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The normalizer splitting conjecture for p-compact groups

by

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Abstract. Let X be a p-compact group, with maximal torus $BT \to BX$, maximal torus normalizer BN and Weyl group W_X. We prove that for an odd prime p, the fibration $BT \to BN \to BW_X$ has a section, which is unique up to vertical homotopy.

1. Introduction. Homotopy Lie groups, or p-compact groups, were introduced by Dwyer and Wilkerson [13] and have since then been investigated closely by a number of people (cf. the surveys [21, 22, 28]). The basic philosophy in the study of these objects, defined purely in terms of homotopy theory, is that they behave like compact Lie groups.

To be more precise, fix a prime p. A *p-compact group* X is a loop space (X, BX, e) (i.e. BX is a pointed space and $e : \Omega BX \simeq X$ is a homotopy equivalence) such that BX is p-complete and X is $\mathbb{F}_p$-finite (i.e. $H^*(X, \mathbb{F}_p)$ is finitely generated as an abelian group). The motivating example is given by the Bousfield–Kan p-completion ([4]) of compact Lie groups: If G is a compact Lie group then $(G^\wedge_p, (BG)^\wedge_p, e)$ is a p-compact group if $\pi_0(G)$ is a finite p-group. Considering the torus $(S^1)^n$ we get the p-compact torus $BT = K(\mathbb{Z}_p, 2)^n$ of rank n.

The main result of Dwyer and Wilkerson [13] is that any p-compact group has a maximal torus, i.e. there is a map $Bi : BT \to BX$ satisfying a certain injectivity and maximality condition, and this is unique up to "conjugacy". Moreover they also construct a Weyl group and a maximal torus normalizer as follows.

We may assume that $BT \to BX$ is a fibration, since otherwise we can just replace it by an equivalent fibration $BT' \to BX$. We then define the Weyl space $\mathcal{W}_T(X)$ as the topological monoid consisting of self-maps of

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BT over BX. This turns out to be homotopically discrete, and the set of components $\pi_0(\mathcal{W}_T(X))$ is a finite group W_X called the *Weyl group* of X. This does not depend on the choice of T. The *maximal torus normalizer* BN is then defined as the Borel construction of the action of $\mathcal{W}_T(X)$ on BT. Thus up to homotopy we have a fibration sequence $BT \to BN \to BW_X$. It should be noted that BN is generally not a p-compact group, since $\pi_1(BN) = W$ need not be a finite p-group.

One would of course like to have a classification of p-compact groups. One approach to this is via the maximal torus normalizer. In the case of a compact connected semisimple Lie group G it was shown by Curtis, Wiederhold and Williams [12] that the normalizer N of the maximal torus T determines G. In the general case of compact connected Lie groups, it was shown by Notbohm [29] using earlier work of Scheerer [32] for the case of simply connected compact Lie groups.

Curtis, Wiederhold and Williams also studied the question of when the extension $0 \to T \to N \to W \to 0$ is split. This question had also been studied earlier by Tits [35]. Believing in the philosophy that p-compact groups behave as compact Lie groups, the conjecture is the following:

CONJECTURE 1.1 [28, 5.20]. *Two connected p-compact groups X and Y are isomorphic if and only if the normalizers of their maximal tori are isomorphic* (*as loop spaces*). *At odd primes the normalizer splits and the Weyl group data are sufficient to distinguish between connected p-compact groups.*

A more precise version is given by Lannes [21, 5.2]. The action of $\mathcal{W}_T(X)$ on BT gives a representation $W_X \to \mathrm{GL}(L_X)$, where $L_X := H_2(BT, \mathbb{Z}_p) \cong (\mathbb{Z}_p)^n$. We will call L_X the *lattice of* X. Since BN fits into a fibration $BT \to BN \to BW_X$, where $BT = K(L_X, 2)$ is an Eilenberg–MacLane space, the obstruction to finding a section is given by an element in $H^3(BW_X, \mathcal{C})$ ([38, IV.6.11]). The local coefficient system $\mathcal{C}$ comes from the action of W_X on $H_2(BT, \mathbb{Z}_p) = L_X$. Thus the obstruction is given by an element $\gamma_X \in H^3(W_X, L_X)$.

The precise version of the conjecture is that up to isomorphism connected p-compact groups X are determined by the triple (W_X, L_X, γ_X) for $p = 2$, and for odd primes that $\gamma_X = 0$ and X is determined by the pair (W_X, L_X). Our main result is the following part of the conjecture:

THEOREM 1.2. *If p is odd and X is a connected p-compact group then the obstruction γ_X vanishes*, *i.e. the fibration $BT \to BN \to BW_X$ has a section. Moreover this section is unique up to vertical homotopy.*

The results of Tits and Curtis, Wiederhold and Williams [35, 12] show that this is true in the case of compact connected Lie groups. Theorem 1.2

has also been proved independently by Dwyer and Wilkerson and by Notbohm, but their proofs have not yet appeared.

Our approach is the following. Recall that an element $w \in \mathrm{GL}_n(\mathbb{Q}_p)$ is called a *pseudoreflection* if $w-1$ has rank 1, that is, w fixes a hyperplane. Dwyer and Wilkerson [13] show that for connected p-compact groups X, $W_X \to \mathrm{GL}(L_X)$ is a faithful representation of W as a finite p-adic *pseudoreflection group* (i.e. the image is generated by pseudoreflections). Since the finite p-adic pseudoreflection groups have been classified [8], we can directly compute the cohomology groups $H^3(W_X, L_X)$. The result is the following:

THEOREM 1.3. *Let* $W \hookrightarrow \mathrm{GL}(L)$ *be a finite irreducible p-adic pseudoreflection group*, *p odd. Then* $H^3(W,L)=0$ *except for the case* $W=\Sigma_3$, $L = L_{\widehat{PU(3)}}$ *and* $p=3$ *where we have* $H^3(\Sigma_3, L_{\widehat{PU(3)}}) = \mathbb{Z}/3$.

The paper is organized as follows. In Section 2 we review the classification of p-adic pseudoreflection groups. This enables us to prove Theorem 1.3, which will be done in Section 3. Finally Section 4 contains the proof of Theorem 1.2.

NOTATION. In the following, p denotes a prime number, $\mathbb{F}_p$ the field with p elements, $\mathbb{Z}_p$ the ring of p-adic integers and $\mathbb{Q}_p$ its quotient field, the field of p-adic rational numbers. If $W \to \mathrm{GL}_n(\mathbb{Z}_p)$ is a representation of the group W, we let $L = (\mathbb{Z}_p)^n$ be the natural W-module. We will say that L is the *lattice associated with the representation*. The reduction map $\mathbb{Z}_p \to \mathbb{Z}_p/(p) = \mathbb{F}_p$ induces a representation $W \to \mathrm{GL}_n(\mathbb{F}_p)$. For short we will write L/p for the W-module $(\mathbb{F}_p)^n \cong L \otimes \mathbb{F}_p$.

