Splitting obstructions and properties of objects in the Nil categories

by

Tadeusz Koźniewski (Warszawa)

Abstract. We show that the objects of Bass–Farrell categories which represent 0 in the corresponding Nil groups are precisely those which are stably triangular. This extends to Waldhausen's Nil group of the amalgamated free product with index 2 factors. Applications include a description of Cappell's special UNil group and reformulations of those splitting and fibering theorems which use the Nil groups.

1. Introduction. We start with a short review of the algebraic sources of different types of Nil groups. The following theorems relate them to the K-groups and L-groups.

THEOREM 1.1 (Bass [Bs], Farrell, Hsiang [F], [FH1], [FH2]). *Let $\alpha : R \to R$ be a ring automorphism and let $R_\alpha[u, u^{-1}]$ be the α-twisted ring of Laurent polynomials. Then there exists a subgroup $\widetilde{\mathrm{Nil}}(R, \alpha) \subset K_1(R_\alpha[u, u^{-1}])$ with the property that $\widetilde{\mathrm{Nil}}(R, \alpha) \oplus \widetilde{\mathrm{Nil}}(R, \alpha^{-1})$ is a direct summand and there is an exact sequence*

$$K_1(R) \xrightarrow{1-\alpha_*} K_1(R) \to \frac{K_1(R_\alpha[u, u^{-1}])}{\widetilde{\mathrm{Nil}}(R, \alpha) \oplus \widetilde{\mathrm{Nil}}(R, \alpha^{-1})} \to K_0(R) \xrightarrow{1-\alpha_*} K_0(R).$$

In particular if $a : H \to H$ is a group automorphism and $G = H \times_a \mathbb{Z}$, then for $R = \mathbb{Z}H$, a induces an automorphism $\alpha : R \to R$, $\mathbb{Z}G \cong R_\alpha[u, u^{-1}]$ and we have an exact sequence

$$\mathrm{Wh}(H) \xrightarrow{1-\alpha_*} \mathrm{Wh}(H) \to \frac{\mathrm{Wh}(G)}{\widetilde{\mathrm{Nil}}(R, \alpha) \oplus \widetilde{\mathrm{Nil}}(R, \alpha^{-1})} \to \widetilde{K}_0(R) \xrightarrow{1-\alpha_*} \widetilde{K}_0(R).$$

Also, for $\alpha = 1$ we have $\widetilde{\mathrm{Nil}}(R, \alpha) \cong NK_1(R)$ of Bass.

THEOREM 1.2 (Waldhausen [W1], [W2]). *For any two R-bimodules B_1, B_2 which are free both as left and as right R-modules there exists an abelian group $\widetilde{\mathrm{Nil}}(R; B_1, B_2)$ with the following property. If H is a subgroup of G_1, G_2 and $G = G_1 *_H G_2$, $R = \mathbb{Z}H$, $B_i = \mathbb{Z}(G_i \backslash H)$ for $i = 1, 2$, then $\widetilde{\mathrm{Nil}}(R; B_1, B_2)$ is a direct summand of $\mathrm{Wh}(G)$ and there is an exact sequence*

$$\mathrm{Wh}(H) \to \mathrm{Wh}(G_1) \oplus \mathrm{Wh}(G_2) \to \frac{\mathrm{Wh}(G)}{\widetilde{\mathrm{Nil}}(R; B_1, B_2)}$$
$$\to \widetilde{K}_0(\mathbb{Z}H) \to \widetilde{K}_0(\mathbb{Z}G_1) \oplus \widetilde{K}_0(\mathbb{Z}G_2).$$

THEOREM 1.3 (Cappell [C1], [C2]). *For any two R-bimodules B_1, B_2 with involution which are free both as left and right R-modules and for each natural number n there exists an abelian group $\mathrm{UNil}_n^h(R; B_1, B_2)$ such that for $G = G_1 *_H G_2$, $R = \mathbb{Z}H$ and $B_i = \mathbb{Z}(G_i \backslash H)$ for $i = 1, 2$ there exists a split monomorphism $\varrho : \mathrm{UNil}_n^h(R; B_1, B_2) \to L_n^h(G)$ with the property that if $H^i(\mathbb{Z}/2; \ker(\widetilde{K}_0(\mathbb{Z}H) \to \widetilde{K}_0(\mathbb{Z}G_1) \oplus \widetilde{K}_0(\mathbb{Z}G_2))) = 0$ for all i then there exists an exact sequence*

$$\ldots \to L_n^h(H) \to L_n^h(G_1) \oplus L_n^h(G_2) \to \frac{L_n^h(G)}{\mathrm{UNil}_n^h(R; B_1, B_2)} \to L_{n-1}^h(H) \to \ldots$$

Cappell also defines a group $\mathrm{UNil}_n^s(R; B_1, B_2)$ which is similarly related to $L_n^s(G)$.

One of the geometric reasons for the introduction of the Nil groups were various kinds of splitting problems. Let Y be a closed n-dimensional manifold with a codimension 1 submanifold X. Let $f : M \to Y$ be a homotopy equivalence from a manifold M to Y. Then f is called *splittable* along X if it is homotopic to a map $g : M \to Y$ which is transverse regular to X, so that $N = g^{-1}(X)$ is a manifold and g restricted to N is also a homotopy equivalence. If f is h-cobordant to such a g we say that f is *h-splittable*. The following theorems give necessary and sufficient conditions for a map to be (h)-splittable.

THEOREM 1.4 ([FH2]). *Let $n \geq 6$ and let $\pi_1(Y) = H \times_a \mathbb{Z}$ with $\pi_1(X) = H$. Then $f : M \to Y$ is splittable along $X \Leftrightarrow p(\tau(f)) = 0 \in \widetilde{\mathrm{Nil}}(R, \alpha)$, where $\tau(f) \in \mathrm{Wh}(H \times_a \mathbb{Z})$ is the torsion of f, $R = \mathbb{Z}H$, $\alpha : R \to R$ is induced by a and $p : \mathrm{Wh}(H \times_a \mathbb{Z}) \to \widetilde{\mathrm{Nil}}(R, \alpha)$ is the projection map.*

THEOREM 1.5 ([C2]). *Let $n \geq 6$ and let $\pi_1(Y) = G_1 *_H G_2$ with $\pi_1(X) = H$. Then for every homotopy equivalence $f : M \to Y$ there exist obstructions $\chi^h(f) \in \mathrm{UNil}_n^h(R; B_1, B_2)$ and $\chi^s(f) \in \mathrm{UNil}_n^s(R; B_1, B_2)$ with the properties:*

(a) *f is h-splittable along $X \Leftrightarrow [\partial(\tau(f))] = 0 \in H^n(\mathbb{Z}/2; \ker(\widetilde{K}_0(\mathbb{Z}H) \to \widetilde{K}_0(\mathbb{Z}G_1) \oplus \widetilde{K}_0(\mathbb{Z}G_2)))$ and $\chi^h(f) = 0$.*

(b) f *is splittable along* $X \Leftrightarrow \tau(f) \in \mathrm{im}(\mathrm{Wh}(G_1) \oplus \mathrm{Wh}(G_2))$ *and* $\chi^s(f) = 0$.

Here, as before, $\tau(f)$ is the torsion of f, $R = \mathbb{Z}H$, $B_i = \mathbb{Z}(G_i \setminus H)$ and ∂ is the boundary map of 1.2.

The UNil groups are closely related to the structure sets of a manifold. Cappell shows that if $\pi_1(Y) = G = G_1 *_H G_2$, then for $x = h, s$ the group $\mathrm{UNil}_n^x(R; B_1, B_2)$ acts freely on the structure set $\mathcal{S}^x(Y)$. In particular if Y is aspherical with universal cover $\mathbb{R}^n$ then free actions of G on $\mathbb{R}^n$ are h-rigid (resp. rigid) only if $\mathrm{UNil}_n^h(R; B_1, B_2) = 0$ ($\mathrm{UNil}_n^s(R; B_1, B_2) = 0$). For some of the newest results on the Nil groups and their applications see [Bd], [CK], [KS], [R].

2. Triangular objects in the Farrell category. In this section we analyze the properties of the Farrell category $\mathbf{Nil}(R, \alpha)$.

Let R be a ring and let $\alpha : R \to R$ be an automorphism of R. We say that a map $\varphi : P \to Q$ of right R-modules is α-*linear* if φ is additive and $\varphi(zr) = \varphi(z)\alpha(r)$ for each $z \in P$, $r \in R$. An object of the category $\mathbf{Nil}(R, \alpha)$ is a pair (P, ν), where P is a stably free, finitely generated right R-module and $\nu : P \to P$ is an α-linear endomorphism which is nilpotent, i.e. $\nu^n = 0$ for some n. Let $R_\alpha[u]$ denote the α-twisted polynomial ring, with $ru = u\alpha(r)$ and let $R_n = R_\alpha[u]/(u^n)$. Then the objects of $\mathbf{Nil}(R, \alpha)$ in which $\nu^n = 0$ are precisely those right R_n-modules which are stably free as R-modules.

A morphism in $\mathbf{Nil}(R, \alpha)$ from (P, ν) to (P', ν') is an R-homomorphism $f : P \to P'$ such that $\nu' f = f\nu$. A sequence of morphisms in $\mathbf{Nil}(R, \alpha)$ is exact if it is exact in the category of R-modules, after we forget about the endomorphisms.

Let $\mathrm{Nil}(R, \alpha) := K_0(\mathbf{Nil}(R, \alpha))$. This means that $\mathrm{Nil}(R, \alpha)$ is the Grothendieck group of the isomorphism classes of objects of $\mathbf{Nil}(R, \alpha)$ modulo the relation generated by short exact sequences. Let $\mathbf{F}_R$ be the category of finitely generated, free right R-modules. The functor $\iota : \mathbf{F}_R \to \mathbf{Nil}(R, \alpha)$ which assigns to a module P the object $(P, 0)$ induces on K_0 a group homomorphism $\iota_* : \mathbb{Z} \to \mathrm{Nil}(R, \alpha)$. One then defines $\widetilde{\mathrm{Nil}}(R, \alpha) := \mathrm{coker}\, \iota_*$.

We say that an object $x = (P, \nu)$ in $\mathbf{Nil}(R, \alpha)$ is *triangular* if there is a filtration $0 = E_0 \subset E_1 \subset \ldots \subset E_n = P$ such that for each i the module E_{i+1}/E_i is free and $\nu(E_i) \subset E_{i-1}$. For example, R_n with the endomorphism given by the right multiplication by u is triangular. Also, if P is a free right R-module, then $P \otimes_R R_n$ with $\nu(\sum p_i u^i) = \sum p_i u^{i+1}$ is triangular.

LEMMA 2.1 (Farrell [F]). *If* x *is triangular, then* $[x] = 0 \in \widetilde{\mathrm{Nil}}(R, \alpha)$.

We call x a *stably triangular object* if there exists a triangular object t such that $x \oplus t$ is triangular.

COROLLARY 2.2. *If x is stably triangular, then $[x] = 0$.*

LEMMA 2.3. *If $0 \to t' \to t \to t'' \to 0$ is an exact sequence of objects of* $\mathbf{Nil}(R, \alpha)$ *and t', t'' are triangular, then t is triangular.*

P r o o f. A triangular filtration of t'' pulled back to t and combined with a triangular filtration of t' provides a triangular filtration of t.

Following Waldhausen [W1], we say that y is a *suspension* of x if there is an exact sequence $0 \to y \to t \to x \to 0$ of objects in $\mathbf{Nil}(R, \alpha)$ with t triangular. If t is R_n-free, then we say that y is an R_n-*suspension* of x.

LEMMA 2.4. (i) *Each object has a suspension.*
(ii) *Triangular objects admit R_n-suspensions which are triangular.*

P r o o f. (i) Let $x = (P, \nu)$. We can assume that P is free, if not we can add to x a free module with a trivial endomorphism and consider the natural projection from the resulting sum to x. Now $P \otimes_R R_n$ is free and maps epimorphically onto x by $\varepsilon(\sum p_i u^i) = \sum \nu^i(p_i)$. So $\ker \varepsilon$ is a suspension of x.

(ii) Let t be a triangular object. We proceed by induction on $l(t)$, the length of the shortest triangular filtration of t. If $l(t) = 1$, then $\ker(\varepsilon : P \otimes_R R_n \to P) = Pu + \ldots + Pu^{n-1}$ is a stably free R_{n-1}-module, and therefore triangular.

If $l(t) = k$, then there is an exact sequence $0 \to t' \to t \to t'' \to 0$ where $l(t'') = 1$ and $l(t') = k - 1$. Because $\otimes_R R_n$ is an exact functor we can use the construction of (i) to get an exact sequence $0 \to u' \to u \to u'' \to 0$ of R_n-suspensions. By the induction hypothesis u' and u'' are now triangular, so u is also triangular, by Lemma 2.3.

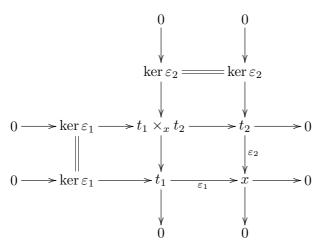LEMMA 2.5. *A suspension of a triangular object is stably triangular.*

P r o o f. Let y be a suspension of x, i.e. $y = \ker \varepsilon_1$ where $0 \to \ker \varepsilon_1 \to t_1 \to x \to 0$ is exact and t_1 is triangular. We have to show that y is stably triangular.

Let z be a triangular R_n-suspension of x, where $n \geq l(x), l(t)$. So $z = \ker \varepsilon_2$ for an exact sequence $0 \to \ker \varepsilon_2 \to t_2 \to x \to 0$ with t_2 R_n-free. Such a z exists by Lemma 2.4(ii).

We define $t_1 \times_x t_2$ in $\mathbf{Nil}(R, \alpha)$ as (P, μ) where

$$P = \{(c_1, c_2) \in T_1 \times T_2 \mid \varepsilon_1(c_1) = \varepsilon_2(c_2)\}, \quad \mu = \nu_1 \times \nu_2|_P.$$

This gives a commutative diagram with exact columns and rows:

$$
\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\ker\varepsilon_2 & =\!=\!= & \ker\varepsilon_2 \\
\downarrow & & \downarrow \\
0 \to \ker\varepsilon_1 \to t_1\times_x t_2 & \to & t_2 \to 0 \\
\| & & \downarrow{\scriptstyle\varepsilon_2} \\
0 \to \ker\varepsilon_1 \xrightarrow{\ \varepsilon_1\ } t_1 & \xrightarrow{} & x \to 0 \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
$$

Since t_1 and $\ker\varepsilon_2$ are triangular, Lemma 2.3 implies that $t_1\times_x t_2$ is triangular. Moreover t_2 is R_n-free, so the row containing $t_1\times_x t_2$ splits, and $t_1\times_x t_2 = \ker\varepsilon_1 \oplus t_2$. We conclude that $\ker\varepsilon_1$ is stably triangular.

LEMMA 2.6. *If an object of* $\mathbf{Nil}(R,\alpha)$ *has a triangular suspension, then it is stably triangular.*

Proof. For any object $x = (P,\nu)$ in $\mathbf{Nil}(R,\alpha)$ let $P^* = \mathrm{Hom}_R(P,R)$ and let $\nu^* : P^* \to P^*$ be given by $\nu^*(\varphi) = \alpha^{-1}\circ\varphi\circ\nu$. Then P^* has the structure of a left R-module $((r\varphi)(c) = r\varphi(c))$ and ν^* is an α^{-1}-linear endomorphism. This gives a (contravariant) functor $\# : \mathbf{Nil}(R,\alpha) \to \mathbf{Nil}'(R,\alpha^{-1})$ where $(P,\nu)^{\#} = (P^*,\nu^*)$ and where $\mathbf{Nil}'(R,\beta)$ denotes the analog of $\mathbf{Nil}(R,\beta)$ with left modules used instead of right modules. Then $x \in \mathbf{Nil}(R,\alpha)$ is triangular $\Leftrightarrow x^{\#}$ is triangular.

Now let x be an object in $\mathbf{Nil}(R,\alpha)$ with a triangular suspension. This means that we have an exact sequence

$$0 \to t_1 \to t \to x \to 0$$

where t and t_1 are triangular. Applying $\#$ we get a sequence

$$0 \to x^{\#} \to t^{\#} \to t_1^{\#} \to 0$$

which is also exact (because our sequences split as sequences of R-modules). From Lemma 2.5 it follows that $x^{\#}$ is stably triangular, so x itself is stably triangular.

As a corollary we get the following

PROPOSITION 2.7. *If in an exact sequence of objects in* $\mathbf{Nil}(R,\alpha)$
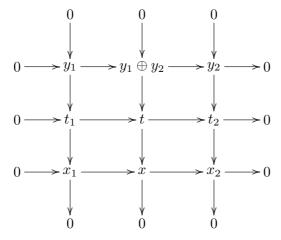
$$0 \to x \to y \to z \to 0$$

two objects are stably triangular, then the third is also stably triangular.

THEOREM 2.8. *If* x_1 *is a suspension of* x_2 *then* $x_1 \oplus x_2$ *is stably triangular.*

P r o o f. Consider exact sequences

$$0 \to x_1 \to x \to x_2 \to 0 \quad \text{and} \quad 0 \to y_i \to t_i \to x_i \to 0$$

for $i = 1, 2$, with x, t_1, t_2 triangular. To show that $x_1 \oplus x_2$ is stably triangular we use the following construction which is due to Waldhausen [W1], [W2]. There exists a commutative diagram of objects in $\mathbf{Nil}(R, \alpha)$ with exact rows:

$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & y_1 & \to & y_1 \oplus y_2 & \to & y_2 & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & t_1 & \to & t & \to & t_2 & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & x_1 & \to & x & \to & x_2 & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & 0 & & 0 & & 0 & &
\end{array}
$$

where for $x = (P_x, \nu_x)$, $x_i = (P_{x_i}, \nu_{x_i})$, $t_i = (P_{t_i}, \nu_{t_i})$, $i = 1, 2$, one defines $t = (P_t, \nu_t)$ by

$$P_t = P_{t_1} \oplus P_{t_2}, \qquad \nu_t = \begin{pmatrix} \nu_{t_1} & c \\ 0 & \nu_{t_2} \end{pmatrix}$$

with $c : P_{t_2} \to P_{t_1}$ specified as follows. We have $P_x \cong P_{x_1} \oplus P_{x_2}$ and $\nu_x = \begin{pmatrix} \nu_{x_1} & \gamma \\ 0 & \nu_{x_2} \end{pmatrix}$ where $\gamma : P_{x_2} \to P_{x_1}$. Then $P_{t_i} = P_{y_i} \oplus P_{x_i}$ for $i = 1, 2$ and $c := \begin{pmatrix} 0 & 0 \\ 0 & \gamma \end{pmatrix}$.

All the rows and columns are exact. Also, since t_1, t_2 are triangular, t is triangular (Lemma 2.3). Now x, t triangular imply that $y_1 \oplus y_2$ is stably triangular (Lemma 2.5). Finally $y_1 \oplus y_2$ and $t_1 \oplus t_2$ stably triangular make $x_1 \oplus x_2$ stably triangular (Lemma 2.6).

We now give another description of $\widetilde{\mathrm{Nil}}(R, \alpha)$. Let N be the set of isomorphism classes of objects in $\mathbf{Nil}(R, \alpha)$. Then N has the structure of an abelian semigroup with addition given by $\oplus$. We introduce an equivalence relation in N by

$$x_1 \sim x_2 \Leftrightarrow \exists t_1, t_2 \text{ triangular such that } x_1 \oplus t_1 = x_2 \oplus t_2.$$

REMARK 2.9. (a) $x \sim 0 \Leftrightarrow x$ is stably triangular.

(b) $N/\sim$ is an abelian group. The inverse of x is given by a suspension of x.

(c) If $0 \to x_1 \to x \to x_2 \to 0$ is exact, then $[x_1] + [x_2] = [x]$ in $N/\sim$.

P r o o f. (a) is clear. To see (b) note that if y is a suspension of x, then (by Theorem 2.8) $x \oplus y$ is stably triangular, so $[x] + [y] = 0$ in $N/\sim$.

(c) Let y_i be a suspension of x_i, $i = 1, 2$. By the construction given in Theorem 2.8 it follows that $y_1 \oplus y_2$ is a suspension of x. So $[x] = -[y_1 \oplus y_2] = -(-[x_1] - [x_2]) = [x_1] + [x_2]$.

COROLLARY 2.10. *The natural epimorphism* $j : N/\sim \to \widetilde{\mathrm{Nil}}(R, \alpha)$ *is an isomorphism.*

As a consequence we get

THEOREM 2.11. *Two objects* x_1, x_2 *of the category* $\mathbf{Nil}(R, \alpha)$ *represent the same element of* $\widetilde{\mathrm{Nil}}(R, \alpha)$ $\Leftrightarrow$ *there exist triangular objects* t_1, t_2 *such that* $x_1 \oplus t_1 \cong x_2 \oplus t_2$.

COROLLARY 2.12. *An object* x *of* $\mathbf{Nil}(R, \alpha)$ *represents* $0 \in \widetilde{\mathrm{Nil}}(R, \alpha)$ $\Leftrightarrow$ x *is stably triangular.*

3. Waldhausen's Nil group. In this section we show how to extend the results of §2 to some other types of Nil categories.

For any two R-bimodules B_1, B_2 which are free both as left and as right R-modules let $\mathbf{Nil}(R; B_1, B_2)$ be the following category. An object of $\mathbf{Nil}(R; B_1, B_2)$ is a quadruple (P, Q, p, q) where P, Q are finitely generated, stably free right R-modules and $p : P \to Q \otimes_R B_1$ and $q : Q \to P \otimes_R B_2$ are R-homomorphisms which satisfy the following nilpotency condition. There exist filtrations $0 = P_0 \subset P_1 \subset \ldots \subset P_n = P$ and $0 = Q_0 \subset Q_1 \subset \ldots \subset Q_n = Q$ so that $p(P_i) \subset Q_{i-1} \otimes_R B_1$ and $q(Q_i) \subset P_{i-1} \otimes_R B_2$ for all $i \geq 1$.

A morphism $(P, Q, p, q) \to (P', Q', p', q')$ is a pair of R-homomorphisms $f : P \to P'$, $g : Q \to Q'$ so that one has the commutative diagrams

$$
\begin{array}{ccc}
P & \xrightarrow{p} & Q \otimes_R B_1 \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g \otimes 1} \\
P' & \xrightarrow{p'} & Q' \otimes_R B_1
\end{array}
\qquad
\begin{array}{ccc}
Q & \xrightarrow{q} & P \otimes_R B_2 \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f \otimes 1} \\
Q' & \xrightarrow{q'} & P' \otimes_R B_2
\end{array}
$$

Forming K_0 of this category we get the natural homomorphism

$$\mathbb{Z} \oplus \mathbb{Z} \xrightarrow{j} K_0(\mathbf{Nil}(R; B_1, B_2))$$

defined by $j(k, l) = (R^k, R^l, 0, 0)$ for $k, l \geq 0$. Waldhausen defines $\widetilde{\mathrm{Nil}}(R; B_1, B_2)$ as the cokernel of j. His notation for this group in [W1] is $\widetilde{C}(R; B_1, B_2)$.

We will discuss the group $\widetilde{\mathrm{Nil}}(R; B_1, B_2)$ in the special case when there are ring automorphisms $\beta_1, \beta_2 : R \to R$ such that:

(a) B_1, B_2 are isomorphic to R as left modules via isomorphisms

$$B_1 \xrightarrow{\Phi_1} R \xleftarrow{\Phi_2} B_2,$$

(b) the map Φ_i is β_i-linear, i.e. $\Phi_i(zr) = \Phi_i(z)\beta_i(r)$ for all $z \in B_i$, $r \in R$ and $i = 1, 2$.

When B_1, B_2 satisfy conditions (a), (b) we write R_{β_i} for B_i and $\widetilde{\mathrm{Nil}}(R, \beta_1, \beta_2)$ for $\widetilde{\mathrm{Nil}}(R; B_1, B_2)$.

The importance of this special case follows from the following remark. Let

$$1 \to C \to \Gamma \xrightarrow{q} D_\infty \to 1$$

be an exact sequence of groups, where $D_\infty = \langle s_1, s_2 : s_1^2 = s_2^2 = 1 \rangle$ is the infinite dihedral group. Set $R = \mathbb{Z}C$, $\Gamma_i = q^{-1}(\langle s_i \rangle)$ and $B_i = \mathbb{Z}(\Gamma_i \setminus C)$ for $i = 1, 2$. Then $\Gamma = \Gamma_1 *_C \Gamma_2$ and B_i is isomorphic to R as a left R-module by the isomorphism $\Phi_i : B_i \to R$, $\Phi_i(x) = xb_i$ for $x \in B_i$, where b_i is some chosen element of $\Gamma_i \setminus C$ for $i = 1, 2$. The elements b_1, b_2 specify two automorphisms β_i of R: $\beta_i(r) = b_i^{-1} r b_i$ for $r \in R$. As a map of right R-modules, Φ_i is β_i-linear. Therefore in this case we get $\widetilde{\mathrm{Nil}}(R; B_1, B_2) = \widetilde{\mathrm{Nil}}(R, \beta_1, \beta_2)$.

We say that an object (P, Q, p, q) of $\mathbf{Nil}(R; B_1, B_2)$ is *triangular* if there exist filtrations $0 = P_0 \subset P_1 \subset \ldots \subset P_n = P$ and $0 = Q_0 \subset Q_1 \subset \ldots \subset Q_n = Q$ so that $p(P_i) \subset Q_{i-1} \otimes_R B_1$ and $q(Q_i) \subset P_{i-1} \otimes_R B_2$ for all $i \geq 1$ with the extra property that for all i the modules P_i/P_{i-1} and Q_i/Q_{i-1} are free. An object x is called *stably triangular* if $x \oplus t$ is triangular for some triangular t. Our goal is to show that an object in $\mathbf{Nil}(R; R_{\beta_1}, R_{\beta_2})$ represents 0 in $\widetilde{\mathrm{Nil}}(R, \beta_1, \beta_2)$ precisely when it is stably triangular. In order to do this we first observe that $\mathbf{Nil}(R; R_{\beta_1}, R_{\beta_2})$ is isomorphic to another category, $\mathbf{Nil}(R, \beta_1, \beta_2)$, which we now describe.

An object of $\mathbf{Nil}(R, \beta_1, \beta_2)$ is a quadruple (P, Q, p, q), where P, Q are finitely generated, stably free right R-modules, $p : P \to Q$ is β_1-linear, $q : Q \to P$ is β_2-linear and $qp : P \to P$ is nilpotent $((qp)^n = 0$ for some $n)$. A morphism from (P, Q, p, q) to (P', Q', p', q') is a pair (f, g), where $f : P \to P'$ and $g : Q \to Q'$ are R-homomorphisms and $p'f = gp$, $q'g = fq$. An object of $\mathbf{Nil}(R, \beta_1, \beta_2)$ is triangular if there exist filtrations $0 = P_0 \subset P_1 \subset \ldots \subset P_n = P$ and $0 = Q_0 \subset Q_1 \subset \ldots \subset Q_n = Q$ so that for all i $p(P_i) \subset Q_{i-1}$, $q(Q_i) \subset P_{i-1}$ and the modules P_i/P_{i-1} and Q_i/Q_{i-1} are free. An object is called stably triangular if it becomes triangular after adding some triangular object.

Note that for any right R-module M there is a β_i-linear isomorphism $M \otimes_R R_{\beta_i} \to M$ given by $x \otimes s \mapsto x\Phi_i(s)$. So any R-homomorphism $P \to Q \otimes_R R_{\beta_1}$ specifies uniquely a β_1-linear homomorphism $P \to Q$. Sim-

ilarly, any R-homomorphism $Q \to P \otimes_R R_{\beta_2}$ determines a β_2-linear homomorphism $Q \to P$. This provides an obvious isomorphism of the categories $\mathbf{Nil}(R; R_{\beta_1}, R_{\beta_2})$ and $\mathbf{Nil}(R, \beta_1, \beta_2)$ under which triangular objects go to triangular objects.

We now give a characterization of the objects representing 0 in $\widetilde{\mathrm{Nil}}(R, \beta_1, \beta_2)$, by showing that for any ring R and any two automorphisms β_1, β_2 of R the following holds.

THEOREM 3.1. *Two objects x, y in $\mathbf{Nil}(R, \beta_1, \beta_2)$ represent the same element in $\widetilde{\mathrm{Nil}}(R, \beta_1, \beta_2) \Leftrightarrow x \oplus t$ is isomorphic to $y \oplus t'$ for some triangular objects t, t'.*

P r o o f. If (P, Q, p, q) is an object of $\mathbf{Nil}(R, \beta_1, \beta_2)$ we can add a triangular object t to it if necessary, so that $x \oplus t = (P', Q', p', q')$ and $P' \cong Q'$ as right R-modules (t has the form $(R^k, 0, 0, 0)$ or $(0, R^k, 0, 0)$). Therefore it is sufficient to prove 3.1 in the full subcategory $\mathbf{Nil}'(R, \beta_1, \beta_2)$ whose objects (P, Q, p, q) satisfy $P \cong Q$.

Let S be the ring $R \times R$. Let $\alpha : S \to S$ be the ring automorphism

$$\alpha(x, y) = (\beta_2(y), \beta_1(x)) \quad \forall (x, y) \in R \times R.$$

Let $e = (1, 0)$ and $f = (0, 1)$ in S. If F is a free right S-module, say $F \cong S^k$, then $P = Fe$ and $Q = Ff$ are right free R-modules and $F = P \oplus Q$.

If $\nu : F \to F$ is an α-linear map, then $\nu(P) \subset Q$, $\nu(Q) \subset P$ and ν can be written as $\begin{pmatrix} 0 & q \\ p & 0 \end{pmatrix} : P \oplus Q \to P \oplus Q$ where $p : P \to Q$ is β_1-linear and $q : Q \to P$ is β_2-linear. Also, if $\nu^{2n} = 0$ then $(pq)^n = 0$.

So an object (F, ν) in $\mathbf{Nil}(S, \alpha)$ determines an object (P, Q, p, q) in $\mathbf{Nil}'(R, \beta_1, \beta_2)$. Conversely, if (P, Q, p, q) is an object in $\mathbf{Nil}'(R, \beta_1, \beta_2)$ then $F = P \oplus Q$ is a stably free right S-module with nilpotent α-linear endomorphism $\nu = \begin{pmatrix} 0 & q \\ p & 0 \end{pmatrix}$.

This gives an equivalence of categories $E : \mathbf{Nil}(S, \alpha) \to \mathbf{Nil}'(R, \beta_1, \beta_2)$. It preserves sums and exact sequences, and $E(t)$ is triangular if t is. Conversely, suppose $E(t) = (P, Q, p, q)$ is triangular. By refining the filtration if necessary, we can assume that P_i/P_{i-1} and Q_i/Q_{i-1} have rank 1 for all $i \geq 1$. This means that $F_i := P_i \oplus Q_i$ is a free $R \times R$-module and so is F_i/F_{i-1}. Moreover, $\nu(F_i) \subset F_{i-1}$. So t itself is triangular.

It follows from the above discussion that E induces an isomorphism $\widetilde{\mathrm{Nil}}(S, \alpha) \cong \widetilde{\mathrm{Nil}}(R, \beta_1, \beta_2)$. Our theorem now follows from 2.11.

COROLLARY 3.2. *An object in $\mathbf{Nil}(R, \beta_1, \beta_2)$ represents 0 in $\widetilde{\mathrm{Nil}}(R, \beta_1, \beta_2) \Leftrightarrow$ it is stably triangular.*

Cappell defines *simple UNil forms* as those UNil forms for which the corresponding objects in $\widetilde{\mathrm{Nil}}(R; B_1, B_2)$ are trivial.

COROLLARY 3.3. *If $G = G_1 *_H G_2$ with $|G_i : H| = 2$, $R = \mathbb{Z}H$, $B_i = \mathbb{Z}(G_i \setminus H)$ for $i = 1, 2$, then the simple $(-1)^k$ UNil forms over (B_1, B_2) are pairs (x_1, x_2) where $x_i = (P_i, \lambda_i, \mu_i)$ is a $(-1)^k$ Hermitian form over B_i (in the sense of Wall [W1], [W2]), $P_2 = P_1^*$ and $(P_1, P_2, \operatorname{ad}\lambda_1, \operatorname{ad}\lambda_2)$ is a stably triangular object of $\mathbf{Nil}(R; B_1, B_2)$.*

4. Applications. We now give examples of some geometric applications of the above theorems. Here is the main idea of getting them.

In the topological theorems which make use of the Nil groups one usually tries to carry out a geometric construction in order to prove a general statement. It turns out that the completion of the construction is not always possible. What one gets is a geometrically defined module with a nilpotent endomorphism. The construction can be completed if and only if this object represents 0 in the appropriate Nil group. One shows that the choices which are made during the attempt of the construction do not alter the class of the module and its endomorphism in the Nil group. This gives a Nil obstruction.

From Corollary 2.12 it now follows that to decide whether the construction can always be completed it is enough to attempt the construction, making any choices, and check if the resulting module with endomorphism is stably triangular. Below we give an example of the above method. It is based on the construction in [F]. Similar examples, related to the splitting theorems 1.4, 1.5, can be obtained using [FH2], [C2], [C3].

Let M be a closed, n-dimensional manifold with $n \geq 6$ and let $f : M \to S^1$ be a continuous map. Assume that $f_\# : \pi_1(M) \to \mathbb{Z}$ is onto and let $H = \ker f_\#$. So $\pi_1(M) = H \times_a \mathbb{Z}$ and a induces an automorphism α of $\mathbb{Z}H$. Let X be the covering space of M corresponding to H. Assume that X has the homotopy type of a finite CW complex. The group $\pi_1(M)/H \cong \mathbb{Z}$ acts on X. A closed, framed, $(n-1)$-dimensional submanifold (N, μ) of M is called a *splitting* of f if it corresponds to f under the Pontryagin–Thom construction. Then $W = M \setminus (\text{a tubular neighborhood of } N)$ is a cobordism on N. Lift (N, μ) to a framed submanifold $(\widehat{N}, \widehat{\mu})$ of X. Then $X \setminus \widehat{N}$ has two components, say A and B, where B denotes the component to which $\widehat{\mu}$ points. Let T be the generator of $\pi_1(M)/H$ for which $A \subset T(A)$. Then T^{-1} induces an α-linear endomorphism T_*^{-1} of $H_i(X, A; \mathbb{Z}H)$. Farrell shows that one can always find (N, μ) so that $H_i(X, A; \mathbb{Z}H) = 0$ for $i \leq 2$ and $H_i(X, B; \mathbb{Z}H) = 0$ for $i \leq n - 3$, in which case $H_3(X, A; \mathbb{Z}H)$ is a finitely generated, projective $\mathbb{Z}H$-module and T_*^{-1} is nilpotent. Pick any such (N, μ) and denote $(H_3(X, A; \mathbb{Z}H), T_*^{-1})$ as (P, ν).

Then Farrell's results can be stated in the following way:

THEOREM 4.1. (a) *There exists a splitting for which W is an h-cobordism $\Leftrightarrow P$ is stably free and (P, ν) is stably triangular.*

Also, if these conditions are satisfied then the class $\tau(f)$ in $\mathrm{Wh}(H)/$ $(1 - \alpha_*)\,\mathrm{Wh}(H)$ of the torsion of such an h-cobordism W does not depend on the choice of the splitting.

(b) *There exists a splitting for which W is a product cobordism* (*equivalently*: *f is homotopic to a locally trivial bundle map*) $\Leftrightarrow$ *the conditions of* (a) *hold and $\tau(f) = 0$.*

References

[Bd] B. B a d z i o c h, *K_1 of twisted polynomial rings*, K-Theory 16 (1999), 29–34.

[Bs] H. B a s s, *Algebraic K-Theory*, Benjamin, New York, 1968.

[C1] S. C a p p e l l, *Unitary nilpotent groups and Hermitian K-theory*, Bull. Amer. Math. Soc. 80 (1974), 1117–1122.

[C2] —, *Manifolds with fundamental group a generalized free product*, ibid. 80 (1974), 1193–1198.

[C3] —, *A splitting theorem for manifolds*, Invent. Math. 33 (1976), 69–170.

[CK] F. C o n n o l l y and T. K o ź n i e w s k i, *Nil groups in K-theory and surgery theory*, Forum Math. 7 (1995), 45–76.

[F] F. T. F a r r e l l, *The obstruction to fibering a manifold over a circle*, Indiana Univ. Math. J. 21 (1971), 315–346.

[FH1] F. T. F a r r e l l and W. C. H s i a n g, *A formula for $K_1 R_\alpha[T]$*, in: Proc. Sympos. Pure Math. 17, Amer. Math. Soc., 1970, 192–218.

[FH2] —, —, *Manifolds with $\pi_1 = G \times_\alpha T$*, Amer. J. Math. 95 (1973), 813–848.

[KS] S. K w a s i k and R. S c h u l t z, *Unitary nilpotent groups and the stability of pseudoisotopies*, Duke Math. J. 71 (1993), 871–887.

[R] A. R a n i c k i, *Lower K- and L-Theory*, Cambridge Univ. Press, 1992.

[W1] F. W a l d h a u s e n, *Whitehead groups of generalized free products*, preprint, 1969.

[W2] —, *Algebraic K-theory of generalized free products*, Ann. of Math. 108 (1978), 135–256.

Institute of Mathematics
Warsaw University
Banacha 2
02-097 Warszawa, Poland
E-mail: t.kozniewski@mimuw.edu.pl