FUNDAMENTA
MATHEMATICAE
162 (1999)

A Lefschetz-type coincidence theorem

by

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Abstract. A Lefschetz-type coincidence theorem for two maps $f, g : X \to Y$ from an arbitrary topological space to a manifold is given: $I_{fg} = \lambda_{fg}$, that is, the coincidence index is equal to the Lefschetz number. It follows that if $\lambda_{fg} \neq 0$ then there is an $x \in X$ such that $f(x) = g(x)$. In particular, the theorem contains well-known coincidence results for (i) X, Y manifolds, f boundary-preserving, and (ii) Y Euclidean, f with acyclic fibres. It also implies certain fixed point results for multivalued maps with "point-like" (acyclic) and "sphere-like" values.

1. Introduction. A Lefschetz-type coincidence theorem states the following. Given a pair of continuous maps $f, g : X \to Y$, the Lefschetz number λ_{fg} of the pair (f, g) is equal to its coincidence index I_{fg}, while I_{fg} is defined in such a way that

$$I_{fg} \neq 0 \Rightarrow f(x) = g(x) \text{ for some } x \in X.$$

Thus, if the Lefschetz number, a computable homotopy invariant of the pair, is not zero, then there is a coincidence. We now consider two ways to define the coincidence index in two different settings.

Case 1. Let M_1, M_2 be closed n-manifolds, X an open subset of M_1, N an open subset of M_1, V an open subset of M_2, $f, g : X \to V \subset M_2$ maps, $\{x \in X : f(x) = g(x)\} \subset N \subset \overline{N} \subset X \subset M_1$. Then the *coincidence index* I_{fg}^X [29, p. 177] is the image of the fundamental class O_{M_1} of M_1 under the composition

$$\begin{aligned} H_n(M_1) &\xrightarrow{\text{inclusion}} H_n(M_1, M_1 \setminus N) \xrightarrow{\text{excision}} H_n(X, X \setminus N) \\ &\xrightarrow{(f,g)_*} H_n(M_2 \times M_2, M_2 \times M_2 \setminus \delta(M_2)) \simeq \mathbb{Q}, \end{aligned}$$

where $\delta(x) = (x, x)$.

1991 *Mathematics Subject Classification*: 55M20, 55H25.

Key words and phrases: Lefschetz coincidence theory, Lefschetz number, coincidence index, fixed point, multivalued map.

This definition represents the original approach to the coincidence problem for closed manifolds due to Lefschetz [23]. It was later generalized to the case of manifolds with boundary (and a boundary-preserving f) by Nakaoka [25], Davidyan [6, 7], Mukherjea [24].

Case 2. Let V be an open subset of n-dimensional Euclidean space, $f : X \to V$ a Vietoris map (i.e., $f^{-1}(y)$ is acyclic for each $y \in V$), $g : X \to K$ a map, $K \subset V$ a finite polyhedron, $N = f^{-1}(K)$. Then the *coincidence index* $I(f, g)$ [15, p. 38] is the image of the fundamental class O_K of K under the composition

$$H_n(V, V \setminus K) \xrightarrow{f_*^{-1}} H_n(X, X \setminus N) \xrightarrow{(f-g)_*} H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \simeq \mathbb{Q}.$$

This coincidence index has evolved from Hopf's fixed point index I_g ($X = V$, $f = \mathrm{Id}_V$); see Brown [3, Chapter IV], Dold [10], [7, VII.5]. This approach was developed by Eilenberg and Montgomery [14], Begle [1], Górniewicz and Granas [15, 17, 18] and others (see [15] for a bibliography). It does not require any knowledge of the topology of X and, for this reason, is especially well suited for the study of fixed points of multivalued maps. For an acyclic-valued multifunction $F : Y \to Y$, we let X be the graph of F and f, g be the projections of X on Y; then f is a Vietoris map, and a coincidence of (f, g) is a fixed point of F. This construction cannot be applied to Case 1 because the graph of a multifunction $F : M_2 \to M_2$ is not, in general, a manifold.

The restrictions on spaces and maps in Case 1 and Case 2 are necessary to ensure the existence of an appropriate homomorphism $f_! : H(Y) \to H(X)$, which we call here the transfer of f. Then the Lefschetz number of $\varphi_{fg} = g_* f_!$ is said to be the Lefschetz number of the pair (f, g). For Case 1,

$$f_! = D_1 f^* D_2^{-1},$$

where D_1 and D_2 are the Poincaré duality isomorphisms for the manifolds M_1 and M_2. For Case 2,

$$f_! = f_*^{-1},$$

with the existence of f_*^{-1} guaranteed by the Vietoris–Begle Mapping Theorem 4.4.

Until now these two ways to treat the same problem have been studied separately. In this paper we provide a unified approach. We define the coincidence index as in Case 1, for arbitrary maps to an n-manifold (with or without boundary), $n \geq 1$, but with no restriction on their domain, as in Case 2. Roughly, we combine

$$\text{Case 1:} \quad n\text{-manifold} \xrightarrow{\text{any map}} n\text{-manifold, and}$$

$$\text{Case 2:} \quad \text{any space} \xrightarrow{\text{Vietoris map}} \text{Euclidean space,}$$

into

$$\text{any space} \xrightarrow{\text{any map}} \text{manifold.}$$

Under the restrictions of Cases 1 and 2, our main theorem reduces to the results mentioned above (see Sections 4 and 10), but it also applies to the case

$$\text{non-manifold} \xrightarrow{\text{any map}} \text{manifold,}$$

as well as

$$m\text{-manifold} \xrightarrow{\text{any map}} n\text{-manifold, } m \neq n$$

(see Section 5). For the sake of simplicity, we limit our attention to the case when Y is a subset of a manifold, although some of the results can be extended to include spaces as general as ANR's.

An important particular situation when the choice of the transfer is obvious (see Corollary 5.1 and Proposition 9.1) occurs if the condition below is satisfied:

(A) $f_* : H_n(X, X \setminus N) \to H_n(V, V \setminus K)$ is nonzero

(note that, in the case of two n-manifolds, the condition simply means that f has nonzero degree). This condition can be fairly easily verified for specific spaces and maps. In particular, we can relax the Vietoris condition on f or assume that f is a fibration (see Section 5). Furthermore, when $g_* = 0$ (in reduced homology) the Lefschetz number of $\varphi_{fg} = g_* f_!$ is equal to 1, so condition (A) implies the existence of a coincidence. For example, $f : (\mathbf{D}^2, \mathbf{S}^1) \to (\mathbf{S}^2, \{*\})$ has a coincidence with any $g : \mathbf{D}^2 \to \mathbf{S}^2$.

Our approach seems to be related to a suggestion made by Dold in [12]. The subject of his paper is coincidences on ENR_B's, Euclidean neighborhood retracts over a space B, and will remain outside the scope of the present paper. At the end of his paper Dold compares his Theorem 2.1 to a result that assumes that one of the maps is Vietoris (Case 2): "It appears less general than 2.1 because 2.1 makes no acyclicity assumption. (...) On the other hand, it has a more general aspect than 2.1 because it doesn't assume an actual fibration (or ENR_B), only a 'cohomology fibration' (with 'point-like' fibres). This comparison suggests a common generalization, namely to general *cohomology fibrations* (...)" (cf. Section 9).

The proofs of our main theorems are self-contained and use some constructions from Górniewicz [15, V.5.1, pp. 38–40] (see also Dold [7, VII.6, pp. 207–211]) and Vick [29, Chapter 6].

The paper is organized as follows. In Section 2 we present our main results (Theorems 2.1–2.3) and in Section 3 we prove Theorem 2.1. In Section 4 we prove our main Theorem 2.3 (a Lefschetz-type coincidence theorem for maps to a manifold with boundary) and obtain Nakaoka's Coincidence Theorem for boundary-preserving maps between manifolds and Górniewicz's

Coincidence Theorem for Vietoris maps. Section 5 is devoted to applications of the main theorem and examples with the emphasis on situations that are not covered by the two traditions discussed above. Sections 6–8 contain the proof of Theorem 2.2 in a slightly more general setting (for maps to an open subset of a manifold). In the last two sections we prove a Lefschetz-type coincidence theorem for generalized Case 2.

2. Main results. Let $E = \{E_q\}$ be a graded $\mathbb{Q}$-module with

$$\dim E_q < \infty, \quad q = 0, \ldots, n, \qquad E_q = 0, \quad q = n+1, \ldots$$

(in other words, E is finitely generated). If $h = \{h_q\}$ is an endomorphism of E of degree 0, then the *Lefschetz number* $L(h)$ of h is defined by

$$L(h) = \sum_q (-1)^q \operatorname{tr}(h_q),$$

where $\operatorname{tr}(h_q)$ is the trace of h_q.

By H we denote the singular homology and by $\check{H}$ the Čech homology with compact carriers and with coefficients in $\mathbb{Q}$. Throughout the paper M is an oriented connected compact closed n-manifold, $n \geq 0$ (although most results remain valid for a nonorientable M if we take the coefficient field to be $\mathbb{Z}_2$).

