Partition properties of subsets of $\mathcal{P}_{\kappa}\lambda$

by

Masahiro Shioya (Tsukuba)

Abstract. Let $\kappa > \omega$ be a regular cardinal and $\lambda > \kappa$ a cardinal. The following partition property is shown to be consistent relative to a supercompact cardinal: For any $f : \bigcup_{n<\omega}[X]^{n}_\kappa \to \gamma$ with $X \subset \mathcal{P}_{\kappa}\lambda$ unbounded and $1 < \gamma < \kappa$ there is an unbounded $Y \subset X$ with $|f^{-1}[Y]^{n}_\kappa| = 1$ for any $n<\omega$.

Let $\kappa$ be a regular cardinal $>\omega$, $\lambda$ a cardinal $\geq \kappa$ and $F$ a filter on $\mathcal{P}_{\kappa}\lambda$. Partition properties of the form $\mathcal{P}_{\kappa}\lambda \rightarrow (F^+)_{2}^{<\omega}$ (see below for the definition) were introduced by Jech [6]. The case where $F$ is the club filter $\mathcal{C}_{\kappa}\lambda$ was particularly studied in connection with a supercompact cardinal: Menas [14] proved $\mathcal{P}_{\kappa}\lambda \rightarrow (\mathcal{C}_{\kappa}\lambda^+)_{2}^{<\omega}$ for a $2^{\lambda^{<\kappa}}$-supercompact $\kappa$ via a normal ultrafilter $U$ with $\mathcal{P}_{\kappa}\lambda \rightarrow (U^+)_{2}^{<\omega}$. As noted by Kamo [9], Menas’ argument can be modified to give the partition property of $\mathcal{P}_{\kappa}\lambda$ for $\kappa$ just $\lambda$-supercompact.

For the converse direction Di Prisco and Zwicker [4] and others refined the global result of Magidor [12]: The partition property of $\mathcal{P}_{\kappa}2^{\lambda^{<\kappa}}$ implies that $\kappa$ is $\lambda$-supercompact.

In [8] Johnson introduced properties of the form $X \rightarrow (F^+)_{2}^{<\omega}$ for $X \in F^+$, which means that for any $f : [X]^{n}_\kappa \to 2$ there is $Y \subset X$ with $|f^{-1}[Y]^{n}_\kappa| = 1$, as well as $F^+ \rightarrow (F^+)_{2}^{<\omega}$, which means $X \rightarrow (F^+)_{2}^{<\omega}$ for any $X \in F^+$. Abe [1] asked whether $\mathcal{F}_{\kappa}\lambda^+ \rightarrow (\mathcal{F}_{\kappa}\lambda^+)_{2}^{<\omega}$ would fail in ZFC, where $\mathcal{F}_{\kappa}\lambda$ denotes the minimal fine filter on $\mathcal{P}_{\kappa}\lambda$.

In this note we answer the question of Abe:

**Theorem.** Let $\kappa$ be a supercompact cardinal and $\lambda$ a cardinal $>\kappa$. Then there is a $\kappa^+ - c.c.$ poset forcing that $\kappa$ is supercompact and $\mathcal{F}_{\kappa}\lambda^+ \rightarrow (\mathcal{F}_{\kappa}\lambda^+)_{\gamma}^{<\omega}$ for any $1 < \gamma < \kappa$.

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Here $F^+ \rightarrow (F^+)\gamma^\omega$ means that for any $f : \bigcup_{n<\omega}\mathcal{X}_n^\omega \rightarrow \gamma$ with $X \in F^+$ there is $Y \in F^+$ with $Y \subseteq X$ and $|f''[Y]^\omega_{\mathcal{X}}| = 1$ for any $n < \omega$. Note that $\kappa$ is Ramsey iff $\mathcal{F}^+_{\kappa^\omega} \rightarrow (\mathcal{F}^+)\gamma^\omega$ for any $1 < \gamma < \kappa$.

