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The order of the Hopf bundle on projective Stiefel manifolds

by

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Abstract. The projective Stiefel manifold $X_{n,k}$ has a canonical line bundle $\xi_{n,k}$, called the Hopf bundle. The order of $c\xi_{n,k}$, the complexification of $\xi_{n,k}$, as an element of (the abelian group) $K(X_{n,k})$, has been determined in [3], [5], [6]. The main result in the present work is that this order equals the order of $\xi_{n,k}$ itself, as an element of $KO(X_{n,k})$, for $n \equiv 0, \pm 1 \pmod 8$, or for k in the "upper range for n" (approximately $k \geq n/2$). Certain applications are indicated.

1. Introduction. Let $\mathbb{F}$ denote either of the fields $\mathbb{R}, \mathbb{C}$, and $K_{\mathbb{F}}(X)$ denote respectively $KO(X)$, $K(X)$. For any $\mathbb{F}$-vector bundle α over a space X, of rank r, the *order* $o(\alpha)$ is the least positive integer m (if such exists) such that $m\alpha$ is stably trivial. Equivalently $m([\alpha] - r) = 0 \in \widetilde{K}_{\mathbb{F}}(X) \subset K_{\mathbb{F}}(X)$, and this is the condition we shall use. It is also convenient to recall that for any $\mathbb{R}$-vector bundle α over X, its complexification $c\alpha$ is the $\mathbb{C}$-vector bundle $\alpha \otimes_{\mathbb{R}} \mathbb{C}$, and that this induces a ring homomorphism $c : KO(X) \to K(X)$.

1.1. Remark. With m (finite) as above, there is no guarantee that $m\alpha$ is actually trivial. However, for X a finite CW-complex, it is usually the case that m is much larger than $\dim(X)$, and $mr > \dim(X)$ suffices to imply the triviality of $m\alpha$ by standard stability properties of vector bundles (cf. [9], Ch. 8, Theorem 1.5). Our main interest is in line bundles ($\mathrm{rank}(\alpha) = 1$) over a finite CW-complex X. Let us start with a famous example (cf. [1]), which will also be important in the present work.

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1.2. Example. Let $X = P^{n-1}$, real projective $(n-1)$-space, and let ξ_{n-1} be its Hopf line bundle. Then $o(c\xi_{n-1}) = 2^{[(n-1)/2]}$ and $o(\xi_{n-1}) = 2^{\phi(n-1)}$, where the function $\phi(n-1)$ is defined to be the number of integers k that satisfy $1 \le k \le n-1$, $k \equiv 0, 1, 2, 4 \pmod 8$.

Since any line bundle ξ over a finite CW-complex X is classified by a map $X \to BO(1) = P^\infty$, which, by cellular approximation, must factor up to homotopy through some finite skeleton $P^M \subset P^\infty$, the above example already implies that the order of ξ, or of $c\xi$, is a power of 2, say respectively $2^b, 2^a$. The well known fact (cf. [1], or [12], Ch. 15) that for the realification and complexification maps the composition $KO(X) \xrightarrow{c} K(X) \xrightarrow{r} KO(X)$ is multiplication by 2, implies that for any real vector bundle α with finite order, the order of α either equals the order of $c\alpha$ or is twice the order of $c\alpha$. Thus $b = a + \theta$, with either $\theta = 0$ or $\theta = 1$.

1.3. Example. For $X = P^{n-1}$ one sees from Example 1.2 that $\theta = 0$ whenever $n \equiv 0, \pm 1 \pmod 8$, and otherwise $\theta = 1$.

1.4. Remarks. (a) Determining θ can sometimes be quite difficult, e.g. part of the Adams Conjecture ([9], Ch. 15, Theorem 14.2(3)) was a question of this type.

(b) Writing $y = [\xi] - 1 \in KO(X)$ for any real line bundle ξ over X, we have $y^2 = -2y$ since $\xi \otimes \xi \approx \varepsilon$, the trivial line bundle, and therefore $y^i = (-2)^{i-1} y$, $1 \le i$. It follows that the order 2^m also gives the (multiplicative) height of y as being $m+1$, as an element in the ring $KO(X)$. A similar statement holds for complexifications of real line bundles over X, which can be seen using the already mentioned fact that complexification c is a ring homomorphism. It is, however, false for arbitrary complex line bundles.

