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On entropy of patterns given by interval maps

by

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Abstract. Defining the complexity of a green pattern exhibited by an interval map, we give the best bounds of the topological entropy of a pattern with a given complexity. Moreover, we show that the topological entropy attains its strict minimum on the set of patterns with fixed eccentricity m/n at a unimodal X-minimal case. Using a different method, the last result was independently proved in [11].

0. Introduction. The aim of this paper is to evaluate the topological entropy of so-called green patterns playing a natural role in one-dimensional dynamics given by continuous interval maps.

In these dynamics, most phenomena are related to the orbit structure of cycles. It is therefore not surprising that many authors investigated various situations involving periods of cycles, their coexistence and coherence between the set of periods and other possible features of such systems.

Because these phenomena often do not depend on a particular scale, instead of a cycle, one can think more generally of a *pattern* as a cyclic permutation and a lot of information can be gained purely by combinatorial methods. Then every continuous interval map *realizes* the patterns by its cycles.

All patterns can be partially ordered [4]: a pattern A forces a pattern B if every continuous interval map which exhibits A also exhibits B.

Of course, one can consider various subclasses of patterns; a *unipattern* is a cyclic permutation which can be divided into two blocks such that elements of the left block move right and elements of the right one move left; its *eccentricity* is a ratio not less than one where the numerator and denominator are the cardinalities of the blocks. Recently the following interesting fact has been discovered [8], [12]: there exists a subclass of unipatterns—we called

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them *X-minimal patterns* (in [8], they are called *twist patterns*)—elements of which are forced by any other pattern. These special cyclic permutations are naturally parameterized by the rational numbers—the eccentricities—and they can be described (hence also constructed) with the aid of *coding* [12]. Moreover, in [11] it was shown that for X-minimal patterns various behaviours are possible. Namely, it was proved that when *modality* increases, *entropy* may stay bounded, but it may also increase to infinity (independently of the eccentricities).

Our paper can be considered as a contribution to the study of properties of *green patterns*, a class of patterns which includes X-minimal ones [8], [12]. A green pattern can be described as follows: for a unipattern, we distinguish green and black elements: all the elements which move into the same block where they lie are called *green* and the others are called *black*. Thus, a green pattern is a unipattern with one block black (containing black elements only), and the corresponding permutation is increasing on the green elements and decreasing on the black ones. By using the notion of *complexity* of a green pattern (to be defined later in detail) we are able to compute the supremum of the topological entropies of green patterns with given complexity. In particular, this yields the least upper bound $\log 3.30075\ldots$ for the entropy of (X-minimal) $2B$-patterns. These patterns have been used in [11] as an example of transitive patterns with arbitrarily large modality and given rotation number.

The paper is organized as follows:

In Section 1 we give some basic notation, definitions and results used throughout the paper. The main results, Theorems A, B and C, are also stated there.

Section 2 is devoted to the proof of Theorem A. An important property concerning the topological entropy of X-minimal patterns is proved in Lemma 2.13.

Section 3 is devoted to the investigation of the green patterns. Their study is based on statements 3.1–3.2 and 3.13. The main result of this section is Proposition 3.14.

In Section 4, using Proposition 3.14, we prove Theorem C. Then we prove Theorem B with the help of Lemma 3.13, Theorem C and Lemma 2.10.

Finally, the Appendix is devoted to the proof of Theorem Ap.1, which is an important tool in proving Lemma 3.9.

1. Definitions and main results. To explain our results—Theorems A, B, C in this section—more rigorously, we have to introduce a few notions concerning the so-called *combinatorial dynamics*. The terminology used here is that of [2].

Consider a pair (P, φ), where $P \subset \mathbb{R}$ is finite and $\varphi : P \to P$.

f_P-map. For a pair (P, φ), we define a continuous map f_P mapping the convex hull $\mathrm{conv}(P)$ into itself, such that $f_P|_P = \varphi$ and $f_P|_J$ is affine for any interval $J \subset \mathrm{conv}(P)$ for which $J \cap P = \emptyset$. The map f_P is called the *P-linear map* given by the pair (P, φ).

Cycle. A pair $(P = \{p_i\}_{i=1}^n, \varphi)$ is a *cycle* if $P = \{\varphi^i(p_1)\}_{i=1}^n$. We usually omit φ and we simply say that P is a cycle. The *period* $\mathrm{per}(P)$ of the cycle P is the number n. If a P-linear map f_P has a unique fixed point, the cycle P is sometimes called a *unicycle*.

The *modality* $\mathrm{mdl}(P)$ of the cycle (P, φ) is defined to be

$$\mathrm{card}\{i : 2 \le i \le n-1,\ (\varphi(p_{i-1}) - \varphi(p_i))(\varphi(p_i) - \varphi(p_{i+1})) < 0\}.$$

If $\mathrm{mdl}(P) = 1$, then (P, φ) is called a *unimodal cycle*.

Pattern. Two cycles (P, φ), (Q, ψ) are *equivalent* if there exists a homeomorphism $h : \mathrm{conv}(P) \to \mathrm{conv}(Q)$ such that $h(P) = Q$ and

$$h \circ \varphi = \psi \circ h|_P.$$

An equivalence class of this relation is called a *pattern*. If A is a pattern and $P \in A$ ($(P, \varphi) \in A$), we say that the cycle P *has the pattern* A (P is a *representative* of A) and we use the symbol $[P]$ to denote the pattern A. Since all representatives of A have the same period and modality, we can speak about the period $\mathrm{per}(A) = \mathrm{per}(P)$ and modality $\mathrm{mdl}(A) = \mathrm{mdl}(P)$ of A for any $P \in A$. We say that A is a *unipattern* if it has a representative which is a unicycle. A pattern A with $\mathrm{mdl}(A) = 1$ is called *unimodal*.

Let $\mathcal{I}$ be the set of all closed finite subintervals of $\mathbb{R}$. We consider the space $C(\mathcal{I})$ of all continuous maps f which are defined on some $I \in \mathcal{I}$ and map it into itself. A function $f \in C(\mathcal{I})$ *has a cycle* (P, φ) if $f|_P = \varphi$. In this case we say that f *exhibits* the pattern $[P]$. The union of all cycles of a map f is denoted by $\mathrm{Per}(f)$. In particular, the set $\mathrm{Fix}(f)$ of all fixed points of f is a subset of $\mathrm{Per}(f)$.

Forcing of patterns. A pattern A *forces* a pattern B if all maps in $C(\mathcal{I})$ exhibiting A also exhibit B.

PROPOSITION 1.1 ([4]). *The forcing relation is a partial order on the set of all patterns and it is an order on the subset of unimodal patterns.*

f-invariant set. Let $f \in C(\mathcal{I})$ be a map defined on $I \in \mathcal{I}$. We say that $Q \subset I$ is *f-invariant* if $f(Q) \subset Q$.

Transitivity. A map $f \in C(\mathcal{I})$ defined on $I \in \mathcal{I}$ is called *transitive* if for some $x \in I$, $\overline{\{f^i(x)\}}_{i=0}^{\infty} = I$, or equivalently, if any closed, f-invariant proper subset of I has an empty interior. A pattern A is said to be *transitive* if $f_P \in C(\mathcal{I})$ is a transitive map for some (and then for any) $P \in A$.

We will deal with entropy [1], [14], [16], [2]. We use Bowen's definition.

Topological entropy. For $f \in C(\mathcal{I})$ defined on $I \in \mathcal{I}$, a set $E \subset I$ is (n, ε)-*separated* with respect to f if, whenever $x, y \in E$ and $x \neq y$ then

$$\max_{0 \leq i \leq n-1} |f^i(x) - f^i(y)| > \varepsilon.$$

The *topological entropy* $\text{ent}(f)$ of f is the quantity

$$\lim_{\varepsilon \to 0_+} \limsup_{n \to \infty} \frac{1}{n} \log s(n, \varepsilon),$$

where $s(n, \varepsilon)$ is the largest cardinality of a subset of I (n, ε)-separated with respect to f. As usual, the *entropy* $\text{ent}(P)$ *of a cycle* P is the minimal topological entropy of a map $f \in C(\mathcal{I})$ with this cycle. It is known [2] that $\text{ent}(P)$ coincides with $\text{ent}(f_P)$ and that for $f \in C(\mathcal{I})$,

$$\text{ent}(f) = \sup\{\text{ent}(P) : f \text{ has the cycle } P\}.$$

Since entropy is a conjugacy invariant, for a pattern A and its representatives $P, Q \in A$, the values $\text{ent}(f_P)$ and $\text{ent}(f_Q)$ are equal. This common value is called the *entropy of the pattern* A. We denote it by $\text{ent}(A)$.

The following fact is an easy consequence of the above definitions.

Proposition 1.2. *If A and B are patterns and A forces B, then* $\text{ent}(A) \geq \text{ent}(B)$.

In the following definition we suppose that a rational number m/n is from the set $\mathbb{R}^+ \setminus \{1\}$.

Eccentricity. A cycle (P, φ) has *eccentricity* m/n with m, n coprime if for any map $f \in C(\mathcal{I})$ with the cycle P there is a fixed point $c \in \text{Fix}(f)$ such that

$$\frac{\text{card}\{x \in P : x < c\}}{\text{card}\{x \in P : x > c\}} = \frac{m}{n}.$$

By our definition, the eccentricity is always different from 1. Thus, it is not defined for a 1-cycle or 2-cycle.

If a cycle P has eccentricity m/n, then the cycle $(h(P), h \circ \varphi \circ h^{-1})$ where $h(x) = -x$ has eccentricity n/m and $[P] = [h(P)]$. So, we define an *eccentricity of a pattern* A as an eccentricity of a representative with an eccentricity greater than one. In accordance with the value of the eccentricity, we talk about an $\frac{m}{n}$-*cycle* and an $\frac{m}{n}$-*pattern*.

Of course, one pattern may have several distinct eccentricities. For a unipattern with a (unique) eccentricity m/n, we use the term $\frac{m}{n}$-*unipattern* (and $\frac{m}{n}$-*unicycle* for its representative). Note that $\text{per}(A) = k(m+n)$ for an $\frac{m}{n}$-unipattern A.

X-minimality. An $\frac{m}{n}$-pattern is *X-minimal* if it does not force any other pattern with the same eccentricity.

Remark 1.3. As already mentioned, the X-minimality was defined in [12]; in [8], Blokh has used for this type of pattern the name of *twist pattern*. He deals with the *rotation number* instead of an eccentricity. For a cycle P, the rotation number of P is the number of points moving to the left divided by the period of the cycle. Thus, if P is a unicycle, our eccentricity m/n is equivalent to the rotation number $n/(m+n)$ (see also [9], [10]).

For $r \in \mathbb{Q}$, denote by $\mathcal{E}_r$ the set of all patterns with an eccentricity greater than or equal to r. Now we are ready to state our first result.

Theorem A. *There is a unique unimodal X-minimal $\frac{m}{n}$-pattern $A \in \mathcal{E}_{m/n}$ such that any other pattern from $\mathcal{E}_{m/n}$ has entropy greater than* $\mathrm{ent}(A)$.

To state Theorems B and C we need to recall the terminology of [3], [19].

Let A be a unipattern and (P,φ) its representative. Define

$$P_{\mathrm{L}} = \{x \in P : x < c\}, \qquad P_{\mathrm{R}} = \{x \in P : x > c\},$$

where c is the unique fixed point of f_P. All points $x \in P$ such that x and $\varphi(x)$ lie on the same side of c are called *green* and all other points of P are called *black*.

Green pattern. A unipattern A is called a *green pattern* if it has a representative (P,φ) such that at least one point is green, the points of P_{R} are black, φ is increasing on the set of green points and decreasing on the set of black ones.

Let A be a green pattern. Its representative P with P_{R} being a black set (containing black points only) is called a *green representative*, or briefly a *g-representative*. Note that for any $(Q,\psi) \in A$, either (Q,ψ) or $(h(Q), h \circ \psi \circ h^{-1})$ where $h(x) = -x$ is a g-representative of A.

Lemma 1.4. *Let A be a green pattern and (P,φ) its g-representative. Then*

(i) $\mathrm{card}\, P_{\mathrm{L}} > \mathrm{card}\, P_{\mathrm{R}}$,

(ii) *the leftmost* (*resp. rightmost*) *point of P_{L} is green* (*resp. black*),

(iii) *for any black point $x \in P_{\mathrm{L}}$, $\varphi^2(x) < x$.*

Proof. The properties (i) and (ii) directly follow from the definitions. It remains to show (iii). If $x \le \varphi^2(x)$ for some black point $x \in P_{\mathrm{L}}$, then since A is green, the set $P \cap [x, \varphi(x)]$ is φ-invariant. This is impossible since $\min P < x$ by (ii). ∎

In this text, we use a *normal partition* of a g-representative P. The set P can be taken as a union of consecutive green and black blocks, i.e. for $j \ge 1$,

$$P_{\mathrm{L}} = \bigcup_{1\le i\le j} P_{2i-1} \cup P_{2i} \quad \& \quad P_{\mathrm{R}} = P_0, \tag{1}$$

where for $1 \le i \le j$ blocks with odd (resp. even) indices contain green (resp. black) points and they are ordered (from the left) according to their label.

Before we explain the definition of complexity of a green pattern, let us recall that for a representative (P,φ) of some unipattern, an interval $[x,y]$ is P-*basic* if $x,y \in P$ and there are no points of P in (x,y). A *switch* of P is a P-basic interval with endpoints of different colour, and the *height* $H(x)$ of a point $x \in P$ is the number of switches between $\varphi^2(x)$ and x (see Lemma 1.4(iii)).

Complexity. Let A be a green pattern and (P,φ) its g-representative. The *complexity* $C(A)$ is defined as the maximum height of black points of P_{L}.

By the previous definitions, for two g-representatives $(P,\varphi),(Q,\psi)$ of a green pattern A there exists an increasing homeomorphism $h : P \to Q$ such that $h(P) = Q$,

$$h \circ \varphi = \psi \circ h$$

and for each $x \in P$, $H(x) = H(h(x))$; thus, the complexity $C(A)$ does not depend on the choice of a g-representative. For a green pattern A and its g-representative P, it follows from Lemma 1.4 that the height of the least black point of P_{L} is greater than or equal to one. Thus, the complexity is always a positive integer. By [11], a lower bound of the entropy of a green pattern A is given by the value $\frac{1}{2}\log C(A)$.