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2. Classification of p-adic pseudoreflection groups. The finite p-adic pseudoreflection groups were first classified (up to conjugacy over $\mathbb{Q}_p$)

by Clark and Ewing [8], based on the earlier classification of the finite complex pseudoreflection groups by Shephard and Todd [33]. Earlier the real reflection groups had been classified by Coxeter [10], who established that they are in bijective correspondence with the so-called Coxeter graphs.

First of all it suffices to consider irreducible pseudoreflection groups, since any finite pseudoreflection group splits as a direct product of finite irreducible pseudoreflection groups. The result of Shephard and Todd is that, up to conjugation, the finite irreducible complex pseudoreflection groups fall into 3 infinite families, denoted by $1-3$, and 34 sporadic cases, denoted by $4-37$.

Assume that $\rho : W \hookrightarrow \mathrm{GL}_n(\mathbb{C})$ is a finite irreducible complex pseudoreflection group. Define the *character field* $\mathbb{Q}(\chi)$ as the field obtained from $\mathbb{Q}$ by adjoining all values of the associated character χ. Obviously a necessary condition for ρ to be equivalent to a representation taking values in $\mathbb{Q}_p$ is that we can embed the character field $\mathbb{Q}(\chi)$ in $\mathbb{Q}_p$. Clark and Ewing show that this is in fact also a sufficient condition.

Thus to classify the finite irreducible p-adic pseudoreflection groups it suffices to compute the character field $\mathbb{Q}(\chi)$ for each of the finite irreducible complex pseudoreflection groups and to determine for each of these the primes p for which we can embed $\mathbb{Q}(\chi)$ in $\mathbb{Q}_p$. The result is shown in Table 1, where the numbering is identical to the numbering of Shephard and Todd. We divide the possible primes into two groups: If $p \nmid |W|$ we say that p is a *nonmodular prime* for the p-adic pseudoreflection group W, and if $p \mid |W|$ we say that p is a *modular prime*. In the table ζ_m denotes a primitive mth root of unity.

It should be emphasized that Table 1 gives the classification of irreducible pseudoreflection representations up to conjugacy in $\mathrm{GL}_n(\mathbb{Q}_p)$. We return to this later. Now we describe some of the groups occurring in the table.

The first family consists of the symmetric groups Σ_{n+1}, $n \geq 1$. The $(n+1)$-dimensional representation obtained by permuting coordinates splits as a direct sum of a one-dimensional representation given by the fixed line spanned by the vector $e_1 + \ldots + e_{n+1}$ and its orthogonal complement consisting of all vectors with sum of coordinates equal to 0. This n-dimensional representation is irreducible, and since the transpositions generate Σ_{n+1} and act as reflections this is actually a pseudoreflection representation. The representation is defined over $\mathbb{Q}$ and is thus among the groups classified by Coxeter. The associated Coxeter graph has the form A_n. It is the Weyl group of the compact Lie groups $U(n+1)$ and $SU(n+1)$ and their central quotients.

The second family consists of the groups $G(m,r,n)$ where $r \mid m$. They are defined as follows. Let $A(m,r,n)$ consist of all diagonal matrices of the form

Table 1. The finite irreducible p-adic pseudoreflection groups

Number	Order	Center	Character field	Primes with $p \nmid \lvert W \rvert$	Primes with $p \mid \lvert W \rvert$
1	$(n+1)!$	1 for $n \geq 2$ 2 for $n = 1$	$\mathbb{Q}$	$p > n+1$	$2 \leq p \leq n+1$
$2a$	$n!m^n/r$	$m(n,r)/r$	$\mathbb{Q}(\zeta_m)$	$p \equiv 1 \pmod{m}$, $p > n$ for $m \geq 3$; $p > n$ for $m = 2$	$p \equiv 1 \pmod{m}$, $p \leq n$ for $m \geq 3$; $p \leq n$ for $m = 2$
$2b$	$2m$	$(2,m)$	$\mathbb{Q}(\zeta_m + \zeta_m^{-1})$	$p \equiv \pm 1 \pmod{m}$ $(p,m) \neq (2,3)$	$m = 3,4,6$ for $p = 2$ $m = 3,6$ for $p = 3$
3	n	n	$\mathbb{Q}(\zeta_n)$	$p \equiv 1 \pmod{n}$ for $p > 2$ or $n = 1$, $p = 2$	$n = 2$, $p = 2$
4	24	2	$\mathbb{Q}(\zeta_3)$	$p \equiv 1 \pmod{3}$	
5	72	6	$\mathbb{Q}(\zeta_3)$	$p \equiv 1 \pmod{3}$	
6	48	4	$\mathbb{Q}(\zeta_{12})$	$p \equiv 1 \pmod{12}$	
7	144	12	$\mathbb{Q}(\zeta_{12})$	$p \equiv 1 \pmod{12}$	
8	96	4	$\mathbb{Q}(\zeta_4)$	$p \equiv 1 \pmod{4}$	
9	192	8	$\mathbb{Q}(\zeta_8)$	$p \equiv 1 \pmod{8}$	
10	288	12	$\mathbb{Q}(\zeta_{12})$	$p \equiv 1 \pmod{12}$	
11	576	24	$\mathbb{Q}(\zeta_{24})$	$p \equiv 1 \pmod{24}$	
12	48	2	$\mathbb{Q}(\sqrt{-2})$	$p \equiv 1,3 \pmod{8}, p \neq 3$	$p = 3$
13	96	4	$\mathbb{Q}(\zeta_8)$	$p \equiv 1 \pmod{8}$	
14	144	6	$\mathbb{Q}(\zeta_3, \sqrt{-2})$	$p \equiv 1,19 \pmod{24}$	
15	288	12	$\mathbb{Q}(\zeta_{24})$	$p \equiv 1 \pmod{24}$	
16	600	10	$\mathbb{Q}(\zeta_5)$	$p \equiv 1 \pmod{5}$	
17	1200	20	$\mathbb{Q}(\zeta_{20})$	$p \equiv 1 \pmod{20}$	
18	1800	30	$\mathbb{Q}(\zeta_{15})$	$p \equiv 1 \pmod{15}$	
19	3600	60	$\mathbb{Q}(\zeta_{60})$	$p \equiv 1 \pmod{60}$	
20	360	6	$\mathbb{Q}(\zeta_3, \sqrt{5})$	$p \equiv 1,4 \pmod{15}$	
21	720	12	$\mathbb{Q}(\zeta_{12}, \sqrt{5})$	$p \equiv 1,49 \pmod{60}$	
22	240	4	$\mathbb{Q}(\zeta_4, \sqrt{5})$	$p \equiv 1,9 \pmod{20}$	
23	120	2	$\mathbb{Q}(\sqrt{5})$	$p \equiv 1,4 \pmod{5}$	
24	336	2	$\mathbb{Q}(\sqrt{-7})$	$p \equiv 1,2,4 \pmod{7}, p \neq 2$	$p = 2$
25	648	3	$\mathbb{Q}(\zeta_3)$	$p \equiv 1 \pmod{3}$	
26	1296	6	$\mathbb{Q}(\zeta_3)$	$p \equiv 1 \pmod{3}$	
27	2160	6	$\mathbb{Q}(\zeta_3, \sqrt{5})$	$p \equiv 1,4 \pmod{15}$	
28	1152	2	$\mathbb{Q}$	$p \neq 2,3$	$p = 2,3$
29	7680	4	$\mathbb{Q}(\zeta_4)$	$p \equiv 1 \pmod{4}, p \neq 5$	$p = 5$
30	14400	2	$\mathbb{Q}(\sqrt{5})$	$p \equiv 1,4 \pmod{5}$	
31	$64 \cdot 6!$	4	$\mathbb{Q}(\zeta_4)$	$p \equiv 1 \pmod{4}, p \neq 5$	$p = 5$
32	$216 \cdot 6!$	6	$\mathbb{Q}(\zeta_3)$	$p \equiv 1 \pmod{3}$	
33	$72 \cdot 6!$	2	$\mathbb{Q}(\zeta_3)$	$p \equiv 1 \pmod{3}$	
34	$108 \cdot 9!$	6	$\mathbb{Q}(\zeta_3)$	$p \equiv 1 \pmod{3}, p \neq 7$	$p = 7$
35	$72 \cdot 6!$	1	$\mathbb{Q}$	$p \neq 2,3,5$	$p = 2,3,5$
36	$8 \cdot 9!$	2	$\mathbb{Q}$	$p \neq 2,3,5,7$	$p = 2,3,5,7$
37	$192 \cdot 10!$	2	$\mathbb{Q}$	$p \neq 2,3,5,7$	$p = 2,3,5,7$