Let X be a topological space, $N \subset X$, M be an oriented connected compact closed n-manifold, $(S, \partial S)$ a connected n-submanifold with (possibly empty) boundary ∂S and interior $\mathring{S} = S \setminus \partial S$. Let

$$f : (X, X \setminus N) \to (S, \partial S), \qquad g : X \to S,$$

be continuous maps with $\operatorname{Coin}(f, g) = \{x \in X : f(x) = g(x)\} \subset N$. Let

$$M^{\times} = (M \times M, M \times M \setminus \delta(M)),$$

where $\delta(x) = (x, x)$ is the diagonal map. Then the map $f \times g : (X, X \setminus N) \times X \to M^{\times}$ is well defined.

Fix an element $\mu \in H_n(X, X \setminus N)$. The *coincidence index* I_{fg} of the pair (f, g) (with respect to μ) is defined by

$$I_{fg} = (f \times g)_* \delta_*(\mu) \in H_n(M^{\times}) \simeq \mathbb{Q}.$$

Let $O_S \in H_n(S, \partial S)$ be the fundamental class of $(S, \partial S)$. The *transfer* of f (with respect to μ) is the homomorphism $f_! : H(S) \to H(X)$ given by

$$f_! = (f^* D^{-1}) \frown \mu,$$

where $D : H^*(S, \partial S) \to H(S)$ is the Poincaré–Lefschetz duality isomorphism. Then we define $\lambda_{fg} = L(g_* f_!)$ to be the *Lefschetz number* of the pair (f, g) (with respect to μ).

The proofs of the three theorems below are located in Sections 3, 8 and 4 respectively.

THEOREM 2.1. *For a pair* $f : (X, X \setminus N) \to (S, \partial S)$, $g : X \to \mathring{S}$,

$$I_{fg} = I_*(\mathrm{Id} \otimes g_* f_!)\delta_*(O_S),$$

where $I : (S, \partial S) \times \mathring{S} \to M^\times$ *is the inclusion.*

THEOREM 2.2. *For any homomorphism* $\varphi : H(S) \to H(\mathring{S})$ *we have*

$$L(\varphi i_*) = I_*(\mathrm{Id} \otimes \varphi)\delta_*(O_S),$$

where $i : \mathring{S} \to S$ *is the inclusion.*

The following is the main theorem of the paper.

THEOREM 2.3 (Lefschetz-type theorem). *For any pair* $f : (X, X \setminus N) \to (S, \partial S)$, $g : X \to S$, *the coincidence index is equal to the Lefschetz number* (*with respect to* μ):

$$I_{fg} = L(g_* f_!).$$

Moreover, *if* $L(g_* f_!) \neq 0$, *then* (f, g) *has a coincidence.*

If $(X, X \setminus N)$ is a manifold with boundary, we get the Lefschetz-type coincidence theorem for Case 1 by letting μ be its fundamental class. To get such a theorem for Case 2 we let $M = \mathbb{R}^n \cup \{\infty\}$, $\mu = f_*^{-1}(O_S)$.

Now we consider these results in detail.

3. The transfer of a map and Theorem 2.1. Recall [29, p. 156] that if $(S, \partial S)$ is a compact oriented n-manifold, then the Poincaré–Lefschetz duality isomorphism

$$D : H^{n-k}(S, \partial S) \to H_k(S)$$

is given by $D(a) = a \frown O_S$. Suppose $f : (S_1, \partial S_1) \to (S_2, \partial S_2)$, where $(S_i, \partial S_i)$, $i = 1, 2$, are n-manifolds, is a map. Following Vick [29, Chapter 6] we could define $f_!$ as follows. If

$$D_i : H^{n-k}(S_i, \partial S_i) \to H_k(S_i), \quad i = 1, 2,$$

denote the duality isomorphisms, we let

$$f_! = D_1 f^* D_2^{-1},$$

so that $f_!$ is the composition of the following maps:

$$H_k(S_2) \xrightarrow{D_2^{-1}} H^{n-k}(S_2, \partial S_2) \xrightarrow{f^*} H^{n-k}(S_1, \partial S_1) \xrightarrow{D_1} H_k(S_1).$$

Similarly we define $f_!$ for $f : (X, X \setminus N) \to (S, \partial S)$, where X is an arbitrary topological space: the transfer of f is the homomorphism $f_! : H(S) \to H(X)$ given by

$$f_! = (f^* D_2^{-1}) \frown \mu,$$

where $D_2 : H^*(S, \partial S) \to H(S)$ is the Poincaré–Lefschetz duality isomorphism.

To prove Theorem 2.1 we use some arguments from Vick [29, pp. 184–186]. Select a basis $\{x_i\}$ for $H^*(S)$ and denote by $\{a_i\}$ the basis for $H(S)$ dual to $\{x_i\}$ under the Kronecker index. Define a basis $\{x'_i\}$ for $H^*(S, \partial S)$ by requiring that $D_2(x'_i) = a_i$, and let $\{a'_i\}$ be the basis for $H(S, \partial S)$ dual to $\{x'_i\}$ under the Kronecker index. Thus we have

$$\langle x_i, a_j\rangle = \langle x'_i, a'_j\rangle = \delta_{ij}, \quad D_2(x'_i) = x'_i \frown O_S = a_i.$$

Similarly, select a basis $\{y'_i\}$ for $H^*(X, X \setminus N)$ and denote by $\{b'_i\}$ the basis for $H(X, X \setminus N)$ dual to $\{y'_i\}$ under the Kronecker index. We define the homomorphism $D_1 : H^*(X, X \setminus N) \to H(X)$ by $D_1(x) = x \frown \mu$ and we let $b_i = D_1(y'_i)$. Next we let $\{y_i\} \subset H^*(X)$ be a collection dual to $\{b_i\}$ under the Kronecker index. Thus we have

$$\langle y_i, b_j\rangle = \langle y'_i, b'_j\rangle = \delta_{ij}, \quad D_1(y'_k) = y'_i \frown \mu = b_i.$$

Lemma 3.1 (cf. Vick [29, Lemma 6.10, p. 185]).

$$\sum_i (\mathrm{Id} \times f_!)(a'_i \times a_i) = \sum_i (f_* \times \mathrm{Id})(b'_i \times b_i).$$

Proof. Since $\{y'_i\}$ and $\{a'_i\}$ are bases, there are representations

$$f^*(x'_i) = \sum_k \gamma_{ik} y'_k \quad \text{and} \quad f_*(b'_j) = \sum_k \lambda_{kj} a'_k.$$

Then

$$\gamma_{ij} = \Big\langle \sum_k \gamma_{ik} y'_k, b'_j \Big\rangle = \langle f^*(x'_i), b'_j\rangle = \langle x'_i, f_*(b'_j)\rangle = \Big\langle x'_i, \sum_k \lambda_{kj} a'_k \Big\rangle = \lambda_{ij},$$

so

$$f_*(b'_i) = \sum_k \gamma_{ki} a'_k.$$

Next,

$$D_1 f^* D_2^{-1}(a_i) = D_1 f^*(x'_i) = D_1 \sum_k \gamma_{ik} y'_k = \sum_k \gamma_{ik} b_k.$$

Therefore,

$$(\mathrm{Id} \times f_!)(a'_i \times a_i) = a'_i \times D_1 f^* D_2^{-1}(a_i) = a'_i \times \sum_k \gamma_{ik} b_k = \sum_k \gamma_{ik}(a'_i \times b_k).$$

On the other hand,

$$(f_* \times \mathrm{Id})(b'_i \times b_i) = f_*(b'_i) \times b_i = \Big(\sum_k \gamma_{ki} a'_k\Big) \times b_i = \sum_k \gamma_{ki}(a'_k \times b_i).$$

Therefore summation over i produces the same result. ∎

Lemma 3.2 (cf. Vick [29, Lemma 6.11, p. 186]).

