FUNDAMENTA
MATHEMATICAE
160 (1999)

Bimorphisms in pro-homotopy and proper homotopy

by

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Abstract. A morphism of a category which is simultaneously an epimorphism and a monomorphism is called a bimorphism. The category is balanced if every bimorphism is an isomorphism. In the paper properties of bimorphisms of several categories are discussed (pro-homotopy, shape, proper homotopy) and the question of those categories being balanced is raised. Our most interesting result is that a bimorphism $f : X \to Y$ of $\mathrm{tow}(H_0)$ is an isomorphism if Y is movable. Recall that $\mathrm{tow}(H_0)$ is the full subcategory of pro-H_0 consisting of inverse sequences in H_0, the homotopy category of pointed connected CW complexes.

0. Introduction. First, let us recall the notions of epimorphism and monomorphism in abstract categories:

0.1. Definition. A morphism $f : X \to Y$ of a category $\mathcal{C}$ is called an *epimorphism* if the induced function $f^* : \mathrm{Mor}(Y,Z) \to \mathrm{Mor}(X,Z)$ is one-to-one for each object Z of $\mathcal{C}$.

A morphism $f : X \to Y$ of a category $\mathcal{C}$ is called a *monomorphism* if the induced function $f_* : \mathrm{Mor}(Z,X) \to \mathrm{Mor}(Z,Y)$ is one-to-one for each object Z of $\mathcal{C}$.

Equivalent, and typically used, is the definition of f being an epimorphism (respectively, monomorphism) of $\mathcal{C}$ as a morphism such that $g \circ f = h \circ f$ (respectively, $f \circ g = f \circ h$) implies $g = h$ for any two morphisms $g, h : Y \to Z$ (respectively, $g, h : Z \to X$).

1991 *Mathematics Subject Classification*: Primary 55P10; Secondary 54C56, 55P65.

Key words and phrases: epimorphism, monomorphism, pro-homotopy, shape, proper homotopy.

The first-named author supported in part by grant DMS-9704372 from NSF and by the Ministry of Science and Education of Spain.

The second-named author supported in part by DGICYT.

0.2. Definition. A morphism $f : X \to Y$ of a category $\mathcal{C}$ is called a *bimorphism* if it is both an epimorphism and a monomorphism of $\mathcal{C}$.

A category $\mathcal{C}$ is called *balanced* if every bimorphism of $\mathcal{C}$ is an isomorphism.

The following fact is well known and useful:

0.3. Proposition. *A monomorphism* (*respectively*, *epimorphism*) *which has a left* (*respectively*, *right*) *inverse is an isomorphism.*

There is considerable literature devoted to the properties of epimorphisms in the homotopy category H_0 of pointed connected CW complexes. We recommend [G] for a nearly complete list and a thorough review of results (see also [C-G]). The question of whether H_0 is balanced has been open for a while with Dyer and Roitberg [Dy-R] resolving it in affirmative and Dydak [D$_2$] giving a simple proof of it. Essentially, both proofs consist in showing that a bimorphism of H_0 satisfies the hypothesis of the Whitehead Theorem (in the case of [Dy-R] the authors prove a version of the Whitehead Theorem involving cohomology with local coefficients). Subsequently, Mukherjee [Mu] generalized [Dy-R] to the equivariant case and Morón–Ruiz del Portal [Mo-P] showed that the shape category of pointed, movable, metric continua is not balanced but every weak isomorphism is a bimorphism.

We believe that the following general question is of interest:

0.4. Problem. *Suppose a category* $\mathcal{C}$ *is balanced. Is the pro-category* pro-$\mathcal{C}$ *balanced?*

[M-S] (see Theorem 1 on p. 107 and Theorem 3 on p. 109) contains an affirmative answer in the case of the category of groups. The purpose of the present paper is to investigate the case of $\mathcal{C} = H_0$. We are also interested in the problem of other categories being balanced, the categories associated with pro-H_0. Examples are: the shape category of pointed metric continua and the proper homotopy category of rayed, strongly locally finite, connected CW complexes connected at infinity (see [Ed-H] for an explanation on how this category is related to pro-H_0). Here are the specific problems we would like to solve:

0.5. Problem. *Suppose $f : X \to Y$ is a bimorphism of* $\mathrm{tow}(H_0)$ (*respectively*, pro-H_0). *Is f is an isomorphism?*

0.6. Problem. *Suppose $f : X \to Y$ is a bimorphism of the* (*strong*) *shape category of pointed metric continua. Is f a weak isomorphism? Is f an isomorphism?*

0.7. Problem. *Suppose $f : X \to Y$ is a bimorphism of the shape category of pointed metric movable continua. Is f is a weak isomorphism?*

0.8. PROBLEM. *Suppose $f : X \to Y$ is a bimorphism of the proper homotopy category of rayed*, *strongly locally finite*, *connected CW complexes K connected at infinity*, *equipped with a ray $r_K : [0,\infty) \to K$. Is f an isomorphism?*

In the case of 0.5 and 0.8 we are able to show that every bimorphism is a weak isomorphism (i.e., induces an isomorphism of homotopy pro-groups). However, it is known that one has difficulty finding an analog of the Whitehead Theorem in those categories and only partial cases are known. Thus we are not able to solve the problem of those categories being balanced completely but we provide additional assumptions to show that bimorphisms are isomorphisms. Problem 0.8 is related to the problem of the strong shape category of metric compacta being balanced as one can see from [Ed-H] and [D-S_2].

The main results of this paper are the following:

2.13. COROLLARY. *If $f : X \to Y$ is a bimorphism of* $\mathrm{tow}(H_0)$, *then it is a weak isomorphism.*

2.14. THEOREM. *Suppose $f : X \to Y$ is a bimorphism of* $\mathrm{tow}(H_0)$. *Then f is an isomorphism if one of the following conditions is satisfied*:

(i) def-dim(Y) *is finite*,
(ii) Y *is movable.*

3.1. THEOREM. *Suppose $f : X \to Y$ is a bimorphism of the shape category of pointed metric continua. If X is movable and* pro-$\pi_k(Y)$ *is stable for each $k \geq 1$, then f is a weak isomorphism.*

4.11. COROLLARY. *If $f : (K, r_K) \to (L, r_L)$ is a bimorphism of P_r, then f induces isomorphisms of proper homotopy groups.*

4.12. COROLLARY. *If $f : (K, r_K) \to (L, r_L)$ is a bimorphism of P_r so that* $\dim(L)$ *is finite*, *then f is an isomorphism.*

The authors would like to express their gratitude to Carles Casacuberta for helpful discussions of the subject, and to Rafael Ayala and Antonio Quintero for sharing their knowledge of proper homotopy theory.

1. Epimorphisms and monomorphisms in pro-categories. For a definition and basic properties of the pro-category pro-$\mathcal{C}$ of any category $\mathcal{C}$ we refer to the book by Mardešić and Segal [M-S].

1.1. DEFINITION. For any category $\mathcal{C}$, the full subcategory of pro-$\mathcal{C}$ whose objects are inverse sequences indexed by natural numbers, is denoted by $\mathrm{tow}(\mathcal{C})$.

Given an object Y of $\mathrm{tow}(\mathcal{C})$, the nth term of Y is denoted by Y_n, the bonding homomorphism from Y_m to Y_n is denoted by $p(Y)^m_n$, and $p(Y)_n : Y \to Y_n$ is the projection morphism.

For readers who are not familiar with pro-categories the following fact should be helpful (the authors had trouble locating a reference for it in the existing literature):

1.2. PROPOSITION. *If $\mathcal{D}$ is a category possessing inverse limits of arbitrary inverse systems* (*respectively*, *inverse sequences*), *then any functor $F : \mathcal{C} \to \mathcal{D}$ extends over* pro-$\mathcal{C}$ (*respectively*, $\mathrm{tow}(\mathcal{C})$).