$$\begin{bmatrix} \theta_1 & 0 & \cdots & 0 \\ 0 & \theta_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \theta_n \end{bmatrix}$$

with $\theta_i^m = 1$, $1 \le i \le n$, and $(\theta_1 \dots \theta_n)^{m/r} = 1$. This is an abelian group of order m^n/r which has an action of Σ_n given by permutation of the entries on the diagonal. Denoting the permutation representation $\Sigma_n \hookrightarrow \mathrm{GL}_n(\mathbb{C})$ by ρ, the group $G(m,r,n)$ is the subgroup of $\mathrm{GL}_n(\mathbb{C})$ generated by $A(m,r,n)$ and $\rho(\Sigma_n)$. In other words, $G(m,r,n)$ is the semidirect product $A(m,r,n) \rtimes \Sigma_n$, and its order is $n!m^n/r$.

If $n = 1$ our group is just a cyclic group of order m/r. These are separated as a special case and they constitute the third infinite family (see below). For $n \ge 2$ the given representation of the group $G(m,r,n)$ is irreducible iff $m \ge 2$ and $(m,r,n) \ne (2,2,2)$ (see [9, 2.4]). The group $G(2,2,2)$ is isomorphic to $C_2 \times C_2$, which is abelian and thus only has irreducible representations of dimension one. To summarize, the allowed parameters are $m \ge 2$, $r \mid m$, $n \ge 2$ and $(m,r,n) \ne (2,2,2)$, and we will always assume these conditions to be satisfied when speaking of groups from this family.

The groups $G(m,r,n)$ consist of generalized signed permutation matrices. In particular if $m = 2$ we have ordinary signed permutation matrices, and we thus recover (as abstract groups) the Weyl groups of the classical compact Lie groups.

The groups $G(2,1,n)$ have Coxeter graphs B_n and are the Weyl groups of the compact Lie groups $SO(2n+1)$ and $Sp(n)$. We also have the groups $G(2,2,n)$ with Coxeter graphs D_n, which are the Weyl groups of the compact Lie groups $SO(2n)$.

Also in the case $n = 2$ and $m = r$ we get a Coxeter group, more precisely $G(m,m,2)$ is the dihedral group D_{2m}, with Coxeter graph $I_2(m)$. For $m = 3$, D_6 is isomorphic to Σ_3, and thus it is the Weyl group of $U(3)$ and $SU(3)$ (and their common quotient $PU(3)$). For $m = 4$, D_8 is the Weyl group of $SO(5)$ and $Sp(2)$. For $m = 6$, D_{12} is the Weyl group of G_2.

It should be noted that the groups of type 2 are divided into the two families $2a$ and $2b$ in the table. This is due to the fact that the character field of $G(m,r,n)$ is $\mathbb{Q}(\zeta_m)$ if $n \ge 3$ or $n = 2$ and $r < m$, while it equals $\mathbb{Q}(\zeta_m + \zeta_m^{-1})$ if $n = 2$ and $m = r$.

The third family consists simply of a cyclic group C_n in the faithful one-dimensional representation which maps a generator of C_n to a primitive nth root of unity ζ_n.

In addition to the above mentioned examples, Coxeter's classification of finite real reflection groups consists only of the following examples. We have the crystallographic groups with numbers 28, 35, 36 and 37 which are respectively the Weyl groups of the compact Lie groups F_4, E_6, E_7 and E_8. Furthermore we have the noncrystallographic groups with numbers 23 and 30 which have Coxeter graphs H_3 and H_4 respectively. They are more closely discussed in for example [18, Section 2.13], [34, pp. 197–198] and [3, p. 80].

As mentioned earlier the results above give a classification of finite p-adic pseudoreflection groups $W \hookrightarrow \mathrm{GL}_n(\mathbb{Q}_p)$ up to conjugacy within $\mathrm{GL}_n(\mathbb{Q}_p)$. What we are really interested in for our purposes is a classification of representations $W \hookrightarrow \mathrm{GL}_n(\mathbb{Z}_p)$ up to conjugacy within $\mathrm{GL}_n(\mathbb{Z}_p)$. It is easy to see that any representation $W \hookrightarrow \mathrm{GL}_n(\mathbb{Q}_p)$ is conjugate within $\mathrm{GL}_n(\mathbb{Q}_p)$ to a representation $W \hookrightarrow \mathrm{GL}_n(\mathbb{Z}_p)$ (cf. [11, 23.16]). However this representation need not be uniquely determined up to conjugacy within $\mathrm{GL}_n(\mathbb{Z}_p)$.

