K-theory, flat bundles and the Borel classes

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Abstract. Using Hausmann and Vogel's homology sphere bundle interpretation of algebraic K-theory, we construct K-theory invariants by a theory of characteristic classes for flat bundles. It is shown that the Borel classes are detected this way, as well as the rational K-theory of integer group rings of finite groups.

1. Introduction. One of the basic problems with algebraic K-theory is the lack of a good interpretation (geometric, algebraic or otherwise) of the elements in the higher K-groups. This makes calculations harder, but perhaps more importantly, it complicates applications, since even in the cases where one can do calculations, the connection to the application is often so indirect that it is hard to translate back.

To give just one example: Borel's calculations show that $K_{4k+1}(\mathbb{Z}) \otimes \mathbb{Q} \approx \mathbb{Q}$ for $k \geq 1$. One consequence is that some homotopy groups of diffeomorphism groups of large discs (fixing the boundary) have rank one. But to construct explicit generators from these calculations seems to be very difficult!

If A is a ring, $K_n(A)$ is defined as $\pi_n(BGL(A)^+)$, i.e. by mapping simple spaces (spheres) into complicated and poorly understood spaces $(BGL(A)^+)$. In this paper we discuss an alternative (but equivalent) definition, where we replace the spheres by more complicated manifolds, but map into spaces that are much better understood (at least from a geometric point of view)—namely the classifying spaces $BGL(A)$. Then the elements have geometric interpretation as (flat) bundles, and one can try to study invariants for these.

The idea to construct geometric invariants for K-theory elements from a flat bundle representation was, I believe, first used by Karoubi. He used a representation by bundles over $K(\pi, 1)$-complexes and developed a theory of simplicial connections and characteristic classes (see [9]). Here, however,

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we shall use an interpretation given by Hausmann and Vogel in [5], using bundles over homology spheres which are actual *manifolds*. In fact, we shall use a differentiable adaptation of Hausmann and Vogel's theory (which works in the PL category). This has the advantage that we can apply directly a differential geometric theory of characteristic classes.

It should be noted, however, that Karoubi's bundle interpretation is slightly more general, since it works in the same manner in all degrees, whereas that of Hausmann and Vogel only works in degrees higher than 4.

Hausmann and Vogel's theory is recalled in Section 2, together with a discussion of the differentiable case ([5] works in the PL category).

In Section 3 we discuss a theory of characteristic classes for flat bundles (essentially due to Kamber and Tondeur), which we apply in Section 4 to give invariants in the cases $A = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. In Section 5 we compare with Borel's calculations, and prove that these can be formulated in terms of the invariants from Section 4. Hence these invariants are highly nontrivial, since they detect the ranks of K-groups of algebraic integers. In fact, they also give a complete set of rational invariants in the case of group rings of finite groups. This case is discussed briefly in Section 6. (For more details, see [6].)

2. Homology sphere bordism and K-theory. The starting point is the "geometric" interpretation of the homotopy of the plus-construction provided by Hausmann and Vogel in [5] (see also [4]):

Let X be a pointed space. We say that two pointed maps $f_i : M_i^n \to X$, $i = 1, 2$, from closed, oriented n-manifolds M_1 and M_2 to X are *homology cobordant* if there exists an oriented, pointed cobordism $F : W \to X$ between f_1 and f_2 such that

$$H_*(W, M_1; \mathbb{Z}) \approx H_*(W, M_2; \mathbb{Z}) \approx 0.$$

(The cobordism is *pointed* if there is an arc in W joining the basepoints in M_1 and M_2 and which is mapped to the basepoint in X by F.)

Let $\Omega_n^{\mathrm{HS}}(X)$ be the set of homology cobordism classes of maps $f : M^n \to X$ where M is an oriented homology sphere. Oriented connected sum gives $\Omega_n^{\mathrm{HS}}(X)$ the structure of an abelian group.

This makes sense in the topological, PL or differentiable category, and if we want to emphasize which category we are in, we write $\Omega_n^{\mathrm{HS}}(X)_{\mathcal{CAT}}$, where $\mathcal{CAT} = $ TOP, PL, or DIFF.

For any space Y, we let Y^+ denote the result of the plus-construction on Y with respect to the maximal locally perfect subgroup $LP\pi_1(Y)$ of $\pi_1(Y)$ ([5]). If W is a cobordism between homology n-spheres M_1 and M_2, then $(W^+; M_1^+, M_2^+) \simeq (\mathrm{S}^n \times I; \partial(\mathrm{S}^n \times I))$, so $(f : M \to X) \mapsto (f^+ : M^+ \to X^+)$ defines a map $\Omega_n^{\mathrm{HS}}(X) \to \pi_n(X^+)$ which is easily seen to be a homomorphism.

In the PL category Hausmann and Vogel prove ([5]):

THEOREM 2.1. (a) $\Omega_n^{\mathrm{HS}}(X)_{\mathrm{PL}} \to \pi_n(X^+)$ *is an isomorphism if* $n \geq 5$.
(b) *There are exact sequences*

$$0 \to \Omega_4^{\mathrm{HS}}(X)_{\mathrm{PL}} \to \pi_4(X^+) \to \Omega_3^{\mathrm{HS}}(\mathcal{F})_{\mathrm{PL}} \to \Omega_3^{\mathrm{HS}}(X)_{\mathrm{PL}} \to \pi_3(X^+) \to 0,$$
$$0 \to \Omega_2^{\mathrm{HS}}(X)_{\mathrm{PL}} \to \pi_2(X^+) \to H_2(LP\pi_1(X); \mathbb{Z}) \to 0,$$

where $\mathcal{F}$ *is the homotopy fiber of the plus-map* $X \to X^+$.

REMARK 2.2. In (b) there is no difference between the PL and differentiable categories, so we get the same statement for $\Omega_n^{\mathrm{HS}}(X)_{\mathrm{DIFF}}$.

In order to get the necessary modifications for the differentiable category (for $n \geq 5$), we investigate the relation between $\Omega_n^{\mathrm{HS}}(X)_{\mathrm{PL}}$ and $\Omega_n^{\mathrm{HS}}(X)_{\mathrm{DIFF}}$.