(a) $$\delta_*(O_S) = \sum_i (a'_i \times a_i),$$

(b)
$$\delta_*(\mu) = \sum_i (b'_i \times b_i).$$

P r o o f. (a) By the Künneth formula, $\{a'_i \times a_j\}$ is a basis of $H((S,\partial S)\times S)$, and $\{x'_i \times x_j\}$ is the dual basis of $H^*((S,\partial S)\times S)$. Then the identity follows from

$$\begin{aligned}\langle x'_j \times x_k, \delta_*(O_S)\rangle &= \langle \delta^*(x'_j \times x_k), O_S\rangle = \langle x'_j \smile x_k, O_S\rangle \\ &= \langle x_k, x'_j \frown O_S\rangle = \langle x_k, a_j\rangle = \delta_{kj}.\end{aligned}$$

(b) It is clear that $\delta_*(\mu)$ belongs to the subspace of $H((X, X\setminus N)\times X)$ spanned by $\{b'_i \times b_i\}$. Then the rest of the proof follows (a). ■

Proof of Theorem 2.1. We have

$$\begin{aligned} I_{fg} = (f\times g)_*\delta_*(\mu) &= I_*(\mathrm{Id}\times g_*)(f_*\times \mathrm{Id})\sum_i (b'_i \times b_i) && \text{by Lemma 3.2} \\ &= I_*(\mathrm{Id}\times g_*)(\mathrm{Id}\times f_!)\sum_i (a'_i \times a_i) && \text{by Lemma 3.1} \\ &= I_*(\mathrm{Id}\times g_* f_!)\delta_*(O_S) && \text{by Lemma 3.2. ■}\end{aligned}$$

Thus I_{fg} is the image of O_S under the composition of the following maps:

$$H(S,\partial S) \xrightarrow{\delta_*} H(S,\partial S)\otimes H(S) \xrightarrow{\mathrm{Id}\otimes\varphi} H(S,\partial S)\otimes H(\mathring{S}) \xrightarrow{I_*} H(M^\times),$$

while $\varphi = g_* f_!$ is defined by the diagram

$$\begin{array}{ccc} H^*(X, X\setminus N) & \xleftarrow{f^*} & H^*(S,\partial S) \\ \downarrow{\frown\mu} & & \downarrow{D_2} \\ H(X) & \xrightarrow{g_*} & H(S) \end{array}$$

It is worth mentioning that Nakaoka [25] defines the *coincidence transfer homomorphism* τ_{fg} for a pair of fiber-preserving maps $f, g : E \to E'$, where E, E' are manifolds with boundary. If the base is trivial then, according to [25, Theorem 5.1(ii)], τ_{fg} is related to φ as follows:

$$\tau_{fg}(1) = L(\varphi_{fg}).$$

4. The main theorem and Cases 1 and 2. The sum of Theorems 2.1 and 2.2 gives us a Lefschetz-type coincidence theorem for $g : X \to \mathring{S}$. To prove the statement for $g : X \to S$ (Theorem 2.3) we need the following fact.

PROPOSITION 4.1 (Topological Collaring Theorem, [29, Theorem 5.2, p. 154]). *Let $(S,\partial S)$ be a manifold. Then there is a manifold $(T,\partial T)$ obtained from $(S,\partial S)$ by attaching a "collar"*:

$$T = S \cup (\partial S \times [0,1]).$$

Next we restate and prove Theorem 2.3.

THEOREM 4.2. *For any pair* $f : (X, X \setminus N) \to (S, \partial S)$, $g : X \to S$, *with* $\mathrm{Coin}(f, g) \subset N$, *the coincidence index is equal to the Lefschetz number* (*with respect to* μ):

$$I_{fg} = L(g_* f_!).$$

Moreover, *if* $L(g_* f_!) \neq 0$, *then* (f, g) *has a coincidence.*

Proof. We attach "collars" to X and S as follows. Let

$$Z = X \cup ((X \setminus N) \times [0, 1]),$$

such that $X \cap ((X \setminus N) \times [0, 1]) = (X \setminus N) \times \{0\}$. Assume also that according to the proposition above

$$T = S \cup (\partial S \times [0, 1]) \subset M$$

is an n-manifold. Then we define $G : Z \to \mathring{T}$ by $G = jgr$ (cf. [7]), where $j : S \to \mathring{T}$ is the inclusion, $r : Z \to X$ is the retraction. We also define $F : (Z, (X \setminus N) \times \{1\}) \to (T, \partial T)$ by

$$\begin{aligned} F(x, t) &= (f(x), t) \quad && \text{if } (x, t) \in (X \setminus N) \times [0, 1], \\ F(x) &= f(x) && \text{if } x \in X. \end{aligned}$$

From Theorems 2.1 and 2.2 we have $I_{FG} = L(G_* F_! i_*)$, where $i : \mathring{T} \to T$ is the inclusion. The inclusions and retractions induce isomorphisms, $F_* = f_*$, $G_* = g_*$, so $L(G_* F_! i_*) = L(g_* f_!)$. Next, the following diagram commutes:

$$\begin{array}{ccc} (X, X \setminus N) & \xrightarrow{(f,g)} & M^{\times} \\ \uparrow{\scriptstyle r} & & \| \\ (Z, (X \setminus N) \times \{1\}) & \xrightarrow{(F,G)} & M^{\times} \end{array}$$

which means that $I_{FG} = I_{fg}$. Thus, $I_{fg} = I_{FG} = L(G_* F_! i_*) = L(g_* f_!)$. ∎

Halpern [20] proves a Lefschetz-type coincidence theorem in an even more general situation: he considers $f, g : X \to Y$, where both X and Y are arbitrary topological spaces. His Lefschetz number is the Lefschetz number of the homomorphism $\varphi : Y \to Y$ given by

$$\varphi(z) = g_*(f^*(b/z) \frown a)$$

for some $a \in H_n(X)$ and $b \in H^n(Y \times Y)$, and he proves that $L(\varphi) \neq 0$ implies $\mathrm{Coin}(f, g) \neq \emptyset$. To compare his result with ours, observe first that he does not define the coincidence index (and it is of independent interest), and second his theorem does not include the Brouwer fixed point theorem.

4.1. *Case 1.* To get a Lefschetz-type coincidence theorem for Case 1, i.e., when $(X, X \setminus N)$ is an n-manifold, we simply let μ be its fundamental class.

COROLLARY 4.3. *Let* $(S_1, \partial S_1), (S_2, \partial S_2)$ *be oriented compact connected* n*-manifolds*, *and let* $f : (S_1, \partial S_1) \to (S_2, \partial S_2)$, $g : S_1 \to S_2$ *be continuous maps. If* $\mathrm{Coin}(f, g) \cap \partial S_1 = \emptyset$, *then the coincidence index with respect to* $\mu = O_{S_1}$ *is equal to the Lefschetz number*:

$$I_{fg} = L(g_* f_!)$$

(*here* $f_!$ *is defined via Poincaré duality for* S_1 *and* S_2). *Moreover*, *if* $L(g_* f_!) \neq 0$ *then* (f, g) *has a coincidence.*

Several authors have dealt with a Lefschetz-type coincidence theorem for manifolds with boundary. Corollary 4.3 provides little additional information in comparison to these results but still has certain advantages.

The Lefschetz–Nakaoka Coincidence Theorem [4, Theorem 3.2] (it is Lemma 8.1 combined with Theorem 5.1 of Nakaoka [25]) is identical to our Corollary 4.3 but applies only to manifolds with nonempty boundary. The reason for this is that in [25] manifolds with boundary are "doubled" (two copies are glued together along the boundary) and then Nakaoka's Lefschetz-type coincidence theorem for closed manifolds is used. Of course, the case of empty boundary follows from the classical Lefschetz coincidence theorem [29, Theorem 6.13], but the case $\partial S_1 = \emptyset$, $\partial S_2 \neq \emptyset$ is still excluded. Bredon [2, VI.14] also considers manifolds with empty and nonempty boundary separately. The Theorem of Davidyan [7] and Theorem 2.1 of Mukherjea [24] use collaring instead of doubling, so they can be specialized to manifolds with empty boundary. But they do not prove that the coincidence index is equal to the Lefschetz number (Davidyan [6] proves this identity only for manifolds without boundary).

Therefore Corollary 4.3 is of some interest, because it opens a possibility of defining a coincidence index for all manifolds with boundary, empty or not. Such an index may be used for a unified Nielsen coincidence theory; see Brown and Schirmer [4, 5], where boundaries are required to be nonempty.

4.2. *Case 2.* We can also obtain a Lefschetz-type coincidence theorem for Case 2 (cf. Theorem 10.5) with an additional assumption.

Recall [15, p. 13] that a map $f : (X, X_0) \to (Y, Y_0)$ is said to be *Vietoris* if

(i) f is proper, i.e., $f^{-1}(B)$ is compact for any compact $B \subset Y$,
(ii) $f^{-1}(Y_0) = X_0$,
(iii) the set $f^{-1}(y)$ is acyclic with respect to the Čech homology for every $y \in Y$.

PROPOSITION 4.4 (Vietoris–Begle Theorem, [15, p. 14]). *If* $f : (X, X_0) \to (Y, Y_0)$ *is a Vietoris map*, *then* $f_* : \check{H}(X, X_0) \to \check{H}(Y, Y_0)$ *is an isomorphism.*

Corollary 4.5. *Suppose X is a topological space, $S \subset \mathbb{R}^n$ is a compact n-manifold. Suppose $f, g : X \to \mathring{S}$ are two continuous maps such that f is Vietoris. If $L(g_* f_*^{-1}) \neq 0$ with respect to the Čech homology over $\mathbb{Q}$, then the pair (f, g) has a coincidence.*

Proof. We put $\mu = f_*^{-1}(O_S)$ and apply Theorem 4.2. Then $I_{fg} = L(g_* f_!)$. But by Proposition 4.4, f_*^{-1} exists, hence by the proposition below, $f_! = f_*^{-1}$. Therefore $I_{fg} = L(g_* f_*^{-1})$. ∎

Proposition 4.6 (cf. [2, Proposition VI.14.1 (6), p. 394]). *If $f_*(\mu) = O_S$ then $f_* f_! = \mathrm{Id}$.*

5. Corollaries and examples. Suppose (X, X') is a topological space, $(S, \partial S)$ is an oriented compact connected n-manifold. Consider the following condition:

(A) $f_* : H_n(X, X') \to H_n(S, \partial S)$ is a nonzero homomorphism.