Proof. We only cover the case of $\mathrm{tow}(\mathcal{C})$; the general case is similar. We define a functor $\overline{F} : \mathrm{tow}(\mathcal{C}) \to \mathcal{D}$ as follows:

(i) $\overline{F}(X) = \mathrm{inv\,lim}\, F(X)$ if X is an object of $\mathrm{tow}(\mathcal{C})$,

(ii) if $f : X \to Y$ is a morphism of $\mathrm{tow}(\mathcal{C})$ and $p : \mathrm{inv\,lim}\, F(X) \to F(X)$ is the projection (and a morphism of pro-$\mathcal{D}$), then the sequence of morphisms $\{F(p(Y)_n) \circ F(f) \circ p : \mathrm{inv\,lim}\, F(X) \to F(Y_n)\}_{n\ge1}$ induces a morphism $\mathrm{inv\,lim}\, F(X) \to \mathrm{inv\,lim}\, F(Y)$ of $\mathcal{D}$ which we denote by $\overline{F}(f)$.

One can easily check that $\overline{F}$ is a functor. ∎

If one views categories with limits as objects of a "category" $\mathcal{CAT\,LIM}$ with "morphisms" being functors preserving limits (we cannot call them morphisms in the classical sense as they do not form a set, in general), then one has the inclusion $i : \mathcal{CAT\,LIM} \to \mathcal{CAT}$ into the "category" $\mathcal{CAT}$ whose objects are arbitrary categories and whose "morphisms" are functors. The meaning of Proposition 1.2 is that $\mathrm{pro} : \mathcal{CAT} \to \mathcal{CAT\,LIM}$ is a right adjoint to i, i.e. $\mathrm{Mor}_{\mathcal{CAT}}(\mathcal{C}, i(\mathcal{D})) \approx \mathrm{Mor}_{\mathcal{CAT\,LIM}}(\text{pro-}\mathcal{C}, \mathcal{D})$.

Traditionally, the concept of being Mittag-Leffler is defined for towers of groups. We generalize it to arbitrary categories:

1.3. DEFINITION. An object of pro-$\mathcal{C}$ is called *Mittag-Leffler* if it is isomorphic to an object Y of $\mathrm{tow}(\mathcal{C})$ such that $p(Y)^{n+1}_n$ is an epimorphism for each n. An object of pro-$\mathcal{C}$ is called *stable* if it is isomorphic to an object Y of $\mathrm{tow}(\mathcal{C})$ such that $p(Y)^{n+1}_n$ is an isomorphism for each n.

If X is stable, then it is isomorphic to an object Y of $\mathrm{tow}(\mathcal{C})$ such that $Y_n = Y_m$ for all n, m and $p(Y)^{n+1}_n$ is the identity morphism for each n.

1.4. PROPOSITION. *A morphism $f : X \to Y$ of* $\mathrm{tow}(\mathcal{C})$ *is an epimorphism of* $\mathrm{tow}(\mathcal{C})$ *iff the induced function $f^* : \mathrm{Mor}(Y, Z) \to \mathrm{Mor}(X, Z)$ is one-to-one for each stable object Z of* $\mathrm{tow}(\mathcal{C})$.

Proof. Suppose the induced function $f^* : \mathrm{Mor}(Y, Z) \to \mathrm{Mor}(X, Z)$ is one-to-one for each stable object Z of $\mathrm{tow}(\mathcal{C})$. Let $g, h : Y \to T$ satisfy

$h\circ f=g\circ f$. Let $p(T)_n:T\to T_n$ be the projection morphism. Since $p(T)_n\circ h\circ f=p(T)_n\circ g\circ f$, we infer $p(T)_n\circ h=p(T)_n\circ g$ for each n, i.e. $h=g$. ∎

In practice, one often uses a special kind of morphisms in pro-categories:

1.5. Definition. A morphism $f:X\to Y$ of $\mathrm{tow}(\mathcal{C})$ is called a *level morphism* if there exist morphisms $f_n:X_n\to Y_n$ such that $p(Y)_n^{n+1}\circ f_{n+1}=f_n\circ p(X)_n^{n+1}$ and $p(Y)_n\circ f=f_n\circ p(X)_n$ for each n.

The morphisms f_n are not uniquely determined by f. We will say that f is *induced* by f_n, $n\ge 1$.

[M-S] (see Theorem 1 on p. 107 and Theorem 3 on p. 109) contains the following characterization of epimorphisms and monomorphisms in the category pro-Gr of pro-groups:

1.6. Proposition. *Suppose $f:X\to Y$ is a level morphism of* tow(Gr) *induced by $f_n:X_n\to Y_n$.*

(a) *f is an epimorphism of* tow(Gr) *iff for each n there is $m>n$ such that* $\mathrm{im}(p(Y)_n^m)\subset\mathrm{im}(f_n)$.

(b) *f is a monomorphism of* tow(Gr) *iff for each n there is $m>n$ such that* $\ker(f_m)\subset\ker(p(X)_n^m)$.

1.7. Lemma. *Suppose $f:X\to Y$ is a morphism of* tow(Gr) *such that $f_*:\mathrm{Mor}(\mathbb{Z},X)\to\mathrm{Mor}(\mathbb{Z},Y)$ is one-to-one*, *where $\mathbb{Z}$ is the group of integers. If X is Mittag-Leffler and Y is stable*, *then f is a monomorphism.*

Proof. Assume Y is a group. It suffices to consider the case of f being a level morphism induced by $f_n:X_n\to Y$ (see [M-S], Theorem 3 on p. 12). Since X is Mittag-Leffler, we may assume that $p(X)_n^m$ is an epimorphism for all $m>n$. Let K_n be the kernel of f_n, $n\ge 1$. We now show that $(p(X)_n^{n+1})^{-1}(K_n)=K_{n+1}$ for each n. Clearly, $p(X)_n^{n+1}(K_{n+1})\subset K_n$. Suppose $p(X)_n^{n+1}(x)\in K_n$. Thus, $f_n(p(X)_n^{n+1}(x))=1$. Since $f_n\circ p(X)_n^{n+1}=f_{n+1}$, one gets $x\in K_{n+1}$.

To prove that f is a monomorphism it suffices to show that $K_n=\{1\}$ for each n. Suppose $x_m\in K_m-\{1\}$ for some m. Define x_r for $r<m$ by $x_r=p(X)_r^m(x_m)$ and define, by induction on $r>m$, $x_r\in K_r$ so that $p(X)_r^s(x_s)=x_r$ if $s>r$. The sequence of homomorphisms $a_n:\mathbb{Z}\to X_n$ defined by $a_n(0)=x_n$ induces a level homomorphism $a:\mathbb{Z}\to X$ so that $f\circ a$ is trivial but a is not trivial, a contradiction. ∎

As one can see in Section 3 later on, it is not easy to generalize results from the case of arbitrary CW complexes to the case of finite complexes. The following problem is the algebraic version of that difficulty:

1.8. Problem. *Let* fgGr *be the category of finitely generated groups. Suppose* $f : X \to Y$ *is a bimorphism of* tow(fgGr). *Is* f *an isomorphism of* tow(fgGr)*?*

2. Epimorphisms and monomorphisms in the pro-homotopy category

2.1. Definition. By CW_0 we denote the topological category of pointed connected CW complexes. By H_0 we denote the homotopy category of pointed connected CW complexes. The homotopy functor $\mathrm{CW}_0 \to H_0$ can be extended to a functor $[\ ] : \mathrm{tow}(\mathrm{CW}_0) \to \mathrm{tow}(H_0)$ so that $[X]_n = X_n$ and $p([X])_n^{n+1} = [p(X)_n^{n+1}]$ for each n.

