k \le n$, where $x_0 = x$. Then clearly $x_n \to x$ (since convergence in the product topology is the coordinatewise convergence).

(2) This follows from the obvious fact that $\Sigma(\Gamma)$ is countably closed in $\mathbb{R}^\Gamma$ for every Γ.

(3) By (2) it is clear that A is countably compact. So the assertion follows from the next lemma. ∎

2.3. Lemma. *Let K be a compact Hausdorff space and $A \subset K$ be a dense countably compact subset of K. Then $G \cap A$ is dense in G whenever $G \subset K$ is G_δ.*

Proof. Let $G \subset K$ be a G_δ set. Then there are open sets $U_n \subset K$ such that $G = \bigcap_{n\in\mathbb{N}} U_n$. Let $x \in G$ and W be an open neighborhood of x. We will show that $W \cap A \cap G \neq \emptyset$. To this end we construct, by an easy induction, open sets V_n, $n \in \mathbb{N}$, such that

$$V_1 \subset W, \quad x \in V_n,\ n \in \mathbb{N}, \quad \overline{V}_{n+1} \subset V_n \cap U_1 \cap \ldots \cap U_n,\ n \in \mathbb{N}.$$

As A is dense in K, we have $V_n \cap A \neq \emptyset$ for every n. Moreover, $\overline{V}_{n+1} \subset V_n$ and A is countably compact, hence $A \cap \bigcap_{n\in\mathbb{N}} V_n \neq \emptyset$. But by construction we have $\bigcap_{n\in\mathbb{N}} V_n \subset G \cap W$, which completes the argument. ∎

2.4. Proposition. *Let K be a compact Hausdorff space and $M \subset K$ be arbitrary. If A, B are two Σ-subsets of K such that $A \cap B \cap M$ is dense in M, then $A \cap M = B \cap M$.*

Proof. Let $x \in A \cap M$. Then $x \in \overline{A \cap B \cap M}$, and hence there is a sequence $x_n \in A \cap B \cap M$ with $x_n \to x$ (by Proposition 2.2(1)). Now, B is countably closed (Proposition 2.2(2)), so $x \in B$. Thus $A \cap M \subset B \cap M$. The inverse inclusion can be proved by interchanging the roles of A and B. ∎

The following characterization of Σ-subsets turns out to be a key observation for the structure of Valdivia compacta (cf. also [K2]). Its consequence in the subsequent proposition will be used in the investigation of continuous images of Valdivia compacta.

2.5. Proposition. *Let K be a compact Hausdorff space and $A \subset K$ be dense. Then A is a Σ-subset of K if and only if A is homeomorphic to a coordinatewise bounded closed subset of $\Sigma(\Gamma)$ for some Γ and $\beta A = K$.*

Proof. By [G, Theorem 2] we have $[0,1]^\Gamma = \beta(\Sigma(\Gamma) \cap [0,1]^\Gamma)$, and it is proved in [C, Theorem 1] that $\Sigma(\Gamma)$ is a normal space. Now it is obvious that $\beta A = \overline{A}^{[0,1]^\Gamma}$ whenever A is a closed subset of $\Sigma(\Gamma)\cap[0,1]^\Gamma$. The result now follows immediately. ∎

2.6. Proposition. *Let K be a compact Hausdorff space and $A \subset K$ be a dense Σ-subset. If U and V are disjoint open subsets of K then $\overline{U}\cap\overline{V}\cap A$ is dense in $\overline{U}\cap\overline{V}$.*

Proof. Without loss of generality we can suppose $K = \overline{U}\cup\overline{V}$. Let $x \in \overline{U}\cap\overline{V}$ and $W \subset K$ be an open neighborhood of x such that $\overline{W}\cap\overline{U}\cap\overline{V}\cap A = \emptyset$. Then $A\cap\overline{W}$ is a dense Σ-subset of $\overline{W}$. The sets $A\cap\overline{W}\cap\overline{U}$ and $A\cap\overline{W}\cap\overline{V}$ are disjoint relatively clopen subsets covering $A\cap\overline{W}$ (as $\overline{W}\cap\overline{U}\cap\overline{V}\cap A = \emptyset$). Let f be the characteristic function of $\overline{W}\cap\overline{U}\cap A$. It is a bounded continuous function on $A\cap\overline{W}$. By Proposition 2.5 it can be continuously extended onto $\overline{W}$. But the point x belongs to both $\overline{\overline{W}\cap\overline{U}\cap A}$ and $\overline{\overline{W}\cap\overline{V}\cap A}$, which is impossible. ∎