We now consider the projective Stiefel manifold $X_{n,k}$, $1 \le k \le n-1$. Recall that $X_{n,k} = V_{n,k}/(\mathbb{Z}/2)$, so there is a Hopf line bundle $\xi_{n,k}$ over $X_{n,k}$, and also a sequence of smooth fibrations

$$X_{n,n-1} \xrightarrow{p} X_{n,n-2} \xrightarrow{p} \dots \xrightarrow{p} X_{n,1} = P^{n-1}$$

with $p^*(\xi_{n,k-1}) \approx \xi_{n,k}$. Let us write $2^{b(n,k)}$ for the order of $\xi_{n,k}$ and $2^{a(n,k)}$ for the order of $c\xi_{n,k}$. The precise values of $a(n,k)$ are known from [3], [5], [6], and we now give these after some preliminary definitions. In the following, for any positive integer m, by $\nu_2(m)$ we mean the highest power of 2 dividing m. Also, let $c = [(n-k)/2]$, and write $n = 2m$ or $n = 2m+1$.

1.5. Definition. (i) For k even or n even,

$$a_0(n,k) = \min\left\{2j - 1 + \nu_2\binom{m}{j} : c < j < m\right\},$$

(ii) for k and n odd,

$$a_0(n,k) = \min\left\{2c + \nu_2\binom{m}{c}, 2j - 1 + \nu_2\binom{m}{j} : c < j < m\right\},$$

(iii) for any n, k, $1 \le k < n$,

$$a(n,k) = \min\{[(n-1)/2], a_0(n,k)\}.$$

As mentioned above, $2^{a(n,k)}$ gives the order of $c\xi_{n,k}$. Therefore the order of $\xi_{n,k}$ equals $2^{a(n,k)+\theta}$, $\theta \in \{0,1\}$. The main purpose of this paper is to give a proof that for "most" values of n, k, in a sense that will be made precise by the next definition, $\theta = 0$ (i.e. the real and complex orders of $\xi_{n,k}$ agree).

1.6. Remark. From Definition 1.5, the following property of $a(n,k)$ is evident:

$$[(n-1)/2] = a(n,1) \ge a(n,2) \ge \ldots \ge a(n,n-1).$$

It is also clear that for k small, $a(n,k) = [(n-1)/2]$, whereas for k close to n, $a(n,k) = a_0(n,k)$. We therefore make the following definition.

1.7. Definition. Whenever $a(n,k) = a_0(n,k)$, we say that k is *in the upper range for* n. Otherwise we say k is *in the lower range for* n (i.e. in case $[(n-1)/2] < a_0(n,k)$).

Of course, Remark 1.6 and this definition imply that if k is in the upper range for n, so are $k+1, k+2, \ldots$ It is difficult to give a precise formula for the smallest k that will be in the upper range; however, it is not hard to see that this number will be slightly larger than $n/2$. For example, for $n = 38, 39, 138$, the upper range starts respectively at $k = 21, 21, 71$. It is also true that $n-1$ is always in the upper range (except for $n = 4$, the only case for which the upper range is empty). We now state the main result.

1.8. Theorem. *If k is in the upper range for n, or if $n \equiv 0, \pm 1 \pmod 8$, the order of $\xi_{n,k}$ equals $2^{a(n,k)}$.*

Notice that the two cases in the hypotheses (which asympotically comprise a little over 2/3 of all possible pairs (n,k)) are not mutually exclusive, and also that the result is definitely false in the lower range as Example 1.2 already shows for the case $k = 1$ (see also 1.11 below). The authors have found two quite different proofs for this theorem, and a proof based on the representation theory of the classical groups is presented in §2 of this paper. A second proof, based on the properties of exterior power operations and fairly involved combinatorial identities, will be submitted elsewhere [15].

Let us now indicate some applications of Theorem 1.8. The first is a straightforward generalization to line bundles over finite CW-complexes.

1.9. THEOREM. *Let ξ be a line bundle over a finite CW-complex X such that $n\xi$ admits at least k independent sections. Also suppose that k is in the upper range for n, or that $n \equiv 0, \pm 1 \pmod 8$. Then the order of ξ is a divisor of $2^{a(n,k)}$.*

Proof. By the universal property of projective Stiefel manifolds for multiples of line bundles (cf. [7], [14]), there exists a map $f : X \to X_{n,k}$ such that $f^*(\xi_{n,k}) \approx \xi$. It follows that the order of ξ divides the order of $\xi_{n,k}$, which by Theorem 1.8 equals $2^{a(n,k)}$. ∎

Theorem 1.8 also has many direct applications to questions such as span, immersions, and embeddings of $X_{n,k}$. These will be explored in detail in the companion paper [15]; we present just a single example here.

