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Spaces of upper semicontinuous multi-valued functions on complete metric spaces

by

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Abstract. Let $X = (X, d)$ be a metric space and let the product space $X \times \mathbb{R}$ be endowed with the metric $\varrho((x,t),(x',t')) = \max\{d(x,x'), |t-t'|\}$. We denote by $\mathrm{USCC_B}(X)$ the space of bounded upper semicontinuous multi-valued functions $\varphi : X \to \mathbb{R}$ such that each $\varphi(x)$ is a closed interval. We identify $\varphi \in \mathrm{USCC_B}(X)$ with its graph which is a closed subset of $X \times \mathbb{R}$. The space $\mathrm{USCC_B}(X)$ admits the Hausdorff metric induced by ϱ. It is proved that if $X = (X, d)$ is uniformly locally connected, non-compact and complete, then $\mathrm{USCC_B}(X)$ is homeomorphic to a non-separable Hilbert space. In case X is separable, it is homeomorphic to $\ell_2(2^{\mathbb{N}})$.

1. Introduction. Let $X = (X, d)$ be a metric space and let the product space $X \times \mathbb{R}$ be endowed with the metric

$$\varrho((x,t),(x',t')) = \max\{d(x,x'), |t-t'|\}.$$

A multi-valued function $\varphi : X \to \mathbb{R}$ is said to be *bounded* if the image $\varphi(X) = \bigcup_{x\in X} \varphi(x)$ is bounded. For any multi-valued function $\varphi : X \to \mathbb{R}$ such that each $\varphi(x)$ is compact, φ is upper semicontinuous (u.s.c.) if and only if the graph of φ is closed in $X \times \mathbb{R}$. Such a φ can be regarded as a closed set in $X \times \mathbb{R}$. We denote by $\mathrm{USCC_B}(X)$ the space of bounded u.s.c. multi-valued functions $\varphi : X \to \mathbb{R}$ such that each $\varphi(x)$ is non-empty, compact and connected, that is, a closed interval. The topology for $\mathrm{USCC_B}(X)$ is induced by the Hausdorff metric

$$\varrho_{\mathrm{H}}(\varphi,\psi) = \max\{\sup_{z\in\varphi} \varrho(z,\psi), \sup_{z\in\psi} \varrho(z,\varphi)\},$$

where $\varrho(z,\psi) = \inf_{z'\in\psi} \varrho(z,z')$. Since φ and ψ are bounded, $\varrho_{\mathrm{H}}(\varphi,\psi) < \infty$ can be defined. In case X is compact, every u.s.c. multi-valued function

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$\varphi : X \to \mathbb{R}$ is bounded, so we write $\mathrm{USCC_B}(X) = \mathrm{USCC}(X)$. Let

$$\mathrm{USCC}(X, \mathbf{I}) = \{\varphi \in \mathrm{USCC_B}(X) \mid \varphi(X) \subset \mathbf{I}\},$$

where $\mathbf{I} = [0, 1]$. In case X is non-compact, as will be seen, the topology for $\mathrm{USCC_B}(X)$ (or $\mathrm{USCC}(X, \mathbf{I})$) depends on the metric d.

Fedorchuk [Fe$_{1,2}$] proved that if X is infinite, locally connected and compact then $\mathrm{USCC}(X, \mathbf{I})$ is homeomorphic to ($\approx$) the Hilbert cube $Q = [-1, 1]^\omega$ and $\mathrm{USCC}(X) \approx Q \setminus \{0\}$ ($\approx Q \times [0, 1)$) (cf. [SU, Appendix]). In this paper, we consider the case where X is non-compact but complete. We say that X is *uniformly* (or *d-uniformly*) *locally connected* if, for each $\varepsilon > 0$, there is $\delta > 0$ such that each pair of points $x, x' \in X$ with $d(x, x') < \delta$ are contained in some connected set in X with diameter $< \varepsilon$. Let m (or ℓ_∞) be the Banach space of bounded sequences in $\mathbb{R}$ with the sup-norm. Note that m is non-separable. Indeed, $m \approx \ell_2(2^{\mathbb{N}})$ [BP, Ch. VII, Theorem 6.1]. By applying Toruńczyk's characterization of Hilbert spaces [To$_3$] (cf. [To$_4$]), we prove the following:

Main Theorem. *If $X = (X, d)$ is a uniformly locally connected, non-compact and complete metric space, then* $\mathrm{USCC}(X, \mathbf{I})$ *and* $\mathrm{USCC_B}(X)$ *are homeomorphic to a non-separable Hilbert space. In case X is separable,*

$$\mathrm{USCC}(X, \mathbf{I}) \approx \mathrm{USCC_B}(X) \approx m \approx \ell_2(2^{\mathbb{N}}).$$

In the above, the word "uniformly" cannot be removed, that is, the Main Theorem is not valid for a locally connected complete metric space X with no isolated points.

Example. *The following closed subspace X of Euclidean plane $\mathbb{R}^2$ is locally path-connected and has no isolated points, but* $\mathrm{USCC}(X, \mathbf{I})$ *and* $\mathrm{USCC_B}(X)$ *are not locally connected, hence they are not ANR's*:

$$X = \mathbb{R} \times \{0\} \cup \bigcup_{n \in \mathbb{N}} \{n, n + 2^{-n}\} \times \mathbf{I} \subset \mathbb{R}^2.$$

Proof. We define a map $f : X \to \mathbf{I}$ by

$$f(s,t) = \begin{cases} 2t & \text{if } s \in \mathbb{N} \text{ and } 0 \le t \le 1/2, \\ 1 & \text{if } s \in \mathbb{N} \text{ and } 1/2 \le t \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

For each $\varepsilon > 0$, choose $n_0 \in \mathbb{N}$ so that $2^{-n_0} < \varepsilon$, and define $g : X \to \mathbf{I}$ by

$$g(s,t) = \begin{cases} 0 & \text{if } s = n_0, \\ 2t & \text{if } s = n_0 + 2^{-n_0} \text{ and } 0 \le t \le 1/2, \\ 1 & \text{if } s = n_0 + 2^{-n_0} \text{ and } 1/2 \le t \le 1, \\ f(s,t) & \text{otherwise.} \end{cases}$$

Then $\varrho_{\mathrm{H}}(f, g) = 2^{-n_0} < \varepsilon$ but g cannot be connected with f by any path in $\mathrm{USCC_B}(X)$ with diameter $< 1/2$. ∎

In the above, $X \approx Y = \mathbb{R} \times \{0\} \cup \mathbb{N} \times \mathbf{I} \subset \mathbb{R}^2$, but $\mathrm{USCC}(X, \mathbf{I}) \not\approx \mathrm{USCC}(Y, \mathbf{I})$ because $\mathrm{USCC}(Y, \mathbf{I}) \approx \ell_2(2^{\mathbb{N}})$ by the Main Theorem.

Throughout the paper, the open ε-ball in $X = (X, d)$ centered at $x \in X$ is denoted by $B(x, \varepsilon)$ (or $B_d(X, \varepsilon)$) and the closure of $B(x, \varepsilon)$ in X by $\overline{B}(x, \varepsilon)$. On the other hand, to avoid confusion, the ε-neighborhood of a subset $F \subset X$ in X is denoted by $N(F, \varepsilon)$ (or $N_d(F, \varepsilon)$), that is,

$$N(F, \varepsilon) = \bigcup_{x \in F} B(x, \varepsilon) = \{y \in X \mid d(y, F) < \varepsilon\} \subset X.$$

For $F \subset X \times \mathbb{R}$ and $A \subset X$, we define $F|A = F \cap \mathrm{pr}_X^{-1}(A) = F \cap A \times \mathbb{R}$ and $F(A) = \mathrm{pr}_{\mathbb{R}}(F|A)$, where $\mathrm{pr}_X : X \times \mathbb{R} \to X$ and $\mathrm{pr}_{\mathbb{R}} : X \times \mathbb{R} \to \mathbb{R}$ are the projections. In case $A = \{x\}$, we write $F|\{x\} = F|x$ and $F(\{x\}) = F(x)$.

1. Relations among $C_{\mathrm{B}}(X)$, $\mathrm{USCC}_{\mathrm{B}}(X)$ and $2^{X \times \mathbb{R}}$. For a metric space $X = (X, d)$, let $(2^X)_m$ denote the hyperspace of non-empty bounded closed subsets of X with the Hausdorff metric d_{H} defined by d (cf. [Ku, p. 214]). If X is complete, then so is $(2^X)_m$ [Ku, p. 407]. In case X is compact, $(2^X)_m$ is the hyperspace $\exp(X)$ of non-empty compact subsets of X. Let 2^X be the totality of non-empty closed subsets of X. When X is unbounded, $2^X \neq (2^X)_m$ and d_{H} is not a metric on the whole 2^X (e.g., $X \notin (2^X)_m$ and $d_{\mathrm{H}}(\{x\}, X) = \infty$ for any $x \in X$), but d_{H} induces the topology on 2^X. In fact, $A \in 2^X$ has a neighborhood base consisting of

$$\{B \in 2^X \mid d_{\mathrm{H}}(A, B) < \varepsilon\} \ (= \{B \in 2^X \mid A \subset N_d(B, \varepsilon),\ B \subset N_d(A, \varepsilon)\}).$$

The spaces $\mathrm{USCC}(X, \mathbf{I}) \subset \mathrm{USCC}_{\mathrm{B}}(X)$ are regarded as subspaces of the hyperspace $2^{X \times \mathbb{R}}$. Note that $\mathrm{USCC}(X, \mathbf{I}) \not\subset (2^{X \times \mathbb{R}})_m$ if X is unbounded, and that ϱ_{H} is not a metric on $2^{X \times \mathbb{R}}$ but it is a metric on $\mathrm{USCC}_{\mathrm{B}}(X)$.

One should remark that a different metric d' on X defines not only a different space $(2^X)_m$ but also a different topology on 2^X even if d' induces the same topology of X as d. However, if d' is uniformly equivalent to d, then d'_{H} induces the same topology on 2^X as d_{H}. Let d^* be the bounded metric on X defined by $d^*(x, y) = \min\{1, d(x, y)\}$. Note that every closed subset on X is bounded with respect to d^*. Since d^*_{H} is a metric on the whole 2^X, the space 2^X is metrizable. Moreover, if d is complete, then so is d^*, hence d^*_{H} is also complete (cf. [Ku, p. 407]).

The following is elementary, but we give a proof for completeness.

1.1. Lemma. *Let $\varphi \in \mathrm{USCC}_{\mathrm{B}}(X)$ and $A \subset X$. If A is connected, then so is the image $\varphi(A)$.*

Proof. Assume that $\varphi(A)$ is disconnected. Then there is $t \in \mathbb{R} \setminus \varphi(A)$ such that $(-\infty, t) \cap \varphi(A) \neq \emptyset$ and $(t, \infty) \cap \varphi(A) \neq \emptyset$, whence $\varphi(x) \subset (-\infty, t)$ or $\varphi(x) \subset (t, \infty)$ for each $x \in A$ because of the connectedness of $\varphi(x)$. Let $U = \{x \in X \mid \varphi(x) \subset (-\infty, t)\}$ and $V = \{x \in X \mid \varphi(x) \subset (t, \infty)\}$. Then

$U \cap V = \emptyset$, $A \subset U \cap V$, $A \cap U \neq \emptyset$ and $A \cap V \neq \emptyset$. Since φ is u.s.c., these U and V are open sets in X. This contradicts the connectedness of A. Hence $\varphi(A)$ is connected. ∎

Without any completeness condition, the following can be proved (cf. [FK, Theorem 3.3(a)]).

1.2. Proposition. *If X is locally connected, then* $\mathrm{USCC_B}(X)$ *is closed in* $2^{X\times\mathbb{R}}$, *hence* $\mathrm{USCC}(X,\mathbf{I})$ *is closed in* $2^{X\times\mathbf{I}}$.

Proof. Let $\varphi \in \mathrm{cl}_{2^{X\times\mathbb{R}}} \mathrm{USCC_B}(X)$. Then, as is easily observed, $\varphi \subset X \times [-a,a]$ for some $a > 0$. If $\varphi(x) = \emptyset$ (i.e., $\varphi \cap \{x\} \times \mathbb{R} = \emptyset$), then $B(x,\varepsilon)\times\mathbb{R}\cap\varphi = \emptyset$ for some $\varepsilon > 0$. For any $\psi \in \mathrm{USCC_B}(X)$, since $\psi(x) \neq \emptyset$, we have $\varrho_{\mathrm{H}}(\psi,\varphi) \geq \varepsilon$, which is a contradiction. Therefore, $\varphi(x) \neq \emptyset$ for every $x \in X$. Since φ is closed in $X \times \mathbb{R}$, it follows that $\varphi : X \to \mathbb{R}$ is u.s.c. We show that each $\varphi(x)$ is connected, which implies that $\varphi \in \mathrm{USCC_B}(X)$.

Assume that some $\varphi(x_0)$ is not connected. Then we can find some $t_1 < t_0 < t_2$ such that $t_1, t_2 \in \varphi(x_0)$ and $t_0 \notin \varphi(x_0)$. Choose $\varepsilon > 0$ so that

$$B(x_0, 2\varepsilon) \times (t_0 - \varepsilon, t_0 + \varepsilon) \cap \varphi = \emptyset,$$

whence $\varrho((x,t_0),\varphi) \geq \varepsilon$ for each $x \in B(x_0,\varepsilon)$. Since X is locally connected, x_0 has a connected neighborhood $U \subset B(x_0,\varepsilon)$. Then U contains some $B(x_0,\delta) \subset U$, whence $\delta \leq \varepsilon$. For each $\psi \in \mathrm{USCC_B}(X)$ with $\varrho_{\mathrm{H}}(\psi,\varphi) < \delta$, we have some $(x_i,s_i) \in \psi$, $i = 1,2$, such that $d(x_i,x_0) < \delta$ and $|t_i - s_i| < \delta$, whence $x_1, x_2 \in U$, $s_1 < t_0$ and $s_2 > t_0$. Since $\psi(U)$ is connected by Lemma 1.1, it follows that $t_0 \in [s_1,s_2] \subset \psi(U)$, that is, $t_0 = \psi(x)$ for some $x \in U \subset B(x_0,\varepsilon)$. Then $\varrho_{\mathrm{H}}(\psi,\varphi) \geq \varrho((x,t_0),\varphi) \geq \varepsilon$, which is a contradiction. Therefore, every $\varphi(x)$ is connected. Thus $\varphi \in \mathrm{USCC_B}(X)$. ∎

By the remark at the beginning of this section, the statement below easily follows from Proposition 1.2.

1.3. Corollary. *If X is complete and locally connected, then* $\mathrm{USCC_B}(X)$ *is complete, hence so is* $\mathrm{USCC}(X,\mathbf{I})$. ∎

Let $C_{\mathrm{B}}(X)$ be the Banach space of bounded continuous real-valued functions on X with the sup-norm (1). Let $C(X,\mathbf{I}) = \{f \in C_{\mathrm{B}}(X) \mid f(X) \subset \mathbf{I}\}$. In case X is compact, every continuous real-valued function on X is bounded, and therefore we write $C_{\mathrm{B}}(X) = C(X)$. For a compact space X, Fedorchuk [$\mathrm{Fe}_{1,2}$] proved that if X is locally connected and has no isolated points then $C(X)$ and $C(X,\mathbf{I})$ are dense in $\mathrm{USCC}(X)$ and $\mathrm{USCC}(X,\mathbf{I})$, respectively. This was generalized in [FK] to non-compact spaces with some completeness condition. Here we give a proof without local connectedness or any completeness condition.

(1) As in [FK, Remark 3.6], although $C_{\mathrm{B}}(X) \subset \mathrm{USCC_B}(X)$, the Banach space $C_{\mathrm{B}}(X)$ is not a subspace of $\mathrm{USCC_B}(X)$ in case X is non-compact.

1.4. Lemma. *For each* $\varphi \in \mathrm{USCC}(X, \mathbf{I})$ *and* $\varepsilon > 0$, *there exists a lower semicontinuous* (*l.s.c.*) *multi-valued function* $\varphi_\varepsilon : X \to \mathbf{I}$ *such that each* $\varphi_\varepsilon(x)$ *is a closed interval*, $\varphi \subset \varphi_\varepsilon$ *and* $\varrho_{\mathrm{H}}(\varphi, \mathrm{cl}_{X\times\mathbf{I}}\, \varphi_\varepsilon) \le \varepsilon$.

Proof. For each $x \in X$, let

$$V_x = (\min \varphi(x) - \varepsilon, \max \varphi(x) + \varepsilon) \cap \mathbf{I}.$$

Since φ is u.s.c., we can choose $\delta_x > 0$ so that $\delta_x \le \varepsilon$ and $\varphi(x') \subset V_x$ if $x' \in B(x, \delta_x)$ (i.e., $d(x, x') < \delta_x$). Let $\psi : X \to \mathbf{I}$ be the multi-valued function defined by

$$\psi(x) = \bigcup \{V_y \mid d(x,y) < \delta_y\} \quad \text{for each } x \in X.$$

We define the multi-valued function $\varphi_\varepsilon : X \to \mathbf{I}$ by $\varphi_\varepsilon(x) = \mathrm{cl}_{\mathbf{I}}\, \psi(x)$. Then $\varphi \subset \psi \subset \varphi_\varepsilon$. As is easily observed, $\varrho_{\mathrm{H}}(\varphi, \mathrm{cl}_{X\times\mathbf{I}}\, \psi) \le \varepsilon$. Since $\mathrm{cl}_{X\times\mathbf{I}}\, \varphi_\varepsilon = \mathrm{cl}_{X\times\mathbf{I}}\, \psi$, we have $\varrho_{\mathrm{H}}(\varphi, \mathrm{cl}_{X\times\mathbf{I}}\, \varphi_\varepsilon) \le \varepsilon$. If $d(x,y) < \delta_y$ then $\varphi(x) \subset V_y$. Since $\varphi(x)$ and V_y are connected, each $\psi(x)$ is connected, hence so is $\varphi_\varepsilon(x)$.

To see that φ_ε is l.s.c., let V be an open set in $\mathbf{I}$ and $x \in X$ such that $\varphi_\varepsilon(x) \cap V \ne \emptyset$. Then we have $t \in \psi(x) \cap V$. By the definition of ψ, we can find $y \in X$ such that $d(x,y) < \delta_y$ and $t \in V_y$. If $d(x,x') < \delta_y - d(x,y)$ then $d(x',y) < \delta_y$, hence $V_y \subset \psi(x') \subset \varphi_\varepsilon(x')$ by the definition. Thus we have $t \in \varphi_\varepsilon(x') \cap V$. Therefore, $\varphi_\varepsilon : X \to \mathbf{I}$ is l.s.c. ∎

Remark. In the above, $\varphi_\varepsilon \ne \mathrm{cl}_{X\times\mathbf{I}}\, \psi$. For example, let $\varphi = \mathbf{I} \times \{0\} \cup [1/2, 1] \times \mathbf{I} \in \mathrm{USCC}(\mathbf{I}, \mathbf{I})$ and $\varepsilon = 1/2$. Then $V_x = [0, 1/2)$ for $x < 1/2$ and $V_x = \mathbf{I}$ for $x \ge 1/2$. Define ψ as above by using

$$\delta_x = \begin{cases} 1/2 - x & \text{if } x < 1/2, \\ 1/2 & \text{if } x \ge 1/2. \end{cases}$$

Observe that $d(0,y) < \delta_y$ implies $y < 1/2$, and that $d(x, 1/2) < \delta_{1/2} = 1/2$ for $x \ne 0, 1$. Therefore, $\psi = \{0\} \times [0, 1/2) \cup (0,1] \times \mathbf{I} = \mathbf{I}^2 \setminus \{0\} \times [1/2, 1]$, hence $\mathrm{cl}_{X\times\mathbf{I}}\, \psi = \mathbf{I}^2$. On the other hand, $\varphi_{1/2} = \{0\} \times [0, 1/2] \cup (0,1] \times \mathbf{I}$ because $\varphi_{1/2}(x) = \mathrm{cl}_{\mathbf{I}}\, \psi(x)$ for each $x \in \mathbf{I}$.

1.5. Theorem. *The following conditions are equivalent for any metric space* $X = (X, d)$:

(a) $C(X, \mathbf{I})$ *is dense in* $\mathrm{USCC}(X, \mathbf{I})$;

(b) $C_{\mathrm{B}}(X)$ *is dense in* $\mathrm{USCC}_{\mathrm{B}}(X)$;

(c) X *has no isolated points.*

Proof. (a)⇒(b). This follows from the fact that each $\varphi \in \mathrm{USCC}_{\mathrm{B}}(X)$ is contained in some $\mathrm{USCC}_{\mathrm{B}}(X, [-a, a])$.

(b)⇒(c). When X has an isolated point x_0, let $\varphi = X \times \{0\} \cup \{x_0\} \times \mathbf{I} \in \mathrm{USCC}_{\mathrm{B}}(X)$. Then, as is easily observed,

$$\varrho_{\mathrm{H}}(\varphi, f) \geq \min\{1/2, d(x_0, X \setminus \{x_0\})\} > 0 \quad \text{for any } f \in C_{\mathrm{B}}(X),$$

which implies that $C_{\mathrm{B}}(X)$ is not dense in $\mathrm{USCC}_{\mathrm{B}}(X)$.

(c)⇒(a). For each $\varphi \in \mathrm{USCC}(X, \mathbf{I})$ and $\varepsilon > 0$, let $\varphi_\varepsilon : X \to \mathbf{I}$ be the l.s.c. multi-valued function obtained by Lemma 1.4. Choose a discrete closed subset D of φ so that $\varrho((x,t), D) < \varepsilon/2$ for any $(x,t) \in \varphi$, whence $\varrho_{\mathrm{H}}(\varphi, D) < \varepsilon/2$. Note that $\mathrm{pr}_X|D$ is finite-to-one and $\mathrm{pr}_X(D)$ is discrete in X. Since φ_ε is l.s.c. and X has no isolated points, for each $(x,t) \in \varphi_\varepsilon$ there are infinitely many $y \in X$ such that

$$d(x,y) < \varepsilon/2 \quad \text{and} \quad \varphi_\varepsilon(y) \cap (t - \varepsilon/2, t + \varepsilon/2) \neq \emptyset.$$

Then we can construct a discrete closed subset f of φ_ε such that $\mathrm{pr}_X|f$ is injective and $\varrho_{\mathrm{H}}(D, f) < \varepsilon/2$, hence $\varrho_{\mathrm{H}}(\varphi, f) < \varepsilon$. Then $A = \mathrm{pr}_X(f)$ is discrete in X and $f : A \to \mathbf{I}$ is a map (2) which is a selection for $\varphi_\varepsilon|A$ (i.e., $f(x) \in \varphi_\varepsilon(x)$ for each $x \in A$). By Michael's Selection Theorem [Mi], we can extend f to $\widetilde{f} \in C(X, \mathbf{I})$ which is a selection for φ_ε. For any $(x,t) \in \varphi$, we have $\varrho((x,t), \widetilde{f}) \leq \varrho((x,t), f) \leq \varrho_{\mathrm{H}}(\varphi, f) < \varepsilon$. Since $\widetilde{f} \subset \mathrm{cl}_{X\times\mathbf{I}}\, \varphi_\varepsilon$ and $\varrho_{\mathrm{H}}(\varphi, \mathrm{cl}_{X\times\mathbf{I}}\, \varphi_\varepsilon) \leq \varepsilon$, it follows that $\varrho((x,t), \varphi) \leq \varepsilon$ for any $(x,t) \in \widetilde{f}$. Thus $\varrho_{\mathrm{H}}(\widetilde{f}, \varphi) \leq \varepsilon$. Consequently, $\varphi \in \mathrm{cl}_{2^{X\times\mathbf{I}}}\, C(X, \mathbf{I})$. ■

Combining Theorem 1.5 with Proposition 1.2, we have the following corollary:

1.6. Corollary. *For any locally connected metric space X with no isolated points*, $\mathrm{USCC}_{\mathrm{B}}(X)$ (*resp.* $\mathrm{USCC}(X, \mathbf{I})$) *is the closure of* $C_{\mathrm{B}}(X)$ (*resp.* $C(X, \mathbf{I})$) *in* $2^{X\times\mathbb{R}}$ (*resp.* $2^{X\times\mathbf{I}}$). ■

One should notice that no completeness is assumed above (cf. [FK, Theorem 3.3(a)]).

2. The AR-property of $\mathrm{USCC}_{\mathrm{B}}(X)$ and $\mathrm{USCC}(X, \mathbf{I})$.

In this section, using Borges' characterization of AR's in [Bo], we prove that $\mathrm{USCC}_{\mathrm{B}}(X)$ and $\mathrm{USCC}(X, \mathbf{I})$ are AR's if $X = (X, d)$ is uniformly locally connected.

Now, we define a new metric d_{c} on X as follows:

$$d_{\mathrm{c}}(x, x') = \begin{cases} \inf\{\mathrm{diam}_d\, C \mid C \in \mathcal{C}(x, x')\} