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Atomic compactness for reflexive graphs

by

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Abstract. A first order structure $\mathfrak{M}$ with universe M is atomic compact if every system of atomic formulas with parameters in M is satisfiable in $\mathfrak{M}$ provided each of its finite subsystems is. We consider atomic compactness for the class of reflexive (symmetric) graphs. In particular, we investigate the extent to which "sparse" graphs (i.e. graphs with "few" vertices of "high" degree) are compact with respect to systems of atomic formulas with "few" unknowns, on the one hand, and are pure restrictions of their Stone–Čech compactifications, on the other hand.

1. Introduction. The definition of atomic compactness goes back to Węglorz [12]. It extends to general first order structures the notion of equational compactness, first considered by I. Kaplansky and J. Łoś for Abelian groups, and later by J. Mycielski for general algebras (see [6]).

Graphs first arose in this context in connection with *Mycielski's question* [6]: *Is every atomic compact algebra a retract of a topologically compact algebra?* This question was settled (negatively) by W. Taylor [8], who produced an atomic compact anti-reflexive graph, the Stone–Čech compactification of which is not anti-reflexive.

The graphs that we consider in this paper are reflexive. Reflexive graphs (and more generally reflexive binary relations) are of particular interest in the study of the relationship between atomic compactness and projection properties (see [3, 13]). The assumption of reflexivity forces a certain behaviour with respect to atomic compactness, related to the fact that there are "many" homomorphisms between reflexive relations; for instance, all constant mappings are homomorphisms, so that in particular, such structures are always weakly atomic compact (i.e. compact with respect to parameter-free systems of atomic formulas).

Here we shall mainly be concerned with systems of atomic formulas for sparse graphs. Section 4 is devoted to the study of (almost) locally finite

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graphs. It is established that a graph with no more than two vertices of infinite degree is always topologically compact, which may no longer be the case with three such vertices; however, a graph with finitely many such vertices is always compact with respect to systems of atomic formulas with finitely many unknowns. Incidentally, we consider a simple process for reducing the systems of atomic formulas, which will also be useful in the subsequent section, where we deal with the threshold of atomic compactness with respect to the number of unknowns. In Section 6, we focus on systems coming from Stone–Čech compactifications. We prove that *every graph with a finite essential degree is pure in its Stone–Čech compactification* (Theorem 6.1); its proof relies on a lemma, concerning the sets of coincidence of finitely many mappings, that may be interesting for its own sake (Lemma 6.2). Then we turn to Mycielski's question; from its negative answer given by Taylor for anti-reflexive graphs, we infer the same answer in our setting (Theorem 6.2). In the last section, we introduce the notion of admissible compactness, which arises when studying projection properties.

2. Preliminaries. Let us fix our notation and recall some basic definitions and general facts about atomic compactness; they appear in [6, 7, 12]. See also the survey [14], and [5, 10, 11] for further related results.

Given a structure $\mathfrak{M} = (M; \ldots)$ with *universe* M, over a first order language $\mathcal{L}$, an *atomic formula with parameters* in M is any formula $t_1 \equiv t_2$ (*equality formula*) or $r(t_1, \ldots, t_n)$, where r is a relation symbol and $t_1, \ldots, t_n$ are terms of the language $\mathcal{L}_M$, obtained from $\mathcal{L}$ by the adjunction of the elements of M as new constant symbols; *positive formulas* are built up from atomic ones using the connectives $\wedge$, $\vee$ and the quantifiers $\exists$, $\forall$; $(\exists, \wedge)$-*formulas* are obtained using only $\wedge$ and $\exists$. A *sentence* is a formula with no free variable.

A *system* of $\mathfrak{M}$ with *unknowns* in X is any set $\Phi(X)$ of formulas with parameters in M and with free variables in X; note that the set of unknowns may be uncountably infinite. The system Φ is an *atomic*, a *positive* or a $(\exists, \wedge)$-*system* if its elements are respectively atomic, positive or $(\exists, \wedge)$-formulas; Φ^{at} and Φ^{+} will denote respectively the sets of atomic and positive elements of Φ. Given cardinals κ and λ, Φ is a κ-*system* (resp. a κ^{-}-*system*) if it has at most (resp. strictly less than) κ elements, it is a $[\lambda]$-*system* (resp. a $[\lambda^{-}]$-*system*) if it has at most (resp. strictly less than) λ unknowns. We shall freely combine these various prefixes.

Given a structure $\mathfrak{N} = (N; \ldots)$ over $\mathcal{L}_M$, a *solution* of Φ in $\mathfrak{N}$ is any family $(c_x : x \in X)$ of elements of N such that $\mathfrak{N}$ satisfies the set of sentences $\Phi[c_x : x \in X]$. The system Φ is *solvable* if it has a solution in $\mathfrak{M}$, and it is *finitely solvable* if each of its finite subsets is solvable.

The structure $\mathfrak{M}$ is *atomic compact* if all its finitely solvable atomic systems are solvable. One similarly defines $\mathfrak{M}$ being compact with respect to any class of systems (e.g. $\mathfrak{M}$ is saturated if and only if it is [1]-card$(M)^-$-compact, where $|M|$ denotes the cardinality of M).

The atomic compactness of $\mathfrak{M}$ can be checked by considering only "small" systems: Indeed, given any infinite cardinal κ, if $\mathfrak{M}$ is κ-atomic compact, then it is κ-$(\exists, \wedge)$-compact, and then it is $(\exists, \wedge)$-compact, provided that κ is greater than or equal to the cardinality of $\mathcal{L}_M$. But as soon as $\mathfrak{M}$ is [1]-$(\exists, \wedge)$-compact, any finitely solvable atomic system can be solved "one unknown after the other" (see [7]).

Given an extension $\mathfrak{N} = (N; \ldots)$ of $\mathfrak{M} = (M; \ldots)$, recall that $\mathfrak{M}$ is a *retract* of $\mathfrak{N}$ if there exists a *retraction* from $\mathfrak{N}$ to $\mathfrak{M}$ (i.e. an $\mathcal{L}$-homomorphism fixing M pointwise); $\mathfrak{N}$ is an *elementary* (resp. a *pure*) *extension* of $\mathfrak{M}$ if $\mathfrak{M}$ and $\mathfrak{N}$ satisfy exactly the same sentences (resp. the same $(\exists, \wedge)$-sentences) with parameters in M.

Observe that if $\mathfrak{M}$ is a retract of $\mathfrak{N}$, then any positive system of $\mathfrak{M}$ with a solution in $\mathfrak{N}$ has a solution in $\mathfrak{M}$. Moreover (by the Compactness Theorem) the finitely solvable systems of $\mathfrak{M}$ are exactly the systems which have a solution in an elementary extension of $\mathfrak{M}$ (they are the so-called incomplete X-types of $\mathfrak{M}$ over M). But for each extension $\mathfrak{N}$ of $\mathfrak{M}$, the set $\mathbf{T}_{\mathcal{L}_N}(\mathfrak{N})$ of $\mathcal{L}_N$-sentences true in $\mathfrak{N}$ can be viewed as a system $\Phi(N \setminus M)$ of $\mathfrak{M}$, with the property that any solution of Φ^{at} defines a retraction of $\mathfrak{N}$ to $\mathfrak{M}$; moreover, Φ (resp. Φ^{at}) is finitely solvable precisely when $\mathfrak{N}$ is an elementary (resp. a pure) extension of $\mathfrak{M}$.

From these observations, together with the fact that homomorphisms onto "preserve" positive formulas (note incidentally that each retract of an atomic compact structure is atomic compact as well), one gets: If $\mathfrak{M}$ is atomic compact, then it is a retract of all its pure extensions, hence it is a retract of all its elementary extensions, and then it is positively compact; thus these four properties are mutually equivalent, and even equivalent to $\mathfrak{M}$ being a retract of all its reduced powers (resp. of all its ultrapowers) [12], as for any finitely solvable atomic system Φ of $\mathfrak{M}$, the system Φ^{at} has a solution in the reduced power $\mathfrak{M}^{[\Phi]}/\mathcal{F}$, where $[\Phi]$ denotes the set of finite subsets of Φ, and $\mathcal{F}$ is any filter (resp. any ultrafilter) on $[\Phi]$ containing $\{v \in [\Phi] : u \subset v\}$ for all u in $[\Phi]$; conversely, recall that any reduced power is a pure extension.

A topology $\mathcal{T}$ on M is *compatible* with $\mathfrak{M}$ if the interpretations of the relation and function symbols are respectively closed and continuous (the powers of M involved being endowed with the product topology); when in addition $\mathcal{T}$ is Hausdorff (in which case the equality relation is also closed), for each atomic system $\Phi(X)$ of $\mathfrak{M}$ and each φ in Φ, the set of solutions of φ in $\mathfrak{M}$ defines a closed subset of M^X. Hence, any topologically compact structure (admitting a Hausdorff-compact compatible topology), and in particular any finite structure is atomic compact [6]. But the converse need not hold (consider for instance $(\mathbb{Q},+)$); in fact, an atomic compact structure need not be a retract of any topologically compact one [8].

Given a *relational* structure $\mathfrak{R} = (E; \mathcal{R}_i : i \in I)$, let $\check{\mathfrak{R}} = (\check{E}; \overline{\mathcal{R}}_i : i \in I)$ denote the structure with universe the Stone–Čech compactification of the discrete space E, each $\overline{\mathcal{R}}_i$ denoting the closure of $\mathcal{R}_i$ in $\check{E}^{n_i}$ (n_i is the arity of $\mathcal{R}_i$). The relational structure $\check{\mathfrak{R}}$ is a topologically compact extension of $\mathfrak{R}$ with the property that every homomorphism from $\mathfrak{R}$ into a topologically compact structure has a (unique) homomorphic extension to $\check{\mathfrak{R}}$; thus $\mathfrak{R}$ is a retract of some topologically compact structure if and only if it is a retract of $\check{\mathfrak{R}}$. We shall consider $\check{E}$ as the set of ultrafilters on E (hence $(\mathcal{U}_1, \ldots, \mathcal{U}_{n_i}) \in \overline{\mathcal{R}}_i$ if and only if for all $U_1 \in \mathcal{U}_1, \ldots, U_{n_i} \in \mathcal{U}_{n_i}$, there are $a_1 \in U_1, \ldots, a_{n_i} \in U_{n_i}$ such that $(a_1, \ldots, a_{n_i}) \in \mathcal{R}_i$).

3. Graphs

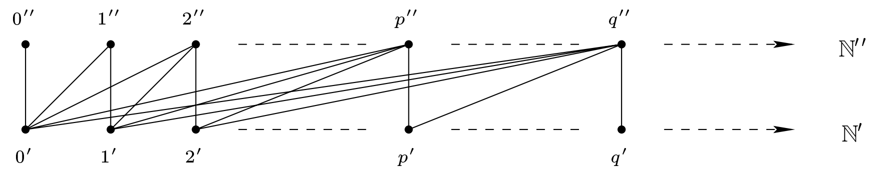
3.1. *Conventions.* The *graphs* that we consider are the binary relations $\mathfrak{G} = (E; \mathcal{G})$ which are *reflexive* and *symmetric*. For any subset F of E, the graph *induced* on F by $\mathfrak{G}$ is $\mathfrak{G}(F) = (F; \mathcal{G} \cap F \times F)$. In handling systems Φ of $\mathfrak{G}$, we adopt the convention to denote by the letters $x, y, z, \ldots$ the free variables (unknowns), by $a, b, c, \ldots$ the vertices (parameters), and by $u, v, w, \ldots$ indiscriminately the unknowns or parameters; the symbol for the binary relation will be $\sim$. So the general forms of atomic formulas with parameters are $u \equiv v$ and $u \sim v$. The notations $u \sim_\Phi v$ and $u \sim_\mathfrak{G} v$ stand respectively for $u \sim v \in \Phi$ and $(u, v) \in \mathcal{G}$. The *neighbourhood* of a vertex a is $\mathcal{G}_a = \{b \in E : a \sim_\mathfrak{G} b\}$ and its *degree* is $\operatorname{card}(\mathcal{G}_a \setminus \{a\})$; the vertex a is a *weak neighbour* (resp. a *strong neighbour*) of the subset A of E if $A \cap \mathcal{G}_a \neq \emptyset$ (resp. if $A \subset \mathcal{G}_a$); in this case, write $a \sim_\mathfrak{G} A$ (resp. $a \approx_\mathfrak{G} A$). The *weak neighbourhood* of A is the set $\mathcal{G}_A = \{a \in E : a \sim_\mathfrak{G} A\}$. In the same way, we write $x \sim_\Phi A$ for: "there is an a in A such that $x \sim_\Phi a$". The graph $\mathfrak{G}$ is *locally finite* (resp. *almost locally finite*) if all its vertices (resp. all but finitely many of its vertices) have a finite degree; an *essential degree* of $\mathfrak{G}$ is any cardinal which bounds the degree of all but finitely many of its vertices.

3.2. *Examples.* All complete graphs and more generally all graphs with a central vertex a (i.e. for which $\mathcal{G}_a = E$) are atomic compact (any atomic system $\Phi(X)$ admits the constant family $(a : x \in X)$ as a solution); they are actually topologically compact (consider the Alexandrov compactification on a of the discrete space $E \setminus \{a\}$).

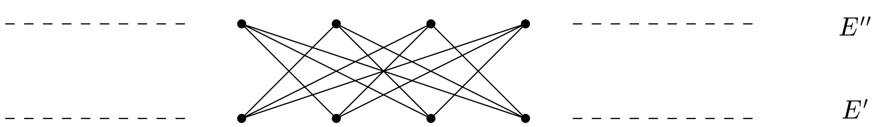
A forest (graph with no cycle) is atomic compact, as each of its elementary extensions is a forest in which there is no path between two vertices previously in distinct connected components.

The "simplest" examples of non-atomic compact graphs are those for which there is a set of vertices A with no strong neighbour while each of its finite subsets has one (consider the [1]-atomic system $\Phi(x) = \{x \sim a : a \in A\}$). Note that in the first example below, no infinite subset of $\mathbb{N}'$ has a strong neighbour; in the second one, every proper subset of E' has a strong neighbour.

EXAMPLE 1. The vertex set of the *half-graph* $\mathfrak{H}_{\mathbb{N}}$ is the disjoint union of two copies $\mathbb{N}' = \{0', 1', 2', \ldots\}$ and $\mathbb{N}'' = \{0'', 1'', 2'', \ldots\}$ of the set of non-negative integers; any two distinct vertices are linked if one is a p', the other is a q'' and $p \leq q$. (One defines similarly $\mathfrak{H}_{\mathfrak{P}}$ for any poset $\mathfrak{P}$.)



EXAMPLE 2. The vertex set of the *Dushnik Miller graph* $\mathfrak{E}_E$ is the disjoint union of two copies $E' = \{a', b', c', \ldots\}$ and $E'' = \{a'', b'', c'', \ldots\}$ of the same infinite set E; any two distinct vertices are linked if they lie in distinct copies and correspond to two different points of E.



Note that each elementary extension of $\mathfrak{E}_E$ is some $\mathfrak{E}_F$, hence it is never atomic compact. The elementary extensions of $\mathfrak{H}_{\mathbb{N}}$ are not atomic compact either, since they are exactly the graphs $\mathfrak{H}_{\mathfrak{C}}$ where $\mathfrak{C}$ is any elementary extension of $(\mathbb{N}, \leq)$ (but such a $\mathfrak{C}$ has no upper bound while each of its finite subsets has one).

REMARK (cf. [2]). Given any infinite cardinal κ, the successor cardinal κ^+ endowed with its order topology has the following property: for any mapping $f : \kappa \to \kappa^+$ and any ultrafilter $\mathcal{F}$ on κ, the image ultrafilter $f(\mathcal{F})$ is convergent in κ^+ (observe that κ^+ having a cofinality strictly greater than κ, $f(\mathcal{F})$ has bounded elements, hence it is "essentially" an ultrafilter on some compact subspace of κ^+).

It follows that the graph $\mathfrak{H}_{\kappa^+}$ is κ-atomic compact (but it is not κ^+-atomic compact: it is obviously not [1]-κ^+-atomic compact). Indeed, the disjoint union topology on $K = (\kappa^+)' \cup (\kappa^+)''$ being compatible, for any ultrafilter $\mathcal{F}$ on κ, $\lim_{\mathcal{F}} : K^\kappa \to K$ yields a retraction from $(\mathfrak{H}_{\kappa^+})^\kappa/\mathcal{F}$ onto

$\mathfrak{H}_{\kappa^+}$; but recall that any finitely solvable atomic system Φ of $\mathfrak{H}_{\kappa^+}$ has a solution in some ultrapower $(\mathfrak{H}_{\kappa^+})^{[\Phi]}/\mathcal{F}$.

4. Almost locally finite graphs. We already know that finite graphs are topologically compact; more generally, it is the case for locally finite ones, and even for those with no more than two vertices of infinite degree; as for general almost locally finite graphs, we shall see that they are always $[\omega^-]$-atomic compact.