Remark. In fact, one can prove that $\bigcap_{n\in\mathbb{N}}\overline{U}_n\cap A$ is dense in $\bigcap_{n\in\mathbb{N}}\overline{U}_n$ whenever A is a dense Σ-subset of K and each U_n is an open subset of K. This is a strengthening of Proposition 2.2(3) but we will not use it.

The following proposition is a refinement of [DG, Proposition III-1]. We include it here for two reasons. First, our proof is much more direct and does not use the notion of retractions, and secondly, it enables us to control where the copy of $[0,\omega_1]$ is embedded in K. This will be used in the proof of our main result. A further refinement of this construction is needed in [K3].

2.7. Proposition. *Let Γ be a set. If K is a compact subset of $\mathbb{R}^\Gamma$ such that $K\cap\Sigma(\Gamma)$ is dense in K and $G \subset K$ is a G_δ set with $G\setminus\Sigma(\Gamma)\neq\emptyset$, then*:

(1) *There is $x \in G$ with* $\operatorname{card}\operatorname{supp} x = \aleph_1$.

(2) *For every $x \in G$ with* $\operatorname{card}\operatorname{supp} x = \aleph_1$ *there exists a homeomorphic injection $\varphi : [0,\omega_1] \to G$ with $\varphi([0,\omega_1)) \subset \Sigma^\kappa(\Gamma)$ and $\varphi(\omega_1) = x$.*

Proof. Choose an arbitrary $x \in G\setminus\Sigma(\Gamma)$. It follows easily from the regularity of K that without loss of generality we can suppose that G is closed. If $\operatorname{card}\operatorname{supp} x = \aleph_1$, put $I = \operatorname{supp} x$, otherwise let $I \subset \operatorname{supp} x$ be some set of cardinality $\aleph_1$. Fix an enumeration $I = \{i_\alpha \mid \alpha < \omega_1\}$. By transfinite induction we will construct $x_\alpha \in G\cap\Sigma(\Gamma)$ and $J_\alpha \subset \Gamma$ for $\alpha < \omega_1$ such that:

(i) $i_\alpha \in J_{\alpha+1}$, $\bigcup_{\beta<\alpha}\operatorname{supp} x_\beta \subset J_\alpha$, J_α is countable;
(ii) $J_\alpha \subset J_{\alpha+1}$, $\operatorname{supp} x_\alpha \cap I \subsetneq J_{\alpha+1}\cap I$;
(iii) $J_\alpha = \bigcup_{\beta<\alpha} J_\beta$ if α is limit;
(iv) $x_\alpha(i) = x(i)$ for every $i \in J_\alpha$;
(v) $x_\alpha = \lim_{\beta<\alpha} x_\beta$ if α is limit.

Put $J_0 = \{i_0\}$ and choose $x_0 \in G \cap \Sigma(\Gamma)$ such that $x_0(i_0) = x(i_0)$. This is possible due to Proposition 2.2(3) as $\{y \in K \mid y(i_0) = x(i_0)\}$ is G_δ.

Suppose that we already have J_α and x_α. As $\operatorname{supp} x_\alpha$ is countable and $\operatorname{card} I = \aleph_1$, there is some $j \in I \setminus \operatorname{supp} x_\alpha$. Put $J_{\alpha+1} = J_\alpha \cup \operatorname{supp} x_\alpha \cup \{i_\alpha, j\}$. Clearly $J_{\alpha+1}$ is countable. Since $\{y \in K \mid y(i) = x(i) \text{ for } i \in J_{\alpha+1}\}$ is a G_δ set, we can (due to Proposition 2.2(3)) choose $x_{\alpha+1} \in G \cap \Sigma(\Gamma)$ such that $x_{\alpha+1}(i) = x(i)$ for every $i \in J_{\alpha+1}$.