Given a functor $F : C \to D$ one can extend it, by Proposition 1.2, to a functor from pro-$\mathcal{C}$ to pro-$\mathcal{D}$. Typically this extension is denoted by pro-F. However, it is easier to use the same symbol F for the extension whenever it is not ambiguous. For example, in the case of the shape category we will continue to use the notation pro-$\pi_k(X)$ and pro-$H_k(X)$ for homotopy and homology pro-groups as those pro-groups are different from the groups $\pi_k(X)$ and $H_k(X)$ if X is a pointed topological space.

2.2. Proposition. *If* $f : X \to Y$ *is an epimorphism of* $\mathrm{tow}(H_0)$, *then* $\pi_1(f) : \pi_1(X) \to \pi_1(Y)$ *is an epimorphism of* tow(Gr).

Proof. Suppose $\alpha, \beta : \pi_1(Y) \to G$ are two morphisms of tow(Gr) such that $\alpha \circ (\pi_1(f)) = \beta \circ (\pi_1(f))$ and G is a group. One can find $a, b : Y \to K(G, 1)$ so that $\pi_1(a) = \alpha$ and $\pi_1(b) = \beta$. Now, $a \circ f = b \circ f$, which implies $a = b$ as f is an epimorphism. By 1.4 we deduce that $\pi_1(f)$ is an epimorphism. ∎

2.3. Definition. A morphism $i : X \to Y$ of $\mathrm{tow}(\mathrm{CW}_0)$ is called an *inclusion* if, for each n, X_n is a subcomplex of Y_n and the following conditions are satisfied ($i_n : X_n \to Y_n$ is the inclusion):

(i) $i_n \circ p(X)_n^{n+1} = p(Y)_n^{n+1} \circ i_{n+1}$ for each n,
(ii) i coincides with the level morphism induced by $\{i_n\}_{n \ge 1}$.

A morphism $f : X \to Y$ of $\mathrm{tow}(H_0)$ is called *inclusion induced* if there is an inclusion $i : \overline{X} \to \overline{Y}$ of $\mathrm{tow}(\mathrm{CW}_0)$ such that $X = [\overline{X}]$, $Y = [\overline{Y}]$, and f coincides with the level morphism induced by $[i_n]$, $n \ge 1$.

2.4. Lemma. *For any morphism* $f : X \to Y$ *of* $\mathrm{tow}(H_0)$ *there exist isomorphisms* $a : \overline{X} \to X$ *and* $b : Y \to \overline{Y}$ *so that* $i = b \circ f \circ a : \overline{X} \to \overline{Y}$ *is inclusion induced. Moreover*, *if* $\pi_1(f)$ *is a monomorphism of* tow(Gr), *then we can ensure that* $\pi_1(i_n)$ *is a monomorphism for each* n, *where* $i_n : \overline{X}_n \to \overline{Y}_n$ *is the inclusion map.*

P r o o f. Without loss of generality we may assume the following (see 1.6 and [M-S], Theorem 3 on p. 12):

(i) f is a level morphism induced by maps $f_n : X_n \to Y_n$,
(ii) if $\pi_1(f)$ is a monomorphism of pro-Gr, then for each n,

$$\ker(\pi_1(f_{n+1})) \subset \ker(\pi_1(p(X)_n^{n+1})).$$

If $\pi_1(f)$ is a monomorphism of pro-Gr, then create Z_n from X_n by attaching 2-cells so that $\ker(\pi_1(f_n))$ is killed. Obviously, one can extend f_n over Z_n and an extension will be denoted by g_n. Notice that $p(X)_n^{n+1}$ extends over Z_{n+1} (see (ii) above) and denote an extension by $r_n : Z_{n+1} \to X_n$. Let $p(Z)_n^{n+1} = j_n \circ r_n$, $j_n : X_n \to Z_n$ being the inclusion. Notice that the inclusion $j : X \to Z$ is an isomorphism of tow(CW_0) (its inverse is induced by r_n, $n \geq 1$). Now, replace $g : Z \to Y$ by an inclusion as follows: let $\overline{Y}_n$ be the reduced mapping cylinder of $g_n : Z_n \to Y_n$. Use Lemma 3 on p. 145 in [M-S] to produce $p(\overline{Y})_n^{n+1}$ for each n. ∎

As seen in [D$_2$] double mapping cylinders are vital in understanding the epimorphisms of H_0. The purpose of the next definition is to extend the concept of double mapping cylinder to inclusions of tow(CW_0):

2.5. Definition. Suppose (K, k_0) is a subcomplex of a pointed CW complex (L, k_0) and $i : (K, k_0) \to (L, k_0)$ is the inclusion map. By DM(i) (the double mapping cylinder of i) we denote the pointed CW complex $(L \times \{0,1\} \cup K \times I)/\{k_0\} \times I$. For simplicity, when discussing DM(i), the space $K \times I/\{k_0\} \times I$ will be denoted by $K \times I$. The two maps $i^0, i^1 : (L, k_0) \to \mathrm{DM}(i)$ are induced by the maps $x \mapsto (x, 0)$ and $x \mapsto (x, 1)$ $(x \in L)$, respectively.

If $i : X \to Y$ is an inclusion of tow(CW_0), then one can easily define DM(i) and $i^0, i^1 : Y \to \mathrm{DM}(i)$.

Below, we write $(X)^\sim$ instead of $\widetilde{X}$ if X is a long expression.

2.6. Lemma. (a) *Suppose that (B, A) is a pair of pointed connected CW complexes such that $\pi_1(i) : \pi_1(A) \to \pi_1(B)$ is a monomorphism of groups. Then* $\mathrm{DM}(\widetilde{i}) \subset (\mathrm{DM}(i))^\sim$.

(b) *Suppose that*, *for $j = 1, 2$, (B_j, A_j) is a pair of pointed connected CW complexes such that $\pi_1(i_j) : \pi_1(A_j) \to \pi_1(B_j)$ is a monomorphism of groups*, *where $i_j : A_j \to B_j$ is the inclusion. If $\beta : B_2 \to B_1$ is a map such that* $\mathrm{im}(\pi_1(\beta)) \subset \mathrm{im}(\pi_1(i_1))$ *and* $\beta(A_2) \subset A_1$, *then the image of* $\widetilde{\gamma} : (\mathrm{DM}(i_2))^\sim \to (\mathrm{DM}(i_1))^\sim$, $\gamma : \mathrm{DM}(i_2) \to \mathrm{DM}(i_1)$ *being induced by* β, *is contained in* $\mathrm{DM}(\widetilde{i}_1)$.

P r o o f. (a) Let $p : (\mathrm{DM}(i))^\sim \to \mathrm{DM}(i)$ be the covering projection. Since $\pi_1(i)$ is one-to-one, $\widetilde{A} \subset \widetilde{B}$, $\mathrm{DM}(\widetilde{i})$ exists and is simply connected. Let $r : \mathrm{DM}(\widetilde{i}) \to (\mathrm{DM}(i))^\sim$ be the lift of the natural map $\mathrm{DM}(\widetilde{i}) \to \mathrm{DM}(i)$.

Notice that r is one-to-one. Indeed, the component of $p^{-1}(B\times 0)$ containing the base point is simply $\widetilde{B}\times 0$, the component of $p^{-1}(B\times 1)$ containing the base point is $\widetilde{B}\times 1$, and the component of $p^{-1}(A\times I)$ containing the base point is $\widetilde{A}\times I$.

(b) As is well known, an element of $(\mathrm{DM}(i_2))^{\sim}$ is the homotopy class rel. 0 of a path $\omega:(I,0)\to\mathrm{DM}(i_2)$.

