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Waldhausen's Nil groups and continuously controlled K-theory

by

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Abstract. Let $\Gamma = \Gamma_1 *_G \Gamma_2$ be the pushout of two groups Γ_i, $i = 1, 2$, over a common subgroup G, and H be the double mapping cylinder of the corresponding diagram of classifying spaces $B\Gamma_1 \leftarrow BG \rightarrow B\Gamma_2$. Denote by ξ the diagram $I \overset{p}{\leftarrow} H \overset{1}{\rightarrow} X = H$, where p is the natural map onto the unit interval. We show that the $\widetilde{\text{Nil}}$ groups which occur in Waldhausen's description of $K_*(\mathbb{Z}\Gamma)$ coincide with the continuously controlled groups $\widetilde{K}^{\text{cc}}_*(\xi)$, defined by Anderson and Munkholm. This also allows us to identify the continuously controlled groups $\widetilde{K}^{\text{cc}}_*(\xi^+)$ which are known to form a homology theory in the variable ξ, with the "homology part" in Waldhausen's description of $K_{*-1}(\mathbb{Z}\Gamma)$. A similar result is also obtained for HNN extensions.

1. Recollections. Let the group Γ be a pushout as in the abstract. By first passing to the corresponding pushout of integral group rings, next applying Theorem 1 of [8], and finally taking homotopy groups, we arrive at the following version of Waldhausen's result concerning $K_*(\mathbb{Z}\Gamma)$.

THEOREM 1.1 (Waldhausen [8]). *For a group Γ as above, there is a chain complex of abelian groups*

$$\ldots \rightarrow K_j(\mathbb{Z}G) \rightarrow K_j(\mathbb{Z}\Gamma_1) \oplus K_j(\mathbb{Z}\Gamma_2) \rightarrow K_j(\mathbb{Z}\Gamma) \rightarrow K_{j-1}(\mathbb{Z}G) \rightarrow \ldots,$$

which is exact except that at each $K_j(\mathbb{Z}\Gamma)$ the homology is $\widetilde{\text{Nil}}_{j-1}(\mathbb{Z}G;B_1,B_2)$ *where $B_i = \mathbb{Z}[\Gamma_i - G]$ as a $\mathbb{Z}G$-bimodule* $(i = 1, 2)$.

The continuously controlled K-theory of Anderson and Munkholm is defined in Section 7 of [4] as a (spectrum-valued) functor $\widetilde{\mathbf{K}}^{\text{cc}}_* : \mathcal{TOP}/\mathcal{CM}^* \rightarrow$

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$\mathcal{SPEC}$. Here $\mathcal{TOP}/\mathcal{CM}^*$ is the category of "diagrams of holink type", i.e., diagrams of the form

$$\eta = (B \xleftarrow{p} H \xrightarrow{q} X),$$

where H and X are Hausdorff spaces, B is a compact metric space, and p, q are continuous maps.

For mnemonic reasons, let us note that the somewhat elaborate name $\mathcal{TOP}/\mathcal{CM}^*$ is supposed to refer to a $\mathcal{TOP}$ological space H over a $\mathcal{C}$ompact $\mathcal{M}$etric space B, all of it "based" (hence the *) over the space X.

Also, to explain why we call this *continuously controlled K-theory*, we note that the spectrum arises ultimately from a category of geometric modules on the open mapping cylinder $\mathrm{cyl}(q) = X \amalg (0,1) \times H$. This cylinder is considered as a space over the cone of B via the map induced by p, and the morphisms of geometric modules are required to be continuously controlled at $B = B \times \{1\}$. Thus X is simply the part of the total space $\mathrm{cyl}(q)$ which sits over the cone point as indicated in Figure 1.

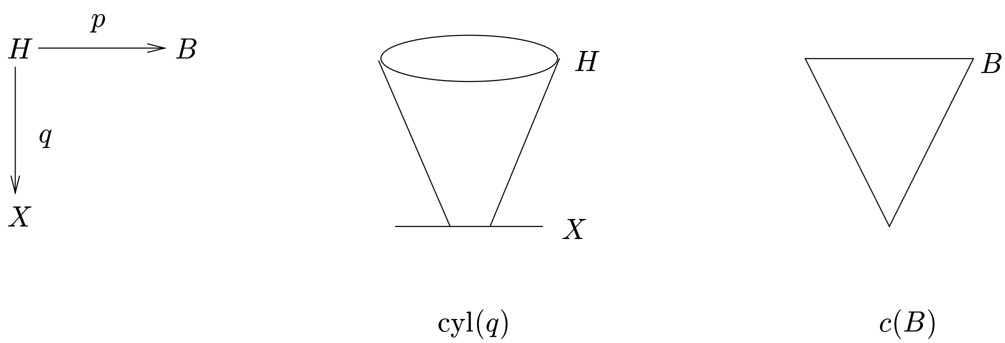

Fig. 1

The continuously controlled groups are the homotopy groups of the spectrum $\widetilde{\mathbf{K}}^{\mathrm{cc}}$. They will here be denoted $\widetilde{K}^{\mathrm{cc}}_*$. To avoid possible confusion, we note that the superscript cc was not used when these groups were introduced in Section 7 of [4].

If η is the above diagram of holink type, one lets

$$\eta^+ = (B_+ \xleftarrow{p \amalg c} H \amalg X \xrightarrow{q \vee 1} X),$$

where the subscript $+$ indicates a disjointly added base point. The inclusion $j : \eta \to \eta^+$ in $\mathcal{TOP}/\mathcal{CM}^*$ induces a map on cc K-theory which is studied in Section 8 (plus Corollary 9.4) of [4] (1) .

(1) We remark that there is a misprint in line -2 of p. 30 of [4]: $\widetilde{K}_* R\pi_1 X$ should not carry the tilde.

PROPOSITION 1.2 (Anderson and Munkholm [4]). *For any η as above, there is a long exact sequence*

$$\ldots \to K_j(\mathbb{Z}\pi_1(X)) \to \widetilde{K}_j^{\mathrm{cc}}(\eta) \to \widetilde{K}_j^{\mathrm{cc}}(\eta^+) \to K_{j-1}(\mathbb{Z}\pi_1(X)) \to \ldots$$

Here, if X is not connected the group ring indicated must be interpreted as a groupoid "ring" as usual (cf., e.g., §21 of [5]). Although we refer the reader to [4] for further details, we do here want to indicate the intuition behind this result: The groups $K_j(\mathbb{Z}\pi_1(X))$ are derived from the category of finitely generated free $\mathbb{Z}\pi_1(X)$-modules. This can also be construed as the category of those geometric modules in $\mathrm{cyl}(q)$ which happen to live in the subspace X and happen to be controlled over the cone point, i.e. to be not controlled at all. This interpretation immediately shows that the composition $K_j(\mathbb{Z}\pi_1(X)) \to \widetilde{K}_j^{\mathrm{cc}}(\eta) \to \widetilde{K}_j^{\mathrm{cc}}(\eta^+)$ must vanish. In fact, when the extra "base point" is added to η the subspace X of $\mathrm{cyl}(q)$ gets augmented to a product $X \times (-1, 0]$ and one can do an Eilenberg swindle (i.e., an alternating infinite sum trick) towards -1 while keeping continuous control at -1. Here, for convenience, we think of $(1, +)$ as -1 as indicated in Figure 2.

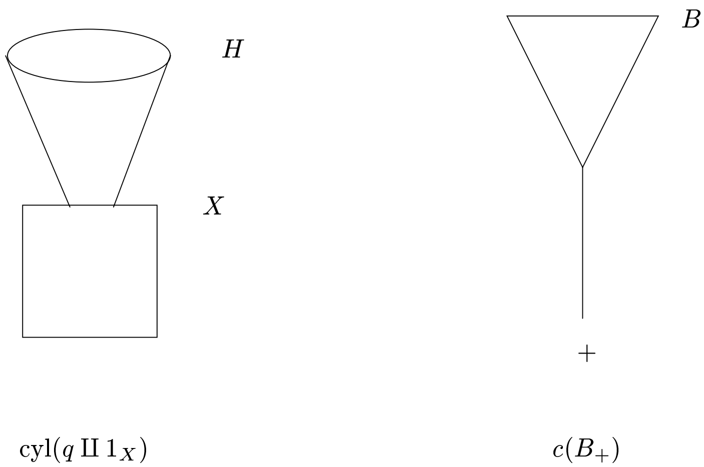

Fig. 2

We also need to recall Theorem 9.1 of [4].

THEOREM 1.3 (Anderson and Munkholm [4]). *If, in the object η, the space $B = |K|$ is a finite simplicial complex and the map $p : H \to |K|$ has an iterated mapping cylinder structure, then the natural map $\widetilde{K}_*^{\mathrm{bc}}(\eta) \to \widetilde{K}_*^{\mathrm{cc}}(\eta)$ is an isomorphism.*

Here, the *boundedly controlled* groups on the left are defined as in [2] (cf. also Section 9 of [4]). Our interest in making the cc → bc substitution comes from the fact that in the situation of the above theorem, it is known from Theorem 4.1 of [2] that the bc K-theory of η^+ is the target of an Atiyah–Hirzebruch spectral sequence with $E^2_{p,q} = H_p(\mathrm{cat}(K), \mathcal{K}_{q-1})$, the homology of K, considered as a category, with coefficients in the functor $\sigma \mapsto K_{q-1}(p^{-1}(\widehat{\sigma}))$ with $\widehat{\sigma}$ the barycenter of the simplex σ.

2. The amalgamated free product case. We can apply the above spectral sequence to ξ^+ where $\xi = (I \xleftarrow{p} H \xrightarrow{1} X = H)$ as in the abstract, and I has the obvious triangulation with one 1-simplex, τ, and two 0-simplices, σ_i, $i = 0, 1$. We get $p^{-1}(\widehat{\tau}) = BG$, $p^{-1}(\widehat{\sigma}_i) = B\Gamma_{i+1}$ so the spectral sequence degenerates to the first long exact sequence in the following commutative diagram:

$$
\begin{array}{ccccccccc}
 & & & & & & \vdots & & \\
 & & & & & & \downarrow & & \\
\longrightarrow & K_q(\mathbb{Z}G) & \longrightarrow & K_q(\mathbb{Z}\Gamma_1) \oplus K_q(\mathbb{Z}\Gamma_2) & \longrightarrow & \widetilde{K}^{\mathrm{cc}}_{q+1}(\xi^+) & \longrightarrow & K_{q-1}(\mathbb{Z}G) & \\
 & {\scriptstyle 1}\downarrow & & {\scriptstyle 1}\downarrow & & \downarrow & & {\scriptstyle 1}\downarrow & \\
\longrightarrow & K_q(\mathbb{Z}G) & \longrightarrow & K_q(\mathbb{Z}\Gamma_1) \oplus K_q(\mathbb{Z}\Gamma_2) & \longrightarrow & K_q(\mathbb{Z}\Gamma) & \longrightarrow & K_{q-1}(\mathbb{Z}G) & \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \widetilde{K}^{\mathrm{cc}}_{q}(\xi) & \longrightarrow & 0 & \\
 & & & & & \downarrow & & & \\
 & & & & & \widetilde{K}^{\mathrm{cc}}_{q}(\xi^+) & & & \\
 & & & & & \downarrow & & & \\
 & & & & & \vdots & & &
\end{array}
$$

Here, the second row is the chain complex of Theorem 1.1, and the vertical column is the exact sequence from Proposition 1.2 (note that $X = H$ has fundamental group Γ). A straightforward diagram chase now shows that the vertical map $\widetilde{K}^{\mathrm{cc}}_{q+1}(\xi^+) \to K_q(\mathbb{Z}\Gamma)$ is monic. Therefore, the second row and the third row have isomorphic homology, i.e. we have proved the following

THEOREM 2.1. *Let $\Gamma = \Gamma_1 *_G \Gamma_2$ be a pushout of two groups Γ_i $(i = 1, 2)$ over a common subgroup G. Also, consider the corresponding diagram of holink type $\xi = (I \leftarrow H \to X = H)$ where H is the double mapping cylinder of $B\Gamma_1 \leftarrow BG \to B\Gamma_2$ parametrized over the unit interval I as usual. Then* *there is a short exact sequence*

$$\widetilde{K}^{\mathrm{cc}}_{*+1}(\xi^{+}) \to K_*(\mathbb{Z}\Gamma) \to \widetilde{K}^{\mathrm{cc}}_{*}(\xi),$$

where $\widetilde{K}^{\mathrm{cc}}_{*+1}(\xi^{+})$ *is the homology part of* $K_*(\mathbb{Z}\Gamma)$, *i.e. the part that fits into the expected Mayer–Vietoris sequence, while*

$$\widetilde{K}^{\mathrm{cc}}_{*}(\xi) \cong \widetilde{\mathrm{Nil}}_{*-1}(\mathbb{Z}G; \mathbb{Z}[\Gamma_1 - G], \mathbb{Z}[\Gamma_2 - G]).$$

3. The HNN extension case. If $\Gamma = A *_C \{t\}$ is an HNN extension defined from the groups A and C and two embeddings $\alpha, \beta : C \to A$, then one constructs an object

$$\zeta = (|K| \xleftarrow{p} H \xrightarrow{1} X = H)$$

as follows. K is the boundary of the standard 2-simplex, and H is an iterated mapping cylinder over (the first derived of) K with

$$p^{-1}(\widehat{\sigma}_i) = p^{-1}(\widehat{\tau}_j) = BC, \quad i = 1, 2,\ j = 0, 1, 2; \quad p^{-1}(\widehat{\sigma}_0) = BA.$$

The maps defining H are as indicated in Figure 3.

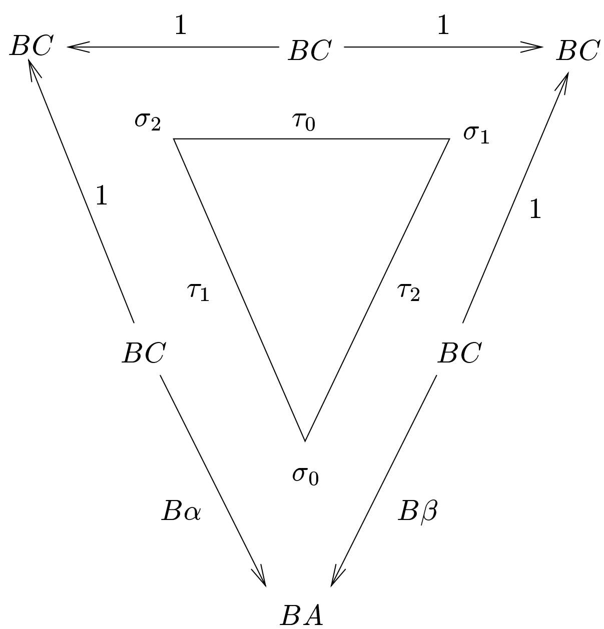

Fig. 3

As is well known, the space H has $\pi_1(H) = \Gamma$. The spectral sequence for ζ_+ has the following E^1-term:

$$\begin{array}{ccc}
E^1_{1,q+1} & = & K_q(\mathbb{Z}C) \oplus K_q(\mathbb{Z}C) \oplus K_q(\mathbb{Z}C) \\
\Big\downarrow d^1_{1,q+1} & & \Big\downarrow \begin{pmatrix} 0 & -\alpha_* & -\beta_* \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \\
E^1_{0,q+1} & = & K_q(\mathbb{Z}A) \oplus K_q(\mathbb{Z}C) \oplus K_q(\mathbb{Z}C)
\end{array}$$

A "base change" by means of the matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad \text{respectively} \quad \begin{pmatrix} 1 & \beta_* & \alpha_* \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

in $E^1_{1,q+1}$, respectively in $E^1_{0,q}$, transforms the differential into the form

$$d^1_{1,q+1} = \begin{pmatrix} \alpha_* - \beta_* & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We can then delete the last two summands $K_q(\mathbb{Z}C)$ and get the exactness of the first row in the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & & & \vdots & & & & \\
 & & & & \big\downarrow & & & & \\
\cdots \longrightarrow & K_q(\mathbb{Z}C) & \xrightarrow{\alpha_* - \beta_*} & K_q(\mathbb{Z}A) & \longrightarrow & \widetilde{K}^{\mathrm{cc}}_{q+1}(\zeta^+) & \longrightarrow & K_{q-1}(\mathbb{Z}C) & \longrightarrow \cdots \\
 & \big\downarrow 1 & & \big\downarrow 1 & & \big\downarrow & & \big\downarrow 1 & \\
\cdots \longrightarrow & K_q(\mathbb{Z}C) & \xrightarrow{\alpha_* - \beta_*} & K_q(\mathbb{Z}A) & \longrightarrow & K_q(\mathbb{Z}\Gamma) & \longrightarrow & K_{q-1}(\mathbb{Z}C) & \longrightarrow \cdots \\
 & & & & \big\downarrow & & & & \\
 & & & & \widetilde{K}^{\mathrm{cc}}_q(\zeta) & & & & \\
 & & & & \big\downarrow & & & & \\
 & & & & \vdots & & & &
\end{array}$$

The second row is analogous to the exact sequence in Theorem 1.1 and is derived from Waldhausen's Theorem 2 of [8]. Thus, it is a chain complex of abelian groups which is exact except that for each q its homology at $K_q(\mathbb{Z}\Gamma)$ is Waldhausen's Nil group

$$\widetilde{\mathrm{Nil}}_{q-1}(\mathbb{Z}C; \mathbb{Z}[A-\alpha(C)], \mathbb{Z}[A-\beta(C)], {}_\alpha\mathbb{Z}A_\beta, {}_\beta\mathbb{Z}A_\alpha).$$

In complete analogy with the proof of Theorem 2.1 one then gets

THEOREM 3.1. *Let $\Gamma = A *_C \{t\}$ as above and let $\zeta = (|K| \xleftarrow{p} H \xrightarrow{1} H = X)$ be the corresponding iterated mapping cylinder of classifying spaces, parametrized over the circle $|K|$. Then there is a short exact sequence*

$$\widetilde{K}^{\mathrm{cc}}_{*+1}(\zeta^+) \to K_*(\mathbb{Z}\Gamma) \to \widetilde{K}^{\mathrm{cc}}_*(\zeta)$$

where $\widetilde{K}^{\mathrm{cc}}_{*+1}(\zeta^+)$ is the homology part of $K_*(\mathbb{Z}\Gamma)$, i.e., the part that fits into the expected exact sequence with $K_*(\mathbb{Z}C)$ and $K_*(\mathbb{Z}A)$, while

$$\widetilde{K}^{\mathrm{cc}}_*(\zeta) \cong \widetilde{\mathrm{Nil}}_{*-1}(\mathbb{Z}C; \mathbb{Z}[A-\alpha(C)], \mathbb{Z}[A-\beta(C)], {}_\alpha\mathbb{Z}[A]_\beta, {}_\beta\mathbb{Z}[A]_\alpha).$$

4. Concluding remarks. Theorems 2.1 and 3.1 fit very well with the well known fact that the functor $\xi \mapsto \widetilde{K}^{\mathrm{cc}}_*(\xi^+)$ is a homology theory. Also, they underscore the importance of the "base point" in ξ^+; in fact in the two theorems the groups $\widetilde{K}^{\mathrm{cc}}_*(\eta)$ and $\widetilde{K}^{\mathrm{cc}}_*(\eta^+)$ are completely "unrelated".

In [1], for $* \le 1$, it is shown that the groups $\widetilde{K}^{\mathrm{cc}}_{*+1}(\eta^+)$ coincide with the "$\varepsilon \to 0$ controlled groups" $K_*(E,p)_{\mathrm{c}}$ in the sense of Quinn [6] and Ranicki and Yamasaki [7]. Here the control indicated by $_{\mathrm{c}}$ takes place in a polyhedron $|K|$ via a map $p : E \to |K|$ with an iterated mapping cylinder structure, and $\eta = (|K| \leftarrow E \to E = X)$.

In fact, for such η, it seems likely that there is a more general, and more global, version of the results presented here: The homotopy fibre of the induced map $\widetilde{\mathbf{K}}(E) \to \widetilde{\mathbf{K}}^{\mathrm{cc}}(\eta)$ should be the Quinn homology spectrum $\mathbb{H}(|K|; \mathbf{K}(p))$. Moreover, a proof should be possible by considerations as above together with the main results of [3]. However, several details have to be checked and/or worked out from scratch, so we leave such a general result for a later, and longer, note.

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