1.10. EXAMPLE. For n even, $X_{n,n-2}$ is known to be parallelizable (cf. [3]). We are now able to prove that for $n \equiv 3 \pmod 4$,

$$\operatorname{span}(X_{n,n-2}) = \dim(X_{n,n-2}) - 2.$$

To see this, first apply Theorem 1.8 to show $4\xi_{n,n-2}$ is stably trivial ($n-2$ is in the upper range except for $n = 3$, but in this case $4\xi_{3,1} = 4\xi_2$ is also stably trivial, cf. Example 1.2). Next, consider the tangent bundle $\tau_{n,n-2}$ and the twisted orthogonal complement bundle $\beta'_{n,n-2}$. We briefly recall the definition of the latter. For any orthonormal k-frame $(a_1, \ldots, a_k) = (-a_1, \ldots, -a_k) \in X_{n,k}$, the fibre of the rank $n-k$ vector bundle $\beta'_{n,k}$ is the $(n-k)$-dimensional real vector space given by

$$\{(a_1, \ldots, a_k, v) : v \in \mathbb{R}^n,\ \langle a_i, v\rangle = 0,\ 1 \le i \le k\},$$

where again $(a_1, \ldots, a_k, v) = (-a_1, \ldots, -a_k, -v)$. Using $\sim$ to denote stable equivalence, it is shown in [10], [11] that for any $X_{n,k}$ one has $\tau_{n,k} \sim nk\xi_{n,k}$ and $\beta'_{n,k} \sim n\xi_{n,k}$. Combining this with $n \equiv 3 \pmod 4$ and the already mentioned fact $4\xi_{n,n-2} \sim 0$, it is easily seen that both $\tau_{n,n-2} \sim 3\xi_{n,n-2}$ and $\beta'_{n,n-2} \sim 3\xi_{n,n-2}$. This gives $\tau_{n,n-2} \sim \beta'_{n,n-2}$, and since $\beta'_{n,n-2}$ has rank 2 we obtain $\operatorname{stable\,span}(X_{n,n-2}) \ge \dim(X_{n,n-2}) - 2$.

On the other hand one easily finds the Stiefel–Whitney class is

$$w(\tau) = w(3\xi) = (1+x)^3 = 1 + x + x^2,$$

where x generates $H^1(X_{n,n-2}; \mathbb{Z}/2)$ and it is known [7] that $x^2 \ne 0$, $x^3 = 0$. Since $w_2 \ne 0$ it follows that

$$\operatorname{stable\,span}(X_{n,n-2}) = \dim(X_{n,n-2}) - 2,$$

and the proof is completed by using the fact ([10], p. 99) that in this case the stable span and span agree.

We close this section with a plausible conjecture, which has been verified for $n \le 8$ and in other cases, but is far from being proved. Note that Example 1.3 above gives the case $k = 1$ of this conjecture.

1.11. Lower Range Conjecture. For $n \not\equiv 0, \pm 1 \pmod 8$ and for k in the lower range for n, we have $\theta = 1$ (i.e. the order of $\xi_{n,k}$ is twice the order of its complexification).

2. Proof of the main theorem

2.1. For a Lie group G, $R_{\mathbb{F}}(G)$ denotes the $\mathbb{F}$-representation ring of G. As is customary, we shall denote the real representation ring of G by $R_{\mathbb{R}}(G)$ or by $RO(G)$, and the complex representation ring by $R_{\mathbb{C}}(G)$ or by $R(G)$. We denote by $c_{\mathrm{rep}} : RO(G) \to R(G)$ and $c_{\mathrm{bun}} : KO(X) \to K(X)$ the complexification maps, which are ring homomorphisms. Note that the map c_{rep} is a monomorphism (cf. Prop. 3.27 of [2]), whereas c_{bun} is not in general a monomorphism.

Let G be a compact simply connected Lie group and let H denote a closed (not necessarily connected) Lie subgroup of G. Denote by M the smooth homogeneous manifold G/H.