Case 1: $\omega(1)\in A_2\times I$. Choose λ in $A_2\times I$ joining $\omega(1)$ and the base point of $\mathrm{DM}(i_2)$. Now, $\omega*\lambda$ is a loop and can be expressed, up to homotopy, as the product of loops in $\mathrm{DM}(i_2)$ which are completely contained in one of the following sets: $A_2\times I$, $B_2\times 0$, $B_2\times 1$. When applying γ to $\omega*\lambda$, the loops in $B_2\times 0$ and $B_2\times 1$ can be homotoped into $A_1\times I$. Thus, $\gamma(\omega)$ has the homotopy class rel. 0 of $\gamma(\omega*\lambda)*\gamma(\lambda^{-1})$, which belongs to $\widetilde{A}_1\times I$.

Case 2: $\omega(1)\in B_2\times 0$. Choose λ in $B_2\times 0$ joining $\omega(1)$ and the base point of $\mathrm{DM}(i_2)$. Now, $\omega*\lambda$ is a loop and can be expressed, up to homotopy, as the product of loops in $\mathrm{DM}(i_2)$ which are completely contained in one of the following sets: $A_2\times I$, $B_2\times 0$, $B_2\times 1$. When applying γ to $\omega*\lambda$, the loops in $B_2\times 1$ can be homotoped into $A_1\times I$ and then into $B_2\times 0$. Thus, $\gamma(\omega)$ has the homotopy class rel. 0 of $\gamma(\omega*\lambda)*\gamma(\lambda^{-1})$, which belongs to $\widetilde{B}_1\times 0$.

Case 3: $\omega(1)\in B_2\times 1$. This case is completely analogous to Case 2. ■

The following lemma characterizes inclusion-induced epimorphisms of $\mathrm{tow}(H_0)$ in terms of the double mapping cylinder:

2.7. Lemma. *Suppose $i:X\to Y$ is an inclusion of* $\mathrm{tow}(\mathrm{CW}_0)$. *Then $[i]:[X]\to[Y]$ is an epimorphism of* $\mathrm{tow}(H_0)$ *iff $[i^0]=[i^1]$, where $i^0,i^1:Y\to\mathrm{DM}(i)$.*

Proof. Since $[i^0\circ i]=[i^1\circ i]$, $[i^0]=[i^1]$ if $[i]$ is an epimorphism. Suppose $[i^0]=[i^1]$ and suppose $f,g:Y\to Z$ are two morphisms of $\mathrm{tow}(H_0)$ such that $f\circ[i]=g\circ[i]$ and Z is a pointed connected CW complex. Choose maps $a,b:Y_n\to Z$ for some n so that $f=[a]\circ[p(Y)_n]$ and $g=[b]\circ[p(Y)_n]$. Since $f\circ[i]=g\circ[i]$, there is $m>n$ such that $a\circ i_n\circ p(X)^m_n$ is homotopic to $b\circ i_n\circ p(X)^m_n$. Using the homotopy one can construct a map $H:\mathrm{DM}(i_m)\to Z$ such that $H\circ i^0_m=a\circ p(Y)^m_n$ and $H\circ i^1_m=b\circ p(Y)^m_n$. Since $[i^0]=[i^1]$, there is $p>m$ such that $i^0_m\circ p(Y)^p_m\approx i^1_m\circ p(Y)^p_m$. Thus $a\circ p(Y)^p_n\approx b\circ p(Y)^p_n$ and $f=g$. Proposition 1.4 says that $[i]$ is an epimorphism of $\mathrm{tow}(H_0)$. ■

2.8. Proposition. *Suppose $f:X\to Y$ is an epimorphism of* $\mathrm{tow}(H_0)$. *If $\pi_1(f)$ is an isomorphism, then $\widetilde{f}:\widetilde{X}\to\widetilde{Y}$ is an epimorphism of* $\mathrm{tow}(H_0)$.

Proof. Without loss of generality (explanation follows) assume that f is induced by an inclusion i of $\mathrm{tow}(\mathrm{CW}_0)$ so that the following conditions are satisfied:

(i) $\pi_1(i_n)$ is a monomorphism for each n,
(ii) $\mathrm{im}(\pi_1(p(Y)_n^{n+1})) \subset \mathrm{im}(\pi_1(i_n))$ for each n,
(iii) $i_n^0 \circ p(Y)_n^{n+1}, i_n^1 \circ p(Y)_n^{n+1} : Y_{n+1} \to \mathrm{DM}(i_n)$ are homotopic for each n.

Indeed, (i) can be ensured by 2.4. (ii) can be ensured with the help of 1.6 by choosing a cofinal subset of integers. Let us provide more details for (iii) so as to see how choosing a cofinal subset works. By 2.7, $[i^0] = [i^1]$, where $i^0, i^1 : Y \to \mathrm{DM}(i)$. Thus, given $n \geq 1$, there is $m > n$ such that $i_n^0 \circ p(Y)_n^m, i_n^1 \circ p(Y)_n^m : Y_m \to \mathrm{DM}(i_n)$ are homotopic. By induction we can choose an increasing sequence $m_1 < m_2 < \ldots$ and rename X_{m_n}, Y_{m_n} as X_n, Y_n so that (iii) is satisfied.

Let $H : Y_{n+2} \to \mathrm{DM}(i_{n+1})$ be a homotopy joining $i_{n+1}^0 \circ p(Y)_{n+1}^{n+2}$ and $i_{n+1}^1 \circ p(Y)_{n+1}^{n+2}$. Let $g : \mathrm{DM}(i_{n+1}) \to \mathrm{DM}(i_n)$ be the natural map induced by $p(Y)_n^{n+1}$. By 2.6, the image of $\widetilde{g}$ is contained in $\mathrm{DM}(\widetilde{i}_n)$. Thus, $\widetilde{g} \circ \widetilde{H} : \widetilde{Y}_{n+2} \to \mathrm{DM}(\widetilde{i}_n)$ is a homotopy joining $i_n^0 \circ p(\widetilde{Y})_n^{n+2}, i_n^1 \circ p(\widetilde{Y})_n^{n+2} : \widetilde{Y}_{n+2} \to \mathrm{DM}(\widetilde{i}_n)$. By 2.7, $\widetilde{f}$ is an epimorphism of $\mathrm{tow}(H_0)$. ■

2.9. Definition. A morphism $f : X \to Y$ of $\mathrm{tow}(H_0)$ is called a *weak isomorphism* if $\pi_k(f)$ is an isomorphism of $\mathrm{tow}(\mathrm{Gr})$ for each $k \geq 1$.

2.10. Theorem. *Suppose $f : X \to Y$ is an epimorphism of* $\mathrm{tow}(H_0)$. *If $\pi_k(f)$ is a monomorphism of* $\mathrm{tow}(\mathrm{Gr})$ *for each $k \geq 1$, then f is a weak isomorphism of* $\mathrm{tow}(H_0)$.

Proof. By 2.2, $\pi_1(f)$ is an epimorphism of $\mathrm{tow}(\mathrm{Gr})$. Thus, $\pi_1(f)$ is an isomorphism of $\mathrm{tow}(\mathrm{Gr})$. Without loss of generality (see 2.4), we may assume that f is induced by an inclusion i of $\mathrm{tow}(\mathrm{CW}_0)$ so that $\pi_1(i_n)$ is a monomorphism for each n.

