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On finite sum theorems for transfinite inductive dimensions

by

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Abstract. We discuss the exactness of estimates in the finite sum theorems for transfinite inductive dimensions trind and trInd. The technique obtained gives an opportunity to repeat and sometimes strengthen some well known results about compacta with trind $\neq$ trInd. In particular we improve an estimate of the small transfinite inductive dimension of Smirnov's compacta $S^{\alpha}, \alpha < \omega_1$, given by Luxemburg [Lu2].

1. Introduction. All our spaces will be metrizable separable. By trind (resp. trInd) we denote Hurewicz's (resp. Smirnov's) transfinite extension of ind (resp. Ind).

It is well known that for any space X one has $\operatorname{ind} X = \operatorname{Ind} X$ and if $X = \bigcup_{i=1}^{\infty} X_i$, where each X_i is closed in X, then $\operatorname{ind} X = \sup\{\operatorname{ind} X_i\}$.

In the transfinite case there exist a compact space X with $\operatorname{trind} X \neq \operatorname{trInd} X$ and a compact space Y which can be represented as the union of two closed subspaces Y_1 and Y_2 such that $\operatorname{trind} Y > \max\{\operatorname{trInd} Y_1, \operatorname{trInd} Y_2\}$. At the same time there exist estimates of $\operatorname{trind} X$ (resp. $\operatorname{trInd} X$) for X being the union of two closed subspaces X_1 and X_2 in terms of $\operatorname{trind} X_1$ and $\operatorname{trind} X_2$ (resp. $\operatorname{trInd} X_1$ and $\operatorname{trInd} X_2$), which are called *finite sum theorems* for trind (resp. trInd) (cf. [E]).

In this paper we show that the estimates for trInd are exact in any class of metrizable compacta containing all Smirnov compacta and their closed subspaces. We improve one of the estimates for trind. The technique obtained gives an opportunity to repeat and sometimes strengthen some well known results of Luxemburg [Lu1, Lu2] about compacta with trind different from trInd. In particular we obtain an estimate of $\operatorname{trind} S^{\alpha}$, where S^{α}, $\alpha < \omega_1$, are Smirnov's compacta [S], better than the estimates given before.

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2. Decompositions of spaces

DEFINITION 2.1. Let X be a metric space. A decomposition

$$X = F \cup \bigcup_{i=1}^{\infty} E_i$$

of X into disjoint sets is called *A-special* (resp. *B-special*) if E_i is clopen in X (resp. E_i is clopen in X and $\lim_{n\to\infty} \delta(E_i) = 0$, where $\delta(A)$ is the diameter of A).

Observe that the product of two spaces admits an A-special decomposition into disjoint nonempty sets if one of the factors does. The one-point compactification of the free union of countably many nonempty compacta admits a B-special decomposition into disjoint nonempty sets.

LEMMA 2.2. *Let X be a compact space and $X = F \cup \bigcup_{i=1}^{\infty} E_i$ be an A-special decomposition. If $\dim F = n \geq 1$, then $X = \bigcup_{k=1}^{n+1} Z_k$, where each Z_k is closed in X and admits a B-special decomposition $Z_k = F \cup \bigcup_{j=1}^{\infty} E_j^k$ with $E_j^k \subset E_i$ for a finite number of indices j for every i.*

Proof. Observe that

$(*)$ for any open nbd OF of F there exists a natural number N such that $E_i \subset OF$ for $i \geq N$.

Let $\varepsilon > 0$. Choose finite systems B_k^{ε}, $k = 1, \ldots, n+1$, consisting of disjoint compact sets with diameter $< \varepsilon$ such that $\bigcup_{k=1}^{n+1} B_k^{\varepsilon}$ contains a nbd OF of F open in X. By $(*)$ there exists a number $N(\varepsilon)$ such that $E_i \subset OF \subset \bigcup_{k=1}^{n+1} B_k^{\varepsilon}$ for $i \geq N(\varepsilon)$.