In other words the problem is that to a rational pseudoreflection representation $W \hookrightarrow \mathrm{GL}_n(\mathbb{Q}_p)$ there might correspond several different nonequivalent integral representations for W. Fortunately these problems have been solved by Notbohm [30]. We shall only need the following result here:

PROPOSITION 2.1. (1) *Let $\Sigma_3 \hookrightarrow \mathrm{GL}_2(\mathbb{Z}_3)$ be an irreducible pseudoreflection representation with associated lattice L. Then either $L \cong L_{SU(3)}$ or $L \cong L_{PU(3)}$, i.e. L is isomorphic to the lattice of $SU(3)$ or its quotient $PU(3)$.*

(2) *Let W denote group number 35 in Table 1, i.e. W is the Weyl group of the compact Lie group E_6, and let $W \hookrightarrow \mathrm{GL}_6(\mathbb{Z}_p)$ be an irreducible pseudoreflection representation with associated lattice L. Then for $p = 3$ either $L \cong L_{E_6}$ or $L \cong L_{PE_6}$, i.e. L is isomorphic to the lattice of E_6 or its quotient PE_6. For $p = 5$ we have a unique lattice, i.e. $L \cong L_{E_6} \cong L_{PE_6}$.*

Proof. This follows directly from [30, 1.2 and 3.2] since for $p = 3$ the lattices of both $SU(3)$ and E_6 are simply connected with center $\mathbb{Z}/3$ and for $p = 5$ the lattices of E_6 and PE_6 are both simply connected and centerfree. ∎

We end this section by the following observation which will be very useful in our cohomology computations. For the proof we quickly recall some invariant theory (for more details see [3, 34] and the references therein).

Let $W \subseteq \mathrm{GL}_n(\mathbb{C})$ be a finite group. Let $V = \mathbb{C}^n$ and $S = S(V^*)$ be the symmetric algebra on V^*. We may identify S with the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$, where $n = \dim V$. Since W acts on V we get an action of W on S given by $(w \cdot f)(v) = f(w^{-1} \cdot v)$. Let $R = S^W$ denote the ring of invariants and let L denote the quotient field of S. Then L also has an action of W. Galois theory shows that the quotient field of R is L^W and that L/L^W is a finite Galois extension with Galois group W.

The main result is the theorem due to Shephard–Todd and Chevalley that the invariant ring R is a polynomial ring exactly if W is a pseudoreflection group. In this case the degrees of the polynomial generators of R are uniquely determined and are called the *degrees of* W.

PROPOSITION 2.2. *Let $W \subseteq \mathrm{GL}_n(\mathbb{Z}_p)$ be an irreducible complex pseudoreflection group. Then the center of W is a cyclic group consisting of scalar matrices, and its order is the one listed in Table* 1.

Proof. That the center $Z(W)$ consists of scalar matrices follows directly from Schur's lemma. Hence it is cyclic and generated by the scalar matrix ζ_m for some m. For a homogeneus polynomial f we have $\zeta_k \cdot f = \zeta_k^{-\deg(f)} f$ so f is fixed by ζ_k exactly if $k \mid \deg(f)$. By the remark above we then see that $\zeta_k \in W$ exactly if $k \mid \deg(f)$ for all invariant polynomials f. Since the ring of invariants is polynomial by the Shephard–Todd–Chevalley theorem, we see that $\zeta_k \in W$ if and only if $k \mid \gcd(d_1, \ldots, d_n)$ where $d_1, \ldots, d_n$ are the degrees of W. Thus the order of the center is precisely $m = \gcd(d_1, \ldots, d_n)$, which may be computed from the tables in [33], [3] or [34]. ■

3. Cohomology of Weyl groups. In this section we compute, for each finite irreducible p-adic pseudoreflection group $W \hookrightarrow \mathrm{GL}(L)$, the low-dimensional cohomology groups $H^*(W, L)$. Throughout this section, p will denote an odd prime. Some good general references for group cohomology are [6, 16, 37].

The main result is that $H^2(W, L) = 0$ for all W, and $H^3(W, L) = 0$ if W is irreducible (with a single exception).

We start by considering trivial coefficients following Notbohm [27]. It should be noted that our proof is basically identical to Notbohm's but somewhat shorter since we can refer directly to a known result for Coxeter groups.

PROPOSITION 3.1. *Let W be a finite p-adic pseudoreflection group. Then there exists a subgroup W' of W such that W' is a Coxeter group and the index $[W : W']$ is coprime to p.*

Proof. Since W splits as a product of irreducible p-adic pseudoreflection groups, it suffices to consider W irreducible. If p does not divide $|W|$ we may just take $W' = 1$. Otherwise we are in the modular case and if W is itself a Coxeter group then obviously we can take $W' = W$. By comparing with Table 1 we see that the only cases left are the groups of type $2a$ and the cases $(G_{12}, p = 3)$, $(G_{29}, p = 5)$, $(G_{31}, p = 5)$ and $(G_{34}, p = 7)$.

The four last cases have been considered by Aguadé [1], and he shows that for each of them it is possible to embed the symmetric group $W' = \Sigma_{n+1}$ (here n denotes the rank of W) in W. By comparing with Table 1 we see that W' has index coprime to p in all four cases.

For the groups of type $2a$ recall that $G(m, r, n) = A(m, r, n) \rtimes \Sigma_n$ has order $n! m^n / r$. As p is odd and we have the restriction $p \equiv 1 \pmod{m}$ for

$m \geq 3$ we see that $p \nmid m$ in all cases. We may thus set $W' = \Sigma_n$ since then $[W : W'] = m^n/r$ is not divisible by p. ∎

THEOREM 3.2 (see [27, 3.1]). *Let W be a finite p-adic pseudoreflection group. Then all the homology and cohomology groups $H_1(W, \mathbb{Z}_p)$, $H_1(W, \mathbb{Z}/p^k)$, $H_2(W, \mathbb{Z}_p)$, $H_2(W, \mathbb{Z}/p^k)$, $H^1(W, \mathbb{Z}_p)$, $H^1(W, \mathbb{Z}/p^k)$, $H^2(W, \mathbb{Z}_p)$, $H^2(W, \mathbb{Z}/p^k)$, $H^3(W, \mathbb{Z}_p)$, $k \geq 1$, with trivial action on the coefficients vanish.*

Proof. By using Proposition 3.1 and a transfer argument we may suppose that W is a Coxeter group. By using the universal coefficient theorems ([37, Theorem 6.1.12, Exercise 6.1.5]) it suffices to show that the groups $H_1(W, \mathbb{Z})$ and $H_2(W, \mathbb{Z})$ do not have p-torsion. In fact by a theorem of Ihara and Yokonuma [19] (see also [17]) both groups are actually elementary abelian 2-groups, so since we only consider p odd we are done. ∎

It should be noted that the Schur multiplier $H_2(W, \mathbb{Z})$ actually has been calculated for all finite pseudoreflection groups [36, 31].