Special Case: $\pi_1(Y_n) = 0$ for each n. In this case, $\pi_1(X_n) = 0$ for each n. Let Y/X be defined by $(Y/X)_n = Y_n/X_n$ and $p(Y/X)_n^m$ be the natural map induced by $p(Y)_n^m$. Since the composition of $X \to Y \to Y/X$ is trivial in $\mathrm{tow}(H_0)$, so is $Y \to Y/X$ as $X \to Y$ is an epimorphism. Notice that $H_k(Y/X)$ is naturally isomorphic to $H_k(Y, X)$ for each $k \geq 1$ (we use the integral homology here). From the homology exact sequence we deduce the exactness of $0 \to H_{k+1}(Y, X) \to H_k(X) \to H_k(Y) \to 0$ for each $k \geq 1$. Suppose $H_k(Y, X) = 0$ for all $k < n$. By the Hurewicz Theorem in pro-H_0 (see [M-S], Theorem 7 on p. 140), the Hurewicz morphism $\phi_k : \pi_k(Y, X) \to H_k(Y, X)$ is an isomorphism for $k = n$ and an epimorphism for $k = n + 1$. Since $\pi_n(Y, X) \to \pi_{n-1}(X)$ is trivial (as $\pi_{n-1}(f)$ is a monomorphism) and $H_n(Y, X) \to H_{n-1}(X)$ is a monomorphism, we get $H_n(Y, X) = 0$. Thus, all homology pro-groups of (Y, X) are trivial, which implies that all homotopy pro-groups of (Y, X) are trivial and f is a weak isomorphism.

General Case. By 2.8 we see that $\widetilde{f}$ is an epimorphism of $\mathrm{tow}(H_0)$. By the Special Case, $\widetilde{f}$ is a weak isomorphism. Since $\pi_k(f) = \pi_k(\widetilde{f})$ for $k > 1$, f is a weak isomorphism. ■

The following characterization of monomorphisms in $\mathrm{tow}(H_0)$ is useful:

2.11. Proposition. *Suppose $f : X \to Y$ is a level morphism of* $\mathrm{tow}(H_0)$ *induced by $f_n : X_n \to Y_n$. Then f is a monomorphism of* $\mathrm{tow}(H_0)$ *iff for each n there is $m > n$ such that given two morphisms $\alpha, \beta : P \to X_m$ of H_0, $f_m \circ \alpha = f_m \circ \beta$ implies $p(X)^m_n \circ \alpha = p(X)^m_n \circ \beta$.*

Proof. Fix $n \ge 1$ and suppose that for each $m > n$ there is a CW complex P_m and two homotopy classes $a_m, b_m : P_m \to X_m$ so that $f_m \circ a_m = f_m \circ b_m$ but $p(X)^m_n \circ a_m \ne p(X)^m_n \circ b_m$. For each $m > n$ let Z_m be the wedge of all P_k, $k \ge m$. If $p > m$, then $p(Z)^p_m : Z_p \to Z_m$ is the inclusion. By defining $\alpha_m, \beta_m : Z_m \to X_m$ via $\alpha_m|P_r = p(X)^r_m \circ a_r$ and $\beta_m|P_r = p(X)^r_m \circ b_r$ one gets $\alpha, \beta : Z \to X$ so that $f \circ \alpha = f \circ \beta$ and $\alpha \ne \beta$, a contradiction. ■

2.12. Corollary. *If $f : X \to Y$ is a monomorphism of* $\mathrm{tow}(H_0)$, *then $\pi_k(f)$ is a monomorphism of* $\mathrm{tow}(\mathrm{Gr})$ *for each $k \ge 1$.*

Proof. Assume f is a level morphism. Apply 2.11 in the case of P being a pointed sphere. Thus, for each n there is $m > n$ such that $\ker(\pi_k(f_m)) \subset \ker(\pi_k(p(X)^m_n)$ for all $k \ge 1$. By 1.6, $\pi_k(f)$ is a monomorphism for each $k \ge 1$. ■

2.13. Corollary. *If $f : X \to Y$ is a bimorphism of* $\mathrm{tow}(H_0)$, *then it is a weak isomorphism.*

Proof. By 2.12, $\pi_k(f)$ is a monomorphism for each $k > 0$. By 2.10, f is a weak isomorphism. ■

2.14. Theorem. *Suppose $f : X \to Y$ is a bimorphism of* $\mathrm{tow}(H_0)$. *Then f is an isomorphism if one of the following conditions is satisfied*:

(i) def-dim(Y) *is finite*,
(ii) Y *is movable.*

Proof. By 2.13, f is a weak isomorphism.

(i) If def-dim(Y) is finite, then Corollary 5.7 of [D_1] says that f has a left inverse. By 0.3, f is an isomorphism.

(ii) We may assume that f is a level morphism induced by f_n, $n \ge 1$. Fix $n \ge 1$. By 2.11 there is $m > n$ so that given two morphisms $\alpha, \beta : P \to X_m$ of H_0, $f_m \circ \alpha = f_m \circ \beta$ implies $p(X)^m_n \circ \alpha = p(X)^m_n \circ \beta$. Since Y is movable, Theorem 5.9 of [D_1] says that f is a weak domination. This means that for each k there is $s > k$ and a morphism $r : Y_s \to X_k$ such that $f_k \circ r = p(Y)^s_k$. Choose $s > m$ and $r : Y_s \to X_m$ so that $f_m \circ r = p(Y)^s_m$. Let $a = r \circ f_s : X_s \to X_m$. Notice that $f_m \circ a = f_m \circ r \circ f_s = p(Y)^s_m \circ f_s = f_m \circ p(X)^s_m$.

Thus, $p(X)_n^m \circ a = p(X)_n^m \circ p(X)_m^s = p(X)_n^s$. Let $b = p(X)_n^m \circ r : Y_s \to X_n$. Notice that $f_n \circ b = p(Y)_n^s$ and $b \circ f_s = p(X)_n^s$. This proves that f is an isomorphism of $\mathrm{tow}(H_0)$. ∎

Our next result improves Theorem 2.14(ii):

2.15. THEOREM. *Suppose* $f : X \to Y$ *is a bimorphism of* $\mathrm{tow}(H_0)$. *If* Z *is movable*, *then the induced function* $f_* : \mathrm{Mor}(Z, X) \to \mathrm{Mor}(Z, Y)$ *is a bijection.*

Proof. It suffices to show that $f_* : \mathrm{Mor}(Z, X) \to \mathrm{Mor}(Z, Y)$ is surjective (it is injective as f is a monomorphism). Suppose $g : Z \to Y$. First, consider the special case of Z so that each $p(Z)_n^{n+1}$ is a domination. This implies the existence of a morphism $r_n : Z_n \to Z$, for each n, so that $p(Z)_n \circ r_n = \mathrm{id}_{Z_n}$.

We may assume that f is a level morphism induced by f_n, $n \geq 1$. By 2.11, for each n there is $m > n$ so that given two morphisms $\alpha, \beta : P \to X_m$ of H_0, $f_m \circ \alpha = f_m \circ \beta$ implies $p(X)_n^m \circ \alpha = p(X)_n^m \circ \beta$. Without loss of generality we may assume $m = n + 1$ for each n. We may also assume that g is a level morphism induced by g_n, $n \geq 1$. By 2.13, f is a weak isomorphism and $[\mathrm{D}_1]$ says that for each n there is a morphism $s_n : Z_n \to X$ so that $f \circ s_n = g \circ r_n$. Let $h_n = p(X)_n \circ s_n : Z_n \to X_n$. Notice that $f_n \circ p(X)^{n+1} \circ h_{n+1} = p(Y)_n^{n+1} \circ f_{n+1} \circ h_{n+1} = f_n \circ h_n \circ p(Z)_n^{n+1}$. Therefore, $p(X)_{n-1}^{n+1} \circ h_{n+1} = p(X)_{n-1}^n \circ h_n \circ p(Z)_n^{n+1}$ and the morphisms $p(X)_n^{n+1} \circ h_{n+1}$, $n \geq 2$, induce a morphism $h : Z \to X$ so that $g = f \circ h$.