First we will consider analogues of some well-known theorems about maps between manifolds without the assumption that the domain of the maps is a manifold. The following is a generalization of Theorem 2.2 of Mukherjea [24].

Corollary 5.1. *Suppose $f : (X, X') \to (S, \partial S)$ satisfies condition* (A) *and $g : X \to S$ induces $g_* = 0$ (in reduced homology). Then (f, g) has a coincidence.*

Proof. First we select $\mu \in H_n(X, X')$ such that $f_*(\mu) = O_S$. Then by Proposition 4.6, we have $f_* f_! = \mathrm{Id}$. Hence $g_* f_! : H_i(S, \partial S) \to H_i(S, \partial S)$ is nonzero for $i = 0$ and zero for $i \neq 0$. Therefore $L(g_* f_!) = 1$, so there is a coincidence by Theorem 2.3. ∎

Corollary 5.1 allows us to use a version of the Vietoris Theorem stronger than Proposition 4.4 (see also Section 9): even if $H_n(f^{-1}(x)) \neq 0$ for some x, while $H_i(f^{-1}(x)) = 0$ for all x and $0 \leq i < n$, we still have an epimorphism $f_* : H_n(X, X \setminus N) \to H_n(S, \partial S)$. Therefore condition (A) is satisfied. For other versions of the Vietoris Theorem see [22].

The following is a generalization of the Kronecker theorem: a map of nonzero degree is onto.

Corollary 5.2. *If $f : (X, X') \to (S, \partial S)$ satisfies condition* (A) *then f is onto.*

Proof. For a given $y \in S$, we define $g : X \to S$ by $g(x) = y$, for all $x \in X$. Therefore $g_* : H(X) \to H(S)$ is a zero homomorphism, hence there is a coincidence by the previous corollary. Therefore $y \in f(X)$, so $f(X) = S$. ∎

A map $f : (S_1, \partial S_1) \to (S_2, \partial S_2)$, where $(S_i, \partial S_i)$, $i = 1, 2$, are manifolds, is called *coincidence-producing* [4, Section 7] if every map $g : S_1 \to S_2$ has a coincidence with f. Brown and Schirmer [4, Theorem 7.1] showed that if S_2 is acyclic, $n \geq 2$, then f is coincidence-producing if and only if $f_* : H_n(S_1, \partial S_1) \to H_n(S_2, \partial S_2)$ is nonzero. We call a map $f : (X, X') \to (S, \partial S)$ *weakly coincidence-producing* if every map $g : X \to S$ with $g_* = 0$ has a coincidence with f. Then Corollary 5.1 takes the following form.

COROLLARY 5.3. *If* $f : (X, X') \to (S, \partial S)$ *satisfies condition* (A) *then* f *is weakly coincidence-producing.*

For an acyclic S, this is a generalization of the "if" part of the Brown–Schirmer statement.

We conclude with a few examples of applications of Corollary 5.3. These examples are not included in either Case 1 or Case 2.

5.1. *Manifolds.* It is hard to come by an example of coincidence that does not involve manifolds. Yet we can consider a pair (X, X') such that X is a manifold (possibly with boundary) but (X, X') is not a manifold with boundary, i.e., X' is not the boundary of X (nor homotopically equivalent to it). This provides a setting not included in Case 1.

EXAMPLE 5.4. Let $f : (\mathbf{D}^2, \partial e \cup \partial e') \to (\mathbf{D}^2, \mathbf{S}^1)$, where e and e' are disjoint cells in $\mathbf{D}^2$.

Then it is a matter of simple computation to check whether condition (A) is satisfied. For examples of acyclic manifolds, see Brown and Schirmer [4, Section 7].

EXAMPLE 5.5. Let $f : (\mathbf{I}, \partial \mathbf{I}) \times \mathbf{S}^1 \to (\mathbf{D}^2, \mathbf{S}^1 \cup \{0\})$, $\mathbf{I} = [0, 1]$, be the map that takes by identification $\{0\} \times \mathbf{S}^1$ to $\{0\}$.

It is clear that condition (A) is satisfied for $X' = \{1\} \times \mathbf{S}^1$, therefore by Corollary 5.3, any map homotopic to f has a coincidence with a map g if $g_* = 0$. But $f^{-1}(0) = \mathbf{S}^1$ is not acyclic, so this example is not covered by Case 2 and Górniewicz's Theorem. On the other hand, even though f maps a manifold to a manifold, it does not map boundary to boundary. Therefore, Case 1 and the results discussed in Section 4 do not include this example.

In a similar fashion we can show that the projection of the torus $\mathbf{T}^2$ on the circle $\mathbf{S}^1$ is a weakly coincidence-producing map. This is an example of a map between manifolds of different dimensions. For a negative example of this kind, take the Hopf map $f : \mathbf{S}^3 \to \mathbf{S}^2$; then for any g, the Lefschetz number and the coincidence index of the pair (f, g) are equal to zero.

5.2. *Nonmanifolds.* Let E be a space that is not acyclic and not a manifold, e.g., the "figure eight". Consider the projection $f : (X, X') = (\mathbf{I}, \partial \mathbf{I}) \times E \to (\mathbf{I}, \partial \mathbf{I})$, $\mathbf{I} = [0, 1]$, onto the first coordinate. Then f clearly satisfies (A). Thus f is weakly coincidence-producing by Corollary 5.3. Observe that $f^{-1}(x) = E$ is not acyclic and X is not a manifold.

A relevant example is given in Kahn [21]. He constructed an infinite-dimensional acyclic space X and an essential map $f : X \to \mathbf{S}^3$. Then f satisfies condition (A) and, since any $g : X \to \mathbf{S}^3$ induces a zero homomorphism in reduced homology, it follows that f is weakly coincidence-producing by Corollary 5.3.

5.3. *Fibrations and UV^n-maps*

COROLLARY 5.6. *Suppose X is a topological space*, *M is an oriented compact closed $(n-1)$-connected n-manifold*, *$f : X \to M$ is a map*, *and*

(A′) *$f_{\#} : \pi_n(X) \to \pi_n(M)$ is onto.*

Then f is weakly coincidence-producing.

Proof. As M is $(n-1)$-connected, the Hurewicz homomorphism $h_n : \pi_n(M) \to H_n(M)$ is onto [2, p. 488]. Hence f_* is onto and (A) is satisfied. ∎

Condition (A′) holds when f is a fibration with $\pi_{n-1}(f^{-1}(y)) = 0$ (this follows from the homotopy sequence of the fibration [2, Theorem VII.6.7, p. 453]).

Condition (A′) also holds when f is onto and for each $y \in Y$, $f^{-1}(y)$ has the *UV^n-property* for each n (see [22, Section 4] and its bibliography): for any neighborhood U of $f^{-1}(x)$ there is a neighborhood $V \subset U$ such that any singular k-sphere in V is inessential, $0 \le k \le n$. Then Theorem 9.2 below gives a version of Theorem 1.2 of Gutev [19]. For related results see also [13, Section 2].

5.4. *m-Acyclic maps.* A multivalued map $\varPhi : Y \to Y$ is called *m-acyclic*, $m \ge 1$, if for each $x \in Y$, $\varPhi(x)$ consists of exactly m acyclic components. Schirmer [28] proved that if Y is locally connected and simply connected then the graph X of $\varPhi$ is a disjoint union of graphs X_i of m acyclic maps $\varPhi_i$, $i = 1, \ldots, m$. Then the projections $f_i : X_i \to Y$ on the first coordinate induce isomorphisms, therefore condition (A) holds for f the projection of X onto Y.

Patnaik [27] defines an *m-map* $\varPhi$ as a multifunction such that $\varPhi(x)$ contains exactly m points for each x. Then he considers the Lefschetz number of a homomorphism similar to $g_* f_!$, although it is not clear how it is related to ours.

5.5. *Spherical maps.* Continuing the discussion from the beginning of this section, what if the acyclicity condition for f fails at degree $n-1$? Then there is no version of the Vietoris Theorem available to ensure condition (A).

DEFINITION 5.7 (cf. [26, 16, 8]). Let $B(A)$, $A \subset \mathbb{R}^n$, denote the bounded component of $\mathbb{R}^n \setminus A$. A closed-valued u.s.c. map $\Phi : \mathbf{D}^n \to \mathbf{D}^n$ is called $(n-1)$-*spherical*, $n > 1$, if

(i) for every $x \in \mathbf{D}^n$, $H(\Phi(x)) = H(\mathbf{S}^{n-1})$ or $H(\text{point})$,

(ii) for every $x \in \mathbf{D}^n$, if $x \in B(\Phi(x))$ then there exists an ε-neighborhood $O_\varepsilon(x)$ of x such that $x' \in B(\Phi(x'))$ for each $x' \in O_\varepsilon(x)$.

COROLLARY 5.8. *An* $(n-1)$-*spherical map* $\Phi : \mathbf{D}^n \to \mathbf{D}^n$, $n > 1$, *has a fixed point.*

Proof. We notice, first, that if Φ has no fixed points and there are no points x such that $x \in B(\Phi(x))$, then by replacing $\Phi(x)$ with $\Phi'(x) = \Phi(x) \cup B(\Phi(x))$ we obtain an acyclic multifunction without fixed points. Therefore we suppose that such an x exists and for simplicity assume that it is 0. Now, if 0 is not a fixed point, then from the upper semicontinuity of Φ and (ii) above, it follows that there is an $\varepsilon > 0$ such that

$$|x| < \varepsilon \Rightarrow x \in B(\Phi(x)) \text{ and } |\Phi(x)| > 2\varepsilon.$$

Let X be the graph of Φ, $K = \{x : |x| \geq 2\varepsilon\}$, f, g projections of X, $X' = f^{-1}(\mathbf{D}^n \setminus K)$. One can see that f is essentially the same as the projection of $(\mathbf{D}^n, \mathbf{S}^{n-1}) \times \mathbf{S}^1$ onto $(\mathbf{D}^n, \mathbf{S}^{n-1})$, and therefore, it induces a surjection

$$f_* : H_n((\mathbf{D}^n, \mathbf{S}^{n-1}) \times \mathbf{S}^1) \to H_n(\mathbf{D}^n, \mathbf{S}^{n-1}).$$

Hence condition (A) is satisfied, so by Corollary 5.3, Φ has a fixed point. ∎

EXAMPLE 5.9 (O'Neill [26]). Let $\Phi : \mathbf{D}^2 \to \mathbf{D}^2$ be given by

$$\Phi(x) = \{y \in \mathbf{D}^2 : |y - x| = \varrho(x)\} \cup \{y \in \mathbf{S}^1 : |y - x| > \varrho(x)\},$$

where $\varrho(x) = 1 - |x| + |x|^2$, $x \in \mathbf{D}^2$.

This example shows that condition (ii) of the above definition is necessary for the existence of a fixed point.

6. The coincidence index of a pair of maps. The next three sections are devoted to the proof of Theorem 2.2, which will be carried out in a setting slightly more general than that of Section 2.

Consider the sets $K \subset V \subset M$. Assume $(M, V, M \setminus K)$ is an excisive triad, i.e., the inclusion $j : (V, V \setminus K) \to (M, M \setminus K)$ induces an isomorphism in homology. Let $i : K \to V$, $I : M \times K \to M^\times$ be the inclusions.

We start by recalling some facts about manifolds. The following definition and propositions are taken from Vick [29, Chapter 5].

PROPOSITION 6.1 [29, Corollary 5.7, p. 136]. *For each* $p \in M$, *the homomorphism*

$$i_{p*} : H_n(M) \to H_n(M, M \setminus \{p\}) = T_p \simeq \mathbb{Q},$$

where $i_p : M \to (M, M \setminus \{p\})$ *is the inclusion, is an isomorphism.*

Definition 6.2 [29, p. 139]. The *fundamental class* of M is an element $z \in H_n(M)$ such that for

$$i_{p*} : H_n(M) \to H_n(M, M \setminus \{p\}) = T_p,$$

$i_{p*}(z)$ is a generator of T_p for each $p \in M$.

Proposition 6.3 [29, Lemma 5.12, p. 143].

$$\xi : \mathbb{Q} \simeq H_0(M) \simeq H_n(M^\times)$$

under the homomorphism ξ *sending the* 0*-chain represented by* $p \in M$ *into the relative class represented by* $l_{p*}(s(p))$, *where* $l_{p*} : H_n(M, M \setminus \{p\}) \to H_n(M^\times)$ *is induced by* $l_p(x) = (x, p)$, $x \in M$, *and* $s : M \to T = \bigcup_{p \in M} T_p$ *is the orientation map of* M, *with* $s(p)$ *a generator of* T_p, *for each* $p \in M$.

Proposition 6.4 [29, Theorem 5.10, p. 140]. *If* s *is the orientation map then there is a unique fundamental class* $O_M \in H_n(M)$ *such that* $i_{p*}(O_M) = s(p)$ *for each* $p \in M$.

Consider

$$M \xrightarrow{k} (M, M \setminus K) \xleftarrow{j} (V, V \setminus K),$$

where k is the inclusion.

Definition 6.5 (cf. [15, p. 16], [7, p. 192]). The *fundamental class* O_K of the pair $(V, V \setminus K)$ is defined by

$$O_K = j_{*n}^{-1} k_{*n}(O_M) \in H_n(V, V \setminus K).$$

Let X be a topological space, N a subset of X, and let $f, g : X \to V$ be continuous maps. Suppose $\mathrm{Coin}(f, g) \subset N$; then the map $f \times g : (X, X \setminus N) \times X \to M^\times$ is well defined.

Fix an element $\mu \in H_n(X, X \setminus N)$.

Definition 6.6. The *coincidence index* I_{fg} of the pair (f, g) (with respect to μ) is defined by

$$I_{fg} = (f \times g)_* \delta_*(\mu) \in H_n(M^\times) \simeq \mathbb{Q}.$$

In the setting of Case 1 this definition turns into the usual one (cf. [29, p. 177]). Another observation (due to the referee): as I_{fg} is defined via a homomorphism from $H_n(X, X \setminus N)$ to $\mathbb{Q}$, it is an element of $H^n(X, X \setminus N)$.

This definition also includes the coincidence index for Case 2, as given in the proposition below.

Proposition 6.7. *Suppose* U *is an open subset of* n*-dimensional Euclidean space*, $f : X \to U$ *a Vietoris map*, $g : X \to K$ *a map*, *where* $K \subset U$ *is a finite polyhedron. Let* $N = f^{-1}(K)$, $\mu = f_*^{-1}(O_K) \in H_n(X, X \setminus N)$. *Then* $I(f, g) = I_{fg}$.

P r o o f. We identify $\mathbb{R}^n$ with a hemisphere of $M = \mathbf{S}^n$. Then from Proposition 6.3 it follows that $H_0(\mathbb{R}^n) \simeq H_n(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n \times \mathbb{R}^n \setminus \delta(\mathbb{R}^n))$ under the homomorphism ξ sending the 0-chain represented by $p \in \mathbb{R}^n$ into the relative class represented by $l_{p*}(s(p))$. Therefore we have a commutative diagram

$$\begin{array}{ccccc}
H_0(\mathbb{R}^n) & \xrightarrow{\ \xi\ } & H_n((\mathbb{R}^n)^\times) & \xrightarrow{\ d_*\ } & H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \\
\downarrow & & \downarrow & \nwarrow^{(f,g)_*} & \uparrow{\scriptstyle (f-g)_*} \\
H_0(M) & \xrightarrow{\ \xi\ } & H_n(M^\times) & \xleftarrow{\ (f,g)_*\ } & H_n(X, X \setminus N)
\end{array}$$

where the first two vertical arrows are isomorphisms induced by inclusions, and $d(x,y) = x - y$. Since d_* is an isomorphism [7, Lemma VII.4.13, p. 200], it follows that

$$I_{fg} = (f,g)_*(\mu) = (f-g)_*(\mu) = (f-g)_* f_*^{-1}(O_K) = I(f,g). \ \blacksquare$$

To justify its name, a coincidence index should have the property below.

LEMMA 6.8. *If* $I_{fg} \neq 0$ (*with respect to some* μ) *then the pair* (f,g) *has a coincidence.*

P r o o f. Suppose not; then $C = \mathrm{Coin}(f,g) = \emptyset$. Hence $H_n((X, X \setminus C) \times X) = 0$. But the following diagram is commutative:

$$\begin{array}{ccc}
H_n((X, X \setminus N) \times X) & \xrightarrow{\ (f\times g)_*\ } & \mathbb{Q} \\
\downarrow{\scriptstyle k_*} & & \| \\
H_n((X, X \setminus C) \times X) & \xrightarrow{\ (f\times g)_*\ } & \mathbb{Q}
\end{array}$$

where k is the inclusion, so $I_{fg} = 0$. $\blacksquare$

7. Generalized Dold's Lemma. In this section we obtain a generalization of Dold's Lemma [7, Lemma VII.6.13, p. 210] (see also [3, p. 153]), which is necessary for our definition of the coincidence index. It is proved for singular homology, but when V is open and K is compact, we can interpret this result for Čech homology with compact carriers, as in [15, I.5.6, p. 17].

We define the following functions:

- the *transposition* $t : V \times K \to K \times V$ by
$$t(x,y) = (y,x);$$
- the *scalar multiplication* $m : \mathbb{Q} \otimes H(V) \to H(V)$ by
$$m(r \otimes v) = r \cdot v;$$

• the *tensor multiplication* $O_K^\times : H(K) \to H(V, V\setminus K) \otimes H(K)$, $O_M^\times : H(K) \to H(M) \otimes H(K)$ by

$$O_K^\times(v) = O_K \otimes v, \quad O_M^\times(v) = O_M \otimes v;$$

• the *projection* $P : H(M^\times) \to H_n(M^\times)$ by

$$P(q) = q_n \quad \text{if } q = \sum_k q_k,\ q_k \in H_k(M^\times).$$

Lemma 7.1 (cf. Dold [7, Lemma VII.6.14, p. 210]). *Suppose that V is an ANR. Then the maps $\psi_0, \psi_1 : (V, V\setminus K) \times K \to M^\times \times V$ given by*

$$\psi_0(v,k) = (v,k,v), \quad \psi_1(v,k) = (v,k,k), \quad v \in V,\ k \in K,$$

induce the same homomorphism in homology: $\psi_{0*} = \psi_{1*}$.

Proof. Let $Q = \mathbf{I} \times D \cup \{0,1\} \times V \times K \subset \mathbf{I} \times V \times K$, where $\mathbf{I} = [0,1]$, $D = \{(v,k) \in V \times K : v = k\}$ is the diagonal of $V \times K$. Note that D is closed since V is Hausdorff. Therefore Q is also closed. Consider a function $\alpha : Q \to V$ given by

$$\alpha(0,v,k) = v, \quad \alpha(1,v,k) = k, \quad \alpha(t,k,k) = k, \quad v \in V,\ k \in K,\ t \in \mathbf{I}.$$

Clearly, α is continuous. Then, since Q is a closed subset of $\mathbf{I} \times V \times K$ and V is an ANR, there is an extension of α to a neighborhood of Q. And since Q contains $\mathbf{I} \times D$, we assume that α is now defined on $\mathbf{I} \times W$, where W is an open neighborhood of D in $V \times K$. Suppose maps

$$\eta_i : (W, W\setminus D) \to M^\times \times V, \quad \varphi_i : (V\times K, (V\times K)\setminus D) \to M^\times \times V, \quad i = 0,1,$$

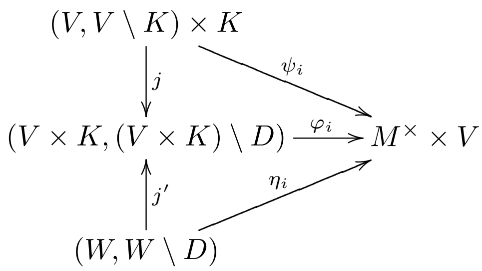
are given by the same formulas as ψ_i:

$$\eta_0(v,k) = \varphi_0(v,k) = (v,k,v), \quad \eta_1(v,k) = \varphi_1(v,k) = (v,k,k).$$

Then η_0 and η_1 are homotopic:

$$\eta_t(v,k) = (v,k,\alpha(t,v,k)), \quad 0 \le t \le 1.$$

Consider the following commutative diagram for $i = 0, 1$:



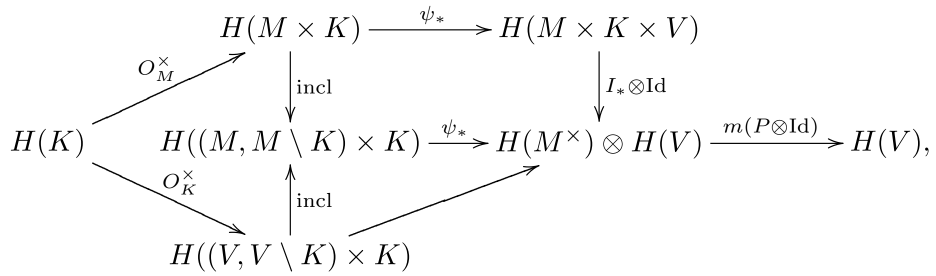
where j, j' are inclusions. Since W is open and D is closed, j'_* is an isomorphism by excision. We also know that $\eta_0 \sim \eta_1$, so $\eta_{0*} = \eta_{1*}$. Therefore $\varphi_{0*} = \varphi_{1*}$. And since $\psi_i = \varphi_i j$, we finally conclude that $\psi_{0*} = \psi_{1*}$. ∎

THEOREM 7.2 (Generalized Dold's Lemma; cf. [7, Lemma VII.6.13, p. 210]). *Suppose that K is arcwise connected and the map $\Phi : H(K) \to H(V)$ is given as a composition of the following homomorphisms*:

$$\Phi : H_i(K) \xrightarrow{O_K^{\times}} H_{n+i}((V, V \setminus K) \times K) \xrightarrow{(\delta\times\mathrm{Id})_*} H_{n+i}((V, V \setminus K) \times V \times K)$$
$$\xrightarrow{(\mathrm{Id}\times t)_*} H_{n+i}((V, V \setminus K) \times K \times V) \xrightarrow{(I\times\mathrm{Id})_*} H_{n+i}(M^{\times} \times V)$$
$$\xrightarrow{P\otimes\mathrm{Id}} H_n(M^{\times}) \otimes H_i(V) \xrightarrow{\mathrm{m}} H_i(V).$$

Then $\Phi = i_*$.

Proof. Consider the following diagram:



where the vertical arrows are induced by inclusions and ψ is given by

$$\psi(v, k) = (v, k, k), \quad v \in M,\ k \in K,$$

so $\psi = (I \times \mathrm{Id})(\mathrm{Id} \times \delta)$. The diagram is commutative, because the left triangle commutes by Definition 6.5. Now, according to Lemma 7.1, the homomorphism $\psi_* : H((V, V \setminus K) \times K) \to H(M^{\times} \times V)$ is also induced by

$$\psi'(v, k) = (v, k, v), \quad v \in V,\ k \in K,$$

so $\psi' = (\mathrm{Id} \times t)(\delta \times \mathrm{Id})$. Then the lower path defines Φ. Therefore so does the upper one. Hence

$$\Phi = m(P \otimes \mathrm{Id})\psi_* O_M^{\times}.$$

Consider $u \in H_i(K)$ and $w = \delta_*(u) \in (H(K) \otimes H(V))_i$. Then

$$w = \sum_{k+l=i} a_k \otimes b_l, \quad a_k \in H_k(K),\ b_l \in H_l(V).$$

But $(\eta \otimes \mathrm{Id})\delta_*(u) = u$, where $\eta : H(K) \to \mathbb{Q}$ is the augmentation. Then

$$u = (\eta \otimes \mathrm{Id})\delta_*(u) = (\eta \otimes \mathrm{Id})w = (\eta \otimes \mathrm{Id}) \sum_{k+l=i} a_k \otimes b_l = \eta(a_0) \otimes b_i.$$

Therefore, $b_i = i_*(u)$ and a_0 is represented by some $p \in K$. Since $O_M \in$

$H_n(M)$, we have

$$\begin{aligned}\Phi(u) &= m(P\otimes \mathrm{Id})\psi_*(O_M\otimes u) = m(PI_*\otimes \mathrm{Id})(O_M\otimes w)\\ &= m(PI_*\otimes \mathrm{Id})\Big(O_M\otimes \sum_{k+l=i}(a_k\otimes b_l)\Big) = m\Big(\sum_{k+l=i} PI_*(O_M\otimes a_k)\otimes b_l\Big)\\ &= m(I_*(O_M\otimes a_0)\otimes b_i) = I_*(O_M\otimes p)\cdot i_*(u).\end{aligned}$$

Finally, we observe that $l_p i_p(x) = I(x,p)$ for any $x\in M$, $p\in K$, so

$$\begin{aligned}\Phi(u) &= l_{p*}i_{p*}(O_M)\cdot i_*(u)\\ &= l_{p*}(s(p))\cdot i_*(u) &&\text{by Proposition 6.4}\\ &= i_*(u) &&\text{by Proposition 6.3. } \blacksquare\end{aligned}$$

In a fashion similar to the proof of Proposition 6.7 the original Dold's Lemma follows from this theorem.

8. The Lefschetz number of a pair and Theorem 2.2. Now we recall some fundamentals of the theory of the Lefschetz number (see [7, pp. 207–208] and [15, pp. 19–20]). Let

$$E_q^* = \mathrm{Hom}(E_{-q},\mathbb{Q}),\qquad E^* = \{E_q^*\},$$
$$(E^*\otimes E)_k = \bigotimes_{q+i=k}(E_q^*\otimes E_i),\qquad E^*\otimes E = \{(E^*\otimes E)_k\}.$$

Now we define the following maps:

$$e : (E^*\otimes E)_0\to\mathbb{Q},\qquad e(u\otimes v) = u(v)\qquad \text{(the evaluation map)},$$

and

$$\theta : (E^*\otimes E)_0\to \mathrm{Hom}(E,E),\qquad [\theta(a\otimes b)](u) = (-1)^{|b|\cdot|u|}a(u)\cdot b,$$

where $|w|$ stands for the degree of w.

Proposition 8.1 [15, Theorem II.1.5, p. 20]. *If $h : E\to E$ is an endomorphism of degree zero of a finitely generated graded module, then*

$$e(\theta^{-1}(h)) = L(h).$$

What follows is an adaptation of Górniewicz's argument [15, pp. 38–40] to the new situation. The next two lemmas are trivial.

Lemma 8.2. *Let $J : H(V, V\setminus K)\to (H(K))^*$ be a homomorphism of degree $-n$ given by*

$$J(u)(v) = I_{*n}(u\otimes v),\qquad u\in H(V,V\setminus K),\ v\in H(K).$$

Then $I_{*n} = e(J\otimes\mathrm{Id})$, so that the following diagram commutes:

$$\begin{array}{ccc}
(H(V,V\setminus K)\otimes H(K))_n & \xrightarrow{J\otimes \mathrm{Id}} & ((H(K))^*\otimes H(K))_0 \\
\Big\downarrow{\scriptstyle I_*} & \swarrow{\scriptstyle e} & \\
H_n(M^\times)\simeq \mathbb{Q} & &
\end{array}$$

Lemma 8.3. *Let*

$$a=(J\otimes \mathrm{Id})(\mathrm{Id}\otimes\varphi)\delta_*(O_K)=(J\otimes\varphi)\delta_*(O_K)\in((H(K))^*\otimes H(K))_0.$$

Then $e(a)=I_*(\mathrm{Id}\otimes\varphi)\delta_*(O_K)$.

Lemma 8.4. *Let* $\varphi: H(V)\to H(K)$ *be a homomorphism. Then the following diagram commutes*:

$$\begin{array}{ccc}
H(V,V\setminus K)\otimes H(V)\otimes H(K) & \xrightarrow{J\otimes\varphi\otimes \mathrm{Id}} & (H(K))^*\otimes H(K)\otimes H(K) \\
\Big\downarrow{\scriptstyle \mathrm{Id}\otimes t_*} & & \Big\downarrow{\scriptstyle \mathrm{Id}\otimes t_*} \\
H(V,V\setminus K)\otimes H(K)\otimes H(V) & \xrightarrow{J\otimes \mathrm{Id}\otimes\varphi} & (H(K))^*\otimes H(K)\otimes H(K) \\
\Big\downarrow{\scriptstyle PI_*\otimes \mathrm{Id}} & & \Big\downarrow{\scriptstyle e\otimes \mathrm{Id}} \\
H_n(M^\times)\otimes H(V) & & \mathbb{Q}\otimes H(K) \\
\Big\downarrow{\scriptstyle m} & & \Big\downarrow{\scriptstyle m} \\
H(V) & \xrightarrow{\varphi} & H(K)
\end{array}$$

Proof. The first square trivially commutes. For the second, consider going $\rightarrow\downarrow$; then we get

$$\begin{aligned}
m(e\otimes \mathrm{Id})(J\otimes \mathrm{Id}\otimes\varphi) &= m(e(J\otimes \mathrm{Id})\otimes\varphi) \\
&= m(I_{*n}\otimes\varphi) && \text{by Lemma 8.2} \\
&= I_{*n}\cdot\varphi && \text{by definition of } m.
\end{aligned}$$

Going $\downarrow\rightarrow$, we get

$$\begin{aligned}
\varphi m(PI_*\otimes \mathrm{Id}) &= \varphi m(I_{*n}\otimes \mathrm{Id}) && \text{by definition of } P \\
&= \varphi(I_{*n}\cdot \mathrm{Id}) && \text{by definition of } m \\
&= I_{*n}\cdot\varphi && \text{by linearity of } \varphi. \ \blacksquare
\end{aligned}$$

The following theorem implies Theorem 2.2.

Theorem 8.5. *Suppose* V *is an ANR*, K *is an arcwise connected space*, $H(K)$ *is finitely generated. Then for any homomorphism* $\varphi: H(V)\to H(K)$ *we have*

$$L(\varphi i_*)=I_*(\mathrm{Id}\otimes\varphi)\delta_*(O_K),$$

where $i: K\to V$ *is the inclusion.*

Proof. We start in the upper left corner of the diagram in Lemma 8.4 with $\delta_*(O_K)\otimes u$, where $u \in H(K)$. Then going ↓→, we get $\varphi i_*(u)$ by Theorem 7.2. Going →↓, we get

$$\begin{aligned}
&m(e\otimes \mathrm{Id})(\mathrm{Id}\otimes t_*)(J\otimes\varphi\otimes \mathrm{Id})(\delta_*(O_K)\otimes u)\\
&\quad = m(e\otimes \mathrm{Id})(\mathrm{Id}\otimes t_*)((J\otimes\varphi)\delta_*(O_K)\otimes u)\\
&\quad = m(e\otimes \mathrm{Id})(\mathrm{Id}\otimes t_*)(a\otimes u) \qquad \text{by Lemma 8.3}\\
&\quad = m(e\otimes \mathrm{Id})(\mathrm{Id}\otimes t_*)\Big(\sum_i a_i\otimes a_i'\otimes u\Big)\\
&\qquad\quad \text{if } a=\sum_i a_i\otimes a_i',\ a_i\in (H(K))^*, a_i'\in H(K)\\
&\quad = m(e\otimes \mathrm{Id})\sum_i (-1)^{|a_i'|\cdot|u|}(a_i\otimes u\otimes a_i')\\
&\quad = \sum_i m((-1)^{|a_i'|\cdot|u|}e(a_i\otimes u)\otimes a_i')\\
&\quad = \sum_i m((-1)^{|a_i'|\cdot|u|}a_i(u)\otimes a_i') \qquad \text{by definition of } e\\
&\quad = \sum_i (-1)^{|a_i'|\cdot|u|}a_i(u)\cdot a_i' \qquad \text{by definition of } m\\
&\quad = \sum_i \theta(a_i\otimes a_i')(u) \qquad \text{by definition of } \theta\\
&\quad = \theta(a)(u).
\end{aligned}$$

Thus $\theta(a)=\varphi i_* : H(K)\to H(K)$. Since $H(K)$ is a finitely generated graded module, Proposition 8.1 applies and we have $L(\varphi i_*) = L(\theta(a)) = e(a)$. Now the statement follows from Lemma 8.3. ∎

9. A Lefschetz-type coincidence theorem for maps to an open subset of a manifold. Recall that if $(V, V\setminus K)$ is a manifold then, according to Theorem 2.1, $\varphi = g_*f_!$ satisfies an identity that connects it to the coincidence index. Our next goal is to show that even if V is not a manifold, under certain purely homological conditions we can construct φ satisfying that identity. This leads to the proof of a Lefschetz-type theorem for Case 2.

Proposition 9.1. *Suppose $f:(X,X\setminus N)\to(V,V\setminus K)$, $g:X\to K$ are continuous maps, and there is $\mu\in H_n(X,X\setminus N)$ satisfying*

(a) $$f_*(\mu)=O_K,$$

where $f_* : H_n(X, X \setminus N) \to H_n(V, V \setminus K)$, *and there is a homomorphism* $\varphi : H(V) \to H(K)$ *of degree* 0 *satisfying*

$$\text{(b)} \qquad \varphi f_* = g_*,$$

where $f_* : H(X) \to H(V)$. *Then*

$$I_{fg} = I_*(\mathrm{Id} \otimes \varphi)\delta_*(O_K),$$

i.e., I_{fg} *is the image of* O_K *under the composition of the following maps*:

$$H(V, V \setminus K) \xrightarrow{\delta_*} H(V, V \setminus K) \otimes H(V) \xrightarrow{\mathrm{Id}\otimes\varphi} H(V, V \setminus K) \otimes H(K) \xrightarrow{I_*} H(M^\times).$$