We next turn to coefficients being the natural action of W on L or L/p.

THEOREM 3.3. *Let $W \hookrightarrow \mathrm{GL}(L)$ be a finite p-adic pseudoreflection group. Then $H^2(W, L) = 0$.*

Proof. For Coxeter groups this is known to be true by [20, proof of 3.5] (see also [23, 5.2]). The general case follows from this by using Proposition 3.1 and a transfer argument. ∎

THEOREM 3.4. *Let $W \hookrightarrow \mathrm{GL}(L)$ be a finite irreducible p-adic pseudoreflection group. If (W, p) does not belong to the following list:*

(1) *$W \cong \Sigma_n$ belonging to the family 1, $n \neq 2, 4$ and $3 \leq p \leq n$,*
(2) *$W \cong D_6 = G(3, 3, 2)$ belonging to the family 2b and $p = 3$,*
(3) *$W \cong W(E_6)$ (number 35) and $p = 3$,*

then $H^*(W, L) = 0$ and $H^*(W, L/p) = 0$.

REMARK 3.5. The special case in (2) of $D_6 = G(3, 3, 2) \cong \Sigma_3$ is in fact already excluded by (1) since the representation is equivalent to that of Σ_3. This repetition is only made to make the proof and statement of the theorem clearer. We give the full answer in this case later (see Theorem 3.7).

Proof (of Theorem 3.4). Notice first of all that $H^0(W, L) = 0$ since W is irreducible.

By Proposition 2.2 and Table 1 we see that in all the cases $4 - 37$ except $W(E_6)$ (number 35), W has a nontrivial center Z consisting of scalar matrices. Moreover we see that $p \nmid |Z|$ since we only consider p odd. Thus $H^{>0}(Z, L) = H^{>0}(Z, L/p) = 0$, and since Z acts without fixed points on

L and L/p we also obtain $H^0(Z, L) = H^0(Z, L/p) = 0$. Considering now the Lyndon–Hochschild–Serre spectral sequence associated with the normal subgroup Z, we immediately obtain $H^*(W, L) = 0$ and $H^*(W, L/p) = 0$.

If $p \nmid |W|$ then obviously $H^{>0}(W, L) = H^{>0}(W, L/p) = 0$ since $|W|$ is invertible in $\mathbb{Z}_p$ and $\mathbb{F}_p$. By considering the long exact sequence in cohomology induced by the short exact sequence $0 \to L \xrightarrow{\cdot p} L \to L/p \to 0$ we also get $H^0(W, L/p) = 0$.

We now only have to consider the modular cases among the families 1, $2a$, $2b$ and 3 and the case of $W(E_6)$ at $p = 5$. Comparing with Table 1 we see that there are no cases to check coming from the family 3 and that the family $2b$ gives the two cases D_6 and D_{12} for $p = 3$. The first case is excluded and since D_{12} contains the central element -1 we are done in this case as well. Concerning cases from the family 1, all of them are excluded except Σ_2 and Σ_4. Since Σ_2 has order 2 this case has already been done above. The case of Σ_4 is handled by observing that Σ_4 and $G(2, 2, 3)$ are both Coxeter groups with the same Coxeter graph $A_3 = D_3$. Therefore the groups are isomorphic and the representations are equivalent. Since $G(3, 3, 2)$ belongs to the family $2a$ which we handle now, this will settle the case of Σ_4.

Thus the only case left among the three infinite families is the family $2a$. In this case we have $p \equiv 1 \pmod m$ and $p \le n$ for $m \ge 3$ and $p \le n$ for $m = 2$. Since p is odd we actually have $p \nmid m$ in both cases. The order of the normal subgroup $A(m, r, n)$, m^n/r, is thus prime to p. Using the Lyndon–Hochschild–Serre spectral sequence we thus get $E_2^{s,t} = 0$ for $t > 0$. Since p is odd and $A(m, r, n)$ contains all diagonal matrices $\operatorname{diag}(1, \ldots, \zeta_m, \ldots, \zeta_m^{-1}, \ldots, 1)$ with all entries equal to 1 except for two which are respectively ζ_m and ζ_m^{-1}, and these act without fixed points on L and L/p, we see that also $E_2^{s,0} = 0$. Thus $H^*(W, L) = 0$ and $H^*(W, L/p) = 0$.

Finally we have to consider $W = W(E_6)$ at $p = 5$. By Proposition 2.1 we have a unique lattice L. An integral representation can be found from the root system given in [18, p. 43]. It may be checked directly that $H^0(W, L/5) = 0$. Using this description we see that W contains the element

$$\sigma = \begin{bmatrix} 1 & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 0 & -1 & -1 & 1 \\ 0 & 2 & 1 & -2 & -1 & 1 \\ 0 & 1 & 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{bmatrix}$$

of order 10. Set $N = 1 + \sigma + \ldots + \sigma^9$; it is easily checked that $\operatorname{Ker}(\sigma - 1) = \operatorname{Im}(N)$ and $\operatorname{Ker}(N) = \operatorname{Im}(\sigma - 1)$ as matrices over $\mathbb{F}_5$. Using the standard formula for the cohomology of cyclic groups ([37, 6.2.2]) we find that

$H^{>0}(\langle\sigma\rangle, L/5) = 0$. Since $|W| = 2^7 \cdot 3^4 \cdot 5$, the index of $\langle\sigma\rangle$ is coprime to 5 and thus we get $H^{>0}(W, L/5) = 0$ using transfer. Therefore $H^*(W, L/5) = 0$.

Consider now the short exact sequence $0 \to L \xrightarrow{\cdot 5} L \to L/5 \to 0$. Since the order of W is $2^7 \cdot 3^4 \cdot 5$, multiplication by 5 is zero on $H^{>0}(W, L)$ but since also $H^*(W, L/5) = 0$ it has to be an isomorphism as well. We conclude that $H^*(W, L) = 0$. ∎

Concerning the excluded cases in the theorem we are able to compute the low-dimensional cohomology groups. We start by considering the symmetric groups, i.e. the cases (1) and (2). The module $\mathbb{Z}_p$ with trivial action will be denoted by L_{triv} and the n-dimensional permutation module for Σ_n will be denoted by L_{perm}.

