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Compositions of simple maps

by

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Abstract. A map (= continuous function) is *of order* $\leq k$ if each of its point-inverses has at most k elements. Following [4], maps of order ≤ 2 are called *simple*.

Which maps are compositions of simple closed [open, clopen] maps? How many simple maps are really needed to represent a given map? It is proved herein that *every closed map of order* $\leq k$ *defined on an* n*-dimensional metric space is a composition of* $(n+1)k-1$ *simple closed maps* (with metric domains). This theorem fails to be true for non-metrizable spaces. An appropriate map on a Cantor cube of uncountable weight is described.

Borsuk and Molski [4] showed that every locally one-to-one map on a compactum ([1]) (= compact metric space) is a composition of a finite number of simple maps between compacta. They asked if there exists a map of finite order which is not such a composition. Sieklucki [16] proved that *every map of finite order defined on a finite-dimensional compactum is a composition of simple maps*. He also constructed an infinite-dimensional counter-example. Dydak [6] answered an analogous question: he showed that, if p is prime, then the map $z \mapsto z^p$ on the unit complex circle is not a composition of locally one-to-one maps of order $\leq p-1$. This map is not a composition of open maps of order $\leq p-1$ (cf. Baildon [1]) either ([2]). Recently, a new proof of the Sieklucki theorem was presented in [12].

This paper aims to extend the Sieklucki theorem to arbitrary finite-dimensional metrizable spaces. (By dimension we mean the covering dimension.) We prove that *every closed map of order* $\leq k$ *with an* n*-dimensional*

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([1]) In [4, 6] such maps are called *elementary*.

([2]) Observe that, if a map $f = f_2 \circ f_1$ is a local homeomorphism, where f_1 is a map onto the domain of f_2, then the following are equivalent: (a) both maps f_1, f_2 are local homeomorphisms, (b) both are open, (c) both are locally one-to-one.

metric domain is a composition of $(n+1)k-1$ *simple closed maps,* whose domains are metric. This upper bound of the number of simple maps is better than those in [12, 16], and in the case $n = 0$ it is the best possible. For maps defined on certain "thin" one-dimensional spaces (arcs, circles, their subspaces, and others) the number is smaller than $2k-1$; namely, it is k, and—under a certain condition—even $k-1$. Also open [clopen] maps on zero-dimensional spaces are represented as compositions of simple open [clopen] maps. As an application we obtain Nagami's result concerning sharpness of the theorem on dimension-raising maps. Finally, we show that the Cantor cube $D^{\aleph_1}$ admits a clopen map of order ≤ 3 which is not composable of simple closed maps.

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0. Preliminaries. Recall the theorem on dimension-raising maps: *If* f *is a closed map of order* $\le k$ *defined on a metric space* X*, then* $\dim f(X) \le \dim X + k - 1$ (cf. Engelking [7], Theorems 4.3.1, 1.12.2). Such a map cannot lower dimension (cf. [7], Theorem 4.3.4). Thus, a simple closed map either preserves dimension or raises it by one. We shall substantially use the following theorem by Morita: *Every* n*-dimensional metric space is the image of a zero-dimensional metric space of the same weight under a closed map of order* $\le n+1$ (see [7], Theorem 4.3.15).

Let us recall that *the image of a metric space under a perfect map is metrizable* (see Engelking [8], Theorem 4.4.15). Therefore, whenever we obtain a composition of surjective simple closed maps, and the first inner domain is metrizable, then so are the subsequent ones.

Given a cover $\mathcal{D}$ of a set X, and a subset A of X, we write $\mathrm{St}(A,\mathcal{D})$ for the *star of* A *with respect to* $\mathcal{D}$, i.e. the union of all $G \in \mathcal{D}$ that intersect A. By $|A| \in \mathbb{N}\cup\{\infty\}$ we denote the number of elements in A. We write $\mathcal{D} \preceq \mathcal{A}$ when $\mathcal{D}$ refines $\mathcal{A}$; then $\mathcal{D} \prec \mathcal{A}$ means that $\mathcal{D} \preceq \mathcal{A}$ and $\mathcal{A} \ne \mathcal{D}$. A *decomposition* of X is a disjoint family of non-empty subsets of X whose union is X. The words upper-semicontinuous, lower-semicontinuous, open-and-closed are abbreviated to *u.s.c., l.s.c.,* and *clopen* respectively. For further terminology see Engelking's monographs [7, 8].

1. The zero-dimensional case. The core of this paper lies in the following special case of our main result.

1.1. THEOREM. *Every closed* [*open*, *clopen*] *map* $f : X \xrightarrow{\text{onto}} Y$ *of order* $\le k$ *defined on a zero-dimensional metric space* X *is a composition* $f_1 \circ \ldots \circ f_{k-1}$ *of* $k-1$ *surjective simple closed* [*open*, *clopen*] *maps* $f_1, \ldots, f_{k-1}$.

Moreover, *the* f_i *can be chosen so that* $(f_1 \circ \ldots \circ f_i)^{-1}(y)$ *has exactly* $\min\{i+1, |f^{-1}(y)|\}$ *elements for all* $y \in Y$ *and* $i = 1, \ldots, k-1$.

Before the proof let us make a few remarks. Firstly, as noted earlier, if the given map is closed, then the space Y and the domains of the maps $f_1, \ldots, f_{k-2}$ are metric. In the other case none of them need be Hausdorff.