Proposition 1.5 ([11]). *If A is a green pattern then*

$$\mathrm{ent}(A) \ge \tfrac{1}{2}\log C(A).$$

In the sequel we use the following notation. For $k \ge 1$,

$$\mathcal{G}_k = \{A : A \text{ is a green pattern and } C(A) \le 2k\},$$

and $\mathcal{X}_k \subset \mathcal{G}_k$ is the set of X-minimal patterns from $\mathcal{G}_k$. We will show in Lemma 2.3 that any X-minimal pattern is green and, on the other hand, for any positive integer k, $\mathcal{G}_k \setminus \mathcal{X}_k \ne \emptyset$ (a consequence of Theorem 2.2).

Now we are ready to formulate our main results on the topological entropy of green patterns. By $\alpha(k)$ we denote a positive root of the polynomial equation (in α)

$$(\alpha+1)^k(1+\sqrt{1+k^2})^k + \alpha^2(\alpha-1)^k k^k(k-\sqrt{1+k^2}) = 0.$$

Theorem B. *Let $A \in \mathcal{G}_k$. Then* $\mathrm{ent}(A) < \log\alpha(k)$.

Theorem C. *For each $k \ge 1$,* $\sup\{\mathrm{ent}(A) : A \in \mathcal{X}_k\} = \log\alpha(k)$.

We will show in Lemma 3.5 that $\alpha(k) > 1$ for each $k \ge 1$, hence all upper bounds in Theorem B are well defined. It is not difficult to show that for each k, the value $\alpha(k)$ is irrational. After a short calculation, we find that it is given by the irrationality of $(1+k^2)^{1/2}$. Here we write six approximate values, $\alpha(1) = 3.30075$, $\alpha(2) = 4.99667$, $\alpha(3) = 6.47283$, $\alpha(4) = 7.81963$, $\alpha(5) = 9.07868$, $\alpha(100) = 80.61520$.

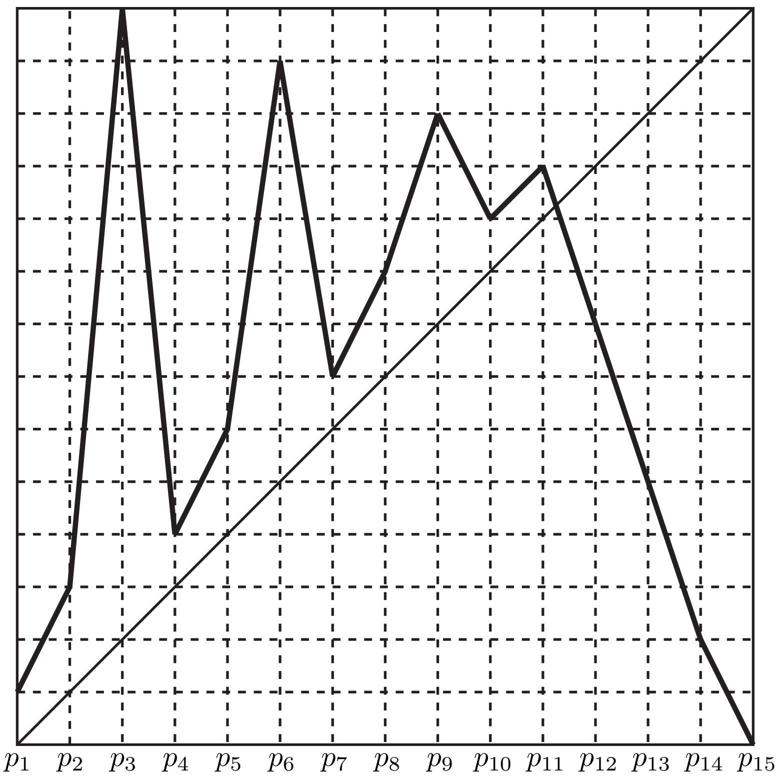

Fig. 1. The P-linear map f_P of a g-representative P of a green pattern $[P]$; $C([P]) = 2$, $\mathrm{ent}([P]) \sim \log 2.78870 < \log \alpha(1) \sim \log 3.30075$; green points: p_1, p_2, p_4, p_5, p_7, p_8, p_{10}

One can ask about the asymptotic behaviour of the sequence $\{\alpha(k)\}_{k\geq 1}$. The next result can be verified by standard methods.

Proposition 1.6. *Let $\alpha(k)$ be the value defined above. Then*

$$\lim_{k\to\infty} \frac{\alpha(k)}{k} = 0 \quad \textit{and} \quad \lim_{k\to\infty} \frac{\alpha(k)}{\sqrt{k}} = \infty.$$

As mentioned in the introduction, in [11] the authors showed that the set of X-minimal patterns provides a rich source of examples for the study of relationships between modality, entropy and eccentricity. Our Theorem B shows that (in a sense) for the entropy of a green pattern (as we already know, this also includes X-minimal patterns), an essential role is played by the complexity. The following two assertions are consequences of Proposition

1.5 and Theorem B. They will be proved after Lemma 2.3 at the beginning of Section 2.

Corollary 1.7. (i) *For a sequence* $\{A_n\}$ *of green patterns, the limit of* $\{\mathrm{ent}(A_n)\}$ *is infinite if and only if it is infinite for* $\{C(A_n)\}$.

(ii) *If* A *is an X-minimal* $\frac{m}{n}$*-pattern, then* $\mathrm{ent}(A) < \log\alpha([(m-n+1)/2])$ ([ ] *denotes the integer part*).

2. Entropy of X-minimal patterns. In this section we prove Corollary 1.7 assuming that Theorem B holds, and Theorem A.

Theorem A. *There is a unique unimodal X-minimal* $\frac{m}{n}$*-pattern* $A \in \mathcal{E}_{m/n}$ *such that any other pattern from* $\mathcal{E}_{m/n}$ *has entropy greater than* $\mathrm{ent}(A)$.

The proof is based on a "code approach" which has been developed in [12]. Therefore we start with a brief description of definitions and results from that article (statements 2.2, 2.3, 2.4, 2.7).

For various questions concerning unipatterns an effective way to describe their properties is given by *coding*.

Code. Let $P = \{p_1 < \ldots < p_{\mathrm{per}(P)}\}$ be an $\frac{m}{n}$-unicycle; denote by c the unique fixed point of f_P. The *code* K_P of the cycle P is a map $K_P : P \to \mathbb{Z}$ such that (see Figure 2)

$$K_P(p_1) = 0, \qquad K_P(f_P(p_i)) = \begin{cases} K_P(p_i) + n & \text{for } p_i < c, \\ K_P(p_i) - m & \text{for } p_i > c. \end{cases}$$

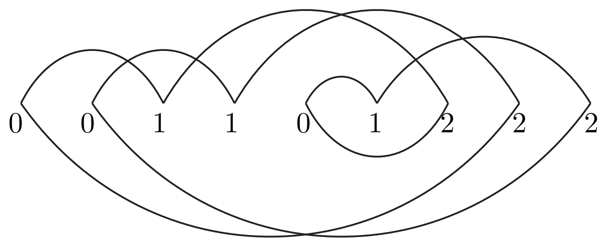

Fig. 2. A $\frac{2}{1}$-unicycle (P,φ); the code K_P of P

Remark 2.1. Notice that if we start from another point $p_i \in P$, i.e. if $K'_P(p_i) = 0$, then $K'_P = K_P - K_P(p_i)$. We will use this "shift" of the code K_P in Section 4.

Monotone code. Let P be an $\frac{m}{n}$-unicycle. The code K_P is called *monotone* if it is increasing on the left part P_{L} and decreasing on the right part P_{R} of P (see Figure 3).

Coding provides us a possibility to check whether a pattern is X-minimal. The next theorem characterizes X-minimality via codes.

THEOREM 2.2. *A pattern A is X-minimal if and only if it has a representative P that is a unicycle with monotone code.*

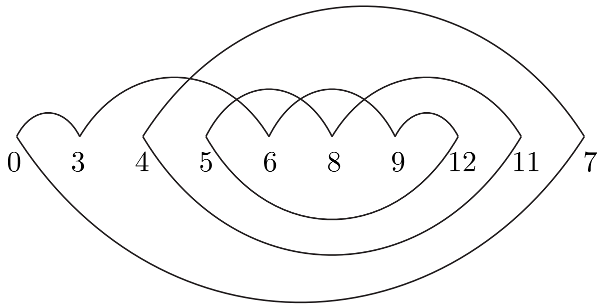

Fig. 3. An X-minimal $\frac{7}{3}$-unicycle (P, φ); the code K_P is monotone

Next we are going to prove that, as mentioned earlier, each X-minimal pattern is green.

LEMMA 2.3. *An X-minimal $\frac{m}{n}$-pattern A is green.*

Proof. Let A be an X-minimal $\frac{m}{n}$-pattern $(m/n>1)$, and (P, φ) its representative that is a unicycle with monotone code. Notice that $\varphi(x) > x$ (resp. $\varphi(x) < x$) for every $x \in P_{\mathrm{L}}$ (resp. $x \in P_{\mathrm{R}}$). Since the code is decreasing on P_{R}, all the points of P_{R} are black and $\operatorname{card} P_{\mathrm{L}} > \operatorname{card} P_{\mathrm{R}}$. Moreover, from the monotonicity of the code it follows that φ is increasing on the green points and decreasing on the black points of P_{L}. In particular, $\varphi(\max P_{\mathrm{L}}) = \min P_{\mathrm{R}}$ and $\varphi(\min P_{\mathrm{R}}) < \max P_{\mathrm{L}}$, hence φ is decreasing on the black points of P. But then (P, φ) is a g-representative of a green pattern. ∎

In fact, Theorem 2.2 and Lemma 2.3 imply that any g-representative of an X-minimal pattern has a monotone code. We are ready to prove Corollary 1.7 assuming that Theorem B holds.

Proof of Corollary 1.7. (i) If $C(A_n)$ tends to infinity, then by Proposition 1.5 so does $\operatorname{ent}(A_n)$. If $C(A_n)$ does not tend to infinity then on a subsequence it is smaller than $2k$ for some k. Then by Theorem B on the same subsequence $\operatorname{ent}(A_n)$ is smaller than $\log \alpha(k)$, so $\operatorname{ent}(A_n)$ does not tend to infinity.

(ii) Notice that since A is an X-minimal $\frac{m}{n}$-pattern, by Theorem 2.2 and Lemma 2.3 it has a representative (P, φ) with monotone code which is also a g-representative of A. By Lemma 1.4 and the definition of complexity, the number of switches between a black point $x \in P_{\mathrm{L}}$ and $\varphi^2(x)$ is less than or equal to $K_P(x) - K_P(\varphi^2(x)) = m - n$. But this shows that $C(A) \le m - n$. Hence, $A \in \mathcal{G}_{[(m-n+1)/2]}$ and by Theorem B,

$$\operatorname{ent}(A) < \log \alpha\left(\left[\frac{m-n+1}{2}\right]\right). \quad ∎$$

Let (P, φ) be an $\frac{m}{n}$-unicycle (then $\mathrm{per}(P) = k(m+n)$ for some $k \in \mathbb{N}$) with monotone code. It follows from the previous lemma that (P, φ) is a g-representative of the green $\frac{m}{n}$-unipattern $[P]$. So if c denotes the unique fixed point of f_P and $P = \{p_1 < \ldots < p_{k(m+n)}\}$, then

$$p_1 < \ldots < p_{km} < c < p_{km+1} < \ldots < p_{k(m+n)}$$

and $\varphi(p_i) < c$ for $i > km$. From (P, φ) we can define a new map $\psi : P^* \to P^*$ where $P^* = \{p_i\}_{i=1}^{km}$ by

$$\psi(p_i) = \begin{cases} \varphi(p_i) & \text{if } \varphi(p_i) \in P^*, \\ \varphi^2(p_i) & \text{if } \varphi(p_i) \notin P^*. \end{cases}$$

It is not difficult to see that the pair (P^*, ψ) is a cycle again. Using the cycle (P^*, ψ) we define another useful type of coding of (P, φ).

Short code. Let $D_P = \langle d_i \rangle_{i=1}^{km}$, where $d_i \in \{0, 1\}$, be a code corresponding to the cycle P in the following way:

$$d_i = \begin{cases} 0 & \text{if } \psi^i(p_1) = \varphi(\psi^{i-1}(p_1)), \\ 1 & \text{if } \psi^i(p_1) = \varphi^2(\psi^{i-1}(p_1)). \end{cases}$$

Since the code K_P is monotone, the code D_P can also be obtained from a cycle (P^*, ψ) if we start at the point p_1 and following the cycle we write 0 if we move right and 1 if we move left.

Note that D_P contains kn ones and $k(m-n)$ zeros. Moreover,

$$K_P(\psi^i(p_1)) = \begin{cases} K_P(\psi^{i-1}(p_1)) + n & \text{if } d_i = 0, \\ K_P(\psi^{i-1}(p_1)) - m + n & \text{if } d_i = 1. \end{cases}$$

Hence we have the following connection between K_P and D_P:

$$K_P(\psi^i(p_1)) = in - m \sum_{j=1}^{i} d_j.$$

Lemma 2.4. *Let (P, φ) be an $\frac{m}{n}$-unicycle with monotone code. Then $\mathrm{per}(P) = m + n$.*

Proof. Assume that $P = \{p_1 < \ldots < p_{k(m+n)}\}$ and $k > 1$. We are going to study the code D_P.

Let i_j be such that $d_{i_j} = 1$ and $\sum_{i=1}^{i_j} d_i = j$ (i_j is the place of the jth one in the sequence D_P).

Since $k > 1$, we have $\psi^{i_n}(p_1) \neq p_1$ and from the monotonicity of the code we have $K_P(\psi^{i_n}(p_1)) > 0$. But $K_P(\psi^{i_n}(p_1)) = ni_n - mn$. Thus $i_n > m$.

Moreover monotonicity of the code yields that no two points from P^* can have the same value of K_P. If there is a part $D^* = \langle d_i \rangle_{i=j+1}^{j+m}$ of D_P such that $\sum_{i=j+1}^{j+m} d_i = n$, then $K_P(\psi^{j+m}(p_1)) = K_P(\psi^j(p_1)) + (m-n)n + n(n-m) = K_P(\psi^j(p_1))$. But $\psi^{j+m}(p_1) \neq \psi^j(p_1)$ ($k > 1$) and so we have a contradiction with the monotonicity of the code. So no part of D_P of length m contains $m-n$ zeros and n ones. Hence $i_n - i_1 \geq m$ (otherwise $\langle d_i \rangle_{i=i_n-m+1}^{i_n}$ contains $m-n$ zeros and n ones).

Therefore $i_1 < i_{n+1} - m + 1$ and using the sequence $\langle d_i \rangle_{i=i_{n+1}-m+1}^{i_{n+1}}$ as above we obtain $i_{n+1} - i_2 \geq m$. Inductively, for all $j \leq (k-1)n$,

$$i_{n+j} - i_{1+j} \geq m.$$

We have $d_1 = 0$ because $K_P(\psi(p_1)) \geq 0$ (monotonicity) and so $1 < i_1 < \ldots < i_{kn-1} < i_{kn} \leq kn$. Using the inequalities above we obtain

$$km \geq 1 + \sum_{j=1}^{k} (i_{jn} - i_{(j-1)n+1}) \geq 1 + \sum_{j=1}^{k} m = 1 + km,$$

which is a contradiction. Thus $k = 1$ and the lemma is proved. ∎

LEMMA 2.5. (i) *There exists a green* $\frac{m}{n}$*-pattern with period* $m+n$ *which is not X-minimal.*

(ii) *For* $n \geq 1$, *there is no non-unimodal X-minimal* $\frac{n+1}{n}$*-pattern.*

(iii) *For fixed* m/n, *there exists a unique unimodal X-minimal* $\frac{m}{n}$*-pattern.*

Proof. (i) Set $P = \{1, \ldots, 7\}$ and define $\varphi : P \to P$ by $\varphi(1) = 2$, $\varphi(2) = 3$, $\varphi(3) = 5$, $\varphi(5) = 6$, $\varphi(6) = 4$, $\varphi(4) = 7$, $\varphi(7) = 1$. Using Theorem 2.2, one can easily verify that the cycle (P, φ) is a g-representative of the green pattern $[P]$ which is not X-minimal.

(ii) Note that by Lemma 2.4 for $n \geq 1$ fixed, an X-minimal $\frac{n+1}{n}$-pattern B has period $2n+1$ and Theorem 2.2 implies that its g-representative $(Q = \{q_1 < \ldots < q_{2n+1}\}, \psi)$ has the code $K_Q(q_1) = 0,\ K_Q(\psi(q_1)) = K_Q(q_{n+1}) = n$ and for $j \in \{1, \ldots, 2n-1\}$,

$$K_Q(\psi^{1+j}(q_1)) = \left(\left[\frac{j+1}{2} \right] + 1 \right) n - \left[\frac{j}{2} \right] (n+1),$$

which is the code of the unimodal Štefan cycle [24].

(iii) follows immediately from the definition of X-minimality and Proposition 1.1. ∎

REMARK 2.6. As a consequence of Lemma 2.4 we see that the set of all X-minimal $\frac{m}{n}$-patterns is finite. It was shown in [11] that there are $\frac{1}{m}\left(\binom{m}{n}\right)$ different X-minimal $\frac{m}{n}$-patterns.

As in Section 1, for $r \in \mathbb{Q}$, the symbol $\mathcal{E}_r$ denotes the set of all patterns with an eccentricity greater than or equal to r. A crucial role of the X-minimal patterns is shown by the following result concerning the forcing relation for patterns.

THEOREM 2.7. *Let* $A \in \mathcal{E}_{m/n}$. *Then* A *forces some X-minimal* $\frac{m}{n}$*-pattern.*

Corollary 2.8. *There is an X-minimal $\frac{m}{n}$-pattern $A \in \mathcal{E}_{m/n}$ such that any other pattern from $\mathcal{E}_{m/n}$ has entropy greater than or equal to* $\mathrm{ent}(A)$.

Proof. Since we know by Remark 2.6 that the set of X-minimal $\frac{m}{n}$-patterns is finite, there exists an X-minimal $\frac{m}{n}$-pattern with minimal entropy. Now the claim follows from Theorem 2.7 and Proposition 1.2. ∎

Corollary 2.8 can be considered as a "weak" version of Theorem A. In order to finish the proof of the latter, we will show that the unique unimodal X-minimal $\frac{m}{n}$-pattern given by Lemma 2.5(iii) has entropy less than any other X-minimal $\frac{m}{n}$-pattern. This is the goal of the rest of this section.

We will require some knowledge of the properties of non-negative matrices. The proofs can be found in [5].

Lemma 2.9. *Let $\mathcal{A} = (a_{ij})$ be a $k \times k$ matrix of non-negative real numbers. Then there exist $\mu \ge 0$ and a non-zero vector $x = (x_j)$ $(j = 1, \ldots, k)$ such that $\mathcal{A}x = \mu x$ and $|\nu| \le \mu$ for any other eigenvalue ν of $\mathcal{A}$.*

Thus, for a non-negative matrix $\mathcal{A}$, its *spectral radius* $\mathrm{r}(\mathcal{A})$ is equal to the maximal eigenvalue. Let the *norm* of a real or complex matrix $\mathcal{A} = (a_{ij})$ be

$$|\mathcal{A}| = \sum_{i,j} |a_{ij}|.$$

It is known that the spectral radius $\mathrm{r}(\mathcal{A})$ of a matrix $\mathcal{A}$ is related to the norm in the following way:

$$\mathrm{r}(\mathcal{A}) = \lim_{n\to\infty} |\mathcal{A}^n|^{1/n}.$$

It follows that for two non-negative $k \times k$ matrices $\mathcal{A} = (a_{ij})$, $\mathcal{B} = (b_{ij})$ the inequality $\mathcal{A} \ge \mathcal{B}$ implies $\mathrm{r}(\mathcal{A}) \ge \mathrm{r}(\mathcal{B})$.

Let $f \in C(\mathcal{I})$ be a map of I into itself, and $Q = \{q_1 < \ldots < q_n\}$ be a finite subset of I (Q need not be f-invariant). The matrix of Q (with respect to f) is the $(n-1) \times (n-1)$ matrix $\mathcal{A}_Q$, indexed by Q-basic intervals and defined by letting $\mathcal{A}_{JK}$ be the largest non-negative integer l such that there are l subintervals $J_1, \ldots, J_l$ of J with pairwise disjoint interiors such that $f(J_i) = K$ for $i = 1, \ldots, l$.

The following lemma is needed in the proof of Lemma 2.13 and in Section 4 in the proof of Theorem B.

Lemma 2.10 ([15]). *Let $f \in C(\mathcal{I})$ be transitive, Q be a finite subset of the ambient interval, and let $\mathcal{A}_Q$ be the matrix of Q with respect to f. Then* $\mathrm{ent}(f) \ge \log \mathrm{r}(\mathcal{A}_Q)$, *with equality if Q is f-invariant and contains the endpoints of the ambient interval, and f is monotone (but not necessarily strictly monotone) on each Q-basic interval.*

Corollary 2.11. *Suppose that f, Q, $\mathcal{A}_Q$ are as in the previous lemma. If $\mathcal{B}$ is a non-negative matrix such that $\mathcal{A}_Q \ge \mathcal{B}$, then* $\mathrm{ent}(f) \ge \log \mathrm{r}(\mathcal{B})$.

We need to compute the spectral radius of the matrix $\mathcal{A}_{2k+2}$ of size $2k+2$ defined for the non-negative integer k by the relations $a_{k+2,k+2}=1$, $a_{1,l}=2$ for $l\in\{k+2,\dots,2k+2\}$, $a_{i,2k+3-i}=1$ for $i\in\{k+2,\dots,2k+2\}$, $a_{i,2k+4-i}=1$ for $i\in\{2,\dots,k+1\}$ and $a_{i,j}=0$ otherwise.

For instance,

$$\mathcal{A}_8=\begin{pmatrix}0&0&0&0&2&2&2&2\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&1&0\\0&0&0&0&0&1&0&0\\0&0&0&1&1&0&0&0\\0&0&1&0&0&0&0&0\\0&1&0&0&0&0&0&0\\1&0&0&0&0&0&0&0\end{pmatrix}.$$

Lemma 2.12. *The sequence* $\{\mathrm{r}(\mathcal{A}_{2k+2})\}_{k=0}^{\infty}$ *is decreasing and*

$$\lim_{k\to\infty}\mathrm{r}(\mathcal{A}_{2k+2})=\sqrt{3}.$$

Proof. Let $\mathcal{D}_{2k+2}(\lambda)=\det(\mathcal{A}_{2k+2}-\lambda\mathcal{E}_{2k+2})$. After a rather laborious computation, we obtain

$$\mathcal{D}_{2k+2}(\lambda)=\frac{1}{\lambda+1}(\lambda^{2k+3}-3\lambda^{2k+1}-2).$$

By Lemma 2.9, $\mathrm{r}(\mathcal{A}_{2k+2})$ is the maximal root of the equation $\lambda^{2k+3}-3\lambda^{2k+1}-2=0$. Now the conclusion can be easily verified. ■

The key lemma follows. Its proof is based on coding.

Lemma 2.13. *Let A be an X-minimal non-unimodal $\frac{m}{n}$-pattern. If*

$$\frac{m}{n}\ge\frac{k+2}{k+1}\qquad\textit{for some non-negative } k,$$

then $\mathrm{ent}(A)>\log\mathrm{r}(\mathcal{A}_{2k+2})$.

Proof. Since by Lemma 2.5(ii) a non-unimodal cycle with an eccentricity $\frac{k+2}{k+1}$ ($k\ge 0$) does not exist, there is a unique non-negative k for which (here the limit is used for the case when $k=0$)

$$\frac{k+2}{k+1}<\frac{m}{n}<\lim_{l\to k_+}\frac{l+1}{l}. \tag{2}$$

As we know from Lemma 2.3, A is green and we can use a normal partition (1) for its g-representative

$$P=\{p_1<\ldots<p_{\mathrm{per}(P)}\}=P_{\mathrm{L}}\cup P_{\mathrm{R}},$$

so

$$P_{\mathrm{L}}=\bigcup_{1\le i\le j}P_{2i-1}\cup P_{2i}\quad\&\quad P_{\mathrm{R}}=P_0.$$

Let us show that in this partition $j \geq 2$, otherwise A would be unimodal.

Suppose to the contrary that $P_{\mathrm{L}} = P_1 \cup P_2$. Lemma 1.4 says that P_1 consists of green elements and P_2 of black ones. But A is green and P is a g-representative, hence f_P is increasing on P_1 and decreasing on $P_2 \cup P_0$. Computing the modality of A we obtain

$$\begin{aligned}&\mathrm{mdl}(A)\\&= \mathrm{card}\{i : 2 \leq i \leq n-1,\ (f_P(p_{i-1}) - f_P(p_i))(f_P(p_i) - f_P(p_{i+1})) < 0\} = 1,\end{aligned}$$

which is impossible.

Thus, the number of blocks of the left part P_{L} of P is at least 4 and $j \geq 2$. Notice that by Lemma 1.4(iii),

$$f_P^2(\max P_{2j}) < \max P_{2j},$$

hence the f_P-preimage of $\max P_{2j}$ from P is the greatest green element from P_{L}, so

$$f_P(\max P_{2j-1}) = \max P_{2j}, \quad \text{i.e.} \quad K_P(\max P_{2j-1}) + n = K_P(\max P_{2j}).$$

The condition (2) can be rewritten as

$$k(m-n) < n < (k+1)(m-n);$$

the definition and monotonicity of K_P imply that for every $i \in \{0, 1, \ldots, k\}$,

$$K_P(\max P_{2j-1}) < K_P(f_P^{2i}(\max P_{2j})) = K_P(\max P_{2j}) + i(n-m),$$

and $K_P(f_P^{2k+2}(\max P_{2j})) < K_P(\max P_{2j-1})$, hence

$$f_P^{2k+2}(\max P_{2j}) < \max P_{2j-1} < f_P^{2i}(\max P_{2j}).$$

Therefore for every $x \in P_{2j}$ and a non-negative integer l the following implication holds:

$$f_P^{2i}(x) \in P_{2j} \text{ for } i = 0, 1, \ldots, l \Rightarrow l \leq k, \tag{3}$$

and we can put $l = k$ for $x = \max P_{2j}$. Thus, the set

$$M_k = \{x \in P_{2j} : f_P^{2i}(x) \in P_{2j} \text{ for } i = 0, 1, \ldots, k\}$$

is non-empty. Moreover, all the points of M_k have their f_P-preimages (with respect to P) at green points of P, otherwise (3) would not be satisfied. Denote by r the least point of M_k whose f_P-preimage is in P_{2j-1} (such a point exists). Next we show that $f_P^{2k+2}(r) < \min P_{2j-1}$. Suppose to the contrary that

$$f_P^{2k+2}(r) \in P_{2j-1}. \tag{4}$$

If we take $s \in P_{2j}$ such that $f_P^2(s) < \min P_{2j-1}$ (P_{L} has 4 blocks at least), $t \in P_{2j}$ and $i \in \{0, 1, \ldots, k\}$ maximal for which $\{f_P^{2l}(t)\}_{l=0}^{i} \subset P_{2j}$ and $f_P^{2i}(t) = s$, then either $i = k$ and by (4), the f_P-preimage of t is less than

$\min P_{2j-1}$, or $i < k$ (then $k > 0$), by (2) we have $n > (i+1)(m-n)$ and from the monotonicity of K_P,

$$K_P(\min P_{2j-1}) > K_P(s) + n - m = K_P(t) + (i+1)(n-m) > K_P(t) - n;$$

the last inequality says that the f_P-preimage of t has to be less than $\min P_{2j-1}$ again. This means that assuming (4) we obtain ($j \geq 2$)

$$P_{2j-2} \cap \bigcup_{i=0}^{\mathrm{per}(P)-1} f_P^i(P_{2j-1}) = \emptyset,$$

which is impossible.

We have seen that $f_P^{2k+2}(r) < \min P_{2j-1}$; by the above,

$$f_P^{2k+2}(r) \leq \max P_{2j-2} < \min P_{2j-1} \leq \max P_{2j-1} < f_P^{2k}(r) \leq r \leq \max P_{2j}$$

and since P is a g-representative,

$$f_P(\max P_{2j-2}) > f_P^{2k+1}(r).$$

Hence, for the intervals

$$J_1 = [\max P_{2j-2}, \max P_{2j-1}], \qquad J_2 = [\max P_{2j-1}, f_P^{2k}(r)]$$

we have $f_P(J_i) \supset [r, f_P^{2k+1}(r)]$, $i = 1, 2$. Putting

$$\begin{aligned} Q &= \{f_P^i(r)\}_{i=0}^{2k+2} \\ &= \{f_P^{2k+2}(r) < f_P^{2k}(r) < \ldots < r < \ldots < f_P^{2k-1}(r) < f_P^{2k+1}(r)\}, \end{aligned}$$

the reader can verify that for the matrix $\mathcal{A}_Q$ of size $2k+2$ indexed by Q-basic intervals (with respect to f_P) (see before Lemma 2.10) we have $\mathcal{A}_Q \geq \mathcal{A}_{2k+2}$, hence by Lemma 2.10 and Corollary 2.11,

$$\mathrm{ent}(A) = \mathrm{ent}(P) = \mathrm{ent}(f_P) \geq \log \mathrm{r}(\mathcal{A}_Q) \geq \log \mathrm{r}(\mathcal{A}_{2k+2}).$$

Since f_P is transitive (see Remark 3.3) and Q is not f_P-invariant, by Lemma 2.10 we even have $\mathrm{ent}(A) > \log \mathrm{r}(\mathcal{A}_{2k+2})$. ∎

Proof of Theorem A. Using Corollary 2.8, it is sufficient to show that on the finite set of X-minimal $\frac{m}{n}$-patterns, the topological entropy of the unique unimodal X-minimal $\frac{m}{n}$-pattern given by Lemma 2.5(iii) is strictly smaller than the entropy of any other X-minimal $\frac{m}{n}$-pattern. By Lemma 2.13, if $\frac{m}{n} > 2 = \frac{0+2}{0+1}$, then a non-unimodal X-minimal $\frac{m}{n}$-pattern A has $\mathrm{ent}(A) > \log \mathrm{r}(\mathcal{A}_2) = \log 2$ and at the same time, $\log 2$ is greater than the entropy of any unimodal pattern [21].

In the case when $m/n < 2$, it follows from Lemmas 2.12 and 2.13 that the entropy of a non-unimodal X-minimal pattern $A \in \mathcal{E}_{m/n}$ is greater than $\frac{1}{2}\log 3$. Hence, it is sufficient to show that the unimodal X-minimal $\frac{m}{n}$-pattern $B \in \mathcal{E}_{m/n}$ (unique by Lemma 2.5(iii)) has entropy less than $\frac{1}{2}\log 3$.

Notice that for a unimodal $\frac{2}{1}$-pattern $[P]$ where P is a 3-cycle, we have $[P] \in \mathcal{E}_{m/n}$ and by Lemma 2.10,

$$\operatorname{ent}([P]) = \log \frac{1+\sqrt{5}}{2} < \frac{1}{2}\log 3.$$

Hence by Theorem 2.7 and Proposition 1.2, $[P]$ forces B and $\operatorname{ent}(B) \le \operatorname{ent}([P]) < \frac{1}{2}\log 3$. ∎

3. Entropy of green patterns. This section is devoted to developing the machinery and preliminary results for proving Theorems B and C in Section 4.

Statements 3.1–3.3 are obtained with the help of the block itineraries of points of a g-representative. For example, Corollary 3.2 will let us recognize when two green patterns are different.

Statements 3.4–3.5 and 3.7–3.9 deal with a one-parameter family of $(k+1)$st order non-homogeneous difference equations with constant coefficients. These equations are constructed to reflect the properties of the patterns from the set $\mathcal{G}_k$. Using Theorem Ap.1 from the appendix, we investigate the least parameter (denoted by $\alpha(k)$) for which the corresponding difference equation has a strictly monotone solution ($\alpha(k)$ is a bifurcation value).

Each strictly monotone solution mentioned above defines a Lipschitz interval map (see Construction, Lemma 3.11) which exhibits any element of the set $\mathcal{G}_k$ (Lemma 3.13). This yields a weaker version of Theorem B (Proposition 3.14).

An important tool for the proof of Theorem B is a description of the trajectories of points of a g-representative by their block itineraries.

Let A be a green pattern and (P, φ) its g-representative, and consider a normal partition of P, i.e.

$$P_{\mathrm{L}} = \bigcup_{1\le i\le j} P_{2i-1} \cup P_{2i} \quad \& \quad P_{\mathrm{R}} = P_0.$$

Define a function $G : P \to \{0, 1, \ldots, 2j\}$ which labels each point of P by the number of its block, i.e. $G(x) = k$ for $x \in P_k$.

For $x \in P$ define a vector

$$v(x) = (G(x), G(\varphi(x)), \ldots, G(\varphi^{\operatorname{per}(A)-1}(x))) \in \mathbb{Z}^{\operatorname{per}(A)}.$$

The usual lexicographical order on the set $\bigcup_{m=2}^{\infty} \mathbb{Z}^m$ is denoted by $\prec$.

The next lemma shows that the block itineraries of distinct points of the g-representative P are different. As we will see in Corollary 3.2 we are able to reconstruct the whole pattern $[P]$ from $v(\min P) \in \mathbb{Z}^{\operatorname{per}(P)}$.

Define a map $\chi : P \to \mathbb{R}$ by $\chi(x) = 0$ for $x \in \bigcup_{i=1}^{j} P_{2i-1}$, $\chi(x) = 1$ for $x \in \bigcup_{i=1}^{j} P_{2i}$ and $\chi(x) = 2$ for $x \in P_0$.

LEMMA 3.1. (i) *Let A be a green pattern and (P,φ) its g-representative. Then $v(x) \prec v(y)$ whenever either $x, y \in P_{\mathrm{L}}$ and $x < y$ or $x, y \in P_{\mathrm{R}}$ and $y < x$.*

(ii) *If for $x, y \in P_{\mathrm{L}}$, $x < y$ and $\chi(\varphi^i(x)) = \chi(\varphi^i(y))$ for each $i \in \{0, 1, \ldots, l\}$, then also $G(\varphi^i(x)) \le G(\varphi^i(y))$ for $i \in \{0, 1, \ldots, l+1\}$.*

Proof. Let $x, y \in P_{\mathrm{L}}$. The conclusion is clear if $G(x) < G(y)$. If x and y are from the same block, then either $x, y \in P_{2i-1}$ and $G(\varphi(x)) \le G(\varphi(y))$, or $x, y \in P_{2i}$, hence $G(\varphi(x)) = G(\varphi(y)) = 0$ and also $G(\varphi^2(x)) \le G(\varphi^2(y))$.

Since the images of x are also the maxima of the blocks, there exists $i \le \mathrm{per}(P) - 1$ such that $G(\varphi^i(x)) \ne G(\varphi^i(y))$. Choosing the first i with this property, we immediately get $G(\varphi^i(x)) < G(\varphi^i(y))$, i.e. $v(x) \prec v(y)$. The case when $x, y \in P_{\mathrm{R}}$ follows similarly.

The property (ii) is clear from the definitions of G and χ. ∎

COROLLARY 3.2. (i) *Let $Q_i = \{q_1^i < q_2^i < \ldots < q_{k_i}^i\}$, $i = 1, 2$, be two g-representatives of different green patterns $[Q_1], [Q_2]$. Then the vectors $v(q_1^1)$ and $v(q_1^2)$ are different.*

(ii) *Let $[Q]$ be a green pattern and (Q,ψ) its g-representative. Then the vector $v(q_1)$ is not periodic.*

Proof. Obviously, it is sufficient to show that if we take the vector

$$v(q_1) = (v_1, \ldots, v_k) \in \mathbb{Z}^k$$

which is given by a g-representative $(Q = \{q_1 < \ldots < q_k\}, \psi)$ of a green pattern $[Q]$, then the pattern $[Q]$ can be reconstructed from $v(q_1)$. In what follows, we will construct its g-representative (P,φ). It is clear that $\mathrm{card}\, P = k$, $\mathrm{card}\, P_{\mathrm{R}} = n$ and $\mathrm{card}\, P_{\mathrm{L}} = k - n$, where n is the number of zeros in $v(q_1)$. Similarly, if we put

$$j = \max\{v_i : i \in \{1, \ldots, k\}\},$$

then for each $i \in \{1, \ldots, j\}$, the number of i's in $v(q_1)$ determines the cardinality of the ith block P_i of a normal partition of P. By rotation of $v(q_1)$ we obtain k vectors

$$(v_{1+l \,(\mathrm{mod}\, k)}, \ldots, v_{k+l \,(\mathrm{mod}\, k)}), \quad l \in \{0, 1, \ldots, k-1\}.$$

It follows from the definition of $v(q_1)$ that they are the block itineraries of the points from P with respect to φ, i.e. for each $l \in \{0, 1, \ldots, k-1\}$,

$$(v_{1+l \,(\mathrm{mod}\, k)}, \ldots, v_{k+l \,(\mathrm{mod}\, k)}) = v(\varphi^l(p_1)).$$

Now, the reader can verify that the knowledge of the order of the block itineraries given by Lemma 3.1 uniquely determines the cycle (P,φ) and the conclusion (i) is proved.

Let $v(q_1)$ be as above and suppose that it is periodic, i.e. there are a positive integer r which divides $k = \mathrm{per}(Q)$ and a vector $u = (u_1, \ldots, u_r) \in$

$\mathbb{Z}^r$ such that

$$v(q_1) = (\underbrace{u_1, \ldots, u_r}_{u}, \ldots, \underbrace{u_1, \ldots, u_r}_{u}).$$

Then

$$v(q_1) = v(\psi^0(q_1)) = v(\psi^r(q_1)) = (v_{1+r\,(\mathrm{mod}\,k)}, \ldots, v_{k+r\,(\mathrm{mod}\,k)}),$$

which contradicts the assertion of Lemma 3.1(i). ∎

REMARK 3.3. Another corollary of Lemma 3.1 is that a green pattern A is transitive. We will not prove this fact in detail (see [20], [8]).

Now we construct a one-parameter family of $(k+1)$st order non-homogeneous difference equations with constant coefficients. The solutions of those equations reflect the properties of the green patterns from $\mathcal{G}_k$.

Notice that for fixed real values $a \neq 0, b, x_0$ and any real value y, each of the equations

$$f(x) = ax + b = y, \qquad g(x) = ax_0 + x = y$$

has a unique real solution x depending on y. Hence, if we choose $\alpha > 1$ and $k \in \mathbb{N}$ and set $w(x) = \alpha - \alpha x$, then for any $k+1$ reals $\gamma_0, \ldots, \gamma_k$ there always exist sequences $\{f_n(x) = \alpha x + \beta_n\}_{n=1}^{\infty}$ and $\{g_n(x) = -\alpha x + \gamma_n\}_{n=0}^{\infty}$ such that

(i) for every $n \geq 0$, the solutions of the equations $g_n(x) = x$ and $f_{n+1}(x) = x$ coincide,

(ii) if for $n \geq 1$, x_n is the solution of the equation $f_n(x) = g_n(x)$, then for each $n \geq 1$, $w(f_{n+k}(x_{n+k})) = x_n$.

REMARK 3.4. To verify the existence of $\{f_n(x)\}_{n=1}^{\infty}$ and $\{g_n(x)\}_{n=0}^{\infty}$ satisfying (i) and (ii), one should start from values $\gamma_0, \ldots, \gamma_k$ which determine the maps $g_0, \ldots, g_k$ and by (i) and (ii) also $f_1, \ldots, f_{k+1}$ and $x_1, \ldots, x_k$.

When we put $n = 1$ in (ii), there is a unique solution x_{k+1} of the equation $w(f_{k+1}(x)) = x_1$. Hence, the value x_{k+1} is known and the equation $f_{k+1}(x_{k+1})) = g_{k+1}(x_{k+1})$ for unknown γ_{k+1} can be solved. This means that the map g_{k+1} is also determined and (i) yields the map f_{k+2}.

Now we can put $n = 2$ and use (ii) and (i) again to compute x_{k+2}, γ_{k+2}, g_{k+2}, f_{k+3}; etc.

The question of explicit expressions of the sequences

$$\{f_n(x) = \alpha x + \beta_n\}_{n=1}^{\infty}, \qquad \{g_n(x) = -\alpha x + \gamma_n\}_{n=0}^{\infty}$$

can be transformed into the task of solving one non-homogeneous $(k+1)$st order difference equation ($n \geq 0$, $k \geq 1$)

$$\gamma_{n+k+1}(\alpha^3+\alpha^2)+\gamma_{n+k}(-\alpha^3+\alpha^2)+\gamma_{n+1}(\alpha+1)+\gamma_n(\alpha-1) = 2(\alpha^3+\alpha^2), \tag{5}$$

with initial condition $\gamma_0, \dots, \gamma_k$. To study the behaviour of the sequence $\{\gamma_n\}_{n=0}^{\infty}$, we need to investigate the distribution of the roots of the characteristic equation of (5),

$$\lambda^{k+1}(\alpha^3+\alpha^2)+\lambda^k(-\alpha^3+\alpha^2)+\lambda(\alpha+1)+\alpha-1=0, \tag{6}$$

in the complex plane. The value α plays a natural role of parameter here. It turns out that from this point of view the most important value is $\alpha(k)$ defined as the root of the polynomial equation (in α)

$$(\alpha+1)^k(1+\sqrt{1+k^2})^k+\alpha^2(\alpha-1)^k k^k(k-\sqrt{1+k^2})=0. \tag{7}$$

LEMMA 3.5. *The equation* (7) *has a unique positive solution and this solution is from the interval* $(2,\infty)$.

Proof. Clearly 1 is not a solution of (7). Write (7) in the form

$$A(\alpha,k)=\frac{1}{\alpha^2}\left(1+\frac{2}{\alpha-1}\right)^k=\frac{k^k(\sqrt{1+k^2}-k)}{(1+\sqrt{1+k^2})^k}=B(k).$$

Since $\sqrt{1+k^2}-k\in(0,1/2)$, we also have $B(k)\in(0,1/2)$. As $|A(\alpha,k)|>1$ for $\alpha\in(0,1)$, a solution of (7) has to be greater than 1 (if it exists). The function $A(\cdot,k)$ is continuous and decreasing on $(1,\infty)$,

$$A(2,k)=3^k/4 \quad\text{and}\quad \lim_{\alpha\to\infty}A(\alpha,k)=0$$

and thus there is a unique value $\alpha(k)\in(2,\infty)$ for which $A(\alpha(k),k)=B(k)$. ∎

In the next lemma we consider a one-parameter family of polynomials $p_\alpha(\lambda)=\sum_{i=0}^n p_i(\alpha)\lambda^i$, where all real functions $p_i(\alpha)$ are from $C^1(J)$ (J is any fixed open subinterval of $\mathbb{R}$). For an open set $G\subset\mathbb{R}$, define

$$M_1=\{\alpha\in J : p_\alpha(\lambda)\text{ has a simple root in }G\}.$$

LEMMA 3.6. *Let M_1 be the set defined above. Then M_1 is open in J.*

Proof. By our definition of M_1, for $\alpha_0\in M_1$ and $F(\alpha,\lambda)=p_\alpha(\lambda)$, $(\alpha,\lambda)\in J\times G$, we have $F(\alpha_0,\lambda_0)=0$ and $\partial F(\alpha_0,\lambda_0)/\partial\lambda\neq 0$ (λ_0 is a simple root). Now, the conclusion follows from the Implicit Function Theorem. ∎

LEMMA 3.7. *Consider the polynomial equation* (6) *for* $\alpha\in(1,\infty)$. *Then*

(i) *all roots lie inside the unit disk*,

(ii) *for* $\alpha\in(1,\alpha(k))$, (6) *has no positive root*; *for* $\alpha=\alpha(k)$, (6) *has a positive root which is a double root.*

Proof. (i) Set

$$\begin{aligned}
\mathfrak{F}(\lambda)&=\lambda^{k+1}(\alpha^3+\alpha^2)+\lambda^k(-\alpha^3+\alpha^2)+\lambda(\alpha+1)+\alpha-1,\\
\mathfrak{G}(\lambda)&=\lambda^k(-\alpha^3+\alpha^2)+\lambda(\alpha+1),\\
\mathfrak{H}(\lambda)&=\lambda^{k+1}(\alpha^3+\alpha^2)+\alpha-1;
\end{aligned}$$

we have $\mathfrak{F} = \mathfrak{G} + \mathfrak{H}$ and for $\lambda \in \{z \in \mathbb{C} : |z| = 1\}$,

$$|\mathfrak{G}(\lambda)| \le \alpha^3 - \alpha^2 + \alpha + 1 < \alpha^3 + \alpha^2 - \alpha + 1 \le |\mathfrak{H}(\lambda)| < \infty,$$

hence by Rouché's Theorem [23] the polynomials $\mathfrak{F}$ and $\mathfrak{H}$ have the same number of zeros (namely, $k+1$) inside the unit disk.

(ii) Set

$$\begin{aligned}
&\mathfrak{F}_1(\lambda) = (k+1)\mathfrak{F}(\lambda) - \lambda\mathfrak{F}'(\lambda), \qquad \mathfrak{F}_2(\lambda) = k\mathfrak{F}(\lambda) - \lambda\mathfrak{F}'(\lambda),\\
&\mathfrak{F}_3(\lambda) = \lambda(\alpha^3 + \alpha^2)\mathfrak{F}_1(\lambda) - (\alpha^3 - \alpha^2)\mathfrak{F}_2(\lambda)\\
&\qquad\quad = (\alpha+1)^2\alpha^2 k\lambda^2 + 2\alpha^2(\alpha^2 - 1)\lambda - k\alpha^2(\alpha-1)^2.
\end{aligned}$$

By the definition of $\mathfrak{F}_1, \mathfrak{F}_2, \mathfrak{F}_3$ we can see that any root of $\mathfrak{F}$ of multiplicity greater than 1 is also a root of $\mathfrak{F}_3$. For each $\alpha \in (1, \infty)$, the quadratic polynomial $\mathfrak{F}_3$ has a unique positive root, which can be explicitly expressed as

$$\lambda_0 = \frac{\alpha - 1}{\alpha + 1} \cdot \frac{\sqrt{1 + k^2} - 1}{k}.$$

Obviously, the equality

$$\mathfrak{F}'(\lambda_0) = 0, \tag{8}$$

which after a rather laborious calculation can be transformed into (7) and considered as an equation in α, provides a necessary condition for the polynomial $\mathfrak{F}$ to have a positive double root. Since

$$\begin{aligned}
\mathfrak{F}_3(\lambda) = {}&(\lambda(\alpha^3 + \alpha^2)(k+1) - k(\alpha^3 - \alpha^2))\mathfrak{F}(\lambda)\\
&+ (\lambda(\alpha^3 - \alpha^2) - \lambda^2(\alpha^3 + \alpha^2))\mathfrak{F}'(\lambda),
\end{aligned}$$

and λ_0 is not a root of $\lambda(\alpha^3 + \alpha^2)(k+1) - k(\alpha^3 - \alpha^2)$, this condition is also sufficient. By Lemma 3.5, the equation (7) has exactly one solution $\alpha = \alpha(k)$ in $(1, \infty)$. By Descartes' rule, $\mathfrak{F}$ can have two or no positive roots (taken with their multiplicities). Writing (6) in the form

$$\lambda \frac{\lambda^k \alpha^2 + 1}{\lambda^k \alpha^2 - 1} = \frac{\alpha - 1}{\alpha + 1},$$

we can see that there exists $\varepsilon > 0$ such that for $\alpha \in (1, 1+\varepsilon)$, the equation (6) has no positive root. Take the least $\alpha \in (1+\varepsilon, \alpha(k)]$ for which $\mathfrak{F}$ has a positive root; by Lemma 3.6 used for $G = \mathbb{R}^+$, this root is of multiplicity greater than 1. As we already know, this implies $\alpha = \alpha(k)$. ∎

Remark 3.8. One can show similarly that for each $k \in \mathbb{N}$ odd, there exists a unique value $\widetilde{\alpha}(k) \in (1, \infty)$ for which the polynomial $\mathfrak{F}$ has a unique negative double root

$$\widetilde{\lambda}_0 = \frac{\widetilde{\alpha}(k) - 1}{\widetilde{\alpha}(k) + 1} \cdot \frac{-\sqrt{1 + k^2} - 1}{k};$$

it can be verified that for $k \in \mathbb{N}$ odd,

$$1 < \widetilde{\alpha}(k) < \alpha(k).$$

If $k \in \mathbb{N}$ is even, then $\mathfrak{F}$ has no negative double root.

Using Lemma 3.7 we are going to determine the limiting behaviour of the sequence $\{\gamma_n\}_{n=0}^{\infty}$. It can be easily seen that the sequence $\{\gamma_n = \alpha\}_{n=0}^{\infty}$ is a particular solution of the complete equation (5) with initial condition $\gamma_0 = \gamma_1 = \ldots = \gamma_k = \alpha$; as we have seen above any imaginary root of $\mathfrak{F}$ is simple. Hence, for $\alpha \notin \{\widetilde{\alpha}(k), \alpha(k)\}$ the solution of (5) has the form

$$\gamma_n = \alpha + \sum_{l=1}^{k+1} c_l \lambda_l^n, \qquad n \geq 0, \tag{9}$$

where $\lambda_1, \ldots, \lambda_{k+1}$ are different simple roots of (6), and for $\alpha = \widetilde{\alpha}(k)$ resp. $\alpha = \alpha(k)$,

$$\gamma_n = \widetilde{\alpha}(k) + c_1 \widetilde{\lambda}_0^n + c_2 n \widetilde{\lambda}_0^n + \sum_{l=3}^{k+1} c_l \lambda_l^n, \qquad n \geq 0, \tag{10a}$$

resp.

$$\gamma_n = \alpha(k) + c_1 \lambda_0^n + c_2 n \lambda_0^n + \sum_{l=3}^{k+1} c_l \lambda_l^n, \qquad n \geq 0, \tag{10b}$$

where

$$\widetilde{\lambda}_0 = \frac{\widetilde{\alpha}(k) - 1}{\widetilde{\alpha}(k) + 1} \cdot \frac{-\sqrt{1+k^2} - 1}{k}, \qquad \lambda_0 = \frac{\alpha(k) - 1}{\alpha(k) + 1} \cdot \frac{\sqrt{1+k^2} - 1}{k} \tag{10c}$$

are the unique double roots of (6) and $\lambda_3, \ldots, \lambda_{k+1}$ are the other (simple) roots of (6). The constants $c_1, \ldots, c_{k+1}$ are determined by the initial condition $\gamma_0, \ldots, \gamma_k$ (since we consider real solutions only, $c_l = \overline{c}_j$ whenever $\lambda_l = \overline{\lambda}_j$).

Lemma 3.9. *Let $\{\gamma_n\}_{n=0}^{\infty}$ be a solution of* (5). *Then*

(i) $\lim_{n\to\infty} \gamma_n = \alpha$,

(ii) *if $\alpha \in (1, \alpha(k))$ then the sequence $\{\gamma_n\}_{n=0}^{\infty}$ is not strictly monotone.*

Proof. (i) follows from Lemma 3.7(i) and from the expressions (9), (10a), (10b).

Lemma 3.7 and Remark 3.8 imply that (6) has no positive root for $1 < \alpha < \alpha(k)$. Applying Theorem Ap.1 from the appendix, we get (ii). ∎

Construction. Denote by Π_k the set of all real vectors $(\alpha_0, \alpha_1, \ldots, \alpha_k, \alpha) \in \mathbb{R}^{k+2}$ for which the difference equation (5) with initial condition $\gamma_0 = \alpha_0, \gamma_1 = \alpha_1, \ldots, \gamma_k = \alpha_k$ has an increasing solution $\{\gamma_n\}_{n=0}^{\infty}$ (here we require strict monotonicity). This solution corresponds to two sequences

$\{f_n(x) = \alpha x + \beta_n\}_{n=1}^{\infty}$, $\{g_n(x) = -\alpha x + \gamma_n\}_{n=0}^{\infty}$ (see (i), (ii) before Remark 3.4). In what follows, for $\pi \in \Pi_k$ we will define a function $F_\pi \in C(\mathcal{I})$ which maps a suitable compact interval $[a_\pi, b_\pi]$ into itself.

Let a_n, x_n, $n \geq 1$, be the solutions of the equations $f_n(x) = x$, $f_n(x) = g_n(x)$. Recall that $w(x) = \alpha - \alpha x$. Since $\pi \in \Pi_k$, we have $a_n < x_n < a_{n+1} < \alpha/(\alpha+1)$ for each $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} a_n = \frac{\alpha}{\alpha+1}.$$

Put

$$F_\pi(x) = \begin{cases} f_n(x), & x \in [a_n, x_n],\ n \geq 2, \\ g_n(x), & x \in [x_n, a_{n+1}],\ n \geq 1; \end{cases}$$

to finish the definition of F_π, set $y = \max\{f_i(x_i) : i = 1, \ldots, k\}$. Now we need to distinguish two possibilities. If $w(y) \geq a_1$ then $a_\pi = a_1$, $b_\pi = w^{-1}(a_1)$ and $F_\pi(x) = f_1(x)$ for $x \in [a_1, x_1]$, $F_\pi(x) = w(x)$ if $x \in [\alpha/(\alpha+1), b_\pi]$.

In the case when $w(y) < a_1$ put $a_\pi = w(y)$, $b_\pi = y$ and $F_\pi(x) = a_\pi$ for $x \in [a_\pi, f_1^{-1}(a_\pi)]$, $F_\pi(x) = f_1(x)$ if $x \in [f_1^{-1}(a_\pi), x_1]$ and $F_\pi(x) = w(x)$ for $x \in [\alpha/(\alpha+1), b_\pi]$.

This finishes the construction.

Finally, denote by Σ_k the set of all possible F_π, i.e.

$$\Sigma_k = \{F_\pi \in C(\mathcal{I}) : \pi \in \Pi_k\}.$$

The finiteness of the entropy of Lipschitz maps of a compact metric space is a well-known fact. We will use it in Proposition 3.11 and then several times more. In the one-dimensional case, the Lipschitz constant determines an upper bound for entropy.

Lemma 3.10 ([18]). *Let $f \in C(\mathcal{I})$ be the Lipschitz map with Lipschitz constant α. Then* $\operatorname{ent}(f) \leq \log \alpha$.

Proposition 3.11. *Let $F_\pi \in \Sigma_k$. Then*

(i) *if $\pi = (\ldots, \alpha)$, then* $\operatorname{ent}(F_\pi) \leq \log \alpha$,
(ii) *for each $n \geq 1$, $F_\pi(x_{n+k}) = x_n$.*

Proof. From the construction, F_π is piecewise linear (with countably many pieces) with slopes $\pm\alpha$ or 0, hence it is a Lipschitz map with constant α and $\operatorname{ent}(F_\pi) \leq \log \alpha$ by Lemma 3.10. The property (ii) follows directly from the fact that the sequences $\{f_n(x) = \alpha x + \beta_n\}_{n=1}^{\infty}$ and $\{g_n(x) = -\alpha x + \gamma_n\}_{n=0}^{\infty}$ are given by the solution $\{\gamma_n\}_{n=0}^{\infty}$ of (5). ∎

When proving the key Lemma 3.13 we will need the notion of a *cyclic sequence*.

Cyclic sequence. For $f \in C(\mathcal{I})$, we say that an interval J *f-covers* (resp. *P-covers*) an interval L if $L \subset f(J)$ (resp. $L \subset f_P(J)$). In this case we write $J \to L$. A sequence $\mathcal{B} = \{I_k\}_{k=0}^{m-1}$ of closed intervals is called *f-cyclic* (resp. *P-cyclic*) if $I_0 \to I_1 \to \ldots \to I_{m-1} \to I_0$.

The intermediate value theorem yields a lot of information from inclusions of subintervals in images of other subintervals.

LEMMA 3.12 ([7]). *For $f \in C(\mathcal{I})$ let $\mathcal{B} = \{I_k\}_{k=0}^{m-1}$ be an f-cyclic sequence. Then there is a point $x \in \mathrm{Per}(f)$ such that $f^k(x) \in I_k$ for $k = 0, \ldots, m-1$ and $f^m(x) = x$.*

Recall that we have denoted by $\mathcal{G}_k$ the set of all green patterns of complexity at most $2k$.

LEMMA 3.13. *Let $A \in \mathcal{G}_k$ and $F \in \Sigma_k$. Then F exhibits the pattern A.*

Proof. Suppose that (P, φ) is g-representative of A, and as in the previous section, consider a normal partition $P = P_{\mathrm{L}} \cup P_{\mathrm{R}}$, i.e. for $j \ge 1$,

$$P_{\mathrm{L}} = \bigcup_{1 \le i \le j} P_{2i-1} \cup P_{2i} \quad \& \quad P_{\mathrm{R}} = P_0;$$

in view of the approach described before Lemma 3.1, the least point of P has an itinerary

$$v(p_1) = (G(p_1) = 1, G(\varphi(p_1)), \ldots, G(\varphi^{\mathrm{per}(P)-1}(p_1))) \in \mathbb{Z}^{\mathrm{per}(P)}.$$

Notice that if we show that F has a cycle $Q = \{q_1 < \ldots\}$ which is a g-representative of some green pattern, then by Corollary 3.2,

$$Q = P \Leftrightarrow v(q_1) = v(p_1).$$

For $i \in \{1, \ldots, j\}$ define

$$I_{2i-1} = \left[a_{k+i}, f_{k+i}^{-1}\left(\frac{\alpha}{\alpha+1}\right)\right], \qquad I_{2i} = \left[x_{k+i}, g_{k+i}^{-1}\left(\frac{\alpha}{\alpha+1}\right)\right].$$

By the definition of F, $F(I_{2l-1}) = [a_{k+l}, \alpha/(\alpha+1)]$ and $F^2(I_{2l}) = [x_l, \alpha/(\alpha+1)]$; since $F \in \Sigma_k$ ($a_n < x_n < a_{n+1} < x_{n+1}$ for each $n \ge 1$), by Proposition 3.11(ii), we have $[x_l, \alpha/(\alpha+1)] \supset \bigcup_{i=1}^{2j} I_i$ if $l < k+1$ and $[x_l, \alpha/(\alpha+1)] \supset I_{2l-2k} \cup \bigcup_{i=2l-2k+1}^{2j} I_i$ if $l \ge k+1$. We know that A has complexity $C(A) \le 2k$, i.e. for a positive even value $G(\varphi^i(p_1))$ always $G(\varphi^i(p_1)) - G(\varphi^{i+2}(p_1)) \le C(A) \le 2k$. Hence we can consider an F-cyclic sequence

$$I_1 \to I_{G(\varphi(p_1))} \to \ldots \to I_{G(\varphi^{\mathrm{per}(P)-1}(p_1))} \to I_1, \tag{11}$$

where for even positive $G(\varphi^i(p_1))$ the sequence has a part

$$\ldots \to I_{G(\varphi^i(p_1))} \to F(I_{G(\varphi^i(p_1))}) \to I_{G(\varphi^{i+2}(p_1))} \to \ldots$$

By Lemma 3.12 this F-cyclic sequence determines a cycle

$$Q = (Q, \psi) = (Q, F_{|Q}).$$

Obviously, Q can be divided into the blocks $Q_i = Q \cap I_i$, $Q_0 = Q_{\mathrm{R}}$ and we can construct the vector $v(q_1)$. Clearly $v(q_1) = v(p_1)$ and since by Corollary 3.2(ii), $v(p_1)$ is not periodic, we even have $\mathrm{per}(Q) = \mathrm{per}(P)$. Let us show that Q is a g-representative of a green pattern. From the definition, ψ is increasing on the set $\bigcup_{i=1}^{j} Q_{2i-1}$ of green points and decreasing on the set $\bigcup_{i=0}^{j} Q_{2i}$ of black points.

The monotonicity on Q_0 is clear; so suppose there exist i_1, i_2 such that $G(\varphi^{i_1}(p_1)), G(\varphi^{i_2}(p_1))$ are both odd, $G(\varphi^{i_1}(p_1)) < G(\varphi^{i_2}(p_1))$ and for some $x \in Q_{G(\varphi^{i_1}(p_1))}$ and $y \in Q_{G(\varphi^{i_2}(p_1))}$ we have $\psi(x) > \psi(y)$ (ψ is not increasing on $\bigcup_{i=1}^{j} Q_{2i-1}$). By Lemma 3.1(ii), the definition of G and (11), necessarily

$$G(\varphi^{i_1+1}(p_1)) = G(\varphi^{i_2+1}(p_1)),$$

i.e. $\psi(x), \psi(y)$ lie in the same block Q_i. Denote by l_0 the positive number defined by

$$l_0 = \max\{l : G(\varphi^{i_1+i}(p_1)) = G(\varphi^{i_2+i}(p_1)) \text{ for } i \in \{1, \ldots, l\}\}.$$

Again by Lemma 3.1(ii),

$$G(\varphi^{i_1+l_0+1}(p_1)) < G(\varphi^{i_2+l_0+1}(p_1))$$

and, on the other hand, since ψ (being a restriction of F) is increasing on each Q_{2i-1} and decreasing on Q_{2i} and Q_0, we have

$$\psi^{l_0+1}(y) < \psi^{l_0+1}(x);$$

at the same time $\psi^{l_0+1}(x) \in I_{G(\varphi^{i_1+l_0+1}(p_1))}$ and $\psi^{l_0+1}(y) \in I_{G(\varphi^{i_2+l_0+1}(p_1))}$ —a contradiction. The case when ψ is not decreasing on $\bigcup_{i=1}^{j} Q_{2i}$ can be disproved analogously. ■

The weak version (the inequality is not strict) of Theorem B is

PROPOSITION 3.14. *For $A \in \mathcal{G}_k$, $\mathrm{ent}(A) \le \log \alpha(k)$.*

Proof. Fix a pattern $A \in \mathcal{G}_k$ and consider the vector

$$\varrho = (\alpha(k) - 1, \alpha(k) - \lambda_0, \ldots, \alpha(k) - \lambda_0^k, \alpha(k)),$$

where λ_0 is defined in (10c). By Lemma 3.7 and (10b), the vector ϱ belongs to Π_k. In particular, we can consider a map $F_\varrho \in \Sigma_k$. Since by Proposition 3.11(i), $\mathrm{ent}(F_\varrho) \le \log \alpha(k)$ and by Lemma 3.13, F_ϱ exhibits A, it follows from the definition of the entropy of a pattern (and its representatives) that

$$\mathrm{ent}(A) \le \log \alpha(k). \ ■$$

We will be able to show later (after Lemma 4.4) that even strict inequality holds.

4. Entropy of green patterns—proof of Theorems B and C. Let us recall

THEOREM C. *For any* $k \geq 1$, $\sup\{\mathrm{ent}(A) : A \in \mathcal{X}_k\} = \log \alpha(k)$.

The key idea of the proof is to construct a sequence $\{A_p\}$ of X-minimal patterns from $\mathcal{G}_k \setminus \mathcal{G}_{k-1}$ such that $\lim_{p\to\infty} \mathrm{ent}(A_p) = \log \alpha(k)$ (see before Lemma 4.4 for the sequence $\{[P_{j_p,p}]\}_{p=1}^{\infty}$). We start with two definitions.

Spiral. Let A be a green pattern and $(P = \{p_1 < \ldots < p_{\mathrm{per}(P)}\}, \varphi) \in A$ its g-representative. A subset $Q = \{\varphi^l(p_1)\}_{l=i}^{j}$ of P is said to be a *spiral* if all points of $Q \setminus \{\varphi^j(p_1)\}$ are black. A spiral Q is called a *maximal spiral* if it is not contained in another spiral different from Q. If $Q = \{\varphi^l(p_1)\}_{l=i}^{j}$ is a maximal spiral, then the *number of threads* is defined as $(j-i)/2$.

Connection. Let A be a green pattern and $(P = \{p_1 < \ldots < p_{\mathrm{per}(P)}\}, \varphi) \in A$ its g-representative. A subset $Q = \{\varphi^l(p_1)\}_{l=i}^{j}$ of P is said to be a *connection* if all points of $Q \setminus \{\varphi^j(p_1)\}$ are green. A connection Q is called a *maximal connection* if it is not contained in another connection different from Q. If $Q = \{\varphi^l(p_1)\}_{l=i}^{j}$ is a connection, then its *length* is defined as $j-i$.

Note that each spiral (resp. connection) is contained in some maximal spiral (resp. connection).

Figure 4 shows a g-representative P of a green $\frac{8}{3}$-pattern with period 11. We can see that the part $\{\varphi^l(p_1) = y_l\}_{l=4}^{8}$ is a maximal spiral of P with 2 threads, and $\{\varphi^l(p_1) = y_l\}_{l=0}^{4}$ is a maximal connection of length 4.

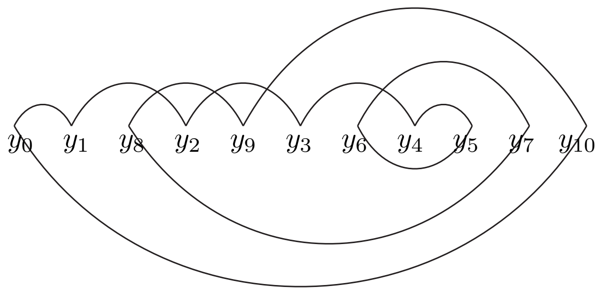

Fig. 4. An $\frac{8}{3}$-unicycle with period 11

The cycles (patterns) constructed below are self-similar in the sense that the maximal spirals (except one) have the same number of threads and the maximal connections (except one) have the same length.

Let $(P = \{p_1 < \ldots < p_{m+n}\}, \varphi)$ be an $\frac{m}{n}$-unicycle with monotone code and consider the short code defined in Section 2; according to Remark 2.1, we can consider the short code D'_P starting from $p_i \in P^*$. The reader can verify that by the monotonicity of the code K'_P, all codes D'_P determine the cycle P uniquely.

Recall that a positive integer k is fixed. For positive integers m, n, j, l we always assume the relations

$$m = j(kl+1) - 1, \quad n = kl + 1, \quad j \geq k(k+1) + 2. \tag{12}$$

Consider a short code $D'_{P_{j,l}} = \langle U_i Z_i \rangle_{i=1}^k$, where U_i (resp. Z_i) is a block of ones (resp. zeros), $\operatorname{card} U_i = l$ for $1 \leq i < k$, $\operatorname{card} U_k = l+1$; if z denotes the least positive integer for which

$$l(n-m) + zn \in \left[\frac{m-n}{k+1}, \frac{m-n}{k}\right) \tag{13}$$

(it exists by (12)), then $\operatorname{card} Z_i = z$ for $1 \leq i < k$. The number $\operatorname{card} Z_k = \widetilde{z}$ is determined by the equality

$$n(n-m) + (k-1)zn + \widetilde{z}n = 0. \tag{14}$$

Lemma 4.1. *The short code $\langle U_i Z_i \rangle_{i=1}^k$ described above determines an X-minimal pattern with eccentricity m/n.*

Proof (see Figure 5). The code $\langle U_i Z_i \rangle_{i=1}^k$ has $n = kl+1$ ones. By our choice of m, n we have the congruence

$$n - m \equiv 1 \pmod{n}$$

and so for $1 \leq v \leq kl+1$ we also have $v(n-m) \equiv v \pmod{n}$. Hence it can be easily verified that the maximal connections and the maximal spirals are disjoint (their points have different codes K'), i.e. the code $\langle U_i Z_i \rangle_{i=1}^k$ really defines a cycle and obviously it defines a unique g-representative with monotone code. Since by (14), $(k-1)z + \widetilde{z} = \sum_{i=1}^k \operatorname{card} Z_i = m - n$ we see that an eccentricity of that g-representative of the X-minimal pattern given by the code $\langle U_i Z_i \rangle_{i=1}^k$ equals m/n. ∎

The cycle from Lemma 4.1 can be written in spatial labeling as

$$P_{j,l} = \{p_1 < \ldots < p_{(j+1)(kl+1)-1}\};$$

the left black points of $P_{j,l}$ are denoted by $\{c_1 < \ldots < c_{kl+1}\}$ ($c_{kl+1} = p_{j(kl+1)-1}$). By $K_P(c_i)$ we mean the code of the point c_i.

Lemma 4.2. *The cycle $P = P_{j,l}$ has the following properties*:

(i) $K_P(c_{i+1}) - K_P(c_i) \geq (m-n)/(k+1)$,

(ii) *P has at least $[j/(k+1)]$ green points between c_i and c_{i+1}, $i = 1, \ldots, kl$ (between $\min P$ and c_1) belonging to the same connection*,

(iii) *a pattern $[P]$ has modality $\operatorname{mdl}([P])$ equal to $2kl+1$*,

(iv) *P has exactly $2k$ switches between $f_P^2(c_i)$ and c_i, $i = k+1, \ldots, kl+1$; for $i = 1, \ldots, k$ the number of switches between $f_P^2(c_i)$ and c_i is less than $2k$.*

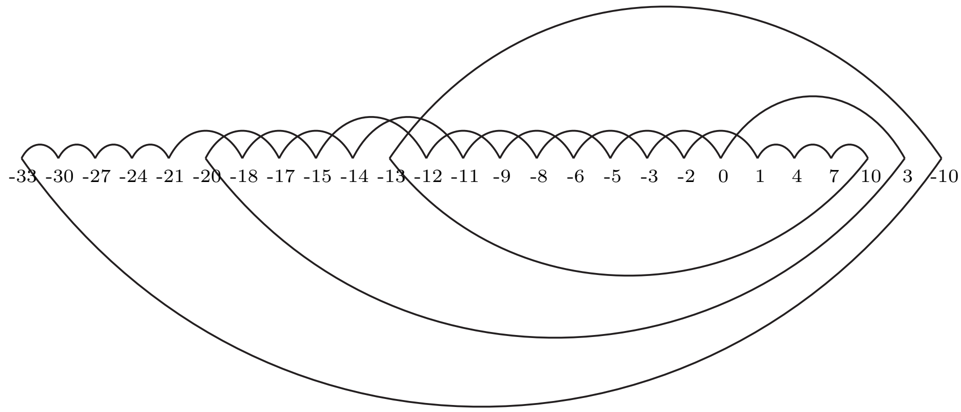

Fig. 5. An X-minimal $\frac{23}{3}$-unicycle (P,φ) given by a short code for $k=2$, $l=1$, $n=3$, $j=8$, $m=23$; $K'_P(p_{20})=0$; $[P]\in\mathcal{G}_2$ and $\operatorname{ent}([P])\sim\log 3.57626<\log\alpha(2)\sim\log 4.99667$

Proof. (i) By the definition of the short code, a left black point x of P has the code $K'_P(x)=r_x l(n-m)+r_x zn+s_x(n-m)$, where either $0\le r_x<k-1$ and $0\le s_x<l$ or $r_x=k-1$ and $0\le s_x\le l$. Of course, if $r_x=r_y$ and $s_x=s_y$, then $x=y$ by the monotonicity of K'_P. Thus, for the difference of the codes of two different left black points x,y we obtain

$$\begin{aligned}|K'_P(x)-K'_P(y)|&=|K_P(x)-K_P(y)|\\&=|(r_x-r_y)(l(n-m)+zn)+(s_x-s_y)(n-m)|\\&\ge||r_x-r_y|(l(n-m)+zn)-|s_x-s_y|(m-n)|.\end{aligned}$$

Notice that if $r_x\ne r_y$ then $1\le|r_x-r_y|\le k-1$. First, suppose that $r_x\ne r_y$ and also $s_x\ne s_y$. Then by (13),

$$|s_x-s_y|(m-n)-|r_x-r_y|(l(n-m)+zn)>(m-n)\left(1-\frac{k-1}{k}\right)>\frac{m-n}{k+1};$$

secondly, either $r_x=r_y$ and

$$|K_P(x)-K_P(y)|=|s_x-s_y|(m-n)\ge m-n>\frac{m-n}{k+1}$$

or $s_x=s_y$ and then by (13) again,

$$|K_P(x)-K_P(y)|=|r_x-r_y|(l(n-m)+zn)\ge\frac{m-n}{k+1},$$

which finishes the proof of (i).

(ii) Note that $f_P^2(c_1)=\min P$, hence by the definition of K_P, $K_P(c_1)-K_P(\min P)=m-n$. By (12) and (i),

$$K_P(c_{i+1})-K_P(c_i)\ge\frac{m-n}{k+1}>\left(\frac{j}{k+1}-1\right)n.$$

Summarizing, by the monotonicity of K_P, between $\min P$ and c_1 there are at least $[(m-n)/n] > j/(k+1)$ green points of a maximal connection which starts from $\min P$ and for every $i \in \{1, \ldots, kl\}$ there is a maximal connection of the cycle P which has at least $[j/(k+1)]$ green points between c_i, c_{i+1} (see the definition of K_P).

(iii) follows from the fact that by (12) and (ii), all $kl+1$ left black points of P are isolated, hence $\mathrm{mdl}([P]) = 2(kl+1) - 1 = 2kl+1$.

(iv) It is clear that for $i = 1, \ldots, k$ the number of switches between $f_P^2(c_i)$ and c_i is less than $2k$ (see Lemma 1.4(iii)). We will show that

$$f_P^2(c_{i+k}) = c_i \quad \text{for } i = 1, \ldots, kl+1-k. \tag{15}$$

By the definition of P, it has k maximal spirals (the ith maximal spiral is given by the block U_i of ones), where the first $k-1$ of them have the number of threads equal to l, and k maximal connections, where the first $k-1$ of them have length z (see (13)); if we consider the blocks U_i of ones in the short code D'_P, the first unity in U_i corresponds to the black point $c_{kl+1-(k-i)}$ and by the definition of K'_P,

$$K'_P(c_{kl+1-(k-i)}) = (i-1)(l(n-m)+zn).$$

In particular, for $i = 1$, $K'_P(c_{kl-k+2}) = 0$, where c_{kl-k+2} is the leftmost black point among those that end maximal connections. This implies that all black points from P less than c_{kl-k+2} have black preimages (with respect to f_P). By (13),

$$K'_P(c_{kl+1}) = (k-1)(l(n-m)+zn) < (m-n)\frac{k-1}{k},$$

hence

$$K'_P(f_P^2(c_{kl+1})) < (m-n)\left(\frac{k-1}{k} - 1\right) < 0,$$

and thus $f_P^2(c_{kl+1}) < c_{kl-k+2}$. Since $[P]$ is green, also for $i \in \{1, \ldots, k\}$,

$$f_P^2(c_{kl+1-(k-i)}) < c_{kl-k+2}.$$

This means that in fact $f_P^2(c_{kl+1}) = c_{kl-k+1}$ (since $n = kl+1 \geq k+1$, the point c_{kl-k+1} always exists) and we have proved the property (15) for $i = kl+1-k$. Now, for i smaller the relation (15) follows from the greenness of $[P]$. ∎

Remark 4.3. If we put $k = 1$ in Lemma 4.1, the X-minimal patterns given by the short code $\langle U_i Z_i \rangle_{i=1}^{1}$ are $2B$-patterns investigated in [11]; so we could say that $[P_{j,l}]$ is a $2kB$-pattern.

In this section we show that there are green patterns in $\mathcal{G}_k \setminus \mathcal{G}_{k-1}$ whose topological entropies do not have an upper bound less than $\log \alpha(k)$.

Put $j_p = (k+1)(k+p)$ and $l_p = p$ and consider the sequence of patterns $\{[P_{j_p,p}]\}_{p=1}^{\infty}$, where the representative $P_{j_p,p}$ is the cycle described in Lemma 4.2. By that lemma

- $\{[P_{j_p,p}]\}_{p=1}^{\infty} \subset \mathcal{G}_k \setminus \mathcal{G}_{k-1}$,
- $P_{j_p,p}$ has at least $k+p$ green points between two consecutive left black points (between $\min P$ and the leftmost black point) belonging to the same connection,
- the modality $\mathrm{mdl}([P_{j_p,p}])$ is equal to $2kp+1$.

Notice that when p grows to infinity, the same is true for the modality of $P_{j_p,p}$ and for the time spent by some green point between any two consecutive left black points.

As we know from Proposition 3.14, the topological entropies

$$\mathrm{ent}(P_{j_p,p}) = \mathrm{ent}(f_{P_{j_p,p}})$$

are bounded by $\log\alpha(k)$. By Remark 3.3, the $P_{j_p,p}$-linear maps $f_{P_{j_p,p}}$ are transitive and we can use Parry's Theorem [22] to renormalize each $f_{P_{j_p,p}}$. Namely, for every $p \geq 1$ there exists a piecewise linear continuous map $f_p : [0,1] \to [0,1]$ with slopes $\pm\exp(\mathrm{ent}(P_{j_p,p}))$ which is conjugate to $f_{P_{j_p,p}}$. Since the entropies (and then the slopes) of the elements of $\{f_p\}_{p=1}^{\infty}$ are bounded by $\log\alpha(k)$ (resp. $\exp(\log\alpha(k))$), these maps from $C(\mathcal{I})$ are equicontinuous and equibounded. Without loss of generality we can assume that

$$\begin{aligned}&\{f_p\}_{p=1}^{\infty} \text{ converges in } C^0\text{-norm to a map } f : [0,1] \to [0,1],\\ &\{\mathrm{ent}(f_p)\}_{p=1}^{\infty} \text{ converges to } \sup_{p\geq 1}\mathrm{ent}(f_p) = \sup_{p\geq 1}\mathrm{ent}(P_{j_p,p}).\end{aligned} \tag{16}$$

In the following lemma we will use the notation

$$\alpha = \exp(\sup_{p\geq 1}\mathrm{ent}(P_{j_p,p})). \tag{17}$$

LEMMA 4.4. *Let $f : [0,1] \to [0,1]$ be the limit of the sequence $\{f_p\}_{p=1}^{\infty}$. Then*

(i) *there exist sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ such that*

$$0 = a_1 < b_1 < a_2 < b_2 < \ldots < \frac{\alpha}{\alpha+1}, \qquad \lim a_n = \lim b_n = \frac{\alpha}{\alpha+1},$$

(ii) *f has a constant slope α on each interval $[a_n, b_n]$ and has slope $-\alpha$ on each $[b_n, a_{n+1}]$, $[\alpha/(\alpha+1), 1]$,*

(iii) *$f(a_n) = a_n$ for $n \geq 1$, $f^2(b_n) = b_{n-k}$ for $n \geq k+1$.*

Proof. It is clear that $f(x) = \alpha - \alpha x$ for $x \in [\alpha/(\alpha+1), 1]$. Write $P_{j_p,p} = P^p$, and let P^p_{L} (resp. P^p_{R}) be its left (resp. right) part with respect to the fixed point of f_{P^p}. Let $\{c_{p,i}\}_{i=1}^{kp+1}$ be the left black points of P^p, and $\{d_{p,i}\}_{i=1}^{kp+1}$ the left green points of the green blocks of P^p, i.e. $d_{p,i} = \min P^p_{2i-1}$

where P^p_{2i-1} is the ith green block of P^p from the left (cf. (1) in Section 1). As already stated, for each $p \geq 1$ there exists an increasing continuous map $h_p : \mathrm{conv}(P^p) \to [0,1]$ such that

$$f_{P^p} = h_p^{-1} \circ f_p \circ h_p.$$

Obviously, the points of the sequences $\{h_p(d_{p,i})\}_{i=1}^{kp+1}$ and $\{h_p(c_{p,i})\}_{i=1}^{kp+1}$ are exactly those at which f_p has its extrema. Now, since the sequence $\{f_p\}_{p=1}^{\infty}$ is convergent the reader can verify (consecutively for $n = 1$, $n = 2$, etc.) that

$$\lim_{p\to\infty} h_p(d_{p,n}) = a_n, \qquad \lim_{p\to\infty} h_p(c_{p,n}) = b_n$$

and by the continuity of f,

$$\lim a_n = \lim b_n = \frac{\alpha}{\alpha+1}.$$

According to Lemma 4.2(ii), (iv), $f(a_n) = a_n$ for $n \geq 1$ and $f^2(b_n) = b_{n-k}$ for $n \geq k+1$. The last property gives us the inequalities $a_n < a_{n+1}$ for $n \geq 1$. ∎

Proof of Theorem C. We know from Lemma 4.2(iv) that all patterns $[P_{j_p,p}]$ are from the set $\mathcal{G}_k$, hence by Proposition 3.14 for α defined in (17) we have $\alpha \leq \alpha(k)$.

On the other hand, for the map f given by (16) we have $f(x) = \alpha - \alpha x$ for $x \in [\alpha/(\alpha+1), 1]$. Using f we can define sequences $\{f_n(x) = \alpha x + \beta_n\}_{n=1}^{\infty}$ and $\{g_n(x) = -\alpha x + \gamma_n\}_{n=0}^{\infty}$ in this manner: if $n \geq 1$, then $f_n(x) = f(x)$ for $x \in [a_n, b_n]$, $g_n(x) = f(x)$ for $x \in [b_n, a_{n+1}]$ and $g_0(x) = -\alpha x$. Extend the domains of the maps f_n and g_n to the real line; by Lemma 4.4 the sequence $\{\gamma_n\}_{n=0}^{\infty}$ is then a solution of the difference equation (5) with the initial condition $\gamma_0 = 0, \gamma_1 = 2, \ldots, \gamma_k$. Using Lemma 4.4 again we can see that $\{\gamma_n\}_{n=0}^{\infty}$ is monotone; by Lemma 3.9, $\alpha \geq \alpha(k)$. This proves the theorem. ∎

Let us recall that a map $f \in C(\mathcal{I})$ defined on $I \in \mathcal{I}$ is transitive if and only if $\overline{\bigcup_{i\geq 0} f^i(J)} = I$ for any interval $J \subset I$ (see Section 0). In the notation of Lemma 2.10 we write $\mathcal{A}_Q(f)$ to emphasize that $\mathcal{A}_Q$ is taken with respect to f.

Proof of Theorem B. Let A be a fixed pattern from $\mathcal{G}_k$ and consider the map $f \in \Sigma_k$ defined by (16) (see the previous proof). The Lipschitz constant of f is equal to $\alpha(k)$, hence by Proposition 3.11, $\mathrm{ent}(f) \leq \log \alpha(k)$. By Lemma 3.13 and Theorem C we even have (see Section 1) $\mathrm{ent}(f) = \log \alpha(k)$.

To show the conclusion of Theorem B, we want to apply Lemma 2.10 to $f \in \Sigma_k$. So, let us show that f is transitive. By Lemma 3.5, $\alpha(k) > 2$, hence by Lemma 4.4 for any interval $J \subset [0,1]$ we have ($|J|$ is the length of J)

$$f(J) \cap \{a_n\}_{n=1}^{\infty} = \emptyset \Rightarrow |f^2(J)| > |J|,$$

where $\{a_n\}_{n=1}^{\infty}$ is the sequence of fixed points defined in Lemma 4.4. Obviously for some n_0,

$$f^{n_0}(J) \cap \{a_n\}_{n=1}^{\infty} \neq \emptyset$$

and then by Lemma 4.4(iii) even $f^{n_1}(J) = I$ for some $n_1 > n_0$. This shows that f is transitive.

By Lemma 3.13, f exhibits A; let f have a cycle P and $P \in A$. Notice that P does not contain $a_1 = 0 \in \mathrm{Fix}(f)$, the left endpoint of the domain of f. Using the definition of the entropy of P, Lemma 2.10 and Corollary 2.11, we obtain

$$\mathrm{ent}(f) = \log \alpha(k) > \log \mathrm{r}(\mathcal{A}_P(f)) \geq \log \mathrm{r}(\mathcal{A}_P(f_P)) = \mathrm{ent}(P) = \mathrm{ent}(A).$$

This proves Theorem B. ∎

Appendix: On strictly monotone solutions of difference equations. In this appendix we prove a necessary and sufficient condition for the existence of a strictly monotone solution of an mth order difference equation with real coefficients. The asymptotic behaviour of solutions of difference equations in partially ordered Banach spaces has been studied by means of spectral analysis in [13].

For $m \in \mathbb{N}$ fixed and $n \geq 0$ let

$$a_0\gamma_{n+m} + a_1\gamma_{n+m-1} + \ldots + a_{m-1}\gamma_{n+1} + a_m\gamma_n = 0 \tag{$A1$}$$

be an mth order difference equation, where all the coefficients a_i are real numbers; the *characteristic equation* of $(A1)$ is

$$a_0\lambda^m + a_1\lambda^{m-1} + \ldots + a_{m-1}\lambda + a_m = 0. \tag{$A2$}$$

Denoting all different roots of $(A2)$ as $\lambda_1, \ldots, \lambda_k$, where $k \leq m$ and each λ_j has a multiplicity m_j, a *general solution* $\{\gamma_n\}_{n\geq 0}$ of $(A1)$ can be expressed as

$$\gamma_n = \sum_{j=1}^{k} \sum_{l=0}^{m_j-1} c_{j,l} n^l \lambda_j^n, \tag{$A3$}$$

where the complex coefficients $c_{j,l}$ are uniquely determined by an initial condition for the values $\gamma_0, \ldots, \gamma_{m-1}$. In particular, the trivial solution (the sequence of zeros) is given by the zero coefficients. Recall that $\{\gamma_n\}$ is real if and only if for all coefficients we have $c_{j(1),l} = \overline{c}_{j(2),l}$ whenever $\lambda_{j(1)} = \overline{\lambda}_{j(2)}$ (see [17]). We will prove the following.

THEOREM Ap.1. *There exists a strictly monotone solution of* $(A1)$ *if and only if either* 1 *is a root of* $(A2)$ *with multiplicity at least* 2 *or* $(A2)$ *has a positive root different from* 1.

When proving our theorem we will need a general result on a topological dynamical system which describes the connection between *minimality* and *strong recurrence*; then this result will be applied to particular maps of the torus.

As usual, by a *topological dynamical system* we mean a pair (X, f), where X is a compact (metric) space and f is a continuous map of X into itself.

Minimality. We say that the topological dynamical system (X, f) is *minimal* if $\overline{\{f^i(x)\}}_{i\geq 0} = X$ for every $x \in X$.

Strong recurrence. Let (X, f) be a topological dynamical system. We say that $x \in X$ is *strongly recurrent* if for any open neighbourhood $U(x)$ of x there is a positive integer N_0 such that for each $j \in \mathbb{N}$,

$$\{f^i(x)\}_{i=j-1}^{j-1+N_0} \cap U(x) \neq \emptyset.$$

The close connection between strong recurrence and minimal sets is brought out in the following result, which was first proved in 1912 by G. D. Birkhoff.

PROPOSITION Ap.2 ([6]). *If the topological dynamical system (X, f) is minimal, then any point $x \in X$ is strongly recurrent.*

Now we are going to recall some results on translations on the torus. For more detailed information, see [18].

Using additive notation, for $j \in \mathbb{N}$ consider as phase space the j-dimensional torus

$$T^j = \underbrace{\mathbb{R}/\mathbb{Z} \times \ldots \times \mathbb{R}/\mathbb{Z}}_{j \text{ times}} = \mathbb{R}^j/\mathbb{Z}^j$$

and for $\omega = (\omega_1, \ldots, \omega_j) \in T^j$, the translation $T_{\omega,j}$ has the form

$$T_{\omega,j}(x_1, \ldots, x_j) = (x_1 + \omega_1, \ldots, x_j + \omega_j) \pmod 1.$$

If all coordinates of ω are rational numbers, then $T_{\omega,j}$ is periodic. Concerning the properties of $T_{\omega,j}$, we have the following result. Note that the 0-dimensional torus is given by a single point.

PROPOSITION Ap.3 ([18]). *For any translation $T_{\omega,j}$ and any $x \in T^j$ the closure*

$$\overline{\{T^i_{\omega,j}(x)\}}_{i\geq 0}$$

of the orbit of x is a finite union of tori of dimension k, $0 \leq k \leq j$, and the restriction of $T_{\omega,j}$ to $\overline{\{T^i_{\omega,j}(x)\}}_{i\geq 0}$ is minimal.

PROPOSITION Ap.4. *For any non-trivial* ($\omega \not\equiv 0 \pmod 1$) *translation* (*rotation*) $T_{\omega,1}$ *of a circle* $T = T^1$ *and any* $x \in T$ *we have*

$$\lim_{m\to\infty} \frac{1}{m} \sum_{n=0}^{m-1} \cos 2\pi T_{\omega,1}^n(x) = 0.$$

Proof. For a non-trivial $\omega \in T$, this follows from the elementary equalities

$$\begin{aligned}\sum_{n=0}^{m-1} \exp(i2\pi T_{\omega,1}^n(x)) &= \exp(i2\pi x) \sum_{n=0}^{m-1} \exp(i2\pi T_{\omega,1}^n(0)) \\ &= \exp(i2\pi x) \frac{\exp(i2\pi T_{\omega,1}^m(0)) - 1}{\exp(i2\pi T_{\omega,1}(0)) - 1}. \ \blacksquare\end{aligned}$$

Proof of Theorem Ap.1. Suppose that $(A2)$ has a positive root λ_+ different from 1. Then it follows from $(A3)$ that the strictly monotone sequence $\{\gamma_n = \lambda_+^n\}_{n\geq 0}$ is a solution of $(A1)$. Similarly, if 1 is a root of $(A2)$ with multiplicity at least 2, then again from $(A3)$ we can see that the increasing sequence $\{\gamma_n = n\}_{n\geq 0}$ is a solution of $(A1)$.

On the other hand, consider the equation $(A1)$ such that $(A2)$ either has 1 as a simple root or has no positive root, and let $\{\gamma_n\}_{n\geq 0}$ be a fixed real non-trivial solution of $(A1)$ given by $(A3)$, i.e.

$$\gamma_n = \sum_{j=1}^{k} \sum_{l=0}^{m_j-1} c_{j,l} n^l \lambda_j^n = \sum_{(j,l)\in S} c_{j,l} n^l \lambda_j^n,$$

where $S = \bigcup_{j=1}^k \{j\} \times \{0, 1, \ldots, m_j - 1\}$. It follows from $(A3)$ that the monotonicity of $\{\gamma_n\}$ depends on the maximal efficient roots in its expression (of maximal modulus with non-zero coefficients) and 1 as a simple root of $(A2)$ has no effect on it. Define $S_0 \subset S$ as $S_0 = \{(j,l) : \lambda_j \text{ is efficient for } \{\gamma_n\}\}$, and for $L = \max_{(j,l)\in S_0} l$ define $S_1 \subset \{1, \ldots, k\}$ by $S_1 = \{j : (j, L) \in S_0\}$. In the sequel we will need another solution $\{\delta_n\}_{n\geq 0}$ of $(A1)$ given by

$$\delta_n = \sum_{j\in S_1} c_{j,L} n^L \lambda_j^n.$$

Note that $\{\delta_n\}$ is a real solution again, i.e. $c_{j(1),L} = \overline{c}_{j(2),L}$ whenever $\lambda_{j(1)} = \overline{\lambda}_{j(2)}$.

Now we will distinguish two cases.

CASE I. First, we will prove the conclusion of our theorem for the solution $\{\delta_n\}_{n\geq 0}$, i.e. we will show that $\{\delta_n\}$ is not strictly monotone. We can then write for $\lambda = |\lambda_j|$, $j \in S_1$,

$$\delta_n = n^L \lambda^n \sum_{j\in S_1} c_{j,L} \exp(i2\pi n\omega_j),$$

and if we set $S_2 = \{j \in S_1 : \omega_j \in (0,1/2]\}$, then also

$$(A4)\qquad \begin{aligned}\delta_n &= n^L\lambda^n \sum_{j\in S_2} a_{j,L}\cos(2\pi n\omega_j) - b_{j,L}\sin(2\pi n\omega_j)\\ &= n^L\lambda^n \sum_{j\in S_2} \sqrt{a_{j,L}^2 + b_{j,L}^2}\cos 2\pi(x_{j,L} + n\omega_j),\end{aligned}$$

where $a_{j,L} = c_{j,L}$ if $\omega_j = 1/2$ (then $b_{j,L} = 0$), $a_{j,L} = 2\Re(c_{j,L})$, $b_{j,L} = 2\Im(c_{j,L})$ for $\omega_j \in (0,1/2)$ and

$$\cos 2\pi x_{j,L} = \frac{a_{j,L}}{\sqrt{a_{j,L}^2 + b_{j,L}^2}}.$$

Since all the coefficients $c_{j,L}$ are non-zero, $\{\delta_n\}$ is not a trivial solution of $(A1)$.

To show that $\{\delta_n\}$ is not strictly monotone, we will use the notation $l = \operatorname{card} S_2$ and

$$(A5)\qquad V(n,x) = \sum_{j\in S_2} \sqrt{a_{j,L}^2 + b_{j,L}^2}\cos 2\pi(x_{j,L} + n\omega_j),$$

which can be rewritten as $V(n,x) = V(0, T_{\omega,l}^n(x))$ for $x,\omega \in T^l$ and the translation $T_{\omega,l} : T^l \to T^l$. Define

$$M_+(x,\omega) = \{n \in \mathbb{N}\cup\{0\} : V(0,T_{\omega,l}^n(x)) > 0\}$$

and $M_-(x,\omega)$ analogously; by $(A4)$, it is sufficient to show that both these sets are non-empty. Suppose to the contrary that $M_-(x,\omega) = \emptyset$ and notice that combining Proposition Ap.2, Proposition Ap.3 and the continuity of the cosine map, we can find a positive integer N_0 and a positive real ε such that for each $j \in \mathbb{N}$,

$$(A6)\qquad \{V(0,T_{\omega,l}^n(x))\}_{n=j-1}^{j-1+N_0} \cap (\varepsilon,\infty) \neq \emptyset,$$

hence we immediately get

$$\liminf_{m\to\infty} \frac{1}{m}\sum_{n=0}^{m-1} V(0,T_{\omega,l}^n(x)) \geq \frac{\varepsilon}{N_0}.$$

On the other hand, Proposition Ap.4 gives us

$$\begin{aligned}&\lim_{m\to\infty} \frac{1}{m}\sum_{n=0}^{m-1} V(0,T_{\omega,l}^n(x))\\ &\qquad = \lim_{m\to\infty} \frac{1}{m}\sum_{n=0}^{m-1}\sum_{j\in S_2} \sqrt{a_{j,L}^2 + b_{j,L}^2}\cos 2\pi(x_{j,L} + n\omega_j)\end{aligned}$$

$$= \sum_{j\in S_2} \sqrt{a_{j,L}^2 + b_{j,L}^2} \lim_{m\to\infty} \frac{1}{m} \sum_{n=0}^{m-1} \cos 2\pi T_{\omega_j,1}^n (x_{j,L}) = 0.$$

The last evaluations yield an obvious contradiction: $M_-(x,\omega) \neq \emptyset$ and similarly for $M_+(x,\omega)$. By $(A4)$ and $(A5)$, the solution $\{\delta_n\}$ is not strictly monotone.

CASE II. To finish the proof for the solution $\{\gamma_n\}_{n\ge 0}$, we define

$$N(\{\delta_n\}) = \{n \in \mathbb{N} : \delta_n \neq 0\}.$$

Notice that by $(A4)$ and $(A6)$, the set $N(\{\delta_n\})$ is infinite. We show that

$$(A7) \qquad \lim_{n\in N(\{\delta_n\})} \gamma_n/\delta_n = 1.$$

Really, for $n \in N(\{\delta_n\})$ we have

$$\begin{aligned}\frac{\gamma_n}{\delta_n} &= \frac{\sum_{(j,l)\in S} c_{j,l} n^l \lambda_j^n}{\sum_{j\in S_1} c_{j,L} n^L \lambda_j^n} = \frac{\sum_{(j,l)\in S\setminus(S_1\times L)} c_{j,l} n^l \lambda_j^n + \sum_{j\in S_1} c_{j,L} n^L \lambda_j^n}{\sum_{j\in S_1} c_{j,L} n^L \lambda_j^n} \\ &= \frac{\sum_{(j,l)\in S\setminus(S_1\times L)} c_{j,l} n^{l-L} (\lambda_j/\lambda)^n + \sum_{j\in S_1} c_{j,L} \exp(i2\pi n\omega_j)}{\sum_{j\in S_1} c_{j,L} \exp(i2\pi n\omega_j)} \\ &= \frac{\sum_{(j,l)\in S\setminus(S_1\times L)} c_{j,l} n^{l-L} (\lambda_j/\lambda)^n}{\sum_{j\in S_1} c_{j,L} \exp(i2\pi n\omega_j)} + 1 = A(n) + 1,\end{aligned}$$

where $\lambda = |\lambda_j|$ for $j \in S_1$; since the numerator of $A(n)$ is a finite sum and by the definition of S_1, for $(j,l) \in S \setminus (S_1 \times L)$, we have the implication

$$|\lambda_j| = \lambda \Rightarrow l < L,$$

we also have

$$\lim_{n\in N(\{\delta_n\})} A(n) = 0,$$

hence $(A7)$ holds. Of course, this means that the sequence $\{\gamma_n\}$ is not strictly monotone, and our theorem is proved. ∎

References

[1] R. L. Adler, A. G. Konheim and M. H. McAndrew, *Topological entropy*, Trans. Amer. Math. Soc. 114 (1965), 309–319.

[2] L. Alsedà, J. Llibre and M. Misiurewicz, *Combinatorial Dynamics and Entropy in Dimension One*, Adv. Ser. Nonlinear Dynam. 5, World Sci., Singapore, 1993.

[3] —, —, —, *Periodic orbits of maps of Y*, Trans. Amer. Math. Soc. 313 (1989), 475–538.

[4] S. Baldwin, *Generalizations of a theorem of Sharkovskii on orbits of continuous real-valued functions*, Discrete Math. 67 (1987), 111–127.

[5] A. Berman and R. J. Plemmons, *Non-Negative Matrices in the Mathematical Sciences*, Academic Press, New York, 1979.

[6] L. S. Block and W. A. Coppel, *Dynamics in One Dimension*, Lecture Notes in Math. 1513, Springer, Berlin, 1992.
[7] L. Block, J. Guckenheimer, M. Misiurewicz and L. S. Young, *Periodic points and topological entropy of one dimensional maps*, in: Global Theory of Dynamical Systems, Lecture Notes in Math. 819, Springer, Berlin, 1980, 18–34.
[8] A. Blokh, *Rotation numbers, twists and a Sharkovskii–Misiurewicz-type order for patterns on the interval*, Ergodic Theory Dynam. Systems 15 (1995), 1–14.
[9] —, *Functional rotation numbers for one dimensional maps*, Trans. Amer. Math. Soc. 347 (1995), 499–514.
[10] —, *On rotation intervals for interval maps*, Nonlinearity 7 (1994), 1395–1417.
[11] A. Blokh and M. Misiurewicz, *Entropy of twist interval maps*, Israel J. Math. 102 (1997), 61–99.
[12] J. Bobok and M. Kuchta, *X-minimal patterns and generalization of Sharkovskii's theorem*, Fund. Math. 156 (1998), 33–66.
[13] J. Bobok and I. Marek, *On asymptotic behaviour of solutions of difference equations in partially ordered Banach spaces*, submitted to Positivity.
[14] R. Bowen, *Entropy for group endomorphisms and homogeneous spaces*, Trans. Amer. Math. Soc. 153 (1971), 401–414.
[15] E. M. Coven and M. C. Hidalgo, *On the topological entropy of transitive maps of the interval*, Bull. Austral. Math. Soc. 44 (1991), 207–213.
[16] M. Denker, Ch. Grillenberger and K. Sigmund, *Ergodic Theory on Compact Spaces*, Lecture Notes in Math. 527, Springer, 1976.
[17] T. Fort, *Finite Difference and Difference Equations in the Real Domain*, Oxford Univ. Press, 1965.
[18] A. Katok and B. Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems*, Encyclopedia Math. Appl. 54, Cambridge Univ. Press, 1995.
[19] M. Misiurewicz, *Minor cycles for interval maps*, Fund. Math. 145 (1994), 281–304.
[20] M. Misiurewicz and Z. Nitecki, *Combinatorial patterns for maps of the interval*, Mem. Amer. Math. Soc. 456 (1990).
[21] M. Misiurewicz and W. Szlenk, *Entropy of piecewise monotone mappings*, Studia Math. 67 (1980), 45–63.
[22] W. Parry, *Symbolic dynamics and transformations of the unit interval*, Trans. Amer. Math. Soc. 122 (1966), 368–378.
[23] W. Rudin, *Real and Complex Analysis*, McGraw-Hill, New York, 1974.
[24] P. Štefan, *A theorem of Sharkovskii on the coexistence of periodic orbits of continuous endomorphisms of the real line*, Comm. Math. Phys. 54 (1977), 237–248.

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