THEOREM 3.6. *Let $\Sigma_n \hookrightarrow \mathrm{GL}(L)$ be any integral representation corresponding to the irreducible pseudoreflection representation $\Sigma_n \hookrightarrow \mathrm{GL}_{n-1}(\mathbb{Q}_p)$ coming from family number* 1. *Then $H^3(\Sigma_n, L) = 0$ if $n \geq 4$.*

P r o o f. By [30, 1.6(2) and 1.2] we have a short exact sequence $0 \to L \to L_{PU(n)} \to F \to 0$, where F is finite with trivial Σ_n-action. By Theorem 3.2 we see that $H^2(\Sigma_n, F) = 0$. Thus from the long exact sequence in cohomology we see that it suffices to prove that $H^3(\Sigma_n, L_{PU(n)}) = 0$ for $n \geq 4$.

To show this consider the short exact sequence $0 \to L_{\mathrm{triv}} \to L_{\mathrm{perm}} \to L_{PU(n)} \to 0$ of Σ_n-modules. Since $L_{\mathrm{perm}} = (L_{\mathrm{triv}}\downarrow_{\Sigma_{n-1}})\uparrow^{\Sigma_n}$ is induced up from the trivial action on $\mathbb{Z}_p$ of the subgroup Σ_{n-1} we get $H^*(\Sigma_n, L_{\mathrm{perm}}) \cong H^*(\Sigma_{n-1}, \mathbb{Z}_p)$ by Shapiro's lemma [37, Section 6.3]. So by Theorem 3.2 we have $H^3(\Sigma_n, L_{\mathrm{perm}}) = 0$. From the short exact sequence we then get an exact sequence

$$0 \to H^3(\Sigma_n, L_{PU(n)}) \to H^4(\Sigma_n, \mathbb{Z}_p) \to H^4(\Sigma_n, L_{\mathrm{perm}}).$$

It is not hard to see that the map

$$H^4(\Sigma_n, \mathbb{Z}_p) \to H^4(\Sigma_n, L_{\mathrm{perm}}) \xrightarrow{\cong} H^4(\Sigma_{n-1}, \mathbb{Z}_p)$$

is induced by the inclusion $\Sigma_{n-1} \hookrightarrow \Sigma_n$. The homology and cohomology groups of the symmetric groups have been computed by Nakaoka [26]. Using these results we conclude that the map above is an isomorphism for $n \geq 4$. Thus $H^3(\Sigma_n, L_{PU(n)}) = 0$ in this case. ∎

In the case of $\Sigma_3 \cong D_6 = G(3, 3, 2)$ the proof above yields some information, but in fact we can actually compute all its homology and cohomology groups.

THEOREM 3.7. *Let $p = 3$ and $W = \Sigma_3$. Let L_{triv} be the trivial module and let $L_{SU(3)}$ and $L_{PU(3)}$ be the irreducible lattices corresponding to the two possible irreducible pseudoreflection representations of Σ_3 in* $\mathrm{GL}_2(\mathbb{Z}_3)$

(cf. Proposition 2.1). *Then W has periodic cohomology with period* 4 *and we have the following table of cohomology and homology groups where $n \in \mathbb{Z}$ is arbitrary. Here $\widehat{H}$ denotes Tate homology/cohomology groups* (see [6, VI.4] or [37, 6.2.4]).

	H^0	H_0	$\widehat{H}^{4n} \cong \widehat{H}_{4n+3}$	$\widehat{H}^{4n+1} \cong \widehat{H}_{4n+2}$	$\widehat{H}^{4n+2} \cong \widehat{H}_{4n+1}$	$\widehat{H}^{4n+3} \cong \widehat{H}_{4n}$
L_{triv}	$\mathbb{Z}_3$	$\mathbb{Z}_3$	$\mathbb{Z}/3$	0	0	0
$L_{\mathrm{triv}}/3$	$\mathbb{Z}/3$	$\mathbb{Z}/3$	$\mathbb{Z}/3$	0	0	$\mathbb{Z}/3$
$L_{SU(3)}$	0	0	0	$\mathbb{Z}/3$	0	0
$L_{SU(3)}/3$	$\mathbb{Z}/3$	0	$\mathbb{Z}/3$	$\mathbb{Z}/3$	0	0
$L_{PU(3)}$	0	$\mathbb{Z}/3$	0	0	0	$\mathbb{Z}/3$
$L_{PU(3)}/3$	0	$\mathbb{Z}/3$	0	0	$\mathbb{Z}/3$	$\mathbb{Z}/3$

P r o o f. The fact that $W \cong D_6 \cong \mathbb{Z}/3 \rtimes \mathbb{Z}/2$ has periodic cohomology with period 4 follows from [6, Exercise VI.9.6]. In fact by direct computation using the Lyndon–Hochschild–Serre spectral sequence we have [37, Example 6.8.5]

$$H_k(W, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } k = 0, \\ \mathbb{Z}/2 & \text{for } k \equiv 1 \pmod 4, \\ \mathbb{Z}/6 & \text{for } k \equiv 3 \pmod 4, \\ 0 & \text{otherwise.} \end{cases}$$

Thus by using the universal coefficient theorems we immediately obtain the results for trivial coefficients.

For the lattices $L_{SU(3)}$ and $L_{PU(3)}$ and their reductions modulo 3 we have short exact sequences

$$0 \to L_{SU(3)} \to L_{\mathrm{perm}} \to L_{\mathrm{triv}} \to 0,$$
$$0 \to L_{\mathrm{triv}} \to L_{\mathrm{perm}} \to L_{PU(3)} \to 0,$$

and similarly after reduction modulo 3. Since $L_{\mathrm{perm}} = (L_{\mathrm{triv}}{\downarrow_{\Sigma_2}}){\uparrow^{\Sigma_3}}$ is equal to the trivial module of the subgroup Σ_2 induced up (and similarly modulo 3), we get $\widehat{H}^*(W, L_{\mathrm{perm}}) = 0$ and $\widehat{H}^*(W, L_{\mathrm{perm}}/3) = 0$ by Shapiro's lemma.

By considering the long exact sequences induced from the above short exact sequences we get isomorphisms $\widehat{H}^n(W, L_{SU(3)}) \cong \widehat{H}^{n-1}(W, L_{\mathrm{triv}})$ and $\widehat{H}^n(W, L_{PU(3)}) \cong \widehat{H}^{n+1}(W, L_{\mathrm{triv}})$ and similarly after reduction modulo 3. Thus the remaining results for $\widehat{H}$ follow from the calculations for trivial coefficients.