We generally follow the terminology of Kanamori [10] with the following exception: For a cardinal $\mu \geq \omega$ we set $[X]^\mu = \{x \in X : |x| = \mu\}$, $[X]^\kappa = \{x \in X : |x| < \kappa\}$ and $\lim A = \{\alpha < \mu : \sup(A \cap \alpha) = \alpha > 0\}$ for $A \subseteq \mu$. We understand $\bigcup a \subseteq \bigcap b$ whenever the union $a \cup b$ of $a, b \in [\mathcal{P}_\kappa\lambda]^\mu$ and $b \in [\mathcal{P}_\kappa\lambda]^\mu$ with $m, n < \omega$ is formed.

We first give two negative partition results, which motivated Abe’s question. In [1] Abe proved $\mathcal{F}^+_{\kappa^\lambda} \not\rightarrow (\mathcal{F}^+)\gamma^\omega$ under $\lambda^{<\kappa} = 2^{\lambda}$. On the other hand, Matet [13], extending a result of Laver (see [7]), got the same conclusion from the opposite assumption:

PROPOSITION 1. Assume $\lambda^\kappa = \lambda$. Then $\mathcal{F}^+_{\kappa} \not\rightarrow (\mathcal{F}^+)\gamma^2$.

**Proof.** First set $\mathcal{P}_\kappa\lambda = \{x_\xi : \xi < \lambda\}$ and $[\mathcal{P}_\kappa\lambda]^\kappa = \{Y_\alpha : \alpha < \lambda\}$. By induction on $\xi < \lambda$ we construct $z_\xi \in \mathcal{P}_\kappa\lambda$ and $\{y^{\xi i}_\alpha : \alpha \in z_\xi \wedge i < 2\}$ so that $x_\xi \subseteq z_\xi$, $z_\xi \neq z_\xi$, $y^{\xi 0}_\alpha \in Y_\alpha$, $y^{\xi 1}_\alpha \not\subseteq z_\xi$ and $y^{\xi 0}_\alpha \neq y^{\xi 1}_\alpha$ for any $\xi < \lambda$, $i < 2$ and $\alpha, \beta \in z_\xi$ as follows: At stage $\xi < \lambda$ by induction on $n < \omega$ build $z_{\xi n} \in \mathcal{P}_\kappa\lambda$ and $\{y^{\xi n i}_\alpha : \alpha \in z_{\xi n} \wedge i < 2\}$ so that $x_\xi \subseteq z_{\xi 0} \not\subseteq \bigcup_{\zeta < \xi} z_\zeta$, $y^{\xi n 0}_\alpha \in Y_\alpha$, $y^{\xi n 0}_\alpha \neq y^{\xi n 1}_\alpha$ and $z_{\xi n} \cup \bigcup\{y^{\xi n i}_\alpha : \alpha \in z_{\xi n} \wedge i < 2\} \not\subseteq z_{\xi n+1}$. Finally set $z_\xi = \bigcup_{n<\omega} z_{\xi n}$. We claim that $f$ defined by $f(\{y^{\xi n i}_\alpha, z_\xi\}) = i$ witnesses $\{z_\xi : \xi < \lambda\} \not\rightarrow (\mathcal{F}^+)\gamma^2$.

Fix an unbounded set $X \subseteq \{z_\xi : \xi < \lambda\}$. We show $f''[X]^\omega_{\mathcal{X}} = 2$. Take $\alpha < \lambda$ with $Y_\alpha \in [X]^\kappa$, and $\xi < \lambda$ with $\alpha \in z_\xi \subseteq X$. Then $f(\{y^{\xi n i}_\alpha, z_\xi\}) = i$ for $i < 2$ by definition, as desired.

The above proof yields in fact for any $\gamma < \kappa$ an unbounded set $X \subseteq \mathcal{P}_\kappa\lambda$ and $f : [X]^\omega_{\mathcal{X}} \rightarrow \gamma$ such that $f''[Y]^\omega_{\mathcal{X}} = \gamma$ for any unbounded $Y \subseteq X$.