We have found the best possible upper bound of the number of simple closed maps. Indeed, the cube $[0,1]^{k-1}$ is the image of the Cantor set under a closed map of order $\le k$ (cf. the Morita theorem, see also [7], Problem 1.7.F). If this map were a composition of less than $k-1$ simple closed maps, the theorem on dimension-raising maps would imply that $\dim [0,1]^{k-1} < k-1$.

The foregoing theorem is a partial answer to Baildon's problem [1]: *Which open maps of finite order are composable of simple open maps?* (Baildon meant maps between compacta. Recall that the map $z \mapsto z^3$ of the unit complex circle is not composable of simple clopen maps.)

1.2. Corollary (Nagami [14]; the separable case: Roberts [15]). *Every metric space Y with $\dim Y \le n+k-1$ is the image of a metric space Z with $\dim Z \le n$ and $w(Z) \le w(Y)$ under a closed map of order $\le k$.*

Proof. Suppose that $\dim Y = n+k-1$. The Morita theorem yields a zero-dimensional metric space X with $w(X) = w(Y)$, and a closed map $f : X \overset{\text{onto}}{\longrightarrow} Y$ of order $\le n+k$. According to Theorem 1.1, this map is a composition $X = X_{n+k} \xrightarrow{f_{n+k-1}} \ldots \xrightarrow{f_1} X_1 = Y$ of surjective simple closed maps. These can be chosen so that $f_1 \circ \ldots \circ f_{k-1}$ is of order $\le k$. The spaces $X_2, \ldots, X_{n+k-1}$ are metrizable and have the same weight as Y. The theorem on dimension-raising maps implies that each of the simple maps raises dimension by one. Hence $\dim X_k = n$. ∎

Theorem 1.1 is a consequence of the following proposition.

1.3. Theorem. *Let X be a zero-dimensional metric space, and $\mathcal{D}_1$ be its decomposition into compact subsets. Then there exist decompositions $\mathcal{D}_2, \mathcal{D}_3, \ldots$ of X into non-empty compact subsets such that*:

(a) *$\mathcal{D}_{n+1}$ refines $\mathcal{D}_n$ for $n \ge 1$.*

(b) *Each set $G \in \mathcal{D}_1$ is covered by exactly $\min\{n, |G|\}$ members of $\mathcal{D}_n$.*

(c) *For every decreasing sequence of sets $G_n \in \mathcal{D}_n$ the intersection $\bigcap_{n\in\mathbb{N}} G_n$ is a single point.*

(d) *If the decomposition $\mathcal{D}_1$ is u.s.c., then so are $\mathcal{D}_n$.*

(e) *If $\mathcal{D}_1$ is l.s.c., then so are $\mathcal{D}_n$.*

Proof of Theorem 1.1. Suppose that $f : X \overset{\text{onto}}{\longrightarrow} Y$ is a map of order $\le k$ defined on a zero-dimensional metric space X. Consider the decomposition $\mathcal{D}_1$ of X into the point-inverses under f, and take the decompositions $\mathcal{D}_2, \mathcal{D}_3, \ldots$ described in Theorem 1.3. We can identify the spaces Y and $X/\mathcal{D}_1$. The assertion (b) of Theorem 1.3 ensures that $\mathcal{D}_k$ consists of singletons, so we identify X and $X/\mathcal{D}_k$. Let $f_n : X/\mathcal{D}_{n+1} \to X/\mathcal{D}_n$ be the quotient projection, i.e. it assigns a set $G \in \mathcal{D}_n$ to each member of $\mathcal{D}_{n+1}$ contained in G. The map f is a composition of the maps $f_1, \ldots, f_{k-1}$ which have the desired properties. ∎

In order to prove Theorem 1.3, we need some preparations. Recall that the *Baire space* $B(\mathfrak{m})$ is the set of all sequences $(x_n)_{n\in\mathbb{N}}$ in a fixed set of cardinality $\mathfrak{m}$; the set $B(\mathfrak{m})$ is equipped with the metric given by

$$\varrho[(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}] = \max\{1/n : x_n \neq y_n\}$$

for any pair of different sequences (cf. [8], Example 4.2.12). Since each Baire space $B(\mathfrak{m})$ is universal for the class of all zero-dimensional metric spaces of weight $\leq \mathfrak{m}$ (see [8], Theorems 7.1.10, 7.3.15), we may assume that the space X of Theorem 1.3 is contained in a $B(\mathfrak{m})$.

1.4. Lemma. *There exists a well-ordered family of decompositions $\mathcal{A}_\alpha$ of $B(\mathfrak{m})$ into non-empty clopen subsets such that*:

(a) $B(\mathfrak{m})$ *is the only member of the first decomposition.*

(b) *In each $\mathcal{A}_\alpha$ there is exactly one member which is the union of two members in $\mathcal{A}_{\alpha+1}$. The other members of $\mathcal{A}_\alpha$ also belong to $\mathcal{A}_{\alpha+1}$.*

(c) *If α is a limit ordinal, and if a sequence of sets $E_\beta \in \mathcal{A}_\beta$, $\beta < \alpha$, is decreasing, then $\bigcap_{\beta<\alpha} E_\beta$ belongs to $\mathcal{A}_\alpha$.*

(d) *All the families $\mathcal{A}_\alpha$ together form a base of the topology for $B(\mathfrak{m})$.*

Proof. In $B(\mathfrak{m})$ two balls of the same radius are either equal or disjoint. Hence, for each $n \in \mathbb{N}$ the family $\mathcal{A}_n^0$ of all such balls of radius $1/n$ is a decomposition of $B(\mathfrak{m})$. Put $\mathcal{A}_0^0 = \{B(\mathfrak{m})\}$; write δ for the least ordinal of cardinality $\mathfrak{m}$. We shall complete the sequence $\mathcal{A}_0^0, \mathcal{A}_1^0, \mathcal{A}_2^0 \ldots$ with decompositions $\mathcal{A}_n^\xi$, $\xi < \delta$, so that the following sequence (read line after line) has the desired properties:

$$\begin{gathered}\mathcal{A}_0^0, \mathcal{A}_0^1, \ldots, \mathcal{A}_0^\xi, \ldots, \\ \mathcal{A}_1^0, \mathcal{A}_1^1, \ldots, \mathcal{A}_1^\xi, \ldots, \\ \mathcal{A}_2^0, \mathcal{A}_2^1, \ldots, \mathcal{A}_2^\xi, \ldots, \\ \vdots\end{gathered}$$

Take $n \in \mathbb{N}$. For every set $E \in \mathcal{A}_n^0$ choose a ball in $\mathcal{A}_{n+1}^0$ which is contained in E. Arrange all the remaining balls from $\mathcal{A}_{n+1}^0$ in a transfinite sequence $A_0, A_1, \ldots, A_\xi, \ldots,\ \xi < \delta$. Then define $\mathcal{A}_n^\xi$ to consist of all the balls A_τ for $\tau < \xi$, and of the sets $E \setminus \bigcup_{\tau<\xi} A_\tau$ for all $E \in \mathcal{A}_n^0$. ∎

With a view to proving the upper-semicontinuity, we shall exercise a condition which is equivalent to it in some cases. Given a decomposition $\mathcal{D}_n$ of the space X in Theorem 1.3, we shall check that the set $\bigcup_{G\in\mathcal{D}_n} G \times G$ is closed in $X \times X$. Clearly, this union is the graph of the multivalued function which assigns $G \in \mathcal{D}_n$ to each point $x \in G$.

1.5. PROPOSITION. *Let $f : X \xrightarrow{\text{onto}} Y$ be a map. If Y is a Hausdorff space, then $\bigcup_{y \in Y} f^{-1}(y) \times f^{-1}(y)$ is closed in $X \times X$. The converse is true whenever f is open.* ■

The following proposition results from a theorem on multivalued functions (cf. [2], Chapter VI, Theorem 7). A direct proof is also straightforward.

1.6. PROPOSITION. *Let $\mathcal{A}$ be an u.s.c. decomposition of a Hausdorff space into compact subsets, and let a decomposition $\mathcal{D}$ refine $\mathcal{A}$. If $\bigcup_{G \in \mathcal{D}} G \times G$ is closed in $X \times X$, then $\mathcal{D}$ is u.s.c.* ■

Proof of Theorem 1.3. (I) Let X be a zero-dimensional metric space, and $\mathcal{D}_1$ be its decomposition into compact subsets. We regard X as a subspace of $B(\mathfrak{m})$. Fix a well-ordered sequence of decompositions $\mathcal{A}_\alpha$ of $B(\mathfrak{m})$, where $\alpha < \gamma$, described in Lemma 1.4. The conditions (b) and (c) imply that, if $\alpha < \beta$, then $\mathcal{A}_\beta$ refines $\mathcal{A}_\alpha$. Write $\mathcal{A}_\gamma$ for the family of all singletons.

Fix $G \in \mathcal{D}_1$. Each decomposition $\mathcal{A}_\alpha$, $\alpha \le \gamma$, induces the decomposition of G that consists of all non-empty sets $G \cap E$, $E \in \mathcal{A}_\alpha$. We can arrange all the induced decompositions in the following sequence (let us agree on $\infty + 1 = \infty$):

$$\mathcal{D}_1^G \succ \mathcal{D}_2^G \succ \ldots \succ \mathcal{D}_{|G|}^G = \mathcal{D}_{|G|+1}^G = \ldots = \mathcal{D}_\infty^G,$$

where $\mathcal{D}_\infty^G$ is the decomposition of G into singletons.

For each $n \in \mathbb{N}$ we define the decomposition $\mathcal{D}_n$ of X as the union of all the families $\mathcal{D}_n^G$, $G \in \mathcal{D}_1$. The properties of the sequence $(A_\alpha)_{\alpha<\gamma}$ guarantee that the assertions (a)–(c) of Theorem 1.3 hold.

(II) We claim that, *if the decomposition $\mathcal{D}_1$ is u.s.c., then each union $\bigcup_{G \in \mathcal{D}_n} G \times G$ is closed in $X \times X$.* To prove this, take arbitrary points $x_0, y_0 \in X$ in different members of $\mathcal{D}_n$. We shall indicate neighbourhoods $U_0 \ni x_0$, $V_0 \ni y_0$ (subsets open in X) such that any two points $x \in U_0$, $y \in V_0$ also belong to different members of $\mathcal{D}_n$. Proposition 1.5 yields that the complement of $\bigcup_{G \in \mathcal{D}_1} G \times G$ is open. Therefore the essential case is when x_0, y_0 lie in the same $G_0 \in \mathcal{D}_1$.