Now suppose that α is limit and we have constructed x_β and J_β for every $\beta < \alpha$. Put $J_\alpha = \bigcup_{\beta<\alpha} J_\beta$. Observe that the net x_β, $\beta < \alpha$, converges to some point, which will be denoted by x_α. Indeed, if $i \in \Gamma \setminus J_\alpha$, then $x_\beta(i) = 0$ for every $\beta < \alpha$. If $i \in J_\alpha$, then $i \in J_\beta$ for some $\beta < \alpha$, and thus $x_\gamma(i) = x(i)$ for $\beta \le \gamma < \alpha$. So the net converges to the point x_α such that $x_\alpha(i) = x(i)$ for $i \in J_\alpha$ and $x_\alpha(i) = 0$ if $i \in \Gamma \setminus J_\alpha$. As G is closed, we have $x_\alpha \in G$.

This completes the construction.

Now put $J = \bigcup_{\alpha<\omega_1} J_\alpha$ and $x_{\omega_1} = \lim_{\alpha<\omega_1} x_\alpha$. As in the limit induction step, it can be shown that this limit exists and that $x_{\omega_1}(i) = x(i)$ for $i \in J$ and $x_{\omega_1}(i) = 0$ otherwise. Further, $J \supset I$ and $\operatorname{card} J = \aleph_1$. Hence $x_{\omega_1} \in G$ and $\operatorname{card} \operatorname{supp} x_{\omega_1} = \aleph_1$, which proves (1).

Moreover, if $\operatorname{card} \operatorname{supp} x = \aleph_1$ then clearly $x_{\omega_1} = x$. Define $\varphi : [0, \omega_1] \to K$ by the formula $\varphi(\alpha) = x_\alpha$. This mapping is one-to-one by (ii) and the continuity follows easily from (v) and the definition of x_{ω_1}. ■

The following lemma is only a technical tool used in our main result.

2.8. Lemma. *Let K be a compact Hausdorff space and $F \subset K$ be a metrizable closed set. Put $L = K \setminus F \cup \{F\}$ endowed with the quotient topology induced by the mapping $Q : K \to L$ defined by*

$$Q(x) = \begin{cases} x, & x \notin F, \\ F, & x \in F. \end{cases}$$

If A is a dense Σ-subset of L, then $Q^{-1}(A)$ is a Σ-subset of K.

Proof. Let $h_0 : L \to \mathbb{R}^\Gamma$ be a homeomorphic injection such that $h_0(A) = h_0(L) \cap \Sigma(\Gamma)$, and $h_1' : F \to \mathbb{R}^{\mathbb{N}}$ be any homeomorphic injection. For $n \in \mathbb{N}$ let $h_1(n)$ be a continuous extension of $h_1'(n)$ on K. Define $h : K \to \mathbb{R}^{\Gamma \cup \mathbb{N}}$ by the formula

$$\begin{aligned} h(x)(\gamma) &= h_0(Q(x))(\gamma), & \gamma \in \Gamma, \\ h(x)(n) &= h_1(x)(n), & n \in \mathbb{N}. \end{aligned}$$

Now it is obvious that h is a homeomorphic injection and $h(Q^{-1}(A)) = h(K) \cap \Sigma(\Gamma \cup \mathbb{N})$. ■

2.9. Remark. Most of the results of this section can also be proved in a more general setting, where we consider $\Sigma_\kappa(\Gamma) = \{x \in \mathbb{R}^\Gamma \mid \operatorname{card} \operatorname{supp} x \le \kappa\}$ instead of $\Sigma(\Gamma)$. Then we have analogues of Propositions 2.2 and 2.7. The latter would say that, if $K \subset \mathbb{R}^\Gamma$ with $K \cap \Sigma_\kappa(\Gamma)$ dense in K but $K \setminus \Sigma_\kappa(\Gamma) \neq \emptyset$, then in a similar way one can embed a copy of $[0, \kappa^+]$ in K. This is useful for example in studying projectional resolutions of the identity on Banach spaces with a large density [K3].

3. Corson compacta and continuous images of Valdivia compacta. The aim of this section is to prove our main result, which is the following theorem, answering the question of M. Valdivia [V3].

3.1. Theorem. *Let K be a compact Hausdorff space. Then the following assertions are equivalent.*

(1) *Every continuous image of K is a Valdivia compactum.*

(2) *Every at most two-to-one continuous image of K is a Valdivia compactum.*

(3) *K is a Corson compactum.*

First we will consider two special cases.

3.2. Proposition. *Let K be a Valdivia compact Hausdorff space which is not super-Valdivia. Then there is an at most two-to-one continuous image of K which is not Valdivia. In fact*, *one can choose a quotient mapping identifying some two points of K.*

Proof. Let $a \in K$ be contained in no dense Σ-subset of K. Then clearly a is not an isolated point. Further, let b be any non-isolated point of K different from a. Such a point exists, otherwise K would be homeomorphic to the one-point compactification of a discrete space and so would be Corson (even Eberlein). Let L be the quotient space obtained from K by identifying the points a and b, i.e. $L = (K \setminus \{a, b\}) \cup \{\{a, b\}\}$, and let

$$Q(x) = \begin{cases} x, & x \neq a, b, \\ \{a, b\}, & x = a, b, \end{cases}$$

be the canonical quotient mapping. If L were Valdivia, there would exist a dense Σ-subset $B \subset L$.

Choose open neighborhoods U and V of a and b respectively with $\overline{U} \cap \overline{V} = \emptyset$. Then $U' = Q(U \setminus \{a\})$ and $V' = Q(V \setminus \{b\})$ are open in L (by the definition of the quotient topology) and clearly $\overline{U'} \cap \overline{V'} = \{\{a, b\}\}$. So, by Proposition 2.6 we get $\{a, b\} \in B$. Lemma 2.8 shows that $Q^{-1}(B)$ is a Σ-subset of K, and clearly $a \in Q^{-1}(B)$, which contradicts the choice of a. ∎

REMARK. The assumption that K is not super-Valdivia in the previous proposition seems to be natural. It can be shown that the conclusion of the proposition does not hold for example if $K = [0,1]^{\Gamma}$ (or $\{0,1\}^{\Gamma}$). However, we do not know whether the condition of K not being super-Valdivia is necessary.

3.3. PROPOSITION. *Let K be a compact Hausdorff space such that there are two disjoint homeomorphic closed nowhere dense subsets $M, N \subset K$ which are not Valdivia compacta. Then there is an at most two-to-one continuous image of K which is not Valdivia.*

Proof. Let $h : M \to N$ be a homeomorphism. Put $L = K \setminus M$ and equip L with the quotient topology induced by the mapping

$$\varphi(x) = \begin{cases} h(x), & x \in M, \\ x, & x \in K \setminus M. \end{cases}$$

Clearly L is compact and it is easy to check that it is Hausdorff. Suppose that L is Valdivia. Then there is a dense Σ-subset $A \subset L$. As K is normal, there are open (in K) sets $U \supset M$, $V \supset N$ such that $\overline{U} \cap \overline{V} = \emptyset$. The sets $U' = \varphi(U \setminus M)$ and $V' = \varphi(V \setminus N)$ are clearly disjoint open sets in L with $\overline{U'} \cap \overline{V'} = N$. Thus, by Proposition 2.6, $N \cap A$ is dense in N, hence N is Valdivia, a contradiction. ■

Now we give two basic examples of non-Valdivia continuous images of Valdivia compacta.

3.4. EXAMPLE. (1) *Let K_1 be the compact space obtained from* $([0,\omega_1] \times \{0\}) \oplus ([0,\omega] \times \{1\})$ *by identifying the points* $(\omega_1, 0)$ *and* $(\omega, 1)$. *Then K_1 is not Valdivia.*

(2) *Let K_2 be the compact space obtained from* $[0,\omega_1] \times \{0,1\}$ *by identifying the points* $(\omega_1, 0)$ *and* $(\omega_1, 1)$. *Then K_2 is not Valdivia.*

Proof. (1) This is proved in [V3] and follows easily from Proposition 2.2.