The same argument as that of Spież [Sp] shows that any movable object Z of $\mathrm{tow}(H_0)$ is dominated by an object T so that each $p(T)_n^{n+1}$ is a domination. Thus, the general case follows from the special one. ∎

3. Bimorphisms in the shape category. This section is devoted to partial answers to Problems 0.6 and 0.7.

3.1. THEOREM. *Suppose* $f : X \to Y$ *is a bimorphism of the shape category of pointed metric continua. If* X *is movable and* $\text{pro-}\pi_k(Y)$ *is stable for each* $k \geq 1$, *then* f *is a weak isomorphism.*

Proof. Assume $X \subset Y$ and f is induced by the inclusion $j : X \to Y$. Notice that $\mathrm{DM}(j)$ is compact and the two maps $j^0, j^1 : Y \to \mathrm{DM}(j)$ are homotopic when restricted to X. Since j induces an epimorphism in the shape category, we have $\mathrm{Sh}(j^0) = \mathrm{Sh}(j^1)$, where Sh is the shape functor from the topological category to the shape category. Express (Y, X) as the inverse limit of an inverse sequence (B, A) of pairs of pointed, connected, and finite CW complexes. Since $\mathrm{Sh}(j^0) = \mathrm{Sh}(j^1)$, we can switch to the inclusion $r : A \to B$ and deduce that $[r^0] = [r^1]$ (here $r^0, r^1 : B \to \mathrm{DM}(r)$). By 2.7, r is an epimorphism of $\mathrm{tow}(H_0)$. Notice that $r_* : \mathrm{Mor}(\mathbb{Z}, \text{pro-}\pi_k(A)) \to \mathrm{Mor}(\mathbb{Z}, \text{pro-}\pi_k(B))$ corresponds to $\mathrm{Mor}(S^k, X) \to \mathrm{Mor}(S^k, Y)$ in the shape category. The latter function is one-to-one as f is a monomorphism. By 1.7, pro-$\pi_k(r)$ is a monomorphism for each k and by 2.10, r is a weak isomorphism. ∎

3.2. Corollary. *Suppose $f : X \to Y$ is a bimorphism of the shape category of pointed metric continua. If X is movable and Y is an FANR, then f is an isomorphism.*

Proof. Since pro-$\pi_k(Y)$ is stable for each $k \geq 1$, 3.1 says that f is a weak isomorphism. By Theorem 6.5 of [D_1], f is an isomorphism. ∎

3.3. Theorem. *Suppose (Y, X, x_0) is a movable triple of metric continua. If the inclusion $i : (X, x_0) \to (Y, x_0)$ is a bimorphism in the shape category of pointed metric movable continua, then i is a shape isomorphism.*

Proof. By Theorem 7.5 of [D-S_2] (see also [Mo-P]) it suffices to show that i is a weak isomorphism. Notice that DM(i) is movable and the reasoning of 2.7 applies to prove that i is an epimorphism of tow(H_0). It remains to show that pro-$\pi_k(i)$ is a monomorphism for each $k \geq 1$. Without loss of generality we may assume that there are shape morphisms $r_{n+1} : X_{n+1} \to X$ and $s_{n+1} : Y_{n+1} \to Y$ so that $p(X)_n \circ r_{n+1} = p(X)_n^{n+1}$, $p(Y)_n \circ s_{n+1} = p(Y)_n^{n+1}$, $i \circ r_{n+1} = s_{n+1} \circ i_{n+1}$ for each n. If $a \in \ker(\pi_k(i_{n+1}))$, then $a = 0$ in X_n. ∎

4. Bimorphisms in the proper homotopy category. Since there is a strong connection between the proper homotopy category and tow(H_0) (see [Ed-H]), the purpose of this section is to apply the results from tow(H_0) to the proper homotopy case. The connection is realized as follows: given a locally compact CW complex K one constructs its end end(K) as $\{K - C : C$ is compact in $K\}$, where the bonding morphisms are induced by inclusions. In the case of K being connected one can express K as an increasing union of its compact subcomplexes K_n with $K_n \subset \operatorname{Int}(K_{n+1})$. Thus, end$(K)$ is equivalent in pro-H to an object of tow(H), namely $\{K - \operatorname{Int}(K_n)\}$.

A CW complex K is locally compact iff it is locally finite. It turns out (see [F-T-W]) that there exists a proper map $f : K \to L$ of locally finite CW complexes which is not properly homotopic to a cellular map (one mapping the nth skeleton of K to the nth skeleton of L). To remedy this one introduces a special class of locally finite CW complexes:

4.1. Definition [F-T-W]. A CW complex K is called *strongly locally finite* if it can be covered by a locally finite family of its finite subcomplexes.

The next definition is a rewording of end(K) being equivalent to a tower of connected CW complexes:

4.2. Definition. A locally finite CW complex K is called *connected at infinity* if for each compact subset C of K there is a compact subset D of K

which contains C in its interior so that $K-D$ is contained in one component of $K-C$.

In the following definition we treat $[0,\infty)$ as a CW complex with integers as vertices and $[n,n+1]$ as 1-cells.

4.3. Definition. Suppose K is a locally finite CW complex. By a *ray* in K we mean an embedding $r:[0,\infty)\to K$ of $[0,\infty)$ onto a subcomplex of K.

Essentially, a ray is an analog of a base point.

Notice that the $\mathrm{end}(K)$ is not an object of pro-H_0. Namely, the base points are not defined yet. The purpose of the next definition is to create an object of $\mathrm{tow}(\mathrm{CW}_0)$ which is equivalent to $\mathrm{end}(K)$ in pro-H once the base points are forgotten.

4.4. Definition. Suppose (K,r_K) is a rayed, connected, locally finite CW complex. By a *preferred end* of (K,r_K) we mean the object $\mathrm{end}(K,r_K)$ of $\mathrm{tow}(CW_0)$ for which the bonding maps are inclusions and $\mathrm{end}(K,r_K)_n = (K_n\cup r_K[0,\infty), r_K(0))$ for each n so that the following conditions are satisfied:

(i) $K_1=K$, each K_n is connected, and each $K-\mathrm{Int}(K_n)$ is compact,
(ii) $K_{n+1}\subset \mathrm{Int}(K_n)$ for each n,
(iii) given a compact subset C of K there is n such that $C\subset K-K_n$,
(iv) for each n there is an integer v_n such that $r_K[0,v_n]\cap K_n$ is a vertex of K_n and $r_K[v_n,\infty)\subset K_n$.

Notice that any two preferred ends of (K,r_K) are equivalent in $\mathrm{tow}(H_0)$. Therefore, one may define the *pro-homotopy groups* pro-$\pi_n(K,r_K)$ of (K,r_K) as $\pi_n(\mathrm{end}(K,r_K))$ (strictly speaking, one should talk about the equivalence class of pro-groups).

4.5. Lemma. *Suppose (K,r_K) is a rayed, locally finite CW complex.*

(a) *If a preferred end of (K,r_K) exists, then K is connected, connected at infinity, and strongly locally finite.*

(b) *If K is connected, connected at infinity, and strongly locally finite, then a preferred end* $\mathrm{end}(K,r_K)$ *exists. Moreover, if $f:(K,r_K)\to(L,r_L)$ is a proper map and* $\mathrm{end}(L,r_L)$ *is a preferred end of (L,r_L), then one can find a preferred end* $\mathrm{end}(K,r_K)$ *of (K,r_K) so that $f(K_n)\subset L_n$ for each n.*

Proof. (a) Suppose a preferred end $\mathrm{end}(K,r_K)$ exists so that $\mathrm{end}(K,r_K)_n=(K_n\cup r_K[0,\infty), r_K(0))$ for each n and the properties in Definition 4.4 are satisfied. Since $K=K_1$, it is connected. Suppose C is a compact subset of K and choose n so that $C\subset K-\mathrm{Int}(K_n)$. Put $D=K-\mathrm{Int}(K_{n+1})$ and notice that $K-D\subset K_n$ is contained in one component of $K-C$. Put $A_n=K_n-\mathrm{Int}(K_{n+1})$ for $n\ge 1$. Notice that A_n,

$n \geq 1$, cover K. Given $x \in K$ there is a compact neighborhood C of $x \in K$. Let m satisfy $C \subset K - \mathrm{Int}(K_m)$. Notice that $C \cap A_n = \emptyset$ for $n \geq m+1$.

(b) Let $\{A_i\}_{i\geq 1}$ be a locally finite cover of K consisting of finite subcomplexes. Put $K_1 = K$ and suppose K_n is given for some n. For each $m > n$ let B_m be the union of those A_i which do not intersect $(K - \mathrm{Int}(K_n)) \cup \bigcup_{k=1}^{m} A_k$. Notice that $K - \mathrm{Int}(B_m)$ is compact for each m.

For each m choose the smallest integer w_m so that $r_K(w_m) \in B_m$ and let $C_m = B_m \cup r_K[w_m, \infty)$. Let D_m be the component of $r_K(w_m)$ in C_m. Notice that $K - \mathrm{Int}(D_m)$ is compact. Our candidate for K_{n+1} is any D_m contained in $\mathrm{Int}(K_n)$. Let us show that such a D_m exists. First of all, there is an integer p such that $r_K[p, \infty) \subset \mathrm{Int}(K_n)$. There exists an integer q such that A_m does not intersect $r_K[0,p] \cup (K - \mathrm{Int}(K_n)) \cup f^{-1}(L - \mathrm{Int}(L_n))$ for $m \geq q$. Notice that D_q can be chosen as K_{n+1}. ■

4.6. Definition. P_r is the proper homotopy category of rayed, strongly locally finite, connected CW complexes K connected at infinity, equipped with a ray $r_K : [0, \infty) \to K$. Morphisms of P_r are proper homotopy classes of ray-preserving proper maps $f : (K, r_K) \to (L, r_L)$ (i.e., $f \circ r_K = r_L$).

4.7. Definition. If $f : (K, r_K) \to (L, r_L)$ is a ray-preserving, proper, cellular map between two rayed locally finite CW complexes, then its *mapping cylinder* (M, r_M) is defined as the quotient of the regular mapping cylinder so that $(r_K(t), s)$ is identified with $r_L(t)$ for all $t \geq 0$, $1 \geq s \geq 0$.

If f is an inclusion, then its *double mapping cylinder* $(\mathrm{DM}(f), r_M)$ is defined as the quotient of the regular double mapping cylinder so that $(r_K(t), s)$ is identified with $(r_L(t), 0)$ for all $t \geq 0$, $1 \geq s \geq 0$.

4.8. Proposition. *If $f : (K, r_K) \to (L, r_L)$ is a ray-preserving, proper, cellular map between two objects of P_r, then its mapping cylinder is an object of P_r. If f is an inclusion, then its double mapping cylinder is an object of P_r.*

Proof. Since f is cellular, M is a connected CW complex. Since f is proper, M is locally finite. Let $p : M(f) \to M$ be the projection from the regular mapping cylinder $M(f)$ of f. Choose preferred ends $\mathrm{end}(K, r_K)$, $\mathrm{end}(L, r_L)$ so that the following conditions are satisfied (see Lemma 4.5):

(i) $\mathrm{end}(K, r_K)_n = (K_n \cup r_K[0, \infty), r_K(0))$ for each n,

(ii) for each n there is an integer v_n such that $r_K[0, v_n] \cap K_n$ is a vertex of K_n and $r_K[v_n, \infty) \subset K_n$,

(iii) $\mathrm{end}(L, r_L)_n = (L_n \cup r_L[0, \infty), r_L(0))$ for each n,

(iv) for each n there is an integer w_n such that $r_L[0, w_n] \cap L_n$ is a vertex of L_n and $r_L[v_n, \infty) \subset L_n$,

(v) $f(K_n) \subset L_n$ for each n.

Since $f(K-\mathrm{Int}(K_n))$ is contained in $L-L_s$ for some s, we may assume that

(vi) $f^{-1}(L_{n+1})\subset \mathrm{Int}(K_n)$ for each n.

Indeed, $f(K-\mathrm{Int}(K_n))\subset L-L_s$ implies $f^{-1}(L_s)\subset \mathrm{Int}(K_n)$ and one may redefine L_{n+1} as L_s.

Notice that $w_n\le v_n$ for each n. Let $f_n: K_n\to L_n$ be induced by f and let $P_n=M(f_n)$. Let $M_n=p(P_n)$. Notice that $r_M[0,w_n]\cap M_n=r_M(w_n)$ and $r_M[w_n,\infty)\subset M_n$. The main purpose of (vi) is to ensure $M_{n+1}\subset \mathrm{Int}(M_n)$. Now, it is easy to check that $\mathrm{end}(M,r_M)_n=(M_n\cup r_M[0,\infty),r_L(0))$ defines a preferred end of (M,r_M). By Lemma 4.5, (M,r_M) is an object of P_r. A similar proof works for the double mapping cylinder. ∎

4.9. Proposition. *If* $f:(K,r_K)\to(L,r_L)$ *is a monomorphism of* P_r, *then* pro-$\pi_k(f)$ *is a monomorphism of pro-groups for each* $k\ge 1$.

Proof. Choose preferred ends $\mathrm{end}(K,r_K)$ of (K,r_K) and $\mathrm{end}(L,r_L)$ of (L,r_L). We may assume that $f(K_n)\subset L_n$ for each n. Fix $k\ge1$ and $m\ge1$. Suppose that for each $p>m$ there is a map $a_p:(S^k,1)\to(K_p\cup r_K[0,\infty),r_K(0))$ so that $f\circ a_p\approx 0$ in $L_p\cup r_L[0,\infty)$ but a_p is not null-homotopic in $K_m\cup r_K[0,\infty)$. Since K_p is a deformation retract of $K_p\cup r_K[0,\infty)$, there is a map $b_p:(S^k,1)\to(K_p,v_p)$ (v_p being the first vertex on the ray which belongs to K_p) so that $f\circ b_p\approx 0$ in L_p but b_p is not null-homotopic in K_m. Define S as $[0,\infty)\times 1\cup\bigcup_{p>m}\{q_p\}\times S^k$ and define $r_S:[0,\infty)\to S$ by $r_S(t)=(t,1)$. Combining the maps b_p, $p>m$, one constructs $b:(S,r_S)\to(K,r_K)$ so that $f\circ b\approx f\circ c$ and b is not properly homotopic to the "constant" map c, a contradiction. The map $c:(S,r_S)\to(K,r_K)$ is defined by $c(\{q_p\}\times S^k)=v_p$ for $p\ge1$. ∎

4.10. Proposition. *If* $f:(K,r_K)\to(L,r_L)$ *is an epimorphism of* P_r, *then* $\mathrm{end}(f):\mathrm{end}(K,r_K)\to\mathrm{end}(L,r_L)$ *is an epimorphism of* $\mathrm{tow}(H_0)$.

Proof. First consider the case of f being an inclusion. Choose preferred ends $\mathrm{end}(K,r_K)$ of (K,r_K) and $\mathrm{end}(L,r_L)$ of (L,r_L). We may assume that $K_n\subset L_n$ for each n (see Lemma 4.5). By 4.8, $(\mathrm{DM}(f),r_M)$ is an object of P_r. Let $i^0,i^1:(L,r_L)\to(\mathrm{DM}(f),r_M)$ be the two inclusions. Since $i^0\circ f$ is properly homotopic to $i^1\circ f$, there is a proper homotopy $H:L\times I\to\mathrm{DM}(f)$ joining i^0 and i^1. Given $m\ge1$, let $C=L-\mathrm{Int}(L_{m+1})$. The image D of $C\times\{0,1\}\cup(K_m\cap C)\times I$ in $\mathrm{DM}(f)$ is compact, so $H^{-1}(D)$ is compact in $L\times I$. There is $p>m$ such that $(L_p\times I)\cap H^{-1}(D)=\emptyset$. This means that $H|L_p\times I:L_p\times I\to\mathrm{DM}(j_m)$ is a homotopy joining $i^0|L_p$ and $i^1|L_p$, where $j_m:K_m\to L_m$ is the inclusion. By 2.7, the inclusion $\mathrm{end}(K,r_K)\to\mathrm{end}(L,r_L)$ is an epimorphism of $\mathrm{tow}(H_0)$.