The analogous problem for the club filter has been solved by Abe [2] via an extension of Magidor’s theorem [12]: $\mathcal{C}^+_{\kappa\lambda} \not\rightarrow (\mathcal{C}^+)\gamma^2$. Let us give a canonical witness to his observation by appealing to Magidor’s idea more directly:

PROPOSITION 2. Let $\mu < \kappa$ be regular. Then $\{x \in \mathcal{P}_\kappa\lambda : \text{cf}(x \cap \kappa) = \mu\} \not\rightarrow (\mathcal{C}^+)\gamma^2$.

**Proof.** Set $S = \{x \in \mathcal{P}_\kappa\lambda : \text{cf}(x \cap \kappa) = \mu\}$ and for $x \in S$ fix an unbounded set $c_x \subseteq x \cap \kappa$ of order type $\mu$. For $x, y \in [S]^\omega_{\mathcal{X}}$ let $f(\{x, y\})$ be 0 when $\min(c_x \Delta c_y) \in c_x$, and 1 otherwise. Fix a stationary set $T \subseteq S$. We show $f''[T]^\omega_{\mathcal{X}} = 2$.

First, we have $\gamma < \kappa$ such that for any $w \in \mathcal{P}_\kappa\lambda$ there are $w \subseteq x, y \subseteq T$ with $\gamma \in c_x - c_y$: Let $g : \kappa \rightarrow \mathcal{P}_\kappa\lambda$ witness the contrary, i.e. $\gamma \in c_x$ iff $\gamma \in c_y$ for any $\gamma < \kappa$ and $g(\gamma) \subseteq x, y \subseteq T$. Take $x, y \in C(g) \cap T$ with $x \cap \kappa < y \cap \kappa$. 

by the stationarity of \( \{ z \cap \kappa : z \in C(g) \cap T \} \) in \( \kappa \). Then \( c_x = c_y \cap x \cap \kappa \) has order type \( \mu \), contradicting the choice of \( c_y \).

Now, let \( \gamma < \kappa \) be minimal as above. Then for \( \alpha < \gamma \) we have \( w_\alpha \in \mathcal{P}_\kappa \lambda \) such that \( \alpha \in c_\gamma \) if \( \alpha \in c_y \) for any \( w_\alpha \subset x \subset y \in T \). Set \( w = \bigcup_{\alpha < \gamma} w_\alpha \in \mathcal{P}_\kappa \lambda \).

Take \( w \subset x \subset y \subset z \) from \( T \) with \( \gamma \in c_x \cap c_z - c_y \). Then \( \min(c_x \Delta c_y) = \min(c_y \Delta c_z) = \gamma \) by \( w_\alpha \subset x \subset y \subset z \) for any \( \alpha < \gamma \), and hence \( f(\{x, y\}) = 0 \) and \( f(\{y, z\}) = 1 \) by definition, as desired. \( \blacksquare \)

The rest of the paper is devoted to establishing our Theorem. We refer to Baumgartner’s expository paper [3] for the rudiments of iterated forcings. We call a poset \( \kappa \)-centered closed when any centered subset of size \( < \kappa \) has a lower bound.

Assume for the moment that \( \kappa \) is a compact cardinal and \( \lambda \leq 2^\kappa \). Fix a coloring \( f : \bigcup_{n<\omega} [S]^\kappa_n \rightarrow \gamma \) with \( S \subset \mathcal{P}_\kappa \lambda \) unbounded and \( 1 < \gamma < \kappa \). Our definition of the poset \( Q_f \) below owes much to Galvin (see [7]), who proved under MA(\( \lambda \)) that for any \( f : [X]^2 \rightarrow 2 \) with \( X \subset [\lambda]^\omega \) cofinal there is a cofinal \( Y \subset X \) with \( |f^*[Y]^2| = 1 \).

Fix a fine ultrafilter \( U \) on \( S \) and define inductively a \( \kappa \)-complete ultrafilter \( U_n \) on \( [S]^\kappa_n \) by \( U_0 = \{ \emptyset \} \) and \( U_{n+1} = \{ X : \{ \alpha : \{ x \cup \alpha \in X \} \in U_n \} \in U \} \). For \( n < \omega \) let \( \beta_n \) be the unique \( \beta < \gamma \) with \( \{ a \in [S]^\kappa_n : f(a) = \beta \} \in U_n \). Let \( Q_f = \{ p \in [S]^{<\kappa} : \forall m, n < \omega \exists a \in [p]^m_n : \{ \{ \beta \in [S]^2 : f(\beta) = \beta_{m+n} \} \in U_n \} \} \), and \( q \leq p \) iff \( q \supset p \) and \( y \not\subset x \) for any \( x \in p \) and \( y \in q - p \). Let us observe some basic properties of \( Q_f \).

First, for a generic filter \( G \subset Q_f \), \( \bigcup G \) is unbounded in \( \mathcal{P}_\kappa \lambda \) by the density of \( \{ q \in Q_f : \exists y \in g \subset y \} \) for any \( x \in \mathcal{P}_\kappa \lambda \), and homogeneous for \( f : f^*[\bigcup G]^\kappa_n = \{ \beta_n \} \) for any \( n < \omega \).

Next, we have the \( \kappa \)-centered closure of \( Q_f \): \( \bigcup D \) is a lower bound of a centered set \( D \in [Q_f]^{<\kappa} \).

Finally, we invoke an argument of Engelking and Karlovič [5] to show that \( Q_f \) is \( \kappa \)-linked. Fix an injection \( \pi : \mathcal{P}_\kappa \lambda \rightarrow \kappa^+ \). For \( A \subset \kappa^+ \) with \( \alpha < \kappa \) set \( Q_{f,A} = \{ p \in Q_f : \{ \pi(x) : \alpha : x \in p \} = A \land ( \pi(z) : z \in \bigcup_{x \in p} \mathcal{P} x ) \) is injective \}. Then \( Q_f = \bigcup \{ Q_{f,A} : \exists \alpha < \kappa ( A \subset \kappa^+ ) \} \) by the inaccessibility of \( \kappa \). To see that \( Q_{f,A} \) is linked, fix \( p, q \in Q_{f,A} \). Then \( x \not\subset y \) for any \( x \in p - q \) and \( y \in q \): Otherwise we would have \( x = z \) for some \( x \in p - q \), \( y \in q \) with \( x \subset y \) and \( z \in q \) with \( \pi(x) = \pi(z) \alpha \). Similarly, \( y \not\subset x \) for any \( x \in p \) and \( y \in q - p \). Thus \( p \cup q \leq p, q \), as desired.

Before starting the proof of our Theorem, we need to generalize a result of Baumgartner [3]:

**Lemma.** Assume \( 2^{<\kappa} = \kappa \). Let \( \{ P_\alpha, Q_\alpha : \alpha < \beta \} \) be a \( \alpha \)-support iteration such that \( \| Q_\alpha \) is \( \kappa \)-centered closed and \( \kappa \)-linked" for any \( \alpha < \beta \). Then \( P_\beta \) is \( \kappa \)-directed closed and \( \kappa^+ \)-c.c.
Proof. It is easily seen that the $\kappa$-centered closure implies the $\kappa$-directed closure, which is preserved by $<\kappa$-support iterations.

To see the $\kappa^+$-c.c., fix $X \in [P_\beta]^{<\kappa}$. For $\alpha < \beta$ let $\forces_\alpha \ "Q_\alpha = \bigcup_{\gamma < \kappa} \dot{Q}_{\alpha\gamma}\"$ with $\dot{Q}_{\alpha\gamma}$ linked for any $\gamma < \kappa$. For $p \in X$ by induction on $\xi < \kappa$ build $p_\xi \leq p$, $\alpha_\xi^p \in \text{supp}(p_\xi)$ and $\gamma_\xi^p < \kappa$ so that $p_\xi \leq p_\zeta$ for any $\zeta < \xi$, $p_{\xi+1}|\alpha_\xi^p \forces_\alpha \ "p_\zeta(\alpha_\zeta^p) \in \dot{Q}_{\alpha_\xi^p\xi_\zeta^p}\"$, and $\{\xi < \kappa : \alpha_\xi^p = \alpha\}$ is unbounded for any $\alpha \in \bigcup_{\xi < \kappa} \text{supp}(p_\xi)$. Take $Y \in [X]^{<\kappa}$ and $\delta < \kappa$ so that $\delta \in \Delta_{\kappa < \delta} \bigcap \{\lim\{\xi < \kappa : \alpha_\zeta^p = \alpha\} : \alpha \in \text{supp}(p_\zeta)\}$ for any $p \in Y$. Note that $\{\alpha_\delta^p : \xi < \delta\} = \bigcup_{\xi < \delta} \text{supp}(p_\xi)$ for any $p \in Y$. Next take $Z \in [Y]^{<\kappa}$ so that $\{\{\alpha_\delta^p : \xi < \delta\} : p \in Z\}$ forms a $\Delta$-system with root $d \in [\beta]^{<\kappa}$. Finally, take $W \in [Z]^{<\kappa}$ and $H \in [\delta \times d \times \kappa]^{<\kappa}$ so that $\{(\xi, \alpha_\delta^p, \gamma_\delta^p) : \xi < \delta \land \alpha_\delta^p \in d\} = H$ for any $p \in W$. We show that $W$ is linked, as desired.

Fix $p, q \in W$. Inductively we build a lower bound $r \in P_\beta$ of $\{p_\xi : \xi < \delta\} \cup \{q_\xi : \xi < \delta\}$ with support $\bigcup_{\xi < \delta} \text{supp}(p_\zeta) \cup \bigcup_{\xi < \delta} \text{supp}(q_\zeta)$. At stage $\alpha < \beta$ we claim that $\{\xi < \delta : r|\alpha \forces_\alpha \ "p_\xi(\alpha) \parallel q_\xi(\alpha)\"\}$ is unbounded, which implies $r|\alpha \forces_\alpha \ "\{p_\xi(\alpha) : \xi < \delta\} \cup \{q_\xi(\alpha) : \xi < \delta\} \text{ is centered}\"$, as desired, since $r|\alpha \forces_\alpha \ "\{p_\xi(\alpha) : \xi < \delta\}$ and $\{q_\xi(\alpha) : \xi < \delta\}$ are descending. Let us concentrate on the nontrivial case where $\alpha \in d = \bigcup_{\xi < \delta} \text{supp}(p_\xi) \cap \bigcup_{\xi < \delta} \text{supp}(q_\xi)$.

Fix $\xi < \delta$ with $\alpha_\delta^p = \alpha$. Then $r|\alpha \leq p_{\xi+1}|\alpha, q_{\xi+1}|\alpha$ forces $"p_\xi(\alpha), q_\xi(\alpha) \in \dot{Q}_{\alpha\gamma}\$ where $(\xi, \alpha, \gamma) \in H$. Now the claim follows, since $\{\xi < \delta : \alpha_\delta^p = \alpha\}$ is unbounded by the choice of $\delta$.

Proof of Theorem. First, we force with the Laver poset [11] for $\kappa$ and then add $\lambda$ Cohen subsets of $\kappa$ to ensure that $\kappa$ is supercompact and $\lambda \leq 2^\kappa$ in the further extensions. Next, we perform a $<\kappa$-support iteration $(P_\alpha, Q_\alpha : \alpha < 2^{<\kappa})$ with $\forces_\alpha "Q_\alpha = Q_f\"$ for some canonical $P_\alpha$-name $f$ for a coloring. The standard inductive argument, together with the $\kappa$-closure and the $\kappa^+$-c.c. of $P_\alpha$, shows that for any $\alpha < 2^{<\kappa}$, $P_\alpha$ is of size $\leq 2^{<\kappa}$, and so is the set of canonical $P_\alpha$-names for colorings, whose union can be identified with that of canonical $P_{\kappa^{<\kappa}}$-names for colorings. Thus the iteration can be arranged so that a homogeneous set for a coloring in the final model by $P_{\kappa^{<\kappa}}$ appears in an intermediate model, which, by absoluteness of $P_\kappa$, remains unbounded, as desired.

References


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Institute of Mathematics
University of Tsukuba
Tsukuba, 305-8571 Japan
E-mail: shioya@math.tsukuba.ac.jp

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