4.1. *Topologically compact graphs*

LEMMA 4.1. *If the graph $\mathfrak{G} = (E, \mathcal{G})$ has a retract $\mathfrak{G}(F)$ which is topologically compact and contains all vertices of infinite degree, then it is topologically compact.*

Proof. Let $\mathcal{T}$ be a Hausdorff-compact topology on F compatible with $\mathfrak{G}(F)$, and $r : E \to F$ a retraction. One easily checks that the following set $\mathcal{B}$ is a base for a Hausdorff-compact topology $\widetilde{\mathcal{T}}$ on E:

$$\mathcal{B} = \{\{a\} : a \in E \setminus F\} \cup \{r^{-1}(U) \setminus A : U \in \mathcal{T},\ A \text{ finite}\}.$$

Let us check that $\mathcal{G}$ is closed. Consider $(a, b) \notin \mathcal{G}$:

• Either $\{a, b\} \subset F$, and then there is a $\mathcal{T}$-open neighbourhood $U \times V$ of (a, b) which is disjoint from $\mathcal{G} \cap F \times F$, i.e. such that there is no edge between U and V; hence there is no edge between $r^{-1}(U)$ and $r^{-1}(V)$ either, and thus $r^{-1}(U) \times r^{-1}(V)$ is a $\widetilde{\mathcal{T}}$-neighbourhood of (a, b) which is disjoint from $\mathcal{G}$.

• Or $a \notin F$ for example, and then $\mathcal{G}_a$ is finite (by hypothesis), hence closed; moreover a is isolated (by construction). Thus $\{a\} \times (E \setminus \mathcal{G}_a)$ is a neighbourhood (a, b) which is disjoint from $\mathcal{G}$. ∎

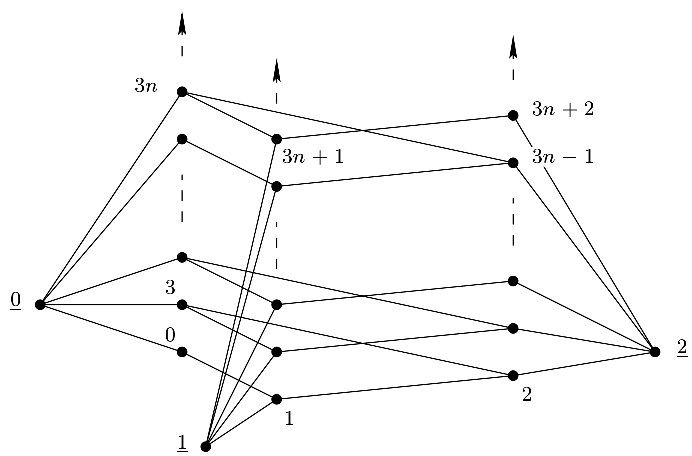
Note that if, in the above lemma, we allow vertices of infinite degree outside F, provided that each such vertex is linked to its image under r, then the same construction yields the topological compactness of $\mathfrak{G}$. From this, it follows for instance that every equivalence relation is a topologically compact graph (take F containing exactly one point in each class).

THEOREM 4.1. *If the graph $\mathfrak{G}$ has at most two vertices of infinite degree, then it is topologically compact.*

Proof. Denote by $d_{\mathfrak{G}} : E \times E \to \mathbb{N} \cup \{\infty\}$ the "shortest path" distance of $\mathfrak{G}$. Suppose that a and b are (possibly equal) vertices with the property that every vertex distinct from a and b has a finite degree. First assume that a and b lie in the same connected component of $\mathfrak{G}$, i.e. $d_{\mathfrak{G}}(a, b) = n \in \mathbb{N}$, and consider a path $a = c_0 \sim_{\mathfrak{G}} c_1 \sim_{\mathfrak{G}} \ldots \sim_{\mathfrak{G}} c_n = b$ of minimal length between a and b. Then observe that the mapping $x \mapsto c_{\min\{d_{\mathfrak{G}}(a,x),n\}}$ is a retraction from $\mathfrak{G}$ onto $\mathfrak{G}(\{c_0, \ldots, c_n\})$. If a and b lie in distinct connected components, just consider r mapping the component of a to a and the other components to b. ■

A graph with exactly 3 vertices of infinite degree may not even be atomic compact, as shown by the next example; note however that it is $[\omega^-]$-atomic compact, according to Theorem 4.2 below.

Example 3. Consider the graph $\mathfrak{C}_{\mathbb{N}}$ with vertex set the disjoint union of $\mathbb{N}$ and $\mathbb{Z}/3\mathbb{Z}$ (respectively the set of non-negative integers and the set of integers modulo 3), and for which any two distinct vertices are linked if and only if one of them is an integer n while the other one is either its successor $n+1$ in $\mathbb{N}$ or its class $\underline{n}$ in $\mathbb{Z}/3\mathbb{Z}$.



Consider the system $\Phi(x_k : k \in \mathbb{N}) = \{x_k \sim x_{k+1} : k \in \mathbb{N}\} \cup \{x_k \sim -\underline{k} : k \in \mathbb{N}\}$. This system can be "represented" in the following way:

$$\left\{\begin{array}{ccccc} & & \vdots & & \vdots \\ -\underline{k+1} \sim x_{k+1} & \sim & x_k \sim -\underline{k} & & (\mathcal{F}_k) \\ & & \vdots & & \vdots \end{array}\right.$$

Observe that if $(a_k : k \in \mathbb{N})$ is a solution of $\mathcal{F}_k$ in $\mathfrak{C}_{\mathbb{N}}$, then a_{k+1} and a_k must be two elements of $\mathbb{N}$ such that $a_{k+1} = a_k - 1$. Then any solution of Φ in $\mathfrak{C}_{\mathbb{N}}$ would be a descending sequence in $\mathbb{N}$. Thus Φ is not solvable; it is however finitely solvable (take a_0 large enough in $3\mathbb{N}$).

4.2. *Reduced atomic systems.* In the sequel, it will prove convenient to restrict our attention to certain kinds of "maximal systems with no non-trivial equality relation".

Let us say that a system $\Phi(X)$ of $\mathfrak{M}$ is *reduced* if, first, it is maximal for inclusion with respect to the property of being a finitely solvable X-atomic system of $\mathfrak{M}$, and second, it contains $u \equiv v$ for no distinct u and v in $M \cup X$. Observe that, given any finitely solvable atomic system $\Phi(X)$ of $\mathfrak{M}$, there is a reduced system $\Theta(X')$ for some $X' \subset X$ with the property that every solution of $\Theta(X')$ in some extension of $\mathfrak{M}$ extends to a solution of $\Phi(X)$ in the same extension: Indeed, first consider a maximal finitely solvable extension $\Psi(X)$ of $\Phi(X)$ (made up of atomic formulas); it is easily seen that "$u \equiv v \in \Psi$" is an equivalence relation on $M \cup X$; so choose a system $X' \subset X$ of representatives of the classes which are disjoint from M, and define $\Theta(X')$ to be the restriction of $\Psi(X)$ to its formulas in which no unknown of $X \setminus X'$ occurs.

From this discussion, it follows that the definitions of atomic compactness, of $[\lambda]$-atomic compactness and of $[\lambda^-]$-atomic compactness can be restricted to reduced systems.

Note also that if a reduced X-system is solvable, then in fact $X = \emptyset$ (indeed the system is then equal to $\mathbf{T}^{\mathrm{at}}_{\mathcal{L}_M}(\mathfrak{M})$, the set of $\mathcal{L}_M$-atomic sentences true in $\mathfrak{M}$).

LEMMA 4.2. *Given a reduced system $\Phi(X)$ of $\mathfrak{M}$, for every x in X and $a_1, \ldots, a_n$ in M, there is a finite subset Ψ of Φ such that no solution of Ψ in $\mathfrak{M}$ maps x to any of $a_1, \ldots, a_n$.*

In particular, if $\Phi(X)$ is a reduced system, then each of its finite subsystems with at least one unknown has infinitely many solutions in $\mathfrak{M}$.