In the general case one may assume f is a cellular map (see [F-T-W]) and then replace f by the inclusion $j:(K,r_K)\to(M,r_M)$ of (K,r_K) into the mapping cylinder of f. Since the inclusion $(L, r_L) \to (M, r_M)$ is an isomorphism of P_r, it induces an isomorphism of preferred ends. ■

4.11. COROLLARY. *If* $f : (K, r_K) \to (L, r_L)$ *is a bimorphism of* P_r, *then* f *induces isomorphisms of proper homotopy groups.*

Proof. Use 4.9, 4.10, and 2.10. ■

4.12. COROLLARY. *If* $f : (K, r_K) \to (L, r_L)$ *is a bimorphism of* P_r *so that* $\dim(L)$ *is finite*, *then* f *is an isomorphism.*

Proof. We may assume that f is a cellular map (see [F-T-W]). It suffices to show that f has a left inverse in P_r.

First, notice that $f : (K, r_K(0)) \to (L, r_L(0))$ is an ordinary homotopy equivalence (the finitness of $\dim(L)$ is not needed here). Indeed, replace f by an inclusion and notice that the proper double mapping cylinder of f is equivalent to the ordinary double mapping cylinder of f. Thus, as in 2.7, f is an epimorphism of H_0. It remains to show (see [D_2]) that $f : (K, r_K(0)) \to (L, r_L(0))$ induces monomorphisms of all homotopy groups. This follows from the fact that $\pi_m(K, r_K(0))$ can be regarded as the set of proper homotopy classes of maps $(S_m, r_S) \to (K, r_K)$, where $S_m = [0, \infty) \times \{1\} \cup \{0\} \times S^m$, 1 being the base point of the m-sphere S^m.

Let (M, r_M) be the mapping cylinder of f. We construct a proper map $H_n : M \times 0 \cup (K \cup M^{(n)}) \times I \to M$ so that $H_n|M \times 0 = \mathrm{id}$, $H_n|K \times I = \mathrm{id}$, and $H_n(M^{(n)} \times 1) \subset K$. Choose ends $\mathrm{end}(K, r_K)$ of (K, r_K) and $\mathrm{end}(M, r_M)$ of (M, r_M) so that $K_m \subset M_m$ for each m. The map H_0 is constructed by homotoping vertices in $M_p - M_{p-1}$ to v_p inside M_p.

Suppose H_{n-1} is given. Since $\mathrm{end}(K, r_K) \to \mathrm{end}(M, r_M)$ induces isomorphisms of pro-homotopy groups, Lemma 8.1.2 of [D_1-S] (see p. 104) says that for each s there is $p(s) > s$ such that $\pi_n(M_{p(s)}, K_{p(s)}) \to \pi_n(M_s, K_s)$ is trivial. This allows extending H_{n-1} to H_n as follows: Given an n-cell σ of M which is not contained in K the homotopy H_{n-1} restricted to $\sigma \times 0 \cup \partial\sigma \times I$ can be extended over $\sigma \times I$ so that the image of $\sigma \times 1$ lies in K. This follows from the fact that K is an ordinary deformation retract of M. However, we require this extension to be done in such a manner that its image lies in K_m with m maximum possible (such an m obviously exists).

Let us show that $H_n : M \times 0 \cup (K \cup M^{(n)}) \times I \to M$ obtained by pasting together such extensions is proper. It suffices to show that $H_n^{-1}(M - \mathrm{Int}(M_s))$ is compact for each s. Suppose there are infinitely many n-cells σ_v, $v \geq 1$, so that $H_n(\sigma_v \times I) \cap (M - \mathrm{Int}(M_s)) \neq \emptyset$. Let $w = p(s + 1)$. Notice that there is v so that $H_{n-1}(\partial\sigma_v \times 1) \subset K_w$ and $H_{n-1}(\partial\sigma_v \times I \cup \sigma_v \times 0) \subset M_w$. Since $\pi_n(M_w, K_w) \to \pi_n(M_{s+1}, K_{s+1})$ is trivial, $H_n(\sigma_v \times I) \subset M_{s+1}$, a contradiction.

Let $\dim(L) = d$. Notice that $H_d|L \times \{1\}$ is a left inverse of $(K, r_K) \to (M, r_M)$ in P_r. ■

References

[B] H. J. Baues, *Foundations of proper homotopy theory*, Draft manuscript, Max-Planck-Institut für Math., 1992.

[Br] E. M. Brown, *Proper homotopy theory in simplicial complexes*, in: Topology Conference (Virginia Polytechnic Institute and State University), R. F. Dickmann Jr. and P. Fletcher (eds.), Lecture Notes in Math. 375, Springer, Berlin, 1974, 41–46.

[C-G] C. Casacuberta and S. Ghorbal, *On homotopy epimorphisms of connective covers*, preprint, 1997.

[D_1] J. Dydak, *The Whitehead and the Smale theorems in shape theory*, Dissertationes Math. 156 (1979).

[D_2] —, *Epimorphism and monomorphism in homotopy*, Proc. Amer. Math. Soc. 116 (1992), 1171–1173.

[D-S_1] J. Dydak and J. Segal, *Shape Theory: An Introduction*, Lecture Notes in Math. 688, Springer, Berlin, 1978.

[D-S_2] —, —, *Strong shape theory*, Dissertationes Math. 192 (1981).

[Dy-R] E. Dyer and J. Roitberg, *Homotopy-epimorphism, homotopy-monomorphism and homotopy-equivalences*, Topology Appl. 46 (1992), 119–124.

[Ed-H] D. A. Edwards and H. M. Hastings, *Čech and Steenrod Homotopy Theories with Applications to Geometric Topology*, Lecture Notes in Math. 542, Springer, Berlin, 1976.

[En_1] R. Engelking, *General Topology*, Heldermann, Berlin, 1989.

[F-T-W] F. T. Farrell, L. R. Taylor and J. B. Wagoner, *The Whitehead theorem in the proper category*, Compositio Math. 27 (1973), 1–23.

[G] S. Ghorbal, *Epimorphisms and monomorphisms in homotopy theory*, PhD Thesis, Université Catholique de Louvain, 1996 (in French).

[H-R] P. Hilton and J. Roitberg, *Relative epimorphisms and monomorphisms in homotopy theory*, Compositio Math. 61 (1987), 353–367.

[H-W] L. Hong and S. Wenhuai, *Homotopy epimorphisms in homotopy pushbacks*, Topology Appl. 59 (1994), 159–162.

[M-S] S. Mardešić and J. Segal, *Shape Theory*, North-Holland, Amsterdam, 1982.

[Mat] M. Mather, *Homotopy monomorphisms and homotopy pushouts*, Topology Appl. 81 (1997), 159–162.

[Mo-P] M. A. Morón and F. R. Ruiz del Portal, *On weak shape equivalences*, ibid. 92 (1999), 225–236.

[Mu] G. Mukherjee, *Equivariant homotopy epimorphisms, homotopy monomorphisms and homotopy equivalences*, Bull. Belg. Math. Soc. 2 (1995), 447–461.

[P] T. Porter, *Proper homotopy theory*, in: Handbook of Algebraic Topology, Elsevier Science, 1995, 127–167.

[S] E. Spanier, *Algebraic Topology*, McGraw-Hill, New York, 1966.

[Sp] S. Spież, *A majorant for the family of all movable shapes*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 21 (1973), 615–620.

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Received 29 September 1998;
in revised form 18 January 1999