Suppose that V is a finite-dimensional $\mathbb{F}$-vector space which affords an $\mathbb{F}$-representation of the Lie group H. We denote by $\alpha_{\mathbb{F}}(V)$ the $\mathbb{F}$-vector bundle over M with projection $G \times_H V \to G/H = M$ and fibre V. The bundle $\alpha_{\mathbb{F}}(V)$ is said to have been obtained from V by the *α-construction* (also called the *mixing construction*).

We now recall some basic facts about the α-construction; for further details cf. [9], Ch. 12, 5.4, also [4], [3], and [8], §9.

2.2. It is well known that if the representation of H on V arises by restriction to H of a representation of G on V, then $\alpha_{\mathbb{F}}(V)$ is isomorphic to a trivial vector bundle over $G/H = M$. The α-construction leads to a well defined ring homomorphism $\alpha_{\mathbb{F}} : R_{\mathbb{F}}(H) \to K_{\mathbb{F}}(M)$. The elements in the image of $\alpha_{\mathbb{F}}$ are said to be *homogeneous*.

We need the following (cf. [9], Ch. 13, Remark 11.2)

2.3. Lemma. *With the above notations one has a commuting diagram*

$$\begin{array}{ccc} RO(H) & \xrightarrow{c_{\mathrm{rep}}} & R(H) \\ \downarrow{\scriptstyle \alpha_{\mathbb{R}}} & & \downarrow{\scriptstyle \alpha_{\mathbb{C}}} \\ KO(M) & \xrightarrow{c_{\mathrm{bun}}} & K(M) \end{array}$$

2.4. Let $I_{\mathbb{F}}(G)$ (or simply $I_{\mathbb{F}}$) denote the ideal of $R_{\mathbb{F}}(H)$ generated by the image of the augmentation ideal of $R_{\mathbb{F}}(G)$ under the restriction homomorphism $\varrho : R_{\mathbb{F}}(G) \to R_{\mathbb{F}}(H)$. Thus, $I_{\mathbb{F}}$ is the ideal of $R_{\mathbb{F}}(H)$ generated by elements of the form $\varrho(x) - \operatorname{rank}(x)$, $x \in R_{\mathbb{F}}(G)$, where $\operatorname{rank} : R_{\mathbb{F}}(G) \to \mathbb{Z}$ is defined by $\operatorname{rank}([V]) = \dim_{\mathbb{F}} V$. Then we have

2.5. Lemma. *For $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ one has $I_{\mathbb{F}} \subset \ker \alpha_{\mathbb{F}}$.*

This is immediate from 2.2.

2.6. Remark. It is known that if H is connected and has maximal rank in G, then $\ker \alpha_{\mathbb{C}} = I_{\mathbb{C}}$. See [4], [13], and [8]. However, this will not apply in the case we eventually consider, the projective Stiefel manifolds.

2.7. Definition. Let x be a torsion element in the additive group $K_{\mathbb{F}}(M)$, where M is as in 2.2. We say that an element $u \in R_{\mathbb{F}}(H)$ *detects the order of* x if $\alpha_{\mathbb{F}}(u) = x$ and $nu \in I_{\mathbb{F}}$, where $n = o(x)$.

2.8. Lemma. *Let $y \in K(M)$ be a torsion element whose order, n, is detected by $v \in R(H)$. Suppose that $nv \in c_{\mathrm{rep}}(I_{\mathbb{R}})$ and that $c_{\mathrm{rep}}(u) = v$. Then the order of $x := \alpha_{\mathbb{R}}(u)$ is detected by u and $o(x) = n$.*

Proof. Let $n = o(y)$, and write $nv = c_{\mathrm{rep}}(w)$, where $w \in I_{\mathbb{R}}$. Clearly $c_{\mathrm{rep}}(nu - w) = 0$. Since c_{rep} is a monomorphism, it follows that $nu = w \in I_{\mathbb{R}}$. Hence $nx = \alpha_{\mathbb{R}}(nu) = \alpha_{\mathbb{R}}(w) = 0$ by Lemma 2.5, as $w \in I_{\mathbb{R}}$. Since $c_{\mathrm{bun}}(x) = y$, and since y has order n, we conclude that $o(x) = n$. ∎

Write $X_{n,k} = \mathrm{Spin}(n)/H_{n,k}$, where $H_{n,k}$ is a certain subgroup of $\mathrm{Spin}(n)$ which contains $\mathrm{Spin}(n-k)$ as an index 2 subgroup. The precise nature of the extension $\mathrm{Spin}(n-k) \hookrightarrow H_{n,k} \to \mathbb{Z}/2$ depends on the parities of n and k (cf. [6]), but we shall not require this here.