For every natural number $p \geq 1$ choose finite systems $B_k^{(p)} = B_k^{1/p}$, $k = 1, \ldots, n+1$, and a number $N_p = N(1/p)$ such that $N_q > N_p$ if $q > p$. Define

$$Z_1 = F \cup \bigcup_{i=1}^{N_1-1} E_i \cup \bigcup_{p=1}^{\infty} \bigcup_{i=N_p}^{N_{p+1}-1} \{B \cap E_i : B \in B_1^{(p)}\},$$

$$Z_k = F \cup \bigcup_{p=1}^{\infty} \bigcup_{i=N_p}^{N_{p+1}-1} \{B \cap E_i : B \in B_k^{(p)}\}, \qquad k = 2, \ldots, n+1.$$

LEMMA 2.3. *Let $X = F \cup \bigcup_{i=1}^{\infty} E_i$ be a B-special decomposition of the metric space X and A, B be disjoint closed subsets of X such that $A \cap F \neq \emptyset$, $B \cap F \neq \emptyset$ and A is compact. If C_F is a partition in F between $A \cap F$*

and $B \cap F$ then there exist a partition C between A and B in X and a natural number m such that

(a) $C = (C \cap F) \cup \bigcup_{i=1}^{m} C_i$, *where C_i is an arbitrary partition in E_i between $A \cap E_i$ and $B \cap E_i$ (C_i is empty if $A \cap E_i$ or $B \cap E_i$ is empty)*;

(b) $C \cap F \subset C_F$.

Proof. Let $f : F \cup A \cup B \to [-1, 1]$ be such that $f^{-1}(-1) = A$, $f^{-1}(0) = C_F$, $f^{-1}(1) = B$. Consider an extension $g : X \to [-1, 1]$ of f. Put $\varepsilon = \delta(A, g^{-1}[0, 1]) > 0$ and choose a natural number m such that $\delta(E_i) < \varepsilon/2$ for all $i > m$. In the clopen subset $Y = X \setminus \bigcup_{i=1}^{m} E_i$ of X take an open set $U = (g^{-1}(0, 1] \cap Y) \cup \bigcup\{E_i : E_i \cap g^{-1}[0, 1] \neq \emptyset \text{ and } i > m\}$. Observe that $\operatorname{Bd} U \subset C_F$. In every set E_i, $i \le m$, consider a partition C_i between $A \cap E_i$ and $B \cap E_i$ (let C_i be empty if at least one of the sets is empty). It is clear that the set $C = \operatorname{Bd} U \cup \bigcup_{i=1}^{m} C_i$ satisfies the required conditions.

3. Finite sum theorems. Recall the definitions of the transfinite inductive dimensions trind and trInd.

DEFINITION. Let X be a space. Then

(i) $\operatorname{trInd} X = -1 \Leftrightarrow X = \emptyset$;

(ii) $\operatorname{trInd} X \le \alpha$, where α is an ordinal number, if for every closed set $A \subset X$ and each open set $V \subset X$ which contains A there exists an open set $U \subset X$ such that $A \subset U \subset V$ and $\operatorname{trInd} \operatorname{Bd} U < \alpha$;

(iii) $\operatorname{trInd} X = \alpha \Leftrightarrow \operatorname{trInd} X \le \alpha$ and the inequality $\operatorname{trInd} X \le \beta$ holds for no $\beta < \alpha$;

(iv) $\operatorname{trInd} X = \infty \Leftrightarrow \operatorname{trInd} X \le \alpha$ holds for no ordinal α.

The definition of trind is obtained by replacing the set A in (ii) with a point of X.

In the sequel, $\alpha = \lambda(\alpha) + n(\alpha)$ is the natural decomposition of the ordinal α into the sum of a limit ordinal $\lambda(\alpha)$ and a nonnegative integer $n(\alpha)$.

The following two finite sum theorems for trind and trInd are due to Toulmin, Levshenko, Landau and Pears (cf. [E]).

THEOREM 3.1. *Let d be* trind *or* trInd. *If a space X is the union of two closed subspaces F_1 and F_2 such that $dF_i \le \alpha_i$, $i = 1, 2$, and $\alpha_2 \ge \alpha_1$, then*

$$dX \le \begin{cases} \alpha_2 & \text{if } \lambda(\alpha_1) < \lambda(\alpha_2), \\ \alpha_2 + n(\alpha_1) + 1 & \text{if } \lambda(\alpha_1) = \lambda(\alpha_2). \end{cases}$$

THEOREM 3.2. *Let d be* trind *or* trInd. *If a space X is the union of two closed subspaces F_1 and F_2 such that $dF_1 \le dF_2 \le \alpha_2$ and $d(F_1 \cap F_2) \le \alpha_1 \le \alpha_2$, then*

$$dX \le \begin{cases} \alpha_2 & \text{if } \lambda(\alpha_1) < \lambda(\alpha_2), \\ \alpha_2 + n(\alpha_1) + 1 & \text{if } \lambda(\alpha_1) = \lambda(\alpha_2). \end{cases}$$

One can ask

QUESTION. *Are these estimates exact?*

In order to answer this question we need some statements.