We are left with determining H_0 and H^0 for the lattices corresponding to $SU(3)$ and $PU(3)$ and their mod-3 reductions. This is done directly. ∎

Note that this shows that Theorem 3.2 cannot be improved in general since all the groups $H_3(\Sigma_3, \mathbb{Z}_3)$, $H_3(\Sigma_3, \mathbb{Z}/3^k)$, $H^4(\Sigma_3, \mathbb{Z}_3)$ and $H^3(\Sigma_3, \mathbb{Z}/3^k)$ are isomorphic to $\mathbb{Z}/3$.

Theorem 3.8. *Let $W = W(E_6)$, and let $W \hookrightarrow \mathrm{GL}(L)$ be any irreducible 3-adic pseudoreflection representation. Then $H^3(W, L) = 0$.*

Remark 3.9. The author is grateful to D. Benson for pointing out that this result may be proved directly using results in his paper [2]. The proof below is our original proof though.

Proof (of Theorem 3.8). By Proposition 2.1 we have either $L \cong L_{E_6}$ or $L \cong L_{PE_6}$. Using the computer algebra system Magma [7] we find that $H^2(W, L/3) = 0$ for both lattices. Consider now the short exact sequence $0 \to L \xrightarrow{\cdot 3} L \to L/3 \to 0$. The induced long exact cohomology sequence shows that multiplication by 3 is injective on $H^3(W, L)$. Since this is a finite abelian 3-group it is trivial. ■

Collecting the above results together we can now prove Theorem 1.3.

Proof of 1.3. Let $W \hookrightarrow \mathrm{GL}(L)$ be a finite irreducible p-adic pseudoreflection group, p odd. By Theorem 3.4 we have $H^3(W, L) = 0$ except for the following cases: $W \cong \Sigma_n$ is a symmetric group, $n \geq 3$ or $W \cong W_{E_6}$, $p = 3$. The case of $W \cong W_{E_6}$ is handled by Theorem 3.8. In the case $W \cong \Sigma_n$, $n \geq 4$ we are done by Theorem 3.6. Finally the case $W \cong \Sigma_3$, $p = 3$ is handled by Theorem 3.7. ■

4. Applications to p-compact groups. To prove our main theorem we need the following result which establishes the existence of product splittings for p-compact groups, analogously to a well-known theorem for compact Lie groups. Let X be a connected p-compact group with Weyl group W_X and associated lattice L_X. We say that X is *simple* if $W_X \to \mathrm{GL}(L_X)$ is an irreducible pseudoreflection representation.

Theorem 4.1 (see [15, 27]). *Let p be an odd prime and X a connected p-compact group. If X is simply connected then there exists a splitting $X \cong X_1 \times \ldots \times X_r$ of X into simple simply connected p-compact groups. We also have corresponding splittings $N \cong N_1 \times \ldots \times N_r$, $W \cong W_1 \times \ldots \times W_r$ and $L \cong L_1 \times \ldots \times L_r$ of the maximal torus normalizer N, the Weyl group W and the associated lattice L for X, such that N_i, W_i and L_i are respectively the maximal torus normalizer, Weyl group and associated lattice for X_i.* ■

Proof of 1.2. Let $Z \to X$ be a central monomorphism. By [25, 4.6] and [24, 3.8] we see that the p-compact group X/Z has maximal torus $T_{X/Z} = T/Z$, Weyl group $W_{X/Z} = W$ and maximal torus normalizer $N_{X/Z} = N/Z$. This gives a commutative diagram

$$\begin{array}{ccccc} T & \longrightarrow & N & \longrightarrow & W \\ \downarrow & & \downarrow & & \| \\ T/Z & \longrightarrow & N/Z & \longrightarrow & W \end{array}$$

Thus the obstruction class $\gamma_{X/Z}$ is the image of the obstruction class γ_X, so by [25, 5.4] it suffices to prove the theorem in the case where X is simply connected.

Assume now that this is the case. By Theorem 4.1 we may also assume that X is simple, i.e. that W is an irreducible pseudoreflection group. In that case Theorem 1.3 shows that $H^3(W,L)=0$ in all cases, except if $p=3$ and X has the same Weyl group data as $\widehat{PU(3)}$. By [5] the 3-compact group $\widehat{PU(3)}$ is determined by its Weyl group data, so $BX \cong B\widehat{PU(3)}$. Since we are assuming that X is simply connected this is a contradiction. Thus for all simple, simply connected X we have $H^3(W_X, L_X)=0$ and in particular $\gamma_X = 0$.

The uniqueness of the section follows from the fact that the set of vertical homotopy classes of sections is in bijective correspondence with the group $H^2(BW_X, \mathcal{C})$ ([38, VI.6.13]), where $\mathcal{C}$ is the local coefficient system from the introduction, coming from the action of W_X on L_X. As $H^2(BW_X,\mathcal{C}) \cong H^2(W_X, L_X) = 0$ by Theorem 3.3, we are done. ∎

Our results also have the following application to the computation of self-maps of p-compact groups. We let $\mathrm{Out}(X)$ denote the group of invertible elements in $[BX, BX]$.

THEOREM 4.2. *Let p be odd and X be a connected p-compact group with maximal torus normalizer N and Weyl group W with associated lattice L. Then there is a natural isomorphism* $\mathrm{Out}(N) \cong N_{\mathrm{GL}(L)}(W)/W$. *In particular if X is totally N-determined in the sense of Møller* [23] *then* $\mathrm{Out}(X) \cong N_{\mathrm{GL}(L)}(W)/W$.

Proof. Denote the discrete approximations of T and N (see [13, 14]) by respectively $\check{T}$ and $\check{N}$. By Theorem 1.2 we see that $\check{N} = \check{T} \rtimes W$ is a semidirect product.

Let $\mathrm{Aut}(W,\check{T})$ denote the subgroup of $\mathrm{Aut}(W) \times \mathrm{Aut}(\check{T})$ consisting of the pairs (χ,φ) such that φ is χ-equivariant, i.e. $\varphi(w\cdot t) = \chi(w)\cdot\varphi(t)$ for all $w \in W$, $t \in \check{T}$. We have a natural homomorphism $\alpha : \mathrm{Aut}(\check{N}) \to \mathrm{Aut}(W,\check{T})$ since $\check{T}$ is a characteristic subgroup of $\check{N}$. Since $\check{N} = \check{T} \rtimes W$ it follows that α is an epimorphism.

Since α is W-equivariant, we obtain an epimorphism $\mathrm{Aut}(\check{N})/W \to \mathrm{Aut}(W,\check{T})/W$. The kernel equals $H^1(W,\check{T}) \cong H^2(W,L)$, which vanishes by Theorem 3.3. Thus we have an isomorphism $\mathrm{Aut}(\check{N})/W \cong \mathrm{Aut}(W,\check{T})/W$.