2.9. Let $V = \mathbb{R}^n$ denote the standard real representation of $\mathrm{SO}(n)$, extended to a representation of $\mathrm{Spin}(n)$ via the double covering projection $\mathrm{Spin}(n) \to \mathrm{SO}(n)$. Let $v_i = [\Lambda^i_{\mathbb{R}}(V)] \in RO(\mathrm{Spin}(n))$ and let $w_i = c_{\mathrm{rep}}(v_i) \in R(\mathrm{Spin}(n))$, $1 \le i \le n/2$. Also let $\Delta^{\pm}_m \in R(\mathrm{Spin}(2m))$ denote the class of the complex half-spin representations of $\mathrm{Spin}(2m)$, and let $\Delta_m = \Delta^+_m + \Delta^-_m$. We regard Δ_m as an element of $R(\mathrm{Spin}(2m+1))$ in the usual manner. Finally, let $z \in RO(H_{n,k})$ denote the class of the one-dimensional representation whose character is the composition $H_{n,k} \to H_{n,k}/\mathrm{Spin}(n-k) \cong C_2 \subset GL(1,\mathbb{R}) = \mathbb{R}^*$, where $C_2 = \{\pm 1\}$. Let $\widetilde{z} = c_{\mathrm{rep}}(z) \in RH_{n,k}$. Note that the Hopf line bundle ξ is isomorphic to $\alpha_{\mathbb{R}}(z)$. Let $y = c_{\mathrm{bun}}(\xi) - 1 = [\xi \otimes_{\mathbb{R}} \mathbb{C}] - 1 \in K(X_{n,k})$.

We now need to know that certain multiples of $\widetilde{z} - 1$ lie in the image of the restriction homomorphism $\varrho : R(\mathrm{Spin}(n)) \to R(H_{n,k})$, as well as certain results about the order of y. These can be found in [3] for $n \equiv 0 \pmod 4$, in [5], [6], for all n; we simply quote them as the next proposition.

2.10. Proposition. *Let $n = 2m$ or $2m+1$, $1 < k < n$. Write $c = [(n-k)/2]$. Let $r = \gcd\left\{2^{2i-1}\binom{m}{i} \,\middle|\, c < i < m\right\}$. Then*

(i) $r(\widetilde{z}-1) = \varrho(P_0) + \widetilde{z}\varrho(P_1)$,
(ii) $2^{[(n-1)/2]}(\widetilde{z}-1) = \varrho(Q_0) + \widetilde{z}\varrho(Q'_0)$,
(iii) $2^{2c}\binom{m}{c}(\widetilde{z}-1) = \varrho(Q_1) + \widetilde{z}\varrho(Q'_1)$ *if both* n *and* k *are odd*,

where $P_i \in Z[w_1,\ldots,w_m] \subset R(\mathrm{Spin}(n))$ *have rank zero*, *and* $Q_j, Q'_j \in R\mathrm{Spin}(n)$ *have rank zero. Furthermore*, *the order of* y *is*

(iv) $\gcd\{r, 2^{[(n-1)/2]}\} = 2^{a(n,k)}$ *if* n *or* k *is even*,
(v) $\gcd\left\{r, 2^m, 2^{2c}\binom{m}{c}\right\} = 2^{a(n,k)}$ *if both* n *and* k *are odd.*

2.11. Remark. By applying the α-construction it is immediate from (i)–(iii) above that $o(y)$ divides $2^{a(n,k)}$. That the order is *equal* to $2^{a(n,k)}$ is a consequence of the nontrivial theorem that $K(X_{n,k})$ is actually isomorphic to $R(H_{n,k}) \otimes_{R(\mathrm{Spin}(n))} \mathbb{Z} \cong R(H_{n,k})/I_{\mathbb{C}}$, where $I_{\mathbb{C}}$ is as in 2.4. Here $\mathbb{Z}$ is regarded as an $R(\mathrm{Spin}(n))$-module via the augmentation map, and $R(H_{n,k})$ via the restriction homomorphism ϱ. Again, this theorem (at least for n,k not both odd) is proved in [3], [6], where it is a consequence of the collapsing of the Hodgkin spectral sequence, but the only results we are using here are those already mentioned in 2.10(iv), (v). We also remark here that $o(y)$ is always even; indeed, a quick check shows that $a(n,k) \ge 1$, $k < n$, always holds.