(2) Suppose $A \subset K_2$ is a dense Σ-subset. It easily follows from Proposition 2.2 that $A = [0,\omega_1) \times \{0,1\}$. But $\chi_{[0,\omega_1) \times \{0\}}$ is a bounded continuous function on A which cannot be extended to K_2, so $K_2 \neq \beta A$, hence by Proposition 2.5 the set A is not a Σ-subset of K_2. So K_2 is not Valdivia. ■

Now we are going to prove a theorem on embedding non-Valdivia compacta into non-Corson super-Valdivia compacta. This theorem completes and strengthens Proposition III-1 of [DG].

3.5. THEOREM. *Let K be a super-Valdivia compact space which is not Corson. Then K contains uncountably many pairwise disjoint nowhere dense homeomorphic copies of the space K_1 from Example* 3.4.

P r o o f. Let P denote the set of all isolated points of K. If P is dense, then by Proposition 2.4 the space K has only one dense Σ-subset, so it is Corson, a contradiction. Now, $K' = \overline{K \setminus \overline{P}}$ is again super-Valdivia non-Corson and has no isolated points. Hence we can suppose without loss of generality that K has no isolated points. Let A be a dense Σ-subset of K. By transfinite induction we will construct $M_\alpha \subset K$, $m_\alpha \in M_\alpha$, $y_\alpha^n \in K$, $n \in \mathbb{N}$, $\alpha < \omega_1$, such that

(i) $M_\alpha \subset K \setminus \bigcup_{\beta<\alpha}(M_\beta \cup \{y_\beta^n \mid n \in \mathbb{N}\})$;

(ii) there is a continuous bijection $\varphi_\alpha : [0, \omega_1] \to M_\alpha$ with $\varphi_\alpha([0, \omega_1)) \subset A$ and $\varphi_\alpha(\omega_1) = m_\alpha$;

(iii) $y_\alpha^n \in K \setminus (M_\alpha \cup \bigcup_{\beta<\alpha}(M_\beta \cup \{y_\beta^n \mid n \in \mathbb{N}\}))$ and $y_\alpha^n \xrightarrow{n} m_\alpha$.

Suppose the construction is already done for $\beta < \alpha$. Then $G = K \setminus \bigcup_{\beta<\alpha}(M_\beta \cup \{y_\beta^n \mid n \in \mathbb{N}\})$ is a dense G_δ subset of K, and so $G \cap A$ is dense in G and $G \setminus A \neq \emptyset$ (otherwise by Proposition 2.4, A would be the only dense Σ-subset and hence K would be Corson). Hence by Proposition 2.7 we can find M_α and m_α satisfying (i) and (ii). The set $G' = G \setminus M_\alpha$ is again dense G_δ. As K is super-Valdivia, there is a dense Σ-subset B of K containing m_α. By Proposition 2.2(3), $B \cap G'$ is dense in K, so $m_\alpha \in \overline{B \cap G'}$, hence by Proposition 2.1(1) we can choose $y_\alpha^n \in B \cap G'$ converging to m_α. This completes the construction.

Now it is enough to observe that $H_\alpha = M_\alpha \cup \{y_\alpha^n \mid n \in \mathbb{N}\}$ is a copy of the space K_1 from Example 3.4 for every $\alpha < \omega_1$. ■

REMARK. In fact, in the previous theorem it is enough to suppose that there are at least two distinct dense Σ-subsets of K. But the proof requires the generalization of Proposition 2.7 described in Remark 2.9.

Proof of Theorem 3.1. (3)⇒(1). This follows from the well known facts that every continuous image of a Corson compact space is again Corson (see e.g. [A, Corollary IV.3.15]) and that every Corson compactum is Valdivia.

(1)⇒(2). This is trivial.

(2)⇒(3). Suppose that K is a Valdivia non-Corson compactum. If K is not super-Valdivia, the result follows from Proposition 3.2. If K is super-Valdivia, we can use Theorem 3.5 and Proposition 3.3. ■

3.6. QUESTION. It is proved in [K1] that there is a Banach space X and a subspace $Y \subset X$ such that (B_{X^*}, w^*) is a Valdivia compactum while (B_{Y^*}, w^*) is not Valdivia. In view of Theorem 3.1 it is natural to ask whether (B_{X^*}, w^*) is Corson provided (B_{Y^*}, w^*) is Valdivia for every subspace Y of X.

4. Other topological properties of Valdivia compacta. In this section we collect results on other topological properties of Valdivia compact spaces which are natural to study and which are, to our knowledge, nowhere published.

It is mentioned in [DG] that the class of Valdivia compact spaces is closed under countable products. In fact, a stronger statement holds.

4.1. THEOREM. (1) *An arbitrary product of Valdivia compact spaces is a Valdivia compactum.*

(2) *An arbitrary product of super-Valdivia compact spaces is a super-Valdivia compactum.*

Proof. This follows easily from the following proposition. ∎

4.2. PROPOSITION. *Let K_α, $\alpha \in I$, be a family of compact Hausdorff spaces and, for each $\alpha \in I$, let A_α be a dense Σ-subset of K_α and $x_\alpha \in A_\alpha$. Then the set*

$$A = \Big\{ y = (y_\alpha)_{\alpha \in I} \in \prod_{\alpha \in I} A_\alpha \,\Big|\, \{\alpha \in I \mid y_\alpha \neq x_\alpha\} \text{ is countable} \Big\}$$

is a dense Σ-subset of the space $K = \prod_{\alpha \in I} K_\alpha$.

Proof. For each $\alpha \in I$ there is a homeomorphic injection $h_\alpha : K_\alpha \to \mathbb{R}^{\Gamma_\alpha}$ such that $h_\alpha(K_\alpha) \cap \Sigma(\Gamma_\alpha) = h_\alpha(A_\alpha)$. It is easy to check that we can ensure that $h_\alpha(x_\alpha) = 0$ for any α. Define $h : K \to \prod_{\alpha \in I} \mathbb{R}^{\Gamma_\alpha} = \mathbb{R}^\Gamma$, where $\Gamma = \{(\gamma, \alpha) \mid \gamma \in \Gamma_\alpha,\ \alpha \in I\}$, by the formula

$$h((y_\alpha)_{\alpha \in I})(\gamma, \beta) = h_\beta(y_\beta)(\gamma).$$

It is obvious that h is a homeomorphic injection, and that $h(A) = h(K) \cap \Sigma(\Gamma)$. ∎

4.3. COROLLARY. *An arbitrary product of Corson compact spaces is a super-Valdivia compactum. In particular, $[0,1]^\Gamma$ and $\{0,1\}^\Gamma$ are super-Valdivia for every set Γ.*

Now we formulate a well known fact ((1)) together with a positive result on certain subspaces of Valdivia compact spaces.

4.4. THEOREM. (1) *Every compact Hausdorff space is homeomorphic to a closed subset of a super-Valdivia compact space.*

(2) *If K is a Valdivia (super-Valdivia) compact space and $F \subset K$ is the closure of a G_δ set (or even the closure of an arbitrary union of G_δ sets), then F is Valdivia (super-Valdivia) as well.*

Proof. (1) It is well known that every compact Hausdorff space is homeomorphic to a closed subset of $[0,1]^\Gamma$ for some set Γ. It remains to use Corollary 4.3.

(2) This follows immediately from Proposition 2.2(3). ∎

Next we give the following result on unions and intersections of Valdivia compacta.

4.5. Theorem. (1) *Any compact Hausdorff space is homeomorphic to the intersection of two super-Valdivia compact subsets of a compact Hausdorff space.*

(2) *Let K and L be two infinite Valdivia compact spaces such that at least one of the following conditions is satisfied.*

(i) *Either K or L is not super-Valdivia.*
(ii) *Neither K nor L is Corson.*

Then there is a non-Valdivia compact Hausdorff space which can be expressed as the union of two subsets, one homeomorphic to K and the other to L.