The left-hand side is equal to $\mathrm{Out}(N)$ and it is easy to see that $\mathrm{Aut}(W,\check{T}) = N_{\mathrm{GL}(L)}(W)$.

Finally, if X is totally N-determined then $\mathrm{Out}(X) \cong \mathrm{Out}(N)$ by [23]. ∎

References

[1] J. Aguadé, *Constructing modular classifying spaces*, Israel J. Math. 66 (1989), 23–40.

[2] D. J. Benson, *Projective modules for the group of twenty-seven lines on a cubic surface*, Comm. Algebra 17 (1989), 1017–1068.

[3] —, *Polynomial Invariants of Finite Groups*, London Math. Soc. Lecture Note Ser. 190, Cambridge Univ. Press, Cambridge, 1993.

[4] A. K. Bousfield and D. M. Kan, *Homotopy Limits, Completions and Localizations*, Lecture Notes in Math. 304, Springer, Berlin, 1987.

[5] C. Broto and A. Viruel, *Homotopy uniqueness of BPU*(3), in: Group Representations: Cohomology, Group Actions and Topology (Seattle, WA, 1996), Proc. Sympos. Pure Math. 63, Amer. Math. Soc., Providence, RI, 1998, 85–93.

[6] K. S. Brown, *Cohomology of Groups*, Grad. Texts in Math. 87, Springer, New York, 1994.

[7] J. Cannon and C. Playoust, *An introduction to Magma*, technical report, School of Math. and Statist., Univ. of Sydney, 1995.

[8] A. Clark and J. R. Ewing, *The realization of polynomial algebras as cohomology rings*, Pacific J. Math. 50 (1974), 425–434.

[9] A. M. Cohen, *Finite complex reflection groups*, Ann. Sci. École Norm. Sup. (4) 9 (1976), 379–436.

[10] H. M. S. Coxeter, *Discrete groups generated by reflections*, Ann. of Math. 35 (1934), 588–621.

[11] C. W. Curtis and I. Reiner, *Methods of Representation Theory*, Vol. I, Wiley, New York, 1990.

[12] M. Curtis, A. Wiederhold and B. Williams, *Normalizers of maximal tori*, in: P. Hilton (ed.), Localization in Group Theory and Homotopy Theory and Related Topics (Seattle, WA, 1974), Lecture Notes in Math. 418, Springer, Berlin, 1974, 31–47.

[13] W. G. Dwyer and C. W. Wilkerson, *Homotopy fixed-point methods for Lie groups and finite loop spaces*, Ann. of Math. (2) 139 (1994), 395–442.

[14] —, —, *The center of a p-compact group*, in: M. Cenkl and H. Miller (eds.), The Čech Centennial (Boston, MA, 1993), Contemp. Math. 181, Amer. Math. Soc., Providence, RI, 1995, 119–157.

[15] —, —, *Product splittings for p-compact groups*, Fund. Math. 147 (1995), 279–300.

[16] L. Evens, *The Cohomology of Groups*, Oxford Univ. Press, New York, 1991.

[17] R. B. Howlett, *On the Schur multipliers of Coxeter groups*, J. London Math. Soc. (2) 38 (1988), 263–276.

[18] J. E. Humphreys, *Reflection Groups and Coxeter Groups*, Cambridge Stud. Adv. Math. 29, Cambridge Univ. Press, Cambridge, 1992.

[19] S. Ihara and T. Yokonuma, *On the second cohomology groups* (*Schur multipliers*) *of finite reflection groups*, J. Fac. Sci. Univ. Tokyo Sect. I 11 (1965), 155–171.

[20] S. Jackowski, J. McClure and R. Oliver, *Homotopy classification of self-maps of BG via G-actions. I*, Ann. of Math. (2) 135 (1992), 183–226.

[21] J. Lannes, *Théorie homotopique des groupes de Lie* (*d'après W. G. Dwyer et C. W. Wilkerson*), Astérisque 227 (1995), 21–45.

[22] J. M. Møller, *Homotopy Lie groups*, Bull. Amer. Math. Soc. (N.S.) 32 (1995), 413–428.

[23] —, *Deterministic p-compact groups*, in: W. Dwyer *et al.* (eds.), Stable and Unstable Homotopy (Toronto, 1996), Fields Inst. Comm. 19, Amer. Math. Soc., Providence, RI, 1998, 255-278.

[24] —, *Normalizers of maximal tori*, Math. Z. 231 (1999), 51–74.

[25] J. M. Møller and D. Notbohm, *Centers and finite coverings of finite loop spaces*, J. Reine Angew. Math. 456 (1994), 99–133.

[26] M. Nakaoka, *Decomposition theorems for homology groups of symmetric groups*, Ann. Math. (2) 71 (1960), 16–42.

[27] D. Notbohm, *Unstable splittings of classifying spaces of p-compact groups*, preprint, 1994.

[28] —, *Classifying spaces of compact Lie groups*, in: I. M. James (ed.), Handbook of Algebraic Topology, North-Holland, Amsterdam, 1995, 1049–1094.

[29] —, *On the "classifying space" functor for compact Lie groups*, J. London Math. Soc. (2) 52 (1995), 185–198.

[30] —, *p-adic lattices of pseudoreflection groups*, in: C. Broto *et al.* (eds.), Algebraic Topology: New Trends in Localization and Periodicity (Sant Feliu de Guíxols, 1994), Progr. Math. 136, Birkhäuser, Basel, 1996, 337–352.

[31] E. W. Read, *On the Schur multipliers of the finite imprimitive unitary reflection groups* $G(m,p,n)$, J. London Math. Soc. (2) 13 (1976), 150–154.

[32] H. Scheerer, *Homotopieäquivalente kompakte Liesche Gruppen*, Topology 7 (1968), 227–232.

[33] G. C. Shephard and J. A. Todd, *Finite unitary reflection groups*, Canad. J. Math. 6 (1954), 274–304.

[34] L. Smith, *Polynomial Invariants of Finite Groups*, Res. Notes in Math. 6, A K Peters, Wellesley, MA, 1995.

[35] J. Tits, *Normalisateurs de tores I. Groupes de Coxeter étendus*, J. Algebra 4 (1966), 96–116.

[36] R. van der Hout, *The Schur multipliers of the finite primitive complex reflection groups*, Nederl. Akad. Wetensch. Proc. Ser. A 80 = Indag. Math. 39 (1977), 101–113.

[37] C. A. Weibel, *An Introduction to Homological Algebra*, Cambridge Stud. Adv. Math. 38, Cambridge Univ. Press, Cambridge, 1994.

[38] G. W. Whitehead, *Elements of Homotopy Theory*, Grad. Texts in Math. 61, Springer, New York, 1978.

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