Proof (cf. Górniewicz [15, pp. 15–16]). The following diagram commutes:

$$\begin{array}{ccccc}
H(V,V\setminus K)\otimes H(V) & \xleftarrow{\mathrm{Id}\otimes f_*} & H(V,V\setminus K)\otimes H(X) & \xrightarrow{\mathrm{Id}\otimes g_*} & H(V,V\setminus K)\otimes H(K) \\
\uparrow{\scriptstyle \alpha_1} & & \uparrow{\scriptstyle \alpha_2} & & \uparrow{\scriptstyle \alpha_3} \\
H((V,V\setminus K)\times V) & \xleftarrow{(\mathrm{Id}\times f)_*} & H((V,V\setminus K)\times X) & \xrightarrow{(\mathrm{Id}\times g)_*} & H((V,V\setminus K)\times K) \\
\uparrow{\scriptstyle \delta_*} & & \uparrow{\scriptstyle (f,\mathrm{Id})_*} & & \downarrow{\scriptstyle I_*} \\
H(V,V\setminus K) & \xleftarrow{f_*} & H(X,X\setminus N) & \xrightarrow{(f,g)_*} & H(M^\times)
\end{array}$$

where $\alpha_1,\ \alpha_2,\ \alpha_3$ are the isomorphisms from the Künneth theorem. If we start with $\mu \in H(X, X \setminus N)$ in the middle of the lower row, then from commutativity of the diagram it follows that

$$\begin{aligned}
&I_*\alpha_3^{-1}(\mathrm{Id}\otimes\varphi)\alpha_1\delta_*(O_K) \\
&\qquad = I_*\alpha_3^{-1}(\mathrm{Id}\otimes\varphi)\alpha_1\delta_* f_*(\mu) && \text{by (a)} \\
&\qquad = I_*\alpha_3^{-1}(\mathrm{Id}\otimes\varphi)(\mathrm{Id}\otimes f_*)\alpha_2(f,\mathrm{Id})_*(\mu) && \text{(the left half)} \\
&\qquad = I_*\alpha_3^{-1}(\mathrm{Id}\otimes\varphi f_*)\alpha_2(f,\mathrm{Id})_*(\mu) \\
&\qquad = I_*\alpha_3^{-1}(\mathrm{Id}\otimes g_*)\alpha_2(f,\mathrm{Id})_*(\mu) && \text{by (b)} \\
&\qquad = I_{fg} && \text{(the right half of the diagram).} \ \blacksquare
\end{aligned}$$

It is obvious that both (a) and (b) are satisfied when f induces isomorphisms $H(X) \simeq H(V)$ and $H_n(X, X\setminus N) \simeq H_n(V, V\setminus K)$. Another example: if the maps $f, g : X \to M$ satisfy $f_{*n} \neq 0$, $g_* = 0$, then we can select $\varphi = 0$ so that $L(\varphi) = 1$.

Now Proposition 9.1 and Theorem 8.5 imply the following.

Theorem 9.2 (Lefschetz-type theorem). *Suppose* X *is a topological space*, $N \subset X$, M *is an oriented connected compact closed* n-*manifold*,

$V \subset M$ is an ANR, $K \subset V$ is an arcwise connected space, $H(K)$ is finitely generated, $(M, V, M \setminus K)$ is an excisive triad. Suppose maps

$$f : (X, X \setminus N) \to (V, V \setminus K), \quad g : X \to K$$

and a homomorphism $\varphi : H(V) \to H(K)$ satisfy the conditions of Proposition 9.1. *Then the coincidence index is equal to the Lefschetz number*:

$$I_{fg} = L(\varphi i_*).$$

Moreover, if $L(\varphi i_*) \neq 0$, then (f, g) has a coincidence.

Remark. We proved the theorem for singular homology, but the proof is algebraic except for generalized Dold's Lemma 7.2. And since it holds for Čech homology, so does the theorem.

The following two examples show limits of applicability of this result.

Example 9.3 (Dranishnikov [13, Lemma 1.9]). There is a multivalued u.s.c. retraction $\Phi : \mathbf{D}^n \to \mathbf{S}^{n-1}$:

$$\Phi(x) = \{y \in \mathbf{S}^{n-1} : |y - x| \geq 4|x|^2 - 3|x|\}.$$

Example 9.4. Let $\mathbf{M}^2$ be the Möbius band, given in cylindrical coordinates by $z = \theta$, $-1 \leq r \leq 1$, $0 \leq \theta \leq \pi$, with the top and bottom edges identified. Let $f : (\mathbf{M}^2, \partial\mathbf{M}^2) \to (\mathbf{D}^2, \mathbf{S}^1)$ be the projection on the horizontal plane and $g : \mathbf{M}^2 \to \mathbf{S}^1$ the projection on the z-axis.

Observe that Φ in Example 9.3 has no fixed points, while g in Example 9.4 is homotopic to a map g' such that the pair (f, g') has no coincidence. This is reflected in the fact that $\Phi(x)$ fails to be acyclic for $|x| \leq 1/2$, while in Example 9.4, f_* does not satisfy condition (a).

10. The generalized Lefschetz number and Case 2. In Corollary 10.5 below we will see how one can use the Vietoris–Begle Theorem 4.4 to avoid the restriction on relative behavior of $f : (X, X \setminus N) \to (V, V \setminus K)$ and consider only $f : X \to V$. For this purpose, we would like to be able to deal with the Lefschetz number of $i_*\varphi_{fg} : H(V) \to H(V)$, instead of $\varphi_{fg} i_* : H(K) \to H(K)$ as before. Then, if $H(V)$ is not finitely generated, we need to define the generalized Lefschetz number $\Lambda(\cdot)$, as in [15, pp. 20–23].

Let $h : E \to E$ be an endomorphism of an arbitrary vector space E. Denote by $h^{(n)} : E \to E$ the nth iterate of h. Then the kernels

$$\ker h \subset \ker h^{(2)} \subset \ldots \subset \ker h^{(n)} \subset \ldots$$

form an increasing sequence of subspaces of E. Let

$$N(h) = \bigcup_n \ker h^{(n)} \quad \text{and} \quad \widetilde{E} = E/N(h).$$

Then h induces an endomorphism $\widetilde{h} : \widetilde{E} \to \widetilde{E}$.

DEFINITION 10.1. Let $h = \{h_q\}$ be an endomorphism of degree 0 of a graded vector space $E = \{E_q\}$ and suppose $\widetilde{E}$ is finitely generated. Then the *generalized Lefschetz number* (*in the sense of Leray*) of h is given by

$$\Lambda(h) = \sum_q (-1)^q \operatorname{tr}(\widetilde{h}_q) = L(\widetilde{h}).$$

PROPOSITION 10.2 [15, II.2.3, p. 22]. *If E is a finitely generated graded vector space then $\Lambda(h) = L(h)$.*

PROPOSITION 10.3 [15, II.2.4, p. 22]. *Let E, E' be graded modules and suppose that the following diagram commutes*:

$$\begin{array}{ccc} E & \xrightarrow{\ k\ } & E' \\ \uparrow{\scriptstyle h} & \nwarrow{\scriptstyle l} & \uparrow{\scriptstyle h'} \\ E & \xrightarrow{\ k\ } & E' \end{array}$$

Then, if $\Lambda(h)$ is defined, so is $\Lambda(h')$ and $\Lambda(h) = \Lambda(h')$.

THEOREM 10.4. *Under the conditions of Theorem* 9.2,

$$I_{fg} = \Lambda(i_*\varphi_{fg}).$$

Proof. This follows from Proposition 10.3 and commutativity of the diagram

$$\begin{array}{ccc} H(K) & \xrightarrow{\ i_*\ } & H(V) \\ {\scriptstyle \varphi_{fg} i_*}\uparrow & \nwarrow{\scriptstyle \varphi_{fg}} & \uparrow{\scriptstyle i_*\varphi_{fg}} \\ H(K) & \xrightarrow{\ i_*\ } & H(V) \end{array} \quad \blacksquare$$

This theorem implies the Coincidence Theorem of Górniewicz [15, p. 38], as follows.

COROLLARY 10.5. *Suppose X is a topological space, $V \subset \mathbb{R}^n$ is open. Suppose $f, g : X \to V$ are two continuous maps such that f is Vietoris and g is compact* (*i.e.*, $\overline{g(X)}$ *is compact*). *If $\Lambda(g_* f_*^{-1}) \neq 0$ with respect to Čech homology over $\mathbb{Q}$, then the pair (f, g) has a coincidence.*

Proof. Since g is a compact map, there is a finite connected polyhedron K such that $g(X) \subset K \subset V$. We can assume that $V \subset M = \mathbf{S}^n$. Then $(M, V, V \setminus K)$ is an excisive triad, V is an ANR, K is an arcwise connected space. Let $N = f^{-1}(K)$; then $\operatorname{Coin}(f, g) \subset N$. Since f_* is an isomorphism by Proposition 4.4, all the conditions of Proposition 9.1 are satisfied. Therefore by Theorem 10.4, $I_{fg} = \Lambda(g_* f_*^{-1})$. ■

See Górniewicz [15, pp. 40–43] for applications of this theorem to the study of fixed points of multivalued maps on polyhedra, ANRs, etc.

Acknowledgments. This paper is a version of Chapter I of my doctoral thesis written under the guidance of M.-E. Hamstrom at the University of Illinois at Urbana-Champaign. I would like to express my deep gratitude to Professor Robert F. Brown who read an early version of the manuscript, suggested ways to turn it into a publication, and always provided advice and encouragement. I also thank Professor R. McCarthy for valuable conversations regarding this work and the referee for a very thorough and thoughtful report.

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Received 15 July 1998