Proof. (1) Let K be any Hausdorff compact space. Let L be a super-Valdivia compactum containing K (Theorem 4.4; it can be $[0,1]^\Gamma$ for some Γ). Put $H = L \times \{0\} \cup (L \setminus K) \times \{1\}$ with the quotient topology induced by the map $F : L \times \{0,1\} \to H$ defined by the formula

$$F(x,i) = \begin{cases} (x,i), & x \in L \setminus K,\ i = 0,1, \\ (x,0), & x \in K,\ i = 0,1. \end{cases}$$

Then clearly H is a compact Hausdorff space, both $F(L \times \{0\})$ and $F(L \times \{1\})$ are homeomorphic to L and hence are super-Valdivia compact spaces, and their intersection is homeomorphic to K.

(2)(i) Without loss of generality suppose that K is not super-Valdivia. Let $x \in K$ be contained in no dense Σ-subset of K and $y \in L$ be an accumulation point of L. Then it easily follows from the proof of Proposition 3.2 that the space obtained from $K \oplus L$ by identifying x and y is not Valdivia.

(2)(ii) Suppose both K and L are super-Valdivia non-Corson compacta. By Theorem 3.5 there are nowhere dense copies $M \subset K$ and $N \subset L$ of the space K_1 from Example 3.4. Now it is easy to construct, in the same way as in Proposition 3.3, a non-Valdivia quotient of $K \oplus L$ which can be expressed as the union of K and L. ∎

It is well known (and follows immediately from Proposition 2.2(1)) that every Corson compactum is a Fréchet–Urysohn space (which is the same as angelic for compact spaces). Valdivia compacta of course need not be angelic, any angelic Valdivia compact space is even clearly Corson. However, an idea in the proof of Theorem 3.5 leads us to the following definition.

4.6. Definition. Let X be a topological space and $\mathcal{R}$ a family of subsets of X. We say that X is *Fréchet–Urysohn with respect to $\mathcal{R}$* if, whenever $x \in X$, $R \in \mathcal{R}$, $x \in \overline{R}$, there is a sequence $x_n \in R$ such that $x_n \to x$.

4.7. Theorem. *Every continuous image of a super-Valdivia compact is Fréchet–Urysohn with respect to G_δ subsets.*

Proof. Let K be super-Valdivia, $x \in K$, and $G \subset K$ a G_δ set with $x \in \overline{G}$. There is a dense Σ-subset A of K containing x. By Proposition 2.2(3) and 2.2(1) there is a sequence $x_n \in G \cap A$ with $x_n \to x$. Finally, it is easy to check that spaces which are Fréchet–Urysohn with respect to G_δ sets are preserved by closed continuous mappings. ∎

4.8. Remarks and questions. (1) Theorem 4.7 generalizes the fact that Corson compacta are angelic. It is easy to check that "super-Valdivia" cannot be replaced by "Valdivia" (for example, the space $[0, \omega_1]$ is a Valdivia compactum which is not even Fréchet–Urysohn with respect to open sets). However, we do not know whether any Valdivia compact space which is Fréchet–Urysohn with respect to G_δ sets is already super-Valdivia.

(2) In view of Theorem 4.5(2) it is natural to ask whether every compact space which can be represented as the union of a Corson compactum and a super-Valdivia compactum is (super-)Valdivia.

Acknowledgements. The author would like to thank L. Zajíček for informing him about the question of M. Valdivia. Thanks are also due to P. Holický for many helpful discussions and comments.

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Department of Mathematical Analysis
Faculty of Mathematics and Physics
Charles University
Sokolovská 83
186 75 Praha 8, Czech Republic
E-mail: kalenda@karlin.mff.cuni.cz

Received 5 August 1998;
in revised form 26 March 1999

Added in proof. A partial positive answer to Quuestion 3.6 (for Banach spaces of the form $C(K)$, where K is a continuous image of a Valdivia compactum) is given in a forthcoming paper *Valdivia compacta and subspaces of* $C(K)$ *spaces*.