2.12. We recall some basic facts about the (half-) spin representations. For details the reader is referred to §12, Ch. 13 of [9]. The complex representation ring of $\mathrm{Spin}(2m)$ (resp. $R(\mathrm{Spin}(2m+1))$) equals the polynomial algebra $\mathbb{Z}[w_1,\ldots,w_{m-2},\Delta_m^+,\Delta_m^-]$ (resp. $\mathbb{Z}[w_1,\ldots,w_{m-1},\Delta_m]$). Recall that a complex representation U of G is called *real* if U is obtained from a real representation by extension of scalars to $\mathbb{C}$, i.e. the class $[U] \in R(G)$ is in the image of c_{rep}. An element of $R(G)$ is said to be *real* if it is in the image of c_{rep}. It is known that Δ_m^+, Δ_m^- are real if $m \equiv 0 \pmod 4$ (which means $2m = n \equiv 0 \pmod 8$) and that Δ_m is real if $n = 2m+1 \equiv \pm 1 \pmod 8$. It is obvious that, in the notation of 2.9, $w_i = c_{\mathrm{rep}}(v_i), 1 \le i \le [n/2]$, are all of real type for all values of n. It follows that for $n \equiv 0, \pm 1 \pmod 8$ *any* representation of $\mathrm{Spin}(n)$ is real. In particular Q_i, Q'_i, $i = 0,1$, are of real type when $n \equiv 0, \pm 1$. Since the elements P_0, P_1 are in the subalgebra $\mathbb{Z}[w_1,\ldots,w_m] \subset R(\mathrm{Spin}(n))$, they are of real type for any n. Notice also that for k in the upper range for n, $r = q \cdot 2^{a(n,k)}$ with q odd.

We are now ready to prove the main theorem of this section.

2.13. Theorem. *Let* $1 \le k < n$. *If* $n \equiv 0, \pm 1 \pmod 8$, *or if* k *is in the upper range*, *then* $\xi_{n,k}$ *has order* $2^{a(n,k)}$.

Proof. As mentioned in §1, this is the same as showing that $x = [\xi]-1 \in KO(X_{n,k})$ has order $2^{a(n,k)}$. We divide the proof into the two (nondisjoint) cases given in the hypotheses. So first consider $n \equiv 0, \pm 1 \pmod 8$, the initial goal is to show that here the order of y is detected by $\widetilde{z}-1 \in R(H_{n,k})$.

Certainly

$$\alpha_{\mathbb{C}}(\widetilde{z}-1)=\alpha_{\mathbb{C}}c_{\mathrm{rep}}(z)-1=c_{\mathrm{bun}}\alpha_{\mathbb{R}}(z)-1=c_{\mathrm{bun}}(\xi)-1=y.$$

The remaining condition for detecting the order of y is immediate from the hypotheses on n and Proposition 2.10(i)–(iii); indeed, these imply

$$r(\widetilde{z}-1),\ 2^{[(n-1)/2]}(\widetilde{z}-1),\ 2^{2c}\binom{m}{c}(\widetilde{z}-1)\in I_{\mathbb{C}}$$

(the latter condition only in case both n,k are odd) so $2^{a(n,k)}(\widetilde{z}-1)\in I_{\mathbb{C}}$.

Secondly, when k is in the upper range for n, write $r=q\cdot 2^{a(n,k)}$, q odd, as in 2.12. As mentioned in 2.11, $a(n,k)\geq 1$, so $2^{a(n,k)}$ is even. There is then a positive integer t with $tq\equiv 1 \pmod{2^{a(n,k)}}$. We now show this implies that $qt(\widetilde{z}-1)$ detects the order of y. Indeed, $\alpha_{\mathbb{C}}(qt(\widetilde{z}-1))=qty=y$, and using 2.10(i) we find $2^{a(n,k)}qt(\widetilde{z}-1)=tr(\widetilde{z}-1)=t\varrho(P_0+yP_1)\in I_{\mathbb{C}}$, proving this statement.

Next, in either case, we find using 2.12 that each time a certain representation used in the above two paragraphs is in $I_{\mathbb{C}}$, it is of real type. So now Lemma 2.8 may be applied. In the first case it shows that $z-1$ detects the order of x, while in the second case it shows that $qt(z-1)$ detects the order of x. Since qt is odd and the order of x must be a power of 2, this implies in either case that $o(x)=2^{a(n,r)}$. ∎

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