The family $\mathcal{D}_n^{G_0}$ has $m = \min\{n, |G_0|\}$ members, and is induced by a decomposition $\mathcal{A}_\alpha$, where $\alpha < \gamma$. Hence $G_0 \subset \bigcup_{i=1}^m E_i$, where $E_i \in \mathcal{A}_\alpha$ for $i = 1, \ldots, m$; and $x_0 \in E_j$, $y_0 \in E_k$ for some $j, k \in \{1, \ldots, m\}$, $j \ne k$. Write W for the union of all $G \in \mathcal{D}_1$ contained in $\bigcup_{i=1}^m E_i$. As $\mathcal{D}_1$ is u.s.c., the set $W \ni x_0, y_0$ is open.

Take $x \in W \cap E_j = U_0$, $y \in W \cap E_k = V_0$, and assume that x, y belong to $G \in \mathcal{D}_1$. By the definition of W we have $G \subset \bigcup_{i=1}^m E_i$, so $\mathcal{A}_\alpha$ induces in G a decomposition into $l \le m$ members. This is $\mathcal{D}_l^G$. Since $\mathcal{D}_n \preceq \mathcal{D}_l$, the points x, y belong to different members of $\mathcal{D}_n$.

Proposition 1.6 and the foregoing claim yield the assertion (d).

(III) We proceed to show (e). Assume that $\mathcal{D}_1$ is l.s.c., and consider an open set $U \subset X$. We shall prove that every point in $\mathrm{St}(U, \mathcal{D}_n)$ has a neighbourhood contained in $\mathrm{St}(U, \mathcal{D}_n)$.

Let $x_0 \in \mathrm{St}(U, \mathcal{D}_n)$. There is a $y_0 \in U$ in the member of $\mathcal{D}_n$ that includes x_0. It sufices to consider the case $y_0 \neq x_0$. Assume that $x_0, y_0 \in G_0 \in \mathcal{D}_1$. The family $\mathcal{D}_n^{G_0}$ consists of exactly n members, or else it would contain only singletons. It is induced by a decomposition $\mathcal{A}_\alpha$, where $\alpha < \gamma$. Hence $G_0 \subset \bigcup_{i=1}^n E_i$, where $E_i \in \mathcal{A}_\alpha$ and $G_0 \cap E_i \neq \emptyset$ for $i = 1, \ldots, n$. Moreover, $x_0, y_0 \in E_j$ for some j. Since $\mathcal{D}_1$ is l.s.c., the following set is an open neighbourhood of the set G_0:

$$W = \mathrm{St}(U \cap E_j, \mathcal{D}_1) \cap \bigcap_{i=1}^{n} \mathrm{St}(E_i, \mathcal{D}_1).$$

Choose $x \in W \cap E_j$, and assume that $x \in G \in \mathcal{D}_1$. By the definition of W there is a point $y \in G \cap U \cap E_j$, and $\mathcal{A}_\alpha$ induces in G at least n members of a decomposition $\mathcal{D}_m^G$, $m \geq n$. As $\mathcal{D}_m \preceq \mathcal{D}_n$, the points x, y belong to a member of $\mathcal{D}_n$, i.e. $x \in \mathrm{St}(U, \mathcal{D}_n)$. ∎

In fact, it is possible to prove the existence of decompositions like those in Theorem 1.3 in a much more general situation. To prove Lemma 1.4 we needed a sequence of decompositions of $B(\mathfrak{m})$ into clopen sets (we used the families of balls of radii $1/n$). The reader perhaps knows that the existence of such a well-ordered transfinite sequence is characteristic of zero-dimensional *linearly uniformizable* (another name: *ω_μ-metrizable*) spaces ([3]).

Further, it suffices to assume (instead of the zero-dimensionality and metrizability of X) that there exists a map $\pi : X \to T$ into a zero-dimensional linearly uniformizable space T, and that the restriction $\pi|G$ is one-to-one for each $G \in \mathcal{D}_1$. In the foregoing proof the sets $E, E_i \in \mathcal{A}_\alpha$ should be replaced by the preimages $\pi^{-1}(E)$, $\pi^{-1}(E_i)$. The assertion (d) of Theorem 1.3 should be replaced by

(d′) *If X is Hausdorff, and if the decomposition $\mathcal{D}_1$ is u.s.c., then so are $\mathcal{D}_n$.*

The effect is that also Theorem 1.1 can be generalized:

([3]) A space Z is called *linearly uniformizable* when its topology comes from a uniformity with a linearly ordered base (with respect to inclusion if uniformity means neighbourhoods of the diagonal, or with respect to refinement if uniformity consists of covers). Such a uniformity also has a well-ordered base of some regular ordinal type ω_μ. If $\mu = 0$, then X is metrizable; if $\mu > 0$, X is either discrete, or non-metrizable and zero-dimensional. In case it is zero-dimensional, the uniformity has a well-ordered base of decompositions into clopen subsets. Hušek and Reichel's paper [11] contains ample bibliographical and historical notes. See also: Frankiewicz and Kulpa [9], Kucia and Kulpa [13].

1.7. Theorem. *Let f be a surjective closed* [*open*, *clopen*] *map of order $\le k$ defined on a Hausdorff* [*arbitrary*, *Hausdorff*] *space X. Suppose that there exists a zero-dimensional linearly uniformizable space T with a map $\pi : X \to T$ such that the map $x \mapsto (f(x), \pi(x))$ is one-to-one. Then the conclusion of Theorem* 1.1 *is satisfied.* ■

2. The finite-dimensional case. We shall obtain compositions $X_{n+1} \xrightarrow{f_n} X_n \xrightarrow{f_{n-1}} \ldots \xrightarrow{f_1} X_1$ with the property that

(∗) *The map f_1 is simple, the maps $f_2, \ldots, f_n$ are surjective, and each point-inverse $(f_1 \circ \ldots \circ f_{i+1})^{-1}(x)$ has at most $|(f_1 \circ \ldots \circ f_i)^{-1}(x)| + 1$ elements, for $x \in X_1$ and $i = 1, \ldots, n-1$.*

It is easily seen that the compositions obtained in Theorems 1.1 and 1.7 satisfy this condition.

The following propositions are obvious.

2.1. Proposition. (a) *If $f_1, \ldots, f_n$ has the property* (∗), *then each f_i is simple and*, *moreover*, *each composition of k successive maps f_i is of order $\le k+1$.*

(b) *If a map is composable of $n+k-2$ closed* [*open*, *clopen*] *maps with the property* (∗), *then it is also composable of two closed* [*open*, *clopen*] *maps*: *the first one of order $\le n$, and the second of order $\le k$.*

(c) *If a map is composable of $(n-1)k$ closed* [*open*, *clopen*] *maps with* (∗), *then it is composable of k closed* [*open*, *clopen*] *maps of order $\le n$.* ■

Given a map f defined on a space Y, let

$$E_k(f) = \{y \in Y : |f^{-1}f(y)| \ge k\}.$$

We need an instrument that will enable the transfer of the method for map decomposition to higher dimensions. Using the following theorem together with Morita's theorem, we obtain our main result.

2.2. Theorem. *Let f be a closed* [*open*] *map. Suppose that there exist a zero-dimensional metric space X and a closed* [*open*] *map $\varphi : X \xrightarrow{\text{onto}} E_2(f)$ such that the composition $f \circ \varphi$ is of order $\le k$. Then f is a composition of $k-1$ simple closed* [*open*] *maps with the property* (∗).

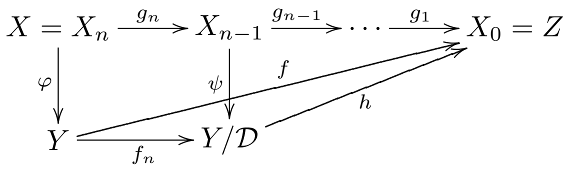
2.3. Main Corollary. *Every closed map f of order $\le k$ whose domain or*, *more generally*, *whose set $E_2(f)$ is n-dimensional and metrizable is a composition of $(n+1)k-1$ simple closed maps that has the property* (∗). ■

Theorem 2.2 follows from Theorem 1.1 and Lemmata 2.4 and 2.5.

2.4. Lemma. *Let $\varphi : X \xrightarrow{\text{onto}} Y$ and $f : Y \to Z$ be closed* [*open*, *clopen*] *maps. If $f \circ \varphi$ is a composition of n closed* [*open*, *clopen*] *maps with the property* (∗), *then so is f.*

P r o o f. The proof is by induction on n. If $n = 1$, i.e. $f \circ \varphi$ is a simple map, then f is simple as well.

Given $n > 1$, assume that $g = f \circ \varphi$ is a composition $X = X_n \xrightarrow{g_n} X_{n-1} \xrightarrow{g_{n-1}} \ldots \xrightarrow{g_1} X_0 = Z$ which satisfies $(*)$. This property implies that each fibre $g^{-1}(z)$ contains at most one pair of distinct elements x_z, y_z such that $g_n(x_z) = g_n(y_z)$. Let $\mathcal{D}$ be the decomposition of Y into all the pairs $\{\varphi(x_z), \varphi(y_z)\}$, when such a pair exists for $z \in Z$, and the remaining singletons. Write f_n for the quotient map $Y \to Y/\mathcal{D}$. If g_n is closed or open, then X_{n-1} is the quotient space of the decomposition of X_n into the fibres under g_n. Since g_n is finer than $f_n \circ \varphi$, there is exactly one map $\psi\colon X_{n-1} \xrightarrow{\text{onto}} Y/\mathcal{D}$ such that $f_n \circ \varphi = \psi \circ g_n$. Likewise, there is exactly one $h\colon Y/\mathcal{D} \xrightarrow{\text{onto}} Z$ such that $f = h \circ f_n$. It is best to draw the diagram:



Check that, if a point $y \in Y$ belongs to $G \in \mathcal{D}$, then

$$G = \varphi g_n^{-1} g_n \varphi^{-1}(y).$$

Hence, for every $F \subset Y$ the set $\varphi g_n^{-1} g_n \varphi^{-1}(F)$ equals $\mathrm{St}(F, \mathcal{D})$. This is why the decomposition $\mathcal{D}$ is u.s.c. [l.s.c.] whenever the given maps are closed [open]. Then also f_n, ψ are closed [open], and so is h.

By the induction hypothesis the map h is a composition of $n - 1$ closed [open, clopen] maps with $(*)$. This completes the proof, for our construction ensures that f_n identifies only the points $\varphi(x_z), \varphi(y_z)$ in the preimage $f^{-1}(z)$. ∎

2.5. Lemma. *Let f be a closed* [*open*, *clopen*] *map. If $f|E_2(f)$ is a composition of n simple closed* [*open*, *clopen*] *maps*, *then so is f. Moreover*, *if the given composition has the property* $(*)$, *then so does the resulting one.*

P r o o f. Let Y denote the domain of f, and $\mathcal{D}$ the decomposition of Y into the fibres under f. We claim that, *if an u.s.c.* [*l.s.c.*] *decomposition $\mathcal{A}$ of $E_2(f)$ refines $\mathcal{D}$ on $E_2(f)$, then the decomposition of Y into all the members of $\mathcal{A}$ and all the remaining single points in $Y \setminus E_2(f)$ is u.s.c.* [*l.s.c.*].

Indeed, write $\mathcal{B}$ for this new decomposition of Y. Take a closed [open] set $F \subset Y$. Since $\mathcal{A}$ is semicontinuous, there is a closed [open] $G \subset Y$ such that $\mathrm{St}(F, \mathcal{A}) = G \cap E_2(f)$. The following formula implies the semicontinuity of $\mathcal{B}$:

$$\mathrm{St}(F, \mathcal{B}) = F \cup [G \cap E_2(f)] = F \cup [G \cap \mathrm{St}(F, \mathcal{D})].$$

The lemma follows, for there is a one-to-one correspondence between closed [open] maps on Y and u.s.c. [l.s.c.] decompositions of Y. ∎

There is also the finite-dimensional analogue of Theorem 1.7.

2.6. THEOREM. *Let f be a closed map of order $\le k$ defined on a Hausdorff space. Suppose that there are an n-dimensional metric space T and a map $\pi : E_2(f) \to T$ such that the map $x \mapsto (f(x), \pi(x))$ is one-to-one. Then f is a composition of $(n+1)k-1$ simple closed maps with the property* $(*)$.

P r o o f. Theorem 3.7.9 of [8] implies that the map $y \mapsto (f(y), \pi(y))$ embeds $E_2(f)$ into the product $f(E_2(f)) \times T$. So, write $Z = f(E_2(f))$, and assume that $E_2(f)$ is a subset of $Z \times T$. According to the Morita theorem, there is a zero-dimensional metric space S, and there is a closed map $\varphi : S \xrightarrow{\text{onto}} T$ of order $\le n+1$. Let

$$X = \{(z,s) \in Z \times S : (z, \varphi(s)) \in E_2(f)\}.$$

The function $\psi : X \xrightarrow{\text{onto}} E_2(f)$ given by $\psi(z,s) = (z, \varphi(s))$ is a closed map of order $\le n+1$. It suffices to apply Theorem 1.7 to the map $g = f|E_2(f) \circ \psi$, and then apply Lemmata 2.4 and 2.5. ∎

3. A particular case in dimension one. Corollary 2.3 seems to overestimate the number of simple maps needed for representation. Examples 4.3 indicate to a degree what upper bound of this number may be expected. For maps defined on a space Y with the following property (γ) we are able to improve our estimation fairly easily:

(γ) *For any boundary set $B \subset Y$ there exist a subspace X of the Cantor set and a simple closed map $\varphi : X \xrightarrow{\text{onto}} Y$ such that each inverse $\varphi^{-1}(y)$ of a point $y \in B$ is a singleton.*

This property is hereditary. Such spaces Y are separable, metrizable, and at most one-dimensional. It is an exercise to show that *the segment, the cirle, and—more generally—finite graphs satisfy* (γ) (4).

3.1. THEOREM. *Let f be a closed map of order $\le k$ defined on a space Y with the property (γ). Then there exists a subset X of the Cantor set with a simple closed map $\varphi : X \xrightarrow{\text{onto}} Y$ such that the composition $f \circ \varphi$ is of order $\le k+1$.*

If, moreover, the interior of the set $E_k(f)$ is discrete, then X and φ can be chosen so that $f \circ \varphi$ is of order $\le k$.

P r o o f. Fix a countable base of Y. Let $(U_n)_{n \in \mathbb{N}}$ be a sequence of all infinite sets in this base. Since f is of finite order, we can choose a sequence of points $a_n \in U_n$ such that the values $f(a_n)$ are all distinct. We apply (γ)

(4) Actually, the condition (γ) is known in another form. It is possible to prove that *a compactum satisfies (γ) if and only if it contains no non-degenerate nowhere dense continuum.* This is eqivalent to Hurewicz's property (α) ([10], p. 74) in the case of one-dimensional compacta.

to the set B of all non-isolated points in $Y \setminus \{a_n\}_{n\in\mathbb{N}}$. There is a subset A of the Cantor set, and there is a simple closed map $\varphi : A \xrightarrow{\text{onto}} Y$ such that each double value under φ is either isolated or in the set $\{a_n\}_{n\in\mathbb{N}}$. Having any isolated double value, we remove a point from its preimage, and, in this way, a closed set X is left in A. The restriction $\varphi|X$ is closed, and the composition $f \circ \varphi|X$ is of order $\le k+1$.

If the interior of $E_k(f)$ is discrete, we can choose the points a_n either isolated or outside $E_k(f)$. Then we obtain a composition $f\circ\varphi$ of order $\le k$. ∎

The following results from Theorems 2.2 and 3.1 ([5]).