Proof. As Φ is maximal finitely solvable and contains $x \equiv a$ for no $a \in M$, there are finite subsets Ψ_k of Φ, $k \in \{1, \ldots, n\}$, such that no $\Psi_k \cup \{x \equiv a_k\}$ is solvable; consider then $\Psi = \Psi_1 \cup \ldots \cup \Psi_n$. ∎

COROLLARY 4.1. *Given a reduced atomic system Φ and a vertex a of a graph $\mathfrak{G}$, if for some x in X, the formula $x \sim a$ belongs to Φ, then the degree of the vertex a is infinite.*

Proof. If $\mathcal{G}_a = \{a_1, \ldots, a_n\}$, then the above lemma gives a finite subset Ψ of Φ such that $\Psi \cup \{x \sim a\}$ is not solvable; hence, as Φ is finitely solvable, it cannot contain $x \sim a$. ∎

Now we can state

THEOREM 4.2. *Almost locally finite graphs are $[\omega^-]$-atomic compact.*

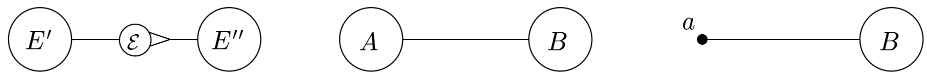
Proof. It follows from the corollary above that any reduced atomic system $\Phi(x_1, \ldots, x_n)$ of such a graph contains only finitely many formulas in which at least one unknown does occur, hence it is solvable. ∎

5. $[n]$-atomic compact graphs

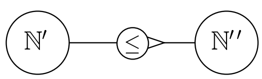
5.1. *Conventions.* Most of the graphs that we give as examples share some common features. They are built according to patterns that we describe before proceeding further.

Given copies E', E_1, $E'_1, \ldots$ of some set E and an element a of E, we shall denote by a', a_1, $a'_1, \ldots$ the elements corresponding to a in each of these copies.

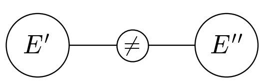
We shall represent graphs by diagrams from which one should be able to recover all the information required. Diagrams are made up of blocks interconnected by labelled lines. To each block is assigned a set of vertices; the names of these vertices shall be clear whenever needed (e.g. $\mathbb{N}'_2 = \{0'_2, 1'_2, \ldots\}$). Each vertex has a loop (unless otherwise specified) and vertices in distinct blocks are linked according to the label of the line between these blocks. The generic primitive patterns correspond to the diagrams below. In the first pattern, E' and E'' are two disjoint subsets of copies of a set E and $\mathcal{E}$ is some subset of $E \times E$; an element a' of E' and an element b'' of E'' are linked when $(a, b) \in \mathcal{E}$. In the second pattern, every element of A is linked to every element of B. In the third one, the vertex a is linked to every element of B.



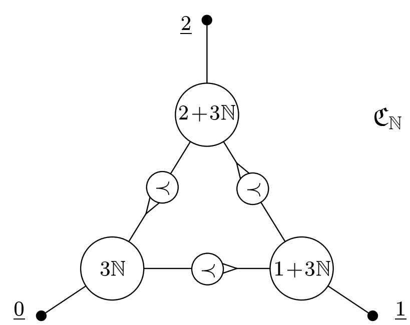
For instance, the three examples above are represented by the following diagrams ($\prec$ denotes the successor relation: $p \prec q$ if and only if $q = p + 1$):



$\mathfrak{H}_{\mathbb{N}}$



$\mathfrak{E}_E$



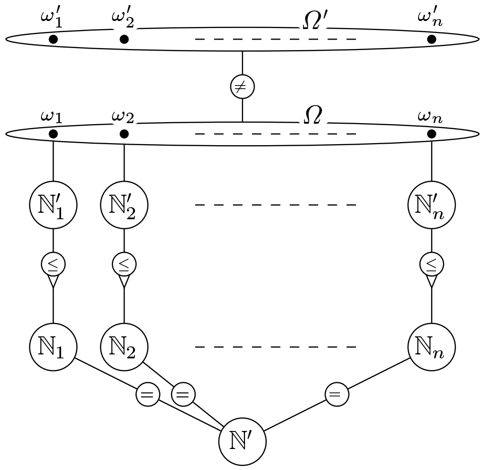
$\mathfrak{C}_{\mathbb{N}}$

5.2. *Generalized half-graphs.* For each integer n, we produce an $[n]$-atomic compact graph $\mathfrak{H}_n$ which fails to be $[n+1]$-atomic compact. Recall that such a graph must have infinitely many vertices of infinite degree (Theorem 4.2).

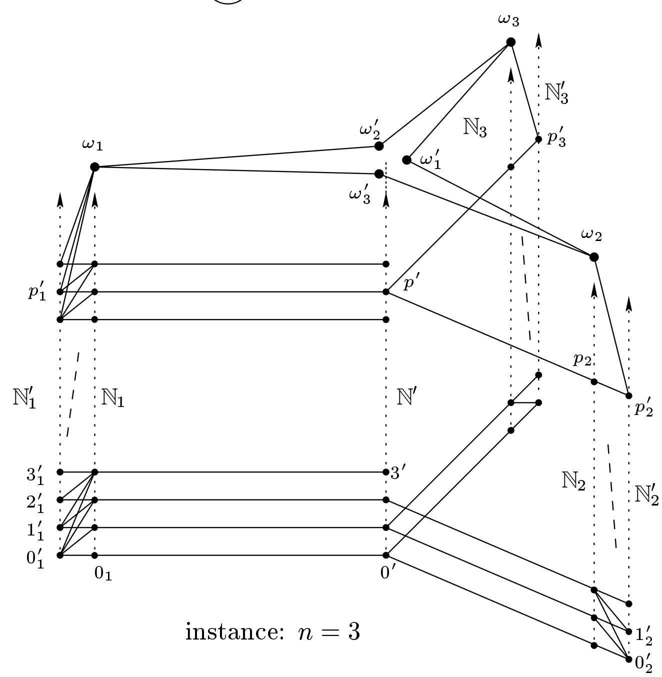
EXAMPLE 4. Consider an integer $n \geq 2$ and the graph $\mathfrak{H}_n$ corresponding to the diagram below (cf. Example 1).

The following $[n+1]$-atomic system is easily seen to be finitely solvable but not solvable:

$$\Phi(x, x_1, \ldots, x_n) = \Phi_1(x_1, \ldots, x_n) \cup \Phi_2(x, x_1, \ldots, x_n) \quad \text{with}$$



$\mathfrak{H}_n$



instance: $n = 3$

$$\Phi_1(x_1, \dots, x_n) = \{x_k \sim p_k' : k \in \{1, \dots, n\},\ p \in \mathbb{N}\},$$
$$\Phi_2(x, x_1, \dots, x_n) = \{x_k \sim x : k \in \{1, \dots, n\}\}.$$

Any finite subset of Φ admits $(p', p_1, \dots, p_n)$ as a solution, for p large enough; but $(\omega_1, \dots, \omega_n)$ is the only solution of Φ_1, and it cannot be extended to any solution of Φ_2.

Now, let us prove that $\mathfrak{H}_n$ is $[n]$-atomic compact. So consider a reduced atomic system $\Phi(X)$ with $|X| \leq n$, and let us show that it is solvable (then recall that, a posteriori, $X = \emptyset$).

For each $k \in \{1, \ldots, n\}$, define $\widetilde{\mathbb{N}}_k = \mathbb{N}'_k \cup \{\omega_k\}$. Recall (Corollary 4.1) that given any $x \in X$, the formula $x \sim d$ can lie in Φ only for some d's of infinite degree, hence for d in $\widetilde{\mathbb{N}}_1 \cup \ldots \cup \widetilde{\mathbb{N}}_n$.

Observe that, given distinct k and l in $\{1, \ldots, n\}$, any vertices a and b such that $a \sim_{\mathfrak{H}_n} \widetilde{\mathbb{N}}_k$, $b \sim_{\mathfrak{H}_n} \widetilde{\mathbb{N}}_l$ and $a \sim_{\mathfrak{H}_n} b$ must lie in the finite set $\Omega \cup \Omega'$; hence, it follows from Lemma 4.2 that, for any x and y such that $x \sim_{\Phi} \widetilde{\mathbb{N}}_k$ and $y \sim_{\Phi} \widetilde{\mathbb{N}}_l$, Φ cannot contain the formula $x \sim y$, and furthermore, x and y have to be distinct.

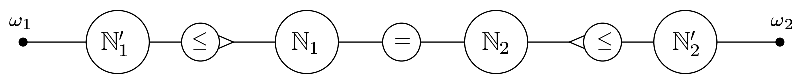
In particular, note that, as $n \geq |X|$, if there happens to be some x for which no $x \sim_{\Phi} \widetilde{\mathbb{N}}_k$ holds, then there must be some k for which no $x \sim_{\Phi} \widetilde{\mathbb{N}}_k$ holds; in that case, let k_0 be such a k.