THEOREM 2.3. *For every* n *there is a functorial split exact sequence*

$$0 \to \Gamma_n \xrightarrow{\iota} \Omega_n^{\mathrm{HS}}(X)_{\mathrm{DIFF}} \xrightarrow{\varrho} \Omega_n^{\mathrm{HS}}(X)_{\mathrm{PL}} \to 0$$

where Γ_n *is the group of concordance classes of differentiable structures on the* n-*sphere.*

P r o o f. For $n \leq 5$ there is no difference between the differentiable and PL case, and $\Gamma_n = 0$. Hence we assume $n \geq 6$. (The argument actually works for $n \geq 5$.)

By [10], every PL homology n-sphere M, $n \neq 3$, is the boundary of a contractible manifold, which by smoothing theory has a (unique) differentiable structure. Therefore M has a differentiable structure, so it follows that $\Omega_n^{\mathrm{HS}}(X)_{\mathrm{DIFF}} \to \Omega_n^{\mathrm{HS}}(X)_{\mathrm{PL}}$ is surjective.

The map $\iota : \Gamma_n \to \Omega_n^{\mathrm{HS}}(X)_{\mathrm{DIFF}}$ takes a homotopy sphere Σ to the constant map $\Sigma \to X$. This is clearly a homomorphism, since addition in Γ_n is also given by connected sum.

ι is injective: Let W be a homology cobordism between Σ^n and (standard) S^n. Then the plus-construction on W can be realized by surgeries on one- and two-handles in the interior. Thus we obtain an h-cobordism between Σ^n and S^n, which is trivial since $n \geq 5$.

It is obvious that $\varrho \circ \iota = 0$, so it remains to prove that $\ker \varrho \subseteq \iota(\Gamma_n)$.

Let $f : M \to X$ represent an element $\alpha \in \ker \varrho$. Then f can be extended to a map $F : W \to X$, where $\partial W = M$ and $H_*(W; \mathbb{Z}) \approx H_*(\mathrm{point}; \mathbb{Z})$. Deleting a small disk in W, we obtain a (PL) homology cobordism $F' : W' \to X$ between f and the constant map $\mathrm{S}^n \to X$. By smoothing theory, the differentiable structure on M can be extended (in fact uniquely) to a differentiable structure on W'. Restricting to the other end, we get a differentiable structure γ on S^n. Then $\alpha = \iota(\gamma)$.

Functoriality of the sequence is obvious. But then the splitting is obtained by mapping the sequence for X to the sequence for a point, since $\Omega_n^{\mathrm{HS}}(*)_{\mathrm{PL}}$ is trivial by the above mentioned result of Kervaire [10]. ∎

COROLLARY 2.4. *For $n \geq 5$ there is a split exact sequence*

$$0 \to \Gamma_n \to \Omega_n^{\mathrm{HS}}(X)_{\mathrm{DIFF}} \to \pi_n(X^+) \to 0.$$

We now specialize to the case we are interested in, namely $X = BGL(A)$ for a ring A. As observed in [5], $E(A)$ (the subgroup of $GL(A)$ generated by elementary matrices) is locally perfect, so $\pi_n(X^+) \approx K_n(A)$. From now on we only consider the *differentiable* homology bordism groups, so we simplify the notation by writing $\Omega_n^{\mathrm{HS}}(X) = \Omega_n^{\mathrm{HS}}(X)_{\mathrm{DIFF}}$.

From the results above, we then have exact sequences

(2.5) $0 \to \Gamma_n \to \Omega_n^{\mathrm{HS}}(BGL(A)) \to K_n(A) \to 0$ $(n \geq 5)$,   and

(2.6) $0 \to \Omega_4^{\mathrm{HS}}(BGL(A)) \to K_4(A) \to \Omega_3^{\mathrm{HS}}(\mathcal{F})$

$$\to \Omega_3^{\mathrm{HS}}(BGL(A)) \to K_3(A) \to 0$$

($\mathcal{F}$ is the homotopy fiber of $BGL(A) \to BGL(A)^+$).

But generators of $\Omega_n^{\mathrm{HS}}(BGL(A))$ can also be thought of as classifying maps for principal $GL(A)$-bundles (or $GL_k(A)$-bundles for large k)—actually covering spaces—over homology n-spheres. Hence we get the following interpretation of $K_n(A)$, $n \geq 5$:

$K_n(A) \approx \{$principal $GL_k(A)$-bundles over smooth homology n-spheres$\}/\sim$

where the equivalence relation $\sim$ is generated by isomorphism, homology cobordism, stabilization (in k), and change of differentiable structure.

REMARKS 2.7. (1) The cases $n = 3$ and $n = 4$ are more mysterious. For $n = 3$ we obviously get a similar interpretation, but with more relations. For $n = 4$ we may not get all of $K_4(A)$. It is conceivable that the exact sequence (2.6) splits into an isomorphism for $n = 4$ and a short exact sequence. However, $\Omega_3^{\mathrm{HS}}(\mathcal{F})$ is definitely not trivial, since it contains $\Omega_3^{\mathrm{HS}}(\mathrm{point}) = \theta_3$—the group of homology 3-spheres—as a direct summand, and this is now known to be big. One might try to use $\Omega_3^{\mathrm{HS}}(\mathcal{F})_{\mathrm{TOP}}$ instead, and an optimistic conjecture would be that this is trivial.

For $n = 1$ and $n = 2$, however, it is easy to see what happens.

Since $K_1(A) = H_1(BGL(A))$, we can describe K_1 by the same generators, but with arbitrary cobordisms as relations.

Similarly, since $K_2(A) = H_2(E(A))$, where $E(A)$ is the subgroup of $GL(A)$ generated by elementary matrices, we can describe K_2 as a cobordism group of $E(A)$-bundles, but this time over arbitrary 2-manifolds. The construction of the corresponding element in $\pi_2 BGL(A)^+$ in this case goes as before, but we have to kill all of $\pi_1 M$, even if it is not (locally) perfect.

(2) For $n \geq 2$ we can replace $GL(A)$ by $E(A)$ or $SL(A)$ when A is commutative. In the following we shall sometimes do this without further comment.

EXAMPLE 2.8. (i) In [7], Jones and Westbury construct examples of flat bundles over 3-dimensional homology spheres, representing elements of $K_3(\mathbb{C})$. For instance, they show that hyperbolic homology spheres naturally give rise to elements of infinite order, and every element of finite order can be represented by flat bundles over Seifert homology spheres of type $\Sigma(p, q, r)$. Such bundles are determined by a complex representation of the fundamental group, and observing that some of these representations are, in fact, defined over integers in certain number fields, one also can represent elements of K-theory of these rings. One particularly nice example is the Poincaré sphere, whose fundamental group is a subgroup of SU_2. This subgroup may be defined over the cyclotomic integers $\mathbb{Z}[\zeta_5]$, and Jones and Westbury show that the resulting element in $K_3(\mathbb{Z}[\zeta_5])$ generates the torsion subgroup (which is isomorphic to $\mathbb{Z}/120$).

(ii) Another interesting case is $X = BS_\infty$, the classifying space of the infinite symmetric group. Then the Barratt–Priddy–Quillen theorem says that $\pi_n(X^+) \approx \pi_n^S$, the stable homotopy groups of spheres, for $n > 0$. Therefore these also have homology bordism interpretation.

The natural inclusion of S_k in $GL_k(\mathbb{Z})$ as the permutation matrices induces a diagram

$$
\begin{array}{ccc}
\Omega_n^{\mathrm{HS}}(BS_\infty) & \longrightarrow & \pi_n^{\mathrm{S}} \\
\downarrow & & \downarrow \\
\Omega_n^{\mathrm{HS}}(BGL(\mathbb{Z})) & \longrightarrow & K_n(\mathbb{Z})
\end{array}
$$

where the right hand vertical map is the standard homomorphism.

For $n = 3$ the horizontal homomorphisms are still surjective, and it would be interesting to have nice descriptions of homology 3-spheres and bundles representing the generators of π_3^{S} and $K_3(\mathbb{Z})$.

With the bundle interpretation of (elements of) K-theory groups, it is natural to look for invariants in terms of characteristic classes. Our aim is to use a simple differential geometric approach related to the Chern–Weil theory, but measuring the incompatibility of the discrete and the metric structure on a flat bundle. We shall do this first in the case of A equal to the real, complex, or quaternionic numbers. For other rings one can use this case to get invariants as follows:

Suppose we have a representation $A \to M_q(F)$ (= $q \times q$-matrices with entries in F), where $F = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. Then we have an induced map $K_*(A) \to$

$K_*(M_q(F)) \approx K_*(F)$ in K-theory. In the bundle interpretation, this takes a
principal $GL_k(A)$-bundle over M to the associated $GL_{kq}(F)$-bundle over M.

$GL_k(F)$-bundles can be thought of as (principal bundles associated with)
flat F-vector bundles, so what we shall do in the next two sections is first
to study a general theory of characteristic classes for flat bundles, and then
apply it to bundles over homology spheres.

3. Characteristic classes for flat bundles. In this section we sketch
the constructions of characteristic classes for *foliated bundles* in [8], Ch. 3,
and apply it to our situation. For details, see [8].

Suppose that we have a differentiable, principal G-bundle $\pi : P \to M$
over a manifold M, where G is a Lie group. Then P has a free right G-action
$\mu : P \times G \to P$ with quotient M. We write $\mu(p, g) = L_p(g) = R_g(p)$—the
same notation as for left and right multiplication in G.

Let $\mathfrak{g}$ be the Lie algebra of G. If $p \in P$, then the differential of L_p at the
unit $e \in G$ is a monomorphism $L_{p*} : \mathfrak{g} \to T_pP$. Any $x \in \mathfrak{g}$ then determines
a vector field $\overline{x}$ on P by $\overline{x}(p) = L_{p*}(x)$.

In Cartan's formalism, a *connection* in the principal bundle is then a
splitting $\overline{\omega}_p : T_pP \to \mathfrak{g}$ of L_{p*} for every $p \in P$ such that

(1) $\overline{\omega}_p$ depends differentiably on p in the sense that they collect to a
differentiable map $TP \to \mathfrak{g}$.

(2) $R_{g^{-1}}\overline{\omega} = \mathrm{Ad}_g\overline{\omega}$ ($\overline{\omega}$ is G-invariant).

$\overline{\omega}$ is a 1-form on P with values in the Lie algebra $\mathfrak{g}$, but we shall rather
think of the connection as the homomorphism

$$\omega : \mathfrak{g}^* \to \Omega^1(P)$$

defined by $\phi \mapsto \phi \circ \overline{\omega}$.

$\mathfrak{g}^*$ and $\Omega^1(P)$ are both the degree one parts of natural differential graded
algebras (DGAs): the Koszul complex $(\Lambda^*(\mathfrak{g}), d_\Lambda)$ which computes the Lie
algebra cohomology of $\mathfrak{g}$, and the de Rham complex $(\Omega^*(P), d_\Omega)$. Clearly,
ω extends to a map of graded algebras

$$\omega : \Lambda^*(\mathfrak{g}) \to \Omega^*(P),$$

but this does *not* in general commute with differentials:

Consider the following diagram:

$$
\begin{array}{ccc}
\mathfrak{g}^* & \xrightarrow{\ \omega\ } & \Omega^1(P) \\
{\scriptstyle d_\Lambda}\downarrow & & \downarrow{\scriptstyle d_\Omega} \\
\Lambda^2(\mathfrak{g}) & \xrightarrow{\ \omega\ } & \Omega^2(P)
\end{array}
$$

A calculation shows that:

(1) If this diagram commutes, then $\omega : \Lambda^*(\mathfrak{g}) \to \Omega^*(P)$ commutes with differentials.

(2) The map $d_\Omega \circ \omega - \omega \circ d_\Lambda : \mathfrak{g}^* \to \Omega^2(P)$ can be identified with the *curvature* $K \in \Omega^2(P; \mathfrak{g})$ of the connection.

It follows that ω extends to a map of DGAs precisely if the connection is flat!

So, if the connection is flat, we obtain a homomorphism from the Lie algebra cohomology $H^*_{\mathrm{LIE}}(\mathfrak{g})$ of $\mathfrak{g}$ to the de Rham cohomology $H^*_{\mathrm{DR}}(P)$ of P. However, this is not yet quite what we want.

By a *flat bundle* we shall mean a bundle with a flat connection. This means that the structure group can be considered with the *discrete* topology. Conversely, a bundle with discrete structure group has a unique flat connection.

Let now H be a closed, connected subgroup of G, and consider the projection $\pi_H : P \to P/H$. This induces a monomorphism $\pi_H^* : \Omega^*(P/H) \to \Omega^*(P)$ with image equal to the set of forms on P which are H-invariant and which vanish on all vector fields of the form $\bar{x}$ for x in the Lie algebra $\mathfrak{h}$ of H. This subspace can be characterized completely in terms of the infinitesimal action of H as

$$(3.1) \qquad \mathrm{im}(\pi_H^*) = \{\tau \in \Omega^*(P) \mid i_{\bar{x}}\tau = 0, \ \mathrm{ad}_{\bar{x}}\tau = 0, \ \text{for all } x \in \mathfrak{h}\}.$$

The right hand side of (3.1) will be denoted by $\Omega^*(P)_{\mathfrak{h}}$. We have similar actions of $\mathfrak{h}$ on $\Lambda^*(\mathfrak{g})$, so we can define

$$\Lambda^*(\mathfrak{g})_{\mathfrak{h}} = \{\tau \in \Lambda^*(\mathfrak{g}) \mid i_x\tau = 0, \ \mathrm{ad}_x\tau = 0, \ \text{for all } x \in \mathfrak{h}\}.$$

It turns out that $\Lambda^*(\mathfrak{g})_{\mathfrak{h}} \subset \Lambda^*(\mathfrak{g})$ is again a DGA, and ω induces a restriction

$$\omega_{\mathfrak{h}} : \Lambda^*(\mathfrak{g})_{\mathfrak{h}} \to \Omega^*(P)_{\mathfrak{h}}.$$

The homology of $\Lambda^*(\mathfrak{g})_{\mathfrak{h}}$ is the relative Lie algebra cohomology and is denoted by $H^*_{\mathrm{LIE}}(\mathfrak{g}; \mathfrak{h})$. Hence we have obtained a homomorphism

$$H^*_{\mathrm{LIE}}(\mathfrak{g}; \mathfrak{h}) \to H^*_{\mathrm{DR}}(P/H).$$

Assume now that the structure group of the original bundle $P \to M$ can be reduced to H, but not necessarily as *flat* bundles. Then there is a section $M \to P/H$ of the bundle $P/H \to M$, and we get a composed homomorphism

$$\alpha = \alpha_P : H^*_{\mathrm{LIE}}(\mathfrak{g}; \mathfrak{h}) \to H^*_{\mathrm{DR}}(P/H) \to H^*_{\mathrm{DR}}(M).$$

REMARKS 3.2. (1) Let $P(H)$ be the principal H-bundle of the reduction. Then $P(H)$ is a sub-H-bundle of P. If the connection can also be reduced

to H, we have a commutative diagram

(3.3)

$$
\begin{array}{ccccc}
\Lambda^*(\mathfrak{g})_{\mathfrak{h}} & \longrightarrow & \Lambda^*(\mathfrak{h})_{\mathfrak{h}} & =\!=\!=\!= & 0 \\
\downarrow{\scriptstyle\omega_{\mathfrak{h}}} & & \downarrow{\scriptstyle\omega_{\mathfrak{h}}} & & \downarrow \\
\Omega^*(P/H) \xrightarrow{\approx} \Omega^*(P)_{\mathfrak{h}} & \longrightarrow & \Omega^*(P(H))_{\mathfrak{h}} & \xrightarrow{\approx} & \Omega^*(M)
\end{array}
$$

The bottom horizontal composition is precisely the splitting of $\Omega^*(M) \to \Omega^*(P/H)$ given by reduction of the structure group, so α is trivial if the bundle has a flat reduction. Hence $\alpha \in \operatorname{Hom}(H^*_{\mathrm{LIE}}(\mathfrak{g};\mathfrak{h}), H^*_{\mathrm{DR}}(M))$ is an obstruction to flat reduction of the structure group.

(2) The obvious application of α would then be to the investigation of flat bundles with (nonflat) reductions on some fixed manifold M. Here, however, we shall fix the pair (G, H) and vary M within its homology type.

For the applications in Section 4, we need the following

PROPOSITION 3.4. *Let W be a homology cobordism between two manifolds M_1 and M_2, and assume that $Q \to W$ is a flat, principal G-bundle with an H-reduction $\sigma : W \to Q/H$. Restriction to M_i then gives principal G-bundles $P_i \to M_i$ with H-reductions $\sigma_i = \sigma|M_i : M_i \to P_i/H$. Then the following diagram commutes*:

$$
\begin{array}{ccc}
H^*_{\mathrm{LIE}}(\mathfrak{g};\mathfrak{h}) & \xrightarrow{\ \alpha_{P_1}\ } & H^*_{\mathrm{DR}}(M_1) \\
{\scriptstyle\alpha_{P_2}}\downarrow & & \downarrow{\scriptstyle\approx} \\
H^*_{\mathrm{DR}}(M_2) & \xrightarrow{\ \approx\ } & H^*_{\mathrm{DR}}(W)
\end{array}
$$

and the two compositions are equal to α_Q.

P r o o f. This follows immediately from the commutative diagram

$$
\begin{array}{ccccc}
\Lambda^*(\mathfrak{g})_{\mathfrak{h}} & \xrightarrow{\ \omega_{\mathfrak{h}}\ } & \Omega^*(P_i/H) & \xrightarrow{\ \sigma_i^*\ } & \Omega^*(M_i) \\
\| & & \downarrow & & \downarrow \\
\Lambda^*(\mathfrak{g})_{\mathfrak{h}} & \xrightarrow{\ \omega_{\mathfrak{h}}\ } & \Omega^*(P/H) & \xrightarrow{\ \sigma^*\ } & \Omega^*(W) \quad\blacksquare
\end{array}
$$

4. Application to K-theory. We now specialize to the case where M is an oriented homology n-sphere and $G = GL_q^0(L)$, where L is a finite-dimensional, semisimple algebra over $\mathbb{R}$ and $GL_q^0(L)$ is the identity component of $GL_q(L)$. (L has a vector space structure such that the algebraic structure on L is smooth and then $GL_q(L)$ also has the structure of a Lie group.) Then we can always reduce the structure group to the maximal compact subgroup $K = K_q$. In fact, $P/K \simeq M$, so $H^*_{\mathrm{DR}}(P/K) \to H^*_{\mathrm{DR}}(M)$ is

uniquely determined (inverse of an isomorphism) and hence α is independent of the choice of reduction.

M has interesting homology only in dimension n, and $H^n_{\mathrm{DR}}(M) \approx \mathbb{R}$, by an isomorphism which is uniquely determined by the orientation of M. Therefore we can think of α as an element in $(H^n_{\mathrm{LIE}}(\mathfrak{g};\mathfrak{k}))^*$. In fact, using the product structure, we see that α must vanish on all decomposables, i.e. on products of at least two elements of positive degree. Hence we can consider α an element of $(H^n_{\mathrm{LIE}}(\mathfrak{gl}_q(L),\mathfrak{k}_q)/\mathcal{D})^*$, where $\mathcal{D}$ is the submodule generated by decomposables.

It follows from Proposition 3.4 that this construction induces a map

$$\beta : \Omega^{\mathrm{HS}}_n(BGL^0_q(L)) \to (H^n_{\mathrm{LIE}}(\mathfrak{gl}_q(L),\mathfrak{k}_q)/\mathcal{D})^*,$$

and this clearly commutes with stabilization in q. Since Γ_n is finite, we then use (2.5) to get

THEOREM 4.1. β *induces a homomorphism*

$$B = B_L : K_n(L) \to \varinjlim_q (H^n_{\mathrm{LIE}}(\mathfrak{gl}_q(L),\mathfrak{k}_q)/\mathcal{D})^*.$$

P r o o f. It remains to prove that β is additive. Suppose that $P \to M$ is the sum of $P_1 \to M_1$ and $P_2 \to M_2$, where $M = M_1 \# M_2 = M^0_1 \cup_{n-1} M^0_2$, $M^0_i = M_i - \mathrm{int}\, D_i$, D_i a small, open disc around the basepoint. Let further $P^0_i = P_i | M^0_i$.

Then we have a diagram of bundle maps

$$
\begin{array}{ccccccccc}
P_1 & \xleftarrow{\;\supset\;} & P^0_1 & \xrightarrow{\;\subset\;} & P & \xleftarrow{\;\supset\;} & P^0_2 & \xrightarrow{\;\subset\;} & P_2 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
M_1 & \xleftarrow{\;\supset\;} & M^0_1 & \xrightarrow{\;\subset\;} & M & \xleftarrow{\;\supset\;} & M^0_2 & \xrightarrow{\;\subset\;} & M_2
\end{array}
$$

We may assume that the reductions to the maximal compact subgroup are compatible, such that the diagram

$$
\begin{array}{ccccccccc}
P_1/H & \xleftarrow{\;\supset\;} & P^0_1/H & \xrightarrow{\;\subset\;} & P/H & \xleftarrow{\;\supset\;} & P^0_2/H & \xrightarrow{\;\subset\;} & P_2/H \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
M_1 & \xleftarrow{\;\supset\;} & M^0_1 & \xrightarrow{\;\subset\;} & M & \xleftarrow{\;\supset\;} & M^0_2 & \xrightarrow{\;\subset\;} & M_2
\end{array}
$$

also commutes, where the vertical maps are sections of the induced bundle maps. Moreover, without loss of generality we may assume that the reductions are flat over D_i, resp. a small product neighborhood $\mathcal{U}$ of $n-1 = M^0_1 \cap M^0_2 \subset M$.

Let ϱ_1, ϱ_2 be maps $M \to M_i$ such that ϱ_i is the identity on M_i^0 and constant outside $M_i \cup \mathcal{U}$. Consider now the diagram

$$
\begin{array}{ccccc}
\Lambda^*(\mathfrak{g})_{\mathfrak{k}} & \longrightarrow & \Omega^*(P_1/H) & \longrightarrow & \Omega^*(M_1) \\
\| & & & & \downarrow{\scriptstyle \varrho_1^*} \\
\Lambda^*(\mathfrak{g})_{\mathfrak{k}} & \longrightarrow & \Omega^*(P/H) & \longrightarrow & \Omega^*(M) \\
\| & & & & \uparrow{\scriptstyle \varrho_2^*} \\
\Lambda^*(\mathfrak{g})_{\mathfrak{k}} & \longrightarrow & \Omega^*(P_2/H) & \longrightarrow & \Omega^*(M_2)
\end{array}
$$

The three horizontal compositions induce α for the three bundles—let us denote them by a_P, a_{P_1} and a_{P_2}. The diagram does not commute, but for every $\phi \in \Lambda^*(\mathfrak{g})_{\mathfrak{k}}$ we have

$$
a_P(\phi)_x = a_{P_i}(\phi)_x \quad \text{if } x \in M_i^0,
$$

and since for every $\tau \in \Omega^*(M_i)$, $\varrho_i^*(\tau) = 0$ outside $M_i \cup \mathcal{U}$, we have

(4.3) $\qquad a_P(\phi)_x = \varrho_1^* a_{P_1}(\phi)_x + \varrho_2^* a_{P_2}(\phi)_x \quad \text{for } x \in M - \mathcal{U}.$

But since the reductions can be assumed to be flat in x and $\varrho_i(x)$ for $x \in \mathcal{U}$, it follows by inspection of diagram (3.3) that both sides of (4.3) are zero in $\mathcal{U}$.

Since ϱ_i are degree one maps and hence canonically identify $H^*_{\mathrm{DR}}(M_i)$ and $H^*_{\mathrm{DR}}(M)$, the assertion follows. ∎

The most important examples are $L = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. Then

$$(G, K) = (GL_q^0(\mathbb{R}), SO_q), \ (GL_q(\mathbb{C}), U_q) \text{ or } (GL_q(\mathbb{H}), Sp_q), \text{ resp.}$$

Define $P_n(F)$, $F = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, by

$$
P_n(F) = \begin{cases} \mathbb{R} & \text{if } n \equiv 1 \pmod 4, \\ \mathbb{R} & \text{if } n \equiv 1 \pmod 2 \text{ and } F = \mathbb{C}, \\ 0 & \text{otherwise.} \end{cases}
$$

Then we have the following calculations:

PROPOSITION 4.2. *Let $\mathfrak{k}_q(F)$ be the maximal compact Lie subalgebra of $\mathfrak{gl}_q(F)$. Then*

$$(H^n_{\mathrm{LIE}}(\mathfrak{g}l_q(F), \mathfrak{k}_q(F))/\mathcal{D})^* \approx P_n(F) \quad \text{for } q \gg n.$$

Moreover, the isomorphisms can be chosen to be compatible with stabilization in q.

P r o o f. We sketch the idea of the proof of this well known fact. Each of the pairs of Lie algebras has a compact dual with the same cohomology. But this is the same as the de Rham cohomology of the corresponding homogeneous spaces. In the three cases $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ we get $H^*_{\mathrm{DR}}(U/SO)$, $H^*_{\mathrm{DR}}(U)$

and $H^*_{\mathrm{DR}}(U/Sp)$ resp., stably. The duals of these cohomology rings mod $\mathcal{D}$ are the modules generated by primitive elements in real homology, which are isomorphic to the homotopy groups tensored by $\mathbb{R}$. But these are known from Bott periodicity computations. ∎

COROLLARY 4.3. *β induces a homomorphism $B_F : K_n(F) \to P_n(F)$ for $n \geq 5$.*

If, more generally, L is a finite-dimensional, semisimple $\mathbb{R}$-algebra, we know from the structure theory that it is isomorphic to a product $L \approx \prod_i M_{d_i}(F_i)$ of matrix algebras over $F_i = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. Therefore, by Morita equivalence, $K_n(L) \approx \prod_i K_n(F_i)$, and hence we get homomorphisms

$$(4.4) \qquad B_L : K_n(L) \to \prod_i P_n(F_i).$$

In Sections 5 and 6, we show that these homomorphisms are closely related to the Borel regulator maps, and hence highly nontrivial.

5. Comparison with Borel's computations. Let k be a number field of degree d, and let A_k be its ring of integers. Then k has r_1 real and r_2 complex Archimedian valuations, where $d = r_1 + 2r_2$.

Each such valuation v determines an embedding i_v of k into the completion k_v, uniquely if $k_v = \mathbb{R}$ and a conjugate pair of embeddings if $k_v = \mathbb{C}$. Choosing one embedding for each conjugate pair, we get a product embedding of k into the ring

$$\widehat{k} = \prod_v k_v = \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}.$$

The important fact is now that A_k is a *lattice* in $\widehat{k}$. In fact, we can think of $\widehat{k}$ as the decomposition of $A_k \otimes_{\mathbb{Z}} \mathbb{R}$ into simple algebras.

Now set $L = \widehat{k}$ in Theorem 4.1. Then B_L in (4.4) reduces to the product of all the homomorphisms B_{k_v} from Corollary 4.3. Hence we have homomorphisms

$$(5.1) \quad K_n(A_k) \to K_n(k) \to K_n(\mathbb{R})^{r_1} \times K_n(\mathbb{C})^{r_2}$$

$$\overset{B}{\to} \begin{cases} \mathbb{R}^{r_1+r_2} & \text{if } n \equiv 1 \pmod 4, \\ \mathbb{R}^{r_2} & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

Call this composed homomorphism B_A. Recall that in [2], Borel computed the rank of $K_n(A_k)$ to be exactly the exponents on the right hand side for n odd > 1 (and 0 for n even). We shall show that Borel's result can be formulated as follows:

THEOREM 5.2. *After tensoring $K_n(A_k)$ with $\mathbb{R}$, B_A becomes an isomorphism.*

In other words, except for torsion, we detect all of $K_n(A_k)$ using the homomorphisms β_F of Corollary 4.3 for all embeddings $A_k \to \mathbb{R}$ and $\mathbb{C}$.

REMARK. We really should not write the right hand side in (5.1) as powers of $\mathbb{R}$, since this involves choosing a basis for each $P_n(k_v)$. One way to do this is to use elements corresponding to integral generators of the homotopy of the homogeneous spaces giving rise to the computations sketched in Proposition 4.2. The comparison of the image of B_A to the resulting basis of $\prod P_n(k_v)$ involves very interesting number theory, and gives rise to the so-called higher regulators ([11], [3]).

Proof of Theorem 5.2. Borel's computations use the fact that all vector spaces involved are finite-dimensional, and the isomorphisms

$$K_n(A_k) \otimes \mathbb{Q} \approx \operatorname{Prim} H_n(SL(A_k); \mathbb{Q}) \approx (H^n(SL(A_k); \mathbb{Q})/\mathcal{D})^* \quad (n > 1),$$

to reduce the problem to that of computing $H^*(SL(A_k); \mathbb{Q})$, or rather $H^*(SL(A_k); \mathbb{R})$. (As above, $\mathcal{D}$ is the module generated by products.)

To do this, he constructs an algebra homomorphism

$$j : I_{G_q} \to H^*(SL_q(A_k); \mathbb{R})$$

where G_q is the group of $\mathbb{R}$-points of an algebraic group defined over $\mathbb{Q}$ and whose $\mathbb{Q}$-points are $SL_q(k)$. But then $G_q \approx SL_q(\widehat{k}) \approx \prod_v SL_q(k_v)$, and $SL_q(A_k) \to G_q$ is the diagonal embedding, composed of all the embeddings $A_k \subset k \to k_v$.

I_G is the algebra of G-invariant forms on the symmetric space of maximal compact subgroups of G. Then the main theorem is that j becomes an isomorphism in degrees below a number that grows to infinity with q. (Borel's result is much more general, but all we need here is this special case.)

But it is a well known fact that $I_G \approx H^*_{\mathrm{LIE}}(\mathfrak{g}; \mathfrak{k})$ where K is a maximal compact subgroup of G, and the computation reduces to the Lie algebra homology computation already mentioned.

Consider the following diagram:

$$(5.3)$$

The lower half is obviously commutative, so we have to prove that the upper half commutes.

The composition

$$h : \Omega_n^{\mathrm{HS}}(BSL_q(A)) \to K_n(A) \to H_n(SL(A); \mathbb{R}) \to (H^n(SL(A); \mathbb{R}))^*$$

takes $\{f : M \to BSL_q(A)\}$ to the class given by $c \mapsto c(f_*([M]))$, which factors through $(H^n(SL_q(A); \mathbb{R}))^*$. It then suffices to prove that

(5.4)
$$
\begin{array}{ccc}
\Omega_n^{\mathrm{HS}}(BSL_q(A)) & \xrightarrow{\ \ h\ \ } & (H^n(SL_q(A); \mathbb{R}))^* \\
\downarrow{\scriptstyle B_A} & & \downarrow{\scriptstyle j^*} \\
(H_{\mathrm{LIE}}^*(\mathfrak{sl}_q(\widehat{k}), \mathfrak{k}))^* & \xrightarrow[\ \ g\ \ ]{\approx} & (I_{G_q})^*
\end{array}
$$

commutes for each q.

We need to make the two compositions very explicit. Let $G = G_q = SL_q^0(\widehat{k})$, K the maximal compact subgroup, and set $X = G/K$. X is a contractible (diffeomorphic to a euclidean space) homogeneous G-space.

Then $I_G = \Omega^*(X)^G \approx \Lambda^*(T_e X) \approx \Lambda^*(\mathfrak{g})_{\mathfrak{k}}$, and the isomorphism $I_G \approx H_{\mathrm{LIE}}^*(\mathfrak{g}; \mathfrak{k})$ is just the fact that differentials are trivial on both sides.

Let $P \to M$ be a principal $SL_q(A)$-bundle representing an element x in $\Omega_n^{\mathrm{HS}}(BSL_q(A))$. Let $\sigma : M \to P_G/K$ be the section associated with a K-reduction of the associated G-bundle, and ω as defined in Section 4. Then

$$gB_A(x)(\phi) = \int\limits_M \sigma^*\omega(\phi).$$

To compare this with the other composition, we need the following de Rham cohomology interpretation of $H^n(SL_q(A); \mathbb{R})$:

LEMMA 5.5. $H^n(SL_q(A); \mathbb{R}) \approx H^n(\Omega^*(X)^{SL_q(A)})$.

With this isomorphism, the map $j : I_G \to H^*(SL_q(A); \mathbb{R})$ is induced by the inclusion $I_G = \Omega^*(X)^G \subset \Omega^*(X)^{SL_q(A)}$.

The statement in the lemma is well known, but we shall need the explicit isomorphism provided by the following

P r o o f. $SL_q(A)$ has a torsion free, normal subgroup Γ of finite index. Then $\Gamma \backslash X$ is a manifold of the homotopy type of the classifying space $B\Gamma$. Since $\Omega^*(\Gamma \backslash X) \approx \Omega^*(X)^\Gamma$, we have $H^*(B\Gamma; \mathbb{R}) \approx H^*(\Omega^*(X)^\Gamma)$ by the de Rham theorem.

Let $i^* : H^*(BSL_q(A); \mathbb{R}) \to H^*(B\Gamma; \mathbb{R})$ be induced by the inclusion, and define tr $: H^*(B\Gamma; \mathbb{R}) \to H^*(BSL_q(A); \mathbb{R})$ to be the transfer divided by the index $[SL_q(A) : \Gamma]$ of Γ in $SL_q(A)$, such that tr $\circ\, i^* = $ id. Then $i^* \circ$ tr is a projection on $H^*(B\Gamma; \mathbb{R})$, with image $i^*(H^*(BSL_q(A); \mathbb{R})) \approx H^*(BSL_q(A); \mathbb{R})$.

Both i^* and tr have natural analogs as maps between $H^n(\Omega^*(X)^{SL_q(A)})$ and $H^n(\Omega^*(X)^\Gamma)$: i^* is again induced by inclusion, and tr by averaging,

using the action of $SL_q(A)/\Gamma$ on Γ. It is then easy to check that the de Rham isomorphism $H^*(B\Gamma; \mathbb{R}) \approx H^*(\Omega^*(X)^\Gamma)$ commutes with the projections $i^* \circ \mathrm{tr}$. Hence it will induce an isomorphism between the two summands $H^*(BSL_q(A); \mathbb{R})$ and $H^*(\Omega^*(X)^{SL_q(A)})$. ∎

REMARK. It actually follows from Borel's computations that i^* (hence also tr) is an isomorphism in low degrees.

Let $\phi \in \Omega^n(X)^{SL_q(A)}$. We can describe the cochain $c(\phi)$ on $BSL_q(A)$ which ϕ determines by this isomorphism as follows:

Let $f : \triangle^n \to BSL_q(A)$ be a singular simplex, and let $\widetilde{f} : \widetilde{\triangle}^n \to B\Gamma$ be the lifting of f to the pullback of $B\Gamma \to BSL_q(A)$ over f. ($\widetilde{\triangle}^n$ consists of $[SL_q(A) : \Gamma]$ copies of $\triangle^n$.) Then

$$c(\phi)(f) = \frac{1}{[SL_q(A) : \Gamma]} \int_{\widetilde{\triangle}^n} \widetilde{f}^* \phi.$$

COROLLARY 5.6. $h : \Omega_n^{\mathrm{HS}}(BSL_q(A)) \to (H^n(SL_q(A); \mathbb{R}))^*$ is given by

$$h(f) = \frac{1}{[SL_q(A) : \Gamma]} \int_{\widetilde{M}_\Gamma} \widetilde{f}^* \phi,$$

where $f : M \to BSL_q(A)$ and $\widetilde{f} : \widetilde{M}_\Gamma \to B\Gamma$ is a lifting of f to the pullback. ($\widetilde{M}_\Gamma \to M$ is then an $[SL_q(A) : \Gamma]$-fold covering space.)

To finish the proof of Theorem 5.2, we now need to give a description of the classifying map for a principal $SL_q(A)$-bundle over a manifold M. In fact, it follows from Corollary 5.6 that it suffices to do this for liftings $\widetilde{M}_\Gamma \to B\Gamma = \Gamma \backslash X$.

So, let $P_A \to M$ be an $SL_q(A)$-bundle, and let $P \to M$ be the associated flat G-bundle. Parallel transport with respect to the flat connection then determines a trivialization of the pullback $\varrho^* P$, where $\varrho : \widetilde{M} \to M$ is the universal covering space—hence also of $\varrho^* P/K$.

The section $\sigma : M \to P/K$ then pulls back to a section $\widetilde{\sigma} : \widetilde{M} \to \varrho^* P/K$, which we compose with the projection λ to the fiber G/K to give

$$\gamma = \lambda \circ \sigma : \widetilde{M} \to X.$$

Note that there is a natural homomorphism $\pi_1 M \to SL_q(A)$ (coming e.g. from parallel transport around loops in $P_A \to M$), and that γ commutes with the actions of $\pi_1 M$ and $SL_q(A)$ on $\widetilde{M}$ and X. Hence we get induced maps

$$\gamma_\Gamma : \widetilde{M}_\Gamma \to \Gamma \backslash X$$

for every $\Gamma \subset SL_q(A)$, where now $\widetilde{M}_\Gamma = \widetilde{M}/\varrho_*^{-1}(\Gamma)$.

But we also have

$$\gamma^*(\phi) \in \Omega^*(\widetilde{M})^{\pi_1 M} \approx \Omega^*(M) \quad \text{if } \phi \in \Omega^*(X)^{SL_q(A)},$$

so we can consider γ^* as a homomorphism

$$\gamma^* : \Omega^*(X)^{SL_q(A)} \to \Omega^*(M).$$

It follows that if $x \in \Omega_n^{\mathrm{HS}}(BSL_q(A))$ is represented by $P_A \to M$, then $h(x)$ is given by

$$h(x)(\phi) = \int_M \gamma^*(\phi).$$

(Hence we do not any longer refer to the finite index subgroup Γ.) It now only remains to observe that

$$\sigma^* \circ \omega = \gamma^* \circ j^* = \widetilde{\sigma}^* \circ \lambda^* : \Lambda^*(\mathfrak{g})_{\mathfrak{k}} \to \Omega^*(M).$$

But this follows from the diagram

$$
\begin{array}{ccccc}
\Omega^*(\widetilde{M}) & \xleftarrow{\ \widetilde{\sigma}^*\ } & \Omega^*(\varrho^* P/K) & \xleftarrow{\ \lambda^*\ } & \Omega^*(X) \\
\big\uparrow{\scriptstyle\cup} & & \big\uparrow{\scriptstyle\cup} & & \big\uparrow{\scriptstyle\cup} \\
\Omega^*(M) & \xleftarrow{\ \sigma^*\ } & \Omega^*(P/K) & \xleftarrow{\ \omega\ } & \Lambda^*(\mathfrak{g})_{\mathfrak{k}}
\end{array}
$$

and the fact that ω is just an infinitesimal version of λ. ∎

REMARK 5.7. Diagram (5.4) and the proof that it is commutative works for all n. Hence, for $n = 3$, we find both that our invariant factors through K_3, and that we have the same interpretation of the Borel classes as in higher degrees. For $n = 2$ or 4 we get zero in both cases, so the identification in Theorem 5.2 is actually valid for all $n > 1$. In fact, using the interpretation in Remark 2.7 for $n = 1$ and GL_q instead of SL_q (actually, it suffices to take $q = 1$), we get a map which we can identify with the usual embedding of $(A_k^*/\text{torsion})$ as a codimension 1 lattice in $\mathbb{R}^{r_1 + r_2}$.

6. The case of group rings. As another illustration, we take the example of group rings. For geometric applications, this is the most important example. We shall see that, properly interpreted, this case is very analogous to the case of rings of integers.

Let π be a finite group. Then $\mathbb{R}[\pi]$ is a semisimple algebra, and

$$(6.1) \qquad \mathbb{R}[\pi] \approx \prod_i M_{d_i}(F_i), \quad \text{where } F_i = \mathbb{R},\ \mathbb{C} \text{ or } \mathbb{H}.$$

The factors are matrix algebras corresponding to the irreducible, real representations of π.

Then the homomorphism

$$B_\pi : K_n(\mathbb{Z}[\pi]) \to K_n(\mathbb{R}[\pi]) \approx \prod_i K_n(F_i) \to \prod_i P_i(F_i)$$

is in many ways similar to B_A for rings of integers, and we have:

THEOREM 6.2. *After tensoring with* $\mathbb{R}$, B_π *becomes an isomorphism for* $n > 1$.

That is, we detect all of $K_n(\mathbb{Z}[\pi])$ except torsion by taking all the irreducible real representations of π and using the idea of Section 4!

The number of factors in the target of B_π is equal to the number r of irreducible real representations of π. By the definition of $P_n(F)$, this means that the rank is r for $n \equiv 1 \pmod 4$ (and > 1), the number of real representations of *complex* type for $n \equiv 3 \pmod 4$ and zero if n is even.

The proof of Theorem 6.2 proceeds very much like the proof of Theorem 5.2, using the results of [6]. In fact, just as in Section 5, we see that B_π is equivalent to the dual of Borel's homomorphism

$$j_{GL_q(\mathbb{Z}[\pi])} : I_{GL_q(\mathbb{R}[\pi])}/\mathcal{D} \to H^*(GL_q(\mathbb{Z}[\pi]))/\mathcal{D}.$$

This is not quite good enough since GL_q is not semisimple, but in [6] it is shown how one can define a semisimple replacement $SL_q(\mathbb{R}[\pi]) \supset SL_q(\mathbb{Z}[\pi])$—essentially by using the reduced norm for each factor of (6.1) and taking the kernel. Then Borel's result applies to give an isomorphism

$$j_{SL(\mathbb{Z}[\pi])} : I_{SL(\mathbb{R}[\pi])}/\mathcal{D} \to H^*(SL(\mathbb{Z}[\pi]))/\mathcal{D}.$$

For general groups we do not have a simple structure theorem such as (6.1), but F-representations $\varrho : \pi \to GL_q(F)$ still induce homomorphisms $\varrho : \mathbb{Z}[\pi] \to \mathbb{R}[\pi] \to M_q(F)$ and hence

$$B_\varrho : K_n(\mathbb{Z}[\pi]) \to K_n(M_q(F)) \to K_n(F) \xrightarrow{B} P_n(F).$$

We can certainly detect elements this way, but in general $\text{rank}(K_n(\mathbb{Z}[\pi]))$ may be nonzero also for *even* n. The simplest example is $\pi = \mathbb{Z}$, where e.g. $K_6(\mathbb{Z}[\pi]) \approx K_6(\mathbb{Z}) \oplus K_5(\mathbb{Z})$ has rank one.

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