3.2. Corollary. *Every closed map f of order $\le k$ defined on a space that satisfies the condition (γ) is composable of k simple closed maps with the property $(*)$. In case the interior of $E_k(f)$ is discrete*, $k-1$ *simple closed maps suffice.* ∎

3.3. Example. The map $z \overset{f}{\mapsto} z^3$ of the unit complex circle $\mathbb{S}_1$ *is not composable of two simple closed maps.* Indeed, suppose that $f = h \circ g$, where g is a simple map into a Hausdorff space. Let $\varepsilon = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$. For each $z \in \mathbb{S}_1$ we have $f^{-1} f(z) = \{z, \varepsilon z, \varepsilon^2 z\}$. Let $F = \{z \in \mathbb{S}_1 : g(z) = g(\varepsilon z)\}$. The sets F, εF, $\varepsilon^2 F$ are closed, and pairwise disjoint as g is simple. Since $\mathbb{S}_1$ is connected, these sets do not cover it. Hence there is $z \in \mathbb{S}_1$ such that $g(z)$, $g(\varepsilon z)$, $g(\varepsilon^2 z)$ are different. Thus h is not simple. ∎

The foregoing example shows that the assumption about $E_k(f)$ in Corollary 3.2 is essential for maps defined on the circle. Is any assumption like this needed in the case of maps on the segment (in order to obtain $k-1$ simple maps)? Does the segment differ from zero-dimensional spaces concerning decomposition of maps into simple ones? The answer to the latter question is "yes". Theorem 1.7 differentiates these spaces. Namely, in [12] we described a finite graph $K \subset \mathbb{R}^2 \times [0,1]$ such that, if we restrict the projection $\mathbb{R}^2 \times [0,1] \to \mathbb{R}^2$ to K, then we obtain a three-to-one map which

([5]) It is worth adding that Theorem 3.1 implies a very special theorem on dimension-raising maps: *If f is a closed map of order $\le k$ $(k \ge 2)$ defined on a complete separable metric space Y with (γ), then* $\dim f(Y) \le k-1$ (cf. Hurewicz [10], Theorem II; also cf.: Bognár [3], Dębski and Mioduszewski [5]). Indeed, there exists a closed subspace $G \subset Y$ such that $f(Y) = f(G)$ and the restriction $f|G$ is *irreducible*, i.e. no proper closed subset $H \subset G$ is carried onto $f(G)$ (cf. [8], Exercise 3.1.C). The set $E_2(f|G)$ is a boundary set: Let $U = U_0$ be a non-empty set open in G. By induction we define non-empty open sets $U_n \subset G$ such that each U_n has diameter less than $1/n$, and

$$U_n \subset \overline{U}_n \subset (f|G)^{-1}[f(G) \setminus f(G \setminus U_{n-1})] \subset U_{n-1}.$$

The only point in $\bigcap_{n\in\mathbb{N}} U_n$ is not in $E_2(f|G)$. Then, by Theorem 3.1 we find a closed subspace X of the Cantor set and a map $\varphi : X \xrightarrow{\text{onto}} G$ such that $f \circ \varphi$ is of order $\le k$. Thus our assertion follows, as X is zero-dimensional and $f \circ \varphi$ is onto $f(Y)$.

is not composable of two simple closed maps. Therefore, $[0,1]$ cannot be the space T in Theorem 1.7. However, there remains

3.4. QUESTION. *Does the segment admit a closed map of order ≤ 3 [of order $\le k$] that is not a composition of two [of $k-1$] simple closed maps?*

4. A counter-example on the Cantor cube $D^{\aleph_1}$. The purpose of this section is to prove that the map $q_F : D^{\aleph_1} \overset{\text{onto}}{\longrightarrow} D^{\aleph_1}/\mathcal{A}_F$ defined in Example 4.2 is not a finite composition of simple closed maps.

Let $G : X \to X$ be a *periodic* homeomorphism, i.e. $G^k = \mathrm{Id}_X$ for a certain k. We shall write $\mathcal{A}_G$ for the decomposition of X into the orbits $\{G^n(x) : n = 1, \ldots, k\}$ of points $x \in X$, and q_G for the natural quotient map $X \to X/\mathcal{A}_G$ that carries a point to its orbit. $\mathcal{A}_G$ and q_G will be called *associated with* G.

4.1. PROPOSITION. *If G is a periodic homeomorphism, then the associated decomposition $\mathcal{A}_G$ is continuous, and the associated map q_G is clopen.* ∎

4.2. EXAMPLE. Having $(D^{\aleph_1})^3 \overset{\text{top}}{=} D^{\aleph_1}$ in mind, we shall define q_F on the former. Let $F : (D^{\aleph_1})^3 \to (D^{\aleph_1})^3$ be the homeomorphism given by

$$F(x,y,z) = (z,x,y) \quad \text{for } x,y,z \in D^{\aleph_1}.$$

Clearly, $F^3 = \mathrm{Id}_{(D^{\aleph_1})^3}$. The associated quotient map q_F is of order ≤ 3 and clopen. Its image has a clopen base of cardinality $\aleph_1$, and hence can be embedded into $D^{\aleph_1}$ (cf. [8], Theorem 6.2.16). ∎

To prove that q_F has the desired property we need Sieklucki's examples:

4.3. EXAMPLE. Consider the complex space $\mathbb{C}^n$ and the unit sphere

$$S_{2n-1} = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : |z_1|^2 + \ldots + |z_n|^2 = 1\}.$$

Define the isometry $G_n : S_{2n-1} \overset{\text{onto}}{\longrightarrow} S_{2n-1}$ by

$$G_n(z_1, \ldots, z_n) = e^{2\pi i/3} \cdot (z_1, \ldots, z_n).$$

We have $G_n^3 = \mathrm{Id}_{S_{2n-1}}$. The quotient map q_{G_n} is of order ≤ 3 and clopen; $S_{2n-1}/\mathcal{A}_{G_n}$ is a compactum. Sieklucki proved that q_{G_n} *is not a composition of $2n$ simple closed maps* ([16], Theorem 2).

Sieklucki's infinite-dimensional counter-example is the sum $\bigoplus_{n=1}^{\infty} S_{2n-1}$ compactified by adding a point p "at infinity". Write S for this space, and define $G : S \overset{\text{onto}}{\longrightarrow} S$ by

$$G(z) = \begin{cases} G_n(z) & \text{for } z \in S_{2n-1}, \\ p & \text{for } z = p. \end{cases}$$

It is readily seen that, although clopen and of order ≤ 3, *the associated quotient map q_G is not a finite composition of simple closed maps.*

We shall combine Dowker's example (see [8], Example 6.2.20) with Sieklucki's foregoing example to obtain an auxiliary space T with a clopen base and with a clopen map q_H of order ≤ 3 which is not composable of simple closed maps. Then we shall embed T into $D^{\aleph_1}$ so that q_F be an extension of q_H.

4.4. Example. We follow the notation of Example 4.3. The space S is representable as the union of an increasing transfinite sequence $(P_\gamma)_{\gamma<\omega_1}$ of zero-dimensional subspaces (Smirnov [17]; see [7], Problem 1.8.J). We can assert that $G(P_\gamma) = P_\gamma$ for each $\gamma < \omega_1$. Let W denote the space of all the ordinals $\le \omega_1$ with the order topology. Consider the product $W \times S$ and its subspaces:

$$T_\alpha = \bigcup_{\gamma\le\alpha} (\{\gamma\} \times P_\gamma), \quad T = \bigcup_{\alpha<\omega_1} T_\alpha, \quad \text{and} \quad T^* = T \cup (\{\omega_1\} \times S).$$

Since $T_\alpha \subset W \times P_\alpha$ for $\alpha < \omega_1$, each T_α has a clopen base. So does T. The homeomorphism $H^* : T^* \to T^*$ of period 3 is defined by

$$H^*(\alpha, z) = (\alpha, G(z)) \quad \text{for } (\alpha, z) \in T^* \subset W \times S.$$

Finally, we define $H : T \to T$ as the restriction of H^* to T. ∎

4.5. Zarelua Lemma (see [7], Lemma 3.3.6). *Let $g : X \to Y$ be a closed map of a completely regular space X to a normal space Y. If g is of order $\le k$, then so is the extension $\beta g : \beta X \to \beta Y$ of g over the Čech–Stone compactifications.* ∎

4.6. Lemma. *Under the notation of Example* 4.4, *the associated quotient map q_H is not a finite composition of simple closed maps.*

Proof. As in [8], Example 6.2.20, we argue that the spaces T and T^* are normal, and that $\beta T^* \supset T$ is the Čech–Stone compactification of T.

Suppose that q_H is a composition of simple closed maps $T = X_{n+1} \xrightarrow{g_n} X_n \xrightarrow{g_{n-1}} \ldots \xrightarrow{g_1} X_1 = T/\mathcal{A}_H$. We can assume that they are surjective. The spaces $X_1, \ldots, X_n$ are normal as closed images of the normal space T. According to the Zarelua lemma, the extensions $\beta g_i : \beta X_{i+1} \xrightarrow{\text{onto}} \beta X_i$ are simple for $i = 1, \ldots, n$. Obviously $\beta q_{H^*} = \beta q_H = \beta g_1 \circ \ldots \circ \beta g_n$. However, βT contains a copy of the space S, namely, $\{\omega_1\} \times S \subset T^*$. The restriction of βq_{H^*} to this copy is composable of closed restrictions of the maps βg_i. A contradiction, as this is just the map q_G of Example 4.3, which is not composable of simple closed maps. ∎

4.7. Theorem. *The clopen map $q_F : D^{\aleph_1} \xrightarrow{\text{onto}} D^{\aleph_1}/\mathcal{A}_F$ of order ≤ 3 defined in Example* 4.2 *is not a composition of any finite number of simple closed maps.*

P r o o f. Referring to T and H, as defined in Example 4.4, we claim that *there exists an embedding* $\eta : T \hookrightarrow (D^{\aleph_1})^3$ *such that* $F \circ \eta = \eta \circ H$. Indeed, T has a clopen base of cardinality $\aleph_1$. Hence there exists a homeomorphic embedding $\delta : T \hookrightarrow D^{\aleph_1}$ (see [8], Theorem 6.2.16). Define $\eta : T \to (D^{\aleph_1})^3$ by

$$\eta = (\delta\,, \delta \circ H^2, \delta \circ H).$$

The desired equality is easily checked:

$$F \circ (\delta\,, \delta \circ H^2, \delta \circ H) = (\delta \circ H, \delta, \delta \circ H^2) = (\delta, \delta \circ H^2, \delta \circ H) \circ H.$$

According to the above claim, we can assume that T is a subspace of $(D^{\aleph_1})^3$, and that $F|T = H$. Thus $q_H = q_F|T$. If q_F were composable of simple closed maps, so would be q_H. This would contradict Lemma 4.6. ∎

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