Now, for each $x \in X$, we can define $d_x \in \Omega \cup \Omega'$ in the following way:

- $d_x = \omega_k$ if and only if $x \sim_{\Phi} \widetilde{\mathbb{N}}_k$,
- $d_x = \omega'_{k_0}$ if and only if for no k does $x \sim_{\Phi} \widetilde{\mathbb{N}}_k$ hold.

We claim that $(d_x : x \in X)$ is a solution of Φ: Indeed, $u \equiv v$ lies in Φ for no distinct u and v. Besides, observe that, if for some x and y, $d_x \not\sim_{\mathfrak{H}_n} d_y$, then (as $\omega'_{k_0} \approx_{\mathfrak{H}_n} \{\omega'_{k_0}\} \cup \{\omega_k : k \neq k_0\}$), $x \sim_{\Phi} \widetilde{\mathbb{N}}_k$ and $y \sim_{\Phi} \widetilde{\mathbb{N}}_l$ for some k and l distinct; but then recall that $x \not\sim_{\Phi} y$.

For $\mathfrak{H}_0$, consider the half-graph $\mathfrak{H}_{\mathbb{N}}$ (first example). Note that, in the case of $\mathfrak{H}_2$, the vertices ω'_1 and ω'_2 can be removed. As for $\mathfrak{H}_1$, consider the following diagram:

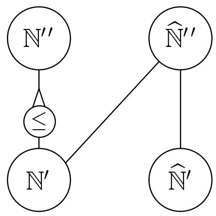


6. Purity. It is known [8] that an atomic compact structure need not be a retract of any topologically compact one. In fact, the first example, given by Taylor, is an atomic compact "anti-reflexive graph" (in fact locally finite and with only finitely many cycles of any given size) which has an infinite chromatic number: For any binary relation $\mathfrak{R}$ with infinite chromatic number, $\check{\mathfrak{R}}$ has loops, hence it is not a pure extension of $\mathfrak{R}$ when the latter is moreover anti-reflexive. In this section, we focus on the property, for a (reflexive) graph, of being a pure restriction of its Stone–Čech compactification.

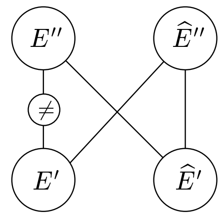
Given a set E, we shall denote by $\widehat{E} = \check{E} \setminus E$ the set of non-principal ultrafilters on E.

6.1. *Examples.* A topologically compact graph $\mathfrak{G}$ is obviously pure in its Stone–Čech extension $\check{\mathfrak{G}}$ (indeed, it is a retract of $\check{\mathfrak{G}}$); but a graph $\mathfrak{G}$ may be pure in $\check{\mathfrak{G}}$ without even being [1]-atomic compact, as shown by the following example:

Example 5. The graphs $\check{\mathfrak{H}}_{\mathbb{N}}$ and $\check{\mathfrak{E}}_E$ (see the first two examples) correspond to the following diagrams:



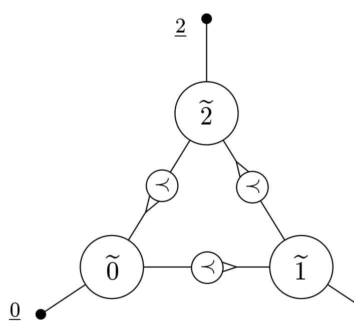
$\check{\mathfrak{H}}_{\mathbb{N}}$



$\check{\mathfrak{E}}_E$

The next example shows that a graph can be atomic compact and pure in its Stone–Čech extension (i.e. a retract of a topologically compact relation) without being topologically compact:

Example 6. Consider the graph $\mathfrak{C}_{\mathbb{Z}}$ with vertex set $\mathbb{Z} \cup \mathbb{Z}/3\mathbb{Z}$ (cf. Example 3 and Section 5.1) defined by the diagram below, with the following convention: the class modulo 3 of an integer $n \in \mathbb{Z}$ is denoted by $\underline{n}$ when it is considered as a vertex of $\mathfrak{C}_{\mathbb{Z}}$ (so $\underline{n} \in \mathbb{Z}/3\mathbb{Z}$), and it is denoted by $\widetilde{n}$ when it is considered as a set of vertices of $\mathfrak{C}_{\mathbb{Z}}$ (so $\widetilde{n} = n + 3\mathbb{Z} \subset \mathbb{Z}$).



$\mathfrak{C}_{\mathbb{Z}}$

The graph $\mathfrak{C}_{\mathbb{Z}}$ satisfying the following two sentences with parameters in $\mathbb{Z}/3\mathbb{Z}$, each of its elementary extensions satisfies them as well:

(α) *each vertex is linked to one and only one element of* $\mathbb{Z}/3\mathbb{Z}$;

(β) *given a vertex* $\underline{n} \in \mathbb{Z}/3\mathbb{Z}$, *every vertex linked to* $\underline{n}$ *and different from* $\underline{n}$ *has exactly two other neighbours*, *one linked to the vertex* $\underline{n-1}$ *and the other linked to the vertex* $\underline{n+1}$.

But any such extension $\mathfrak{G} = (E, \mathcal{G})$ has a retraction onto $\mathfrak{C}_{\mathbb{Z}}$ (for each connected component C of $\mathfrak{G}(E \setminus \mathbb{Z}/3\mathbb{Z})$ distinct from $\mathbb{Z}$, there is an isomorphism i_C from $\mathfrak{G}(C \cup \mathbb{Z}/3\mathbb{Z})$ onto $\mathfrak{C}_{\mathbb{Z}}$ fixing $\mathbb{Z}/3\mathbb{Z}$ pointwise; the union of these i_C's with $\mathrm{id}_{\mathbb{Z}}$ yields a retraction of $\mathfrak{G}$ onto $\mathfrak{C}_{\mathbb{Z}}$). Hence $\mathfrak{C}_{\mathbb{Z}}$ is atomic compact; but it is not topologically compact:

Assume that $\mathcal{T}$ is a Hausdorff-compact compatible topology and consider the *shift automorphism* σ, which maps each n to $n+1$ and each vertex $\underline{n}$ to the vertex $\underline{n+1}$. For each vertex $\underline{n}$, $\mathcal{G}_{\underline{n}}$ is closed. So for any $\mathcal{U} \in \check{\tilde{n}}$, $\lim_{\mathcal{T}} \mathcal{U} \in \mathcal{G}_{\underline{n}} = \tilde{n} \cup \{\underline{n}\}$; in particular, as $\sigma(\mathcal{U}) \in \check{\widetilde{n+1}}$, $\lim_{\mathcal{T}} \sigma(\mathcal{U}) \in \widetilde{n+1} \cup \{\underline{n+1}\}$; but moreover, for all $p \in \mathcal{U}$, $\sigma(p) \sim_{\mathfrak{C}_{\mathbb{Z}}} p$, hence $\lim_{\mathcal{T}} \sigma(\mathcal{U}) \sim_{\mathfrak{C}_{\mathbb{Z}}} \lim_{\mathcal{T}} \mathcal{U}$, which now forces $\lim_{\mathcal{T}} \mathcal{U} \in \tilde{n}$ and $\lim_{\mathcal{T}} \sigma(\mathcal{U}) = \sigma(\lim_{\mathcal{T}} \mathcal{U})$ (indeed, any vertex $a \in \tilde{n} \cup \{\underline{n}\}$ linked to a vertex $b \in \widetilde{n+1} \cup \{\underline{n+1}\}$ must lie in $\tilde{n}$, and furthermore, b must equal $\sigma(a)$). Now take any $\mathcal{U} \in \hat{\tilde{0}}$; as for each $k \in \mathbb{Z}$, $\lim_{\mathcal{T}} \sigma^k(\mathcal{U}) = \sigma^k(\lim_{\mathcal{T}} \mathcal{U})$, every $p \in \mathbb{Z}$ is the $\mathcal{T}$-limit of a non-principal ultrafilter (namely, if $\lim_{\mathcal{T}} \mathcal{U} = l$, then p is equal to $\lim_{\mathcal{T}} \sigma^{p-l}(\mathcal{U})$); hence each element of $\mathbb{Z}$ is a $\mathcal{T}$-accumulation point; but, according to Baire's Category Theorem, a countable Hausdorff-compact space must have at least one, and then infinitely many isolated points.

It is easy to check directly that $\mathfrak{C}_{\mathbb{Z}}$ is a retract of $\check{\mathfrak{C}}_{\mathbb{Z}}$ ($\check{\mathfrak{C}}_{\mathbb{Z}}$ has the properties (α) and (β) above, as any ultrafilter $\mathcal{U}$ on $\tilde{n}$ is linked exactly to itself, to $\underline{n}$, to $\sigma(\mathcal{U})$ and to $\sigma^{-1}(\mathcal{U})$). This is also a consequence of the next theorem.

6.2. *Graphs with a finite essential degree.* In this subsection, we prove the theorem below.

THEOREM 6.1. *Every graph with a finite essential degree is pure in its Stone–Čech compactification.*

This statement is optimal in some respects, as an almost locally finite graph need not be pure in its Stone–Čech extension (see Example 8 below). Before turning to the proof, let us establish two general lemmas.

Given a set $\mathcal{I}$ of self-mappings of some set E, let, for every element a of E and for every subset A of E, $\mathcal{I}(a)$ and $\mathcal{I}^{-1}(A)$ denote $\{f(a) : f \in \mathcal{I}\}$ and $\bigcup\{f^{-1}(A) : f \in \mathcal{I}\}$ respectively; for each positive integer l, let $\mathcal{I}_l$ denote the set of all products (under composition) of at most l factors from $\mathcal{I}$.

LEMMA 6.1. *For a graph $\mathfrak{G} = (E, \mathcal{G})$, let $\mathcal{U}$ be an ultrafilter on E, H an element of $\mathcal{U}$, $\mathcal{I} = \{f_0, \ldots, f_d\}$ a finite set of self-mappings of E and K a subset of E. If $\mathcal{I}(a) = \mathcal{G}_a \cap K$ for all a in H, then $\mathcal{I}(\mathcal{U}) = \overline{\mathcal{G}}_{\mathcal{U}} \cap \check{K}$ (i.e. $\{f_0(\mathcal{U}), \ldots, f_d(\mathcal{U})\}$ is the set of ultrafilters $\mathcal{V}$ containing K such that $\mathcal{V} \sim_{\check{\mathfrak{G}}} \mathcal{U}$).*

P r o o f. As obviously $\mathcal{I}(\mathcal{U}) \subset \check{K}$, we have to check that for every $\mathcal{V}$ in $\check{K}$, $\mathcal{V} \in \overline{\mathcal{G}}_{\mathcal{U}}$ if and only if $\mathcal{V} \in \mathcal{I}(\mathcal{U})$. So let $\mathcal{V}$ be an ultrafilter on E containing K:

$$\begin{aligned}
\mathcal{V} \sim_{\check{\mathfrak{G}}} \mathcal{U} \ &\Leftrightarrow\ \forall U \in \mathcal{U},\ \forall V \in \mathcal{V},\ V \cap \mathcal{G}_U \neq \emptyset \\
&\Leftrightarrow\ \forall U \in \mathcal{U},\ \forall V \in \mathcal{V},\ (V \cap K) \cap \mathcal{G}_{U \cap H} \neq \emptyset \\
&\Leftrightarrow\ \forall U \in \mathcal{U},\ \forall V \in \mathcal{V},\ V \cap (K \cap \mathcal{G}_{U \cap H}) \neq \emptyset \\
&\Leftrightarrow\ \forall U \in \mathcal{U},\ \forall V \in \mathcal{V},\ V \cap (f_0(U \cap H) \cup \ldots \cup f_d(U \cap H)) \neq \emptyset \\
&\Leftrightarrow\ \forall U \in \mathcal{U},\ \forall V \in \mathcal{V},\ V \cap (f_0(U) \cup \ldots \cup f_d(U)) \neq \emptyset,
\end{aligned}$$

which is easily seen to precisely characterize (the ultrafilter) $\mathcal{V}$ belonging to (the finite set of ultrafilters) $\{f_0(\mathcal{U}), \ldots, f_d(\mathcal{U})\}$. ∎

LEMMA 6.2. *Given a non-principal ultrafilter $\mathcal{U}_0$ on some set F, consider a finite set $\mathcal{I}$ of self-mappings of F. Assume that the cardinalities of the sets $\mathcal{I}^{-1}(a)$ are all less than or equal to some integer γ. Then for every positive integer l, there is an element U of $\mathcal{U}_0$ such that any two elements of $\mathcal{I}_l$ either coincide pointwise on U or map it onto two disjoint sets.*

Proof. As $\mathcal{I}_l$ satisfies the hypothesis of the lemma (with $\gamma + \ldots + \gamma^l$ replacing γ), the general statement follows from the case $l = 1$, which we handle now.

First consider $U_0 \in \mathcal{U}_0$ such that any two elements of $\mathcal{I}$ coincide either at every point or at no point of U_0. First consider $U_0 \in \mathcal{U}_0$ such that any two elements of $\mathcal{I}$ agree either at every point or at no point of U_0 (as $\mathcal{I}$ is finite, the following equivalence relation on F has only finitely many classes, one of which must then belong to $\mathcal{U}_0$: two elements a and b of F are equivalent when, for any elements f and g of $\mathcal{I}$, $f(a) = g(a)$ if and only if $f(b) = g(b)$).

Then consider a sequence $(V_p)_{p\in\mathbb{N}}$ of subsets of U_0 such that, for every $p \in \mathbb{N}$, V_p is a maximal subset of $U_0 \setminus \bigcup\{V_q : q < p\}$ with the property that, for any distinct elements a and b of V_p, $\mathcal{I}(b) \cap \mathcal{I}(a) = \emptyset$, i.e. $b \notin \mathcal{I}^{-1}(\mathcal{I}(a))$.

Observe that, given $p \in \mathbb{N}$, any two elements of $\mathcal{I}$ either coincide pointwise on V_p or map it onto two disjoint sets (incidentally observe also that every element of $\mathcal{I}$ is one-to-one on V_p); furthermore, for every $q < p$ and $a \in U_0 \setminus \bigcup\{V_{q'} : q' < p\}$, $\mathcal{I}^{-1}(\mathcal{I}(a)) \cap V_q \neq \emptyset$. In particular, as the V_q's are pairwise disjoint and for every $a \in F$, $|\mathcal{I}^{-1}(\mathcal{I}(a))| \leq \gamma \times |\mathcal{I}|$, it follows that $U_0 = V_0 \cup \cdots \cup V_{\gamma\times|\mathcal{I}|}$. Now U can be taken among $V_0, \ldots, V_{\gamma\times|\mathcal{I}|}$. ∎

Proof of Theorem 6.1. Let $c_1, \ldots, c_e$ be the vertices of infinite degree of the graph $\mathfrak{G} = (E, \mathcal{G})$ and let F denote the set $E \setminus \{c_1, \ldots, c_e\}$. Consider a finite set $\mathcal{I}$ of self-mappings of F with the property that, for every $a \in F$, $\mathcal{I}(a) = \mathcal{G}_a \cap F$; observe that, for every $\mathcal{U} \in \widehat{E}$ and $h \in \mathcal{I}$, $F \in h(\mathcal{U})$ on the one hand and $h(\mathcal{U}) \in \widehat{E}$ on the other hand, since for each $a \in F$, $h^{-1}(\{a\}) \subset \mathcal{G}_a$, which is finite, and thus does not belong to $\mathcal{U}$. Then it follows from Lemma 6.1 that, for each $\mathcal{U}$ in $\widehat{E}$, $\overline{\mathcal{G}}_{\mathcal{U}} \cap \widehat{E} = \mathcal{I}(\mathcal{U})$ (take H and K both equal to F).

Now let $A \subset E$ and $B = \{\mathcal{U}_0, \ldots, \mathcal{U}_q\} \subset \widehat{E}$ be two finite sets. We must prove that there is a family $(b_{\mathcal{U}})_{\mathcal{U}\in B}$ in E^B such that:

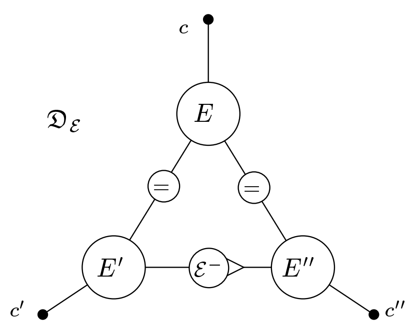
- for any $\mathcal{U}$ and $\mathcal{V}$ in B, $\mathcal{U} \sim_{\check{\mathfrak{G}}} \mathcal{V} \Rightarrow b_{\mathcal{U}} \sim_{\mathfrak{G}} b_{\mathcal{V}}$,
- for any a in A and $\mathcal{U}$ in B, $a \sim_{\check{\mathfrak{G}}} \mathcal{U} \Rightarrow a \sim_{\mathfrak{G}} b_{\mathcal{U}}$.

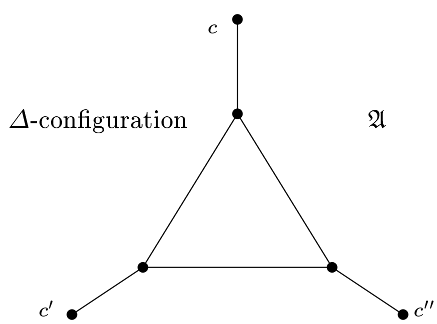
As we can deal with each connected component of $\check{\mathfrak{G}}(B)$ independently, assume that $\check{\mathfrak{G}}(B)$ is connected; so, for each $\mathcal{U} \in B$, we can consider an $h_{\mathcal{U}} \in \mathcal{I}_q$ such that $\mathcal{U} = h_{\mathcal{U}}(\mathcal{U}_0)$. Lemma 6.2 with $l = q + 1$ provides us with some $U \in \mathcal{U}_0$ that we restrict in such a way that, for each $a \in A$ and $\mathcal{U} \in B$, if $\mathcal{G}_a \in \mathcal{U} = h_{\mathcal{U}}(\mathcal{U}_0)$ (i.e. $a \sim_{\check{\mathfrak{G}}} \mathcal{U}$), then $h_{\mathcal{U}}(U) \subset \mathcal{G}_a$ (i.e. $a \approx_{\mathfrak{G}} h_{\mathcal{U}}(U)$): just intersect it with the corresponding $h_{\mathcal{U}}^{-1}(\mathcal{G}_a)$'s.

We claim that for any b in U, the family $(h_{\mathcal{U}}(b))_{\mathcal{U}\in B}$ meets the requirements: For any $\mathcal{U}$ and $\mathcal{V}$ in B, if $\mathcal{U} \sim_{\check{\mathfrak{G}}} \mathcal{V}$, then there is an h in $\mathcal{I}$ for which $\mathcal{U} = h(\mathcal{V})$, so that $h_{\mathcal{U}}(\mathcal{U}_0) = h \circ h_{\mathcal{V}}(\mathcal{U}_0)$, hence $h_{\mathcal{U}}(U) \cap h \circ h_{\mathcal{V}}(U) \neq \emptyset$; but then, as $h_{\mathcal{U}}$ and $h \circ h_{\mathcal{V}}$ both lie in $\mathcal{I}_l$, they must coincide on U, and finally $h_{\mathcal{U}}(b) = h \circ h_{\mathcal{V}}(b) \sim_{\mathfrak{G}} h_{\mathcal{V}}(b)$. ■

6.3. *Mycielski's question for reflexive graphs.* The object of this section is to extend to reflexive graphs the result of Taylor (op. cit.) about the existence of atomic compact anti-reflexive graphs which are retracts of no topologically compact relation (see also [9]). Indeed, with each anti-reflexive binary relation $(E, \mathcal{E})$, we associate a graph $\mathfrak{D}_{\mathcal{E}}$ which is not pure in its Stone–Čech extension as soon as $(E, \mathcal{E})$ has an infinite chromatic number, and which is atomic compact as soon as $(E, \mathcal{E})$ is symmetric and atomic compact.

Given a set E and a subset $\mathcal{E}$ of $E \times E$, consider the graph $\mathfrak{D}_{\mathcal{E}}$ corresponding to the diagram on the left hand side of the figure below, with $\mathcal{E}^-$ denoting the set $\mathcal{E} \setminus \{(x, x) : x \in E\}$. Note that the chromatic number of $\mathfrak{D}_{\mathcal{E}}$ is at most 3.

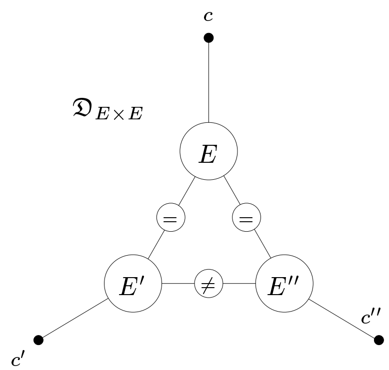


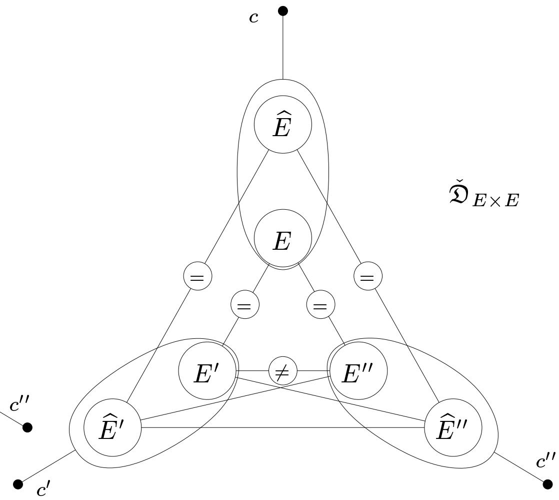


Say that the Δ-configuration *occurs* in a graph $\mathfrak{G}$ containing c, c' and c'' as vertices if there is a morphism from $\mathfrak{A}$ (represented above, on the right) to $\mathfrak{G}$ (fixing c, c' and c''). Observe that the Δ-configuration does not occur in $\mathfrak{D}_{\mathcal{E}}$; however, it may occur in $\check{\mathfrak{D}}_{\mathcal{E}}$, which, in this case, is obviously not a pure extension of $\mathfrak{D}_{\mathcal{E}}$:

EXAMPLE 7. $\mathfrak{D}_{E\times E}$ and $\check{\mathfrak{D}}_{E\times E}$ correspond to the diagrams on p. 114.

LEMMA 6.3. *If the binary relation* $(E, \mathcal{E}^-)$ *has an infinite chromatic number*, *then* $\check{\mathfrak{D}}_{\mathcal{E}}$ *is not a pure extension of* $\mathfrak{D}_{\mathcal{E}}$.





Proof. Indeed, the Δ-configuration occurs in $\check{\mathfrak{D}}_{\mathcal{E}}$, every time there is a $\mathcal{U}$ in $\check{E}$ each element of which "contains" an edge of $(E, \mathcal{E}^-)$: $\mathcal{U}'$ and $\mathcal{U}''$ are then linked in $\check{\mathfrak{D}}_{\mathcal{E}}$ (besides being both linked to $\mathcal{U}$); this happens precisely when the chromatic number χ of the binary relation $(E, \mathcal{E}^-)$ is infinite (cf. [8]): when χ is infinite, the ideal $\mathcal{I}$ of subsets of E with a finite chromatic number is proper, hence some $\mathcal{U} \in \check{E}$ is disjoint from $\mathcal{I}$. Conversely, when χ is finite, every ultrafilter contains an independent set. ∎

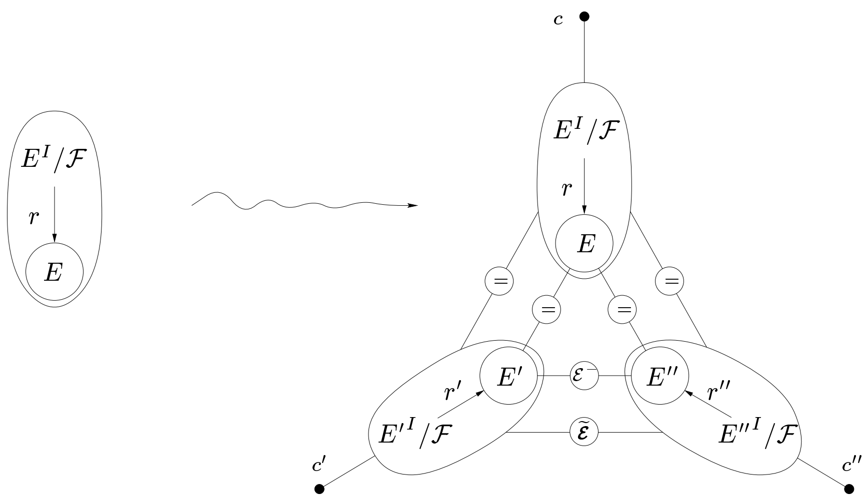
Example 8. Consider $(\mathbb{N}, \mathcal{D})$ with $\mathcal{D} = \{(p,q) : p < q \le 2p\}$. It has an infinite chromatic number since it contains complete subsets of arbitrarily large finite size (namely the intervals of the form $[p, 2p]$). Indeed, $\mathfrak{D}_{\mathcal{D}}$ is an almost locally finite graph which is not pure in its Stone–Čech extension (see the comment following the statement of Theorem 6.1).

Lemma 6.4. *If the binary relation $\mathfrak{R} = (E, \mathcal{E}^-)$ is symmetric and is atomic compact, then $\mathfrak{D}_{\mathcal{E}}$ is atomic compact as well.*

Proof. Given any ultrafilter $\mathcal{F}$ on some set I, every retraction $r : \mathfrak{R}^I/\mathcal{F} \to \mathfrak{R}$ gives rise to a retraction $s : \mathfrak{D}_{\mathcal{E}}^I/\mathcal{F} \to \mathfrak{D}_{\mathcal{E}}$ as follows.

Let $(E^I/\mathcal{F}, \widetilde{\boldsymbol{\mathcal{E}}})$ denote the relation $\mathfrak{R}^I/\mathcal{F}$. Note that $\mathfrak{D}_{\mathcal{E}}^I/\mathcal{F}$ canonically identifies with $\mathfrak{D}_{\widetilde{\boldsymbol{\mathcal{E}}}}$ (denoting by $\widetilde{\mathbf{a}} \in E^I/\mathcal{F}$ the class modulo $\mathcal{F}$ of $\mathbf{a} = (a_i : i \in I) \in E^I$, the class $\widetilde{\mathbf{a}}' \in {E'}^I/\mathcal{F}$ of $\mathbf{a}' = (a_i' : i \in I) \in {E'}^I$ identifies with $\widetilde{\mathbf{a}}' \in (E^I/\mathcal{F})'$; in the same way, $\widetilde{\mathbf{a}}'' \in {E''}^I/\mathcal{F}$ identifies with $\widetilde{\mathbf{a}}'' \in (E^I/\mathcal{F})''$). Now consider for s the extension of r, r' and r'', where $r' : {E'}^I/\mathcal{F} \to E'$ and $r'' : {E''}^I/\mathcal{F} \to E''$ correspond to r: for every $\widetilde{\mathbf{a}} \in E^I$, $r'(\widetilde{\mathbf{a}}') = (r(\widetilde{\mathbf{a}}))'$ and $r''(\widetilde{\mathbf{a}}'') = (r(\widetilde{\mathbf{a}}))''$ (see the figure below). The map s is a homomorphism since:

$$\widetilde{\mathbf{a}} \sim_{\mathfrak{D}^I_{\mathcal{E}}/\mathcal{F}} \widetilde{\mathbf{b}}' \;\Leftrightarrow \widetilde{\mathbf{a}} = \widetilde{\mathbf{b}} \Rightarrow r(\widetilde{\mathbf{a}}) = r(\widetilde{\mathbf{b}}) \Leftrightarrow r(\widetilde{\mathbf{a}}) \sim_{\mathfrak{D}_{\mathcal{E}}} (r(\widetilde{\mathbf{b}}))' = r'(\widetilde{\mathbf{b}}'),$$
$$\widetilde{\mathbf{a}} \sim_{\mathfrak{D}^I_{\mathcal{E}}/\mathcal{F}} \widetilde{\mathbf{b}}'' \Rightarrow r(\widetilde{\mathbf{a}}) \sim_{\mathfrak{D}_{\mathcal{E}}} r''(\widetilde{\mathbf{b}}''),$$
$$\widetilde{\mathbf{a}}' \sim_{\mathfrak{D}^I_{\mathcal{E}}/\mathcal{F}} \widetilde{\mathbf{b}}'' \Leftrightarrow \widetilde{\mathbf{a}} \sim_{\mathfrak{R}^I/\mathcal{F}} \widetilde{\mathbf{b}} \Rightarrow r(\widetilde{\mathbf{a}}) \sim_{\mathfrak{R}} r(\widetilde{\mathbf{b}}) \Leftrightarrow \underbrace{(r(\widetilde{\mathbf{a}}))'}_{r'(\widetilde{\mathbf{a}}')} \sim_{\mathfrak{D}_{\mathcal{E}}} \underbrace{\big(r(\widetilde{\mathbf{b}})\big)''}_{r''(\widetilde{\mathbf{b}}'')}.$$

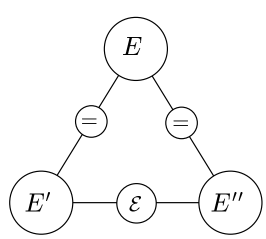


■

Recall that the atomic compact "anti-reflexive graphs" which are retracts of no topologically compact binary relation given in [8] have an infinite chromatic number, and in particular, they yield atomic compact (reflexive) graphs which are retracts of no topologically compact binary relation. Now, we can state:

THEOREM 6.2. *There exist atomic compact* (*reflexive*) *graphs which are retracts of no topologically compact relation.* ■

REMARK. Given an atomic compact anti-reflexive symmetric binary relation $(E, \mathcal{E})$ with infinite chromatic number, consider the anti-reflexive symmetric binary relation $\mathfrak{R}$ corresponding to the following diagram. It is atomic



compact, has a finite chromatic number and is a retract of no topologically compact relation, as $\breve{\mathfrak{R}}$ contains triangles while $\mathfrak{R}$ does not.

7. Admissible compactness. We briefly discuss the connection of atomic compactness with infinite projection properties.

Given a graph $\mathfrak{G}$ and an atomic system Φ of $\mathfrak{G}$, let Φ_P denote the set of elements of Φ of the form $x \sim a$ or $x \equiv a$; Φ is *ω-admissible* if Φ_P is countable and for each finite subset Ψ of Φ_P, $(\Phi \setminus \Phi_P) \cup \Psi$ is solvable (in $\mathfrak{G}$); the graph $\mathfrak{G}$ is *ω-admissible compact* if all its ω-admissible systems are solvable. (In the case of a general structure $\mathfrak{M} = (M, \ldots)$, Φ_P would be the set of formulas $\varphi(x_1, \ldots, x_p, a_1, \ldots, a_q)$ from Φ satisfying: for every $(b_1, \ldots, b_p) \in M^p$, there is $(c_1, \ldots, c_q) \in M^q$ such that $\mathfrak{M} \models \varphi(b_1, \ldots, b_p, c_1, \ldots, c_q)$ (see [3]).)

Obviously, atomic compact graphs are ω-admissible compact. Moreover, ω-admissible compact graphs are $[\omega^-]$-ω-compact, but the converse need not hold, as shown by the graph $\mathfrak{C}_{\mathbb{N}}$ (recall that it is $[\omega^-]$-compact by Theorem 4.2, but observe that the system of Example 3 is ω-admissible). Let us also mention the following surprising fact about shift-graphs:

EXAMPLE 9. Recall that the *shift-graph* of a linear order $\mathfrak{R} = (E, \leq)$ is the graph $\mathfrak{S}_{\mathfrak{R}}$ with vertices the pairs of distinct elements of E, and such that two distinct pairs are linked by an edge if and only if the greater element of one of these pairs equals the smaller element of the other pair.

PROPOSITION 7.1 (see [2]). *For every linear order $\mathfrak{R}$, every induced subgraph of $\mathfrak{S}_{\mathfrak{R}}$ is $[\omega^-]$-atomic compact, but $\mathfrak{S}_{\mathfrak{R}}$ is atomic compact if and only if $\mathfrak{R}$ is finite or isomorphic to one of the following lexicographical sums: $2+\mathbb{Z}+2$, $\mathbb{Z}_+ + 2$ or $2+\mathbb{Z}_-$, and in this case, it is topologically compact; in the opposite case, and provided that $\mathfrak{R}$ is countable, $\mathfrak{S}_{\mathfrak{R}}$ is not ω-admissible.* ■

Our reason for introducing admissible compactness is the following: Given a cardinal κ, a structure $\mathfrak{M} = (M, \ldots)$ has the *κ-projection property* (κ-PP for short) if the only retractions of $\mathfrak{M}^\kappa$ onto $\mathfrak{M}$ (diagonally embedded) are the projections [1]. (Note that, being precisely a retract of its ultrapowers, an atomic compact structure has the κ-PP for no infinite κ.) Conversely, a structure with the finite-projection property (n-PP for every integer n) has the countable-projection property (ω-PP) provided it is not ω-admissible compact and satisfies some simple additional necessary conditions [3]; for instance, a graph has the ω-PP provided it has the 2-PP, has a finite diameter and is not ω-admissible compact. In particular $\mathfrak{D}_E$, $\mathfrak{H}_n$ and $\mathfrak{C}_{\mathbb{N}}$ have the 2-PP (they are connected, free of triangles and free of dangling trees; see [4]), so they have the countable-projection property as well. In fact, we wonder whether ω-admissible compact graphs are $[\omega]$-atomic compact too. In the case of an affirmative answer, a countable graph would have the countable-projection property if and only if it has the 2-PP, has a finite diameter and is not atomic compact.

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