LEMMA 3.3. *Let X be a space with* $\operatorname{trInd} X = \alpha$, $n(\alpha) \ge 1$. *Then*

(a) $X \neq \bigcup_{i=1}^{n(\alpha)} P_i$ *for any P_i closed, and* $\operatorname{trInd} P_i \le \lambda(\alpha)$.

If, in addition, $X = \bigcup_{i=1}^{n(\alpha)+1} Z_i$, *where each Z_i is closed and* $\operatorname{trInd} Z_i \le \lambda(\alpha)$, *then*

(b) $\operatorname{trInd}(Z_1 \cup \ldots \cup Z_{k+1}) = \lambda(\alpha) + k$ *for any k with* $0 \le k \le n(\alpha)$;

(c) $\operatorname{trInd}((Z_1 \cup \ldots \cup Z_{i+1}) \cap (Z_{i+2} \cup \ldots \cup Z_{i+j+2})) = \lambda(\alpha) + \min\{i, j\}$ *for any nonnegative integers i, j such that $i + j + 1 \le n(\alpha)$.*

Proof. (a) If $X = \bigcup_{i=1}^{n(\alpha)} P_i$ apply Theorem 3.1 consecutively $n(\alpha) - 1$ times to get $\operatorname{trInd} \bigcup_{i=1}^{n(\alpha)} P_i \le \alpha - 1$, a contradiction.

(b) By Theorem 3.1 we have $\operatorname{trInd}(Z_1 \cup \ldots \cup Z_{k+1}) \le \lambda(\alpha) + k$. If $\operatorname{trInd}(Z_1 \cup \ldots \cup Z_{k+1}) < \lambda(\alpha) + k$ apply Theorem 3.1 to the union $(Z_1 \cup \ldots \cup Z_{k+1}) \cup (Z_{k+2} \cup \ldots \cup Z_{n(\alpha)+1})$. We again get $\operatorname{trInd} \bigcup_{i=1}^{n(\alpha)+1} Z_i \le \alpha - 1$.

(c) Apply (b) and Theorem 3.2.

Applying Lemmas 2.2, 2.3 and Theorem 3.2 one easily shows the following lemma.

LEMMA 3.4. (a) *Let $X = F \cup \bigcup_{i=1}^{\infty} E_i$ be a B-special decomposition and α be an ordinal. If* $\sup\{\operatorname{trind} F, \operatorname{trind} E_i\} \le \alpha$ *then* $\operatorname{trind} X \le \alpha$, *and if X is compact and* $\sup\{\operatorname{trInd} F, \operatorname{trInd} E_i\} \le \alpha$ *then* $\operatorname{trInd} X \le \alpha$.

(b) *Let $X = F \cup \bigcup_{i=1}^{\infty} E_i$ be an A-special decomposition of the compact space X, α be a limit ordinal and d be* trind *or* trInd. *If* $\dim F \le n$ *and* $\sup\{dE_i\} \le \alpha$ *then* $X = \bigcup_{k=1}^{n+1} Z_k$, *where Z_k is closed in X and $dZ_k \le \alpha$.*

Observe that in the case of trind the statement of Lemma 3.4(a) is almost the same as Lemma 3.4 from [Lu2] for $k = 1$ (cf. also [E], Problem 7.1.G(c)).

Recall that *Smirnov's compacta* $S^0, S^1, \ldots, S^\alpha, \ldots$, $\alpha < \omega_1$, are defined by transfinite induction (see [S]): S^0 is a one-point space, $S^\alpha = S^\beta \times I$ for $\alpha = \beta + 1$, and if α is a limit ordinal, then $S^\alpha = \{*_\alpha\} \cup \bigcup_{\beta<\alpha} S^\beta$ is the one-point compactification of the free union of all the previously defined S^β's, where $*_\alpha$ is the compactification point. It is well known that $\operatorname{trInd} S^\alpha = \alpha$ for every $\alpha < \omega_1$.

In [Le] Levshenko proved that $S^{\omega_0+1} = Z_1 \cup Z_2$, where Z_i is closed in S^{ω_0+1} and $\operatorname{trInd} Z_i = \omega_0$. Now we prove a generalization of this fact.

LEMMA 3.5. *Let α be an ordinal $< \omega_1$. Then*

(a) $S^\alpha = \bigcup_{i=1}^{n(\alpha)+1} Z_i$, *where each Z_i is closed in S^α,*

$$\operatorname{trInd}(Z_1 \cup \ldots \cup Z_{k+1}) = \lambda(\alpha) + k$$

for any k with $0 \le k \le n(\alpha)$, and

$$\operatorname{trInd}((Z_1 \cup \ldots \cup Z_{i+1}) \cap (Z_{i+2} \cup \ldots \cup Z_{i+j+2})) = \lambda(\alpha) + \min\{i, j\}$$

for any nonnegative integers i, j such that $i + j + 1 \le n(\alpha)$;

(b) $S^\alpha \ne \bigcup_{i=1}^{n(\alpha)} P_i$ *for any P_i closed in S^α with* $\operatorname{trInd} P_i \le \lambda(\alpha)$.

Proof. Observe that

$$\begin{aligned} S^\alpha &= \{*_{\lambda(\alpha)}\} \times I^{n(\alpha)} \cup \bigcup\{S^\beta \times I^{n(\alpha)} : \beta < \lambda(\alpha)\} \\ &= \{*_{\lambda(\alpha)}\} \times I^{n(\alpha)} \cup \bigcup\{S^{\beta+n(\alpha)} : \beta < \lambda(\alpha)\} \end{aligned}$$

is an A-special decomposition with $\dim(\{*_{\lambda(\alpha)}\} \times I^{n(\alpha)}) = n(\alpha)$ and with $\sup\{\operatorname{trInd} S^{\beta+n(\alpha)}\} \le \lambda(\alpha)$. Now apply Lemmas 3.3 and 3.4.

From Lemma 3.5 we obtain a complement to Theorems 3.1 and 3.2 (the case of trInd) showing the exactness of the estimates:

THEOREM 3.6. *For any infinite countable ordinal α with $n(\alpha) \ge 1$ there exists a compact space X_α with* $\operatorname{trInd} X_\alpha = \alpha$ *such that for any nonnegative integers p, q with $p + q = n(\alpha) - 1$ there exist closed subsets X_p and X_q of X such that $X_\alpha = X_p \cup X_q$,* $\operatorname{trInd} X_p = \lambda(\alpha) + p$, $\operatorname{trInd} X_q = \lambda(\alpha) + q$ *and* $\operatorname{trInd}(X_q \cap X_p) = \lambda(\alpha) + \min\{p, q\}$.

In order to improve Theorem 3.1 (the case of trind) we need the following two statements. The first one is evident, the proof of the second is left to the reader.

LEMMA 3.7. *Let $X = X_1 \cup X_2$. If* $\operatorname{Int} X_1 \cup \operatorname{Int} X_2 = X$ *and* $\operatorname{trind} X_i \le \alpha_i$, $i = 1, 2$, *then* $\operatorname{trind} X \le \max\{\alpha_1, \alpha_2\}$.

LEMMA 3.8. *Let $X = F_1 \cup F_2$, where F_i is closed in X. Let A and B be two disjoint closed subsets of X, and C_i be a partition in F_i between $A \cap F_i$ and $B \cap F_i$. Then there exists a partition C in X between A and B such that $C \subset C_1 \cup C_2 \cup (F_1 \cap F_2)$.*

Observe that Lemma 3.8 is a particular case of a more general result (see Lemma 2 of [Ch]) which was communicated to me by Pasynkov some years ago.

Now we are ready to consider a revision of Theorem 3.1 (the case of trind):

THEOREM 3.9. *Let $X = X_1 \cup X_2$, where X_i is closed in X and* $\operatorname{trind} X_i = \alpha_i$, $i = 1, 2$. *Then*

(a) *for any two closed subsets A and B of X there exists a partition C between A and B such that* $\operatorname{trind} C \le \max\{\alpha_1, \alpha_2\}$;

(b) $\max\{\alpha_1, \alpha_2\} \le \operatorname{trind} X \le \max\{\alpha_1, \alpha_2\} + 1$.

Proof. (a) If A or B is disjoint from X_i for some $i = 1, 2$, then one can find a partition C in X between A and B such that $C \cap X_i = \emptyset$. So we have $\operatorname{trind} C \le \max\{\operatorname{trind} X_1, \operatorname{trind} X_2\}$. Assume now that $A \cap X_i \ne \emptyset$ and $B \cap X_i \ne \emptyset$ for each $i = 1, 2$. Choose a partition C_1 in X_1 between $A \cap X_1$ and $B \cap X_1$. Let $X_1 \setminus C_1 = U_1 \cup V_1$, where U_1, V_1 are open in X_1 and disjoint, and $A \cap X_1 \subset U_1$. Choose a partition C_2 in X_2 between $A \cap X_2$ and $((C_1 \cup V_1) \cup B) \cap X_2$. Observe that

$$Y = C_1 \cup C_2 \cup (X_1 \cap X_2) = Y_1 \cup Y_2,$$

where $Y_i = C_i \cup (X_1 \cap X_2)$. Moreover $\operatorname{Int} Y_1 \cup \operatorname{Int} Y_2 = Y$, $\operatorname{trind} Y_i \le \alpha_i$ (recall that $Y_i \subset X_i$). So $\operatorname{trind} Y \le \max\{\alpha_1, \alpha_2\}$ by Lemma 3.7. Observe that by Lemma 3.8 there exists a partition C between A and B such that $C \subset Y$. Consequently, $\operatorname{trind} C \le \max\{\alpha_1, \alpha_2\}$.

(b) The statement follows from (a).

Corollary 3.10. *Let X be a space and α be an ordinal.*

(a) *If $X = \bigcup_{k=1}^{n+1} X_k$, where each X_k is closed in X, $0 \le n \le 2^m - 1$ for some integer m and* $\max\{\operatorname{trind} X_k\} \le \alpha$ *then* $\operatorname{trind} X \le \alpha + m$.

(b) *If* $\operatorname{trind} X = \alpha + n$, $n \ge 1$ *then $X \ne \bigcup_{i=1}^{k} P_i$, where each P_i is closed in X*, $\operatorname{trind} P_i \le \alpha$ *and* $k \le 2^{n-1}$.

(c) *If $X = X_1 \cup X_2$, where each X_i is closed in X*, $\operatorname{trInd} X = \alpha$, $n(\alpha) \ge 2$ *and* $\max\{\operatorname{trind} X_k\} \le \alpha - 2$ *then* $\operatorname{trind} X < \operatorname{trInd} X$.

Proof. (a) Let $n = 2^m - 1$. For every integer j such that $1 \le j \le 2^{m-1}$ put $X_j^{(1)} = X_{2j-1} \cup X_{2j}$. Theorem 3.9 yields $\operatorname{trind} X_j^{(1)} \le \alpha + 1$. For every integer p such that $1 \le p \le 2^{m-2}$ put $X_p^{(2)} = X_{2p-1}^{(1)} \cup X_{2p}^{(1)}$. Theorem 3.9 shows $\operatorname{trind} X_p^{(2)} \le \alpha + 2$, and so on. Observe that $X = X_1^{(m)}$. It is clear that $\operatorname{trind} X \le \alpha + m$.

(b) Apply the proof of (a).

Corollary 3.11. *Let X be a compact space and λ be a limit ordinal.*

(a) *If $X = F \cup \bigcup_{i=1}^{\infty} E_i$ is an A-special decomposition such that* $\dim F = n \ge 1$, $\sup\{\operatorname{trind} E_i\} \le \lambda$ *and* $n \le 2^m - 1$ *for some integer m then* $\operatorname{trind} X \le \lambda + m$.

(b) *If F is a closed subset of X such that* $\dim F = n \ge 1$, $\sup\{\operatorname{trind}_x X : x \in X \setminus F\} \le \lambda$ *and* $n \le 2^m - 1$ *for some integer m then* $\operatorname{trind} X \le \lambda + m + 1$.

Proof. (a) By Lemma 3.4(b) we have $X = \bigcup_{k=1}^{n+1} Z_k$, where each Z_k is closed in X and $\operatorname{trind} Z_k \le \lambda$ for every $k = 1, \ldots, n + 1$. Now apply Corollary 3.10(a).

(b) It is clear that the compactum X can be written as the union of two closed subsets $X_1,\ X_2$ such that each X_i has a decomposition as in (a). So $\operatorname{trind} X_i \le \lambda + m, i = 1,2$. Now apply Theorem 3.9.

REMARK 3.12. Recall that $\sup\{\operatorname{trind} S^\alpha : \alpha < \omega_1\} = \omega_1$ (see [Le]). So the estimates from Theorem 3.9 are exact in any class of metrizable compacta containing all Smirnov compacta and their closed subspaces.

REMARK 3.13. Observe that the estimates of trind from Theorem 3.9(a) and Corollary 3.10(a), (b) are also valid for regular T_1-spaces [Ch-K].

4. Estimates of $\operatorname{trind} S^\alpha$**,** $\alpha < \omega_1$**.** In [Lu1] Luxemburg proved that $\operatorname{trind} S^{\omega_0+2} = \operatorname{trind} S^{\omega_0+3} = \omega_0 + 2$. It was the first example of a metrizable compact space with noncoinciding transfinite trind and trInd. Observe that four years earlier Filippov [F] constructed the first example of a nonmetrizable compact space with noncoinciding finite ind and Ind. Recall that $S^{\omega_0+3} = S^{\omega_0+2} \times [0,1]$. So it was also an example where $\operatorname{trind}(X \times [0,1]) < \operatorname{trind} X + 1$ (recall that in the finite-dimensional case for any metrizable compact space X we always have equality). Later on Luxemburg [Lu2] also obtained an estimate of trind for all Smirnov compacta. Namely, $\operatorname{trind} S^\alpha \le \lambda(\alpha) + [(n(\alpha)+2)/2]$ for every infinite ordinal $\alpha < \omega_1$.

We now have the following estimate of $\operatorname{trind} S^\alpha,\ \alpha < \omega_1$.

THEOREM 4.1. *If α is an infinite ordinal and $n(\alpha) \le 2^m - 1$ for some integer m then* $\operatorname{trind} S^\alpha \le \lambda(\alpha) + m$. *In particular* $\operatorname{trind} S^{\omega_0+3} \le \omega_0 + 2$.

Proof. Apply Corollary 3.11(a).

REMARK 4.2. One can easily prove that $\operatorname{trind} S^{\omega_0+2} \ge \omega_0 + 2$ ($\operatorname{trind} Y \le \omega_0$ implies $\operatorname{trInd} Y \le \omega_0$ for any compact space Y). So $\operatorname{trind} S^{\omega_0+2} = \omega_0 + 2$. Now recall that $S^{\omega_0+2} \subset S^{\omega_0+3}$, which gives $\operatorname{trind} S^{\omega_0+3} = \omega_0 + 2$.

THEOREM 4.3. (a) *Let n be a natural number and $m = \min\{k : n+3 \le 2^k\}$. Then $n + 2 > m$.*

(b) $\operatorname{trind}(S^{\omega_0+2} \times X) < \operatorname{trind} S^{\omega_0+2} + \dim X$ *for any finite-dimensional space X such that* $\dim X \ge 1$ (Theorem 7.2 of [Lu2]).

Proof. (a) Apply induction.

(b) Let $\dim X = n \ge 1$ and Y be a compactification of X such that $\dim Y = n$ (cf. [E]). Observe that

$$\begin{aligned} S^{\omega_0+2} \times Y &= (\{*_{\omega_0}\} \times I^2 \times Y) \cup \bigcup \{I^k \times I^2 \times Y : k < \omega_0\} \\ &= (\{*_{\omega_0}\} \times I^2 \times Y) \cup \bigcup \{I^{k+2} \times Y : k < \omega_0\} \end{aligned}$$

is an A-special decomposition with $\dim(\{*_{\omega_0}\} \times I^2 \times Y) = n + 2$ and $\sup\{\operatorname{trind}(I^{k+2} \times Y)\} \le \omega_0$. By Corollary 3.11(a) we have $\operatorname{trind}(S^{\omega_0+2} \times Y) \le$

$\omega_0 + m$, where $m = \min\{k : n+2 \le 2^k - 1\}$. By (a) we get

$$\begin{aligned}\operatorname{trind}(S^{\omega_0+2} \times Y) &\le \omega_0 + m < \omega_0 + (n+2) = (\omega_0 + 2) + n \\ &= \operatorname{trind} S^{\omega_0+2} + \dim Y.\end{aligned}$$

Observe that $S^{\omega_0+2} \times X \subset S^{\omega_0+2} \times Y$. This yields the assertion.

CONJECTURE. *If α is an infinite ordinal and $n(\alpha) = 2^{m-1}$ for some integer $m \ge 1$ then* $\operatorname{trind} S^\alpha = \lambda(\alpha) + m$.

Observe that by Theorem 4.1 the proof of the conjecture would solve a long standing problem of computing the small transfinite dimension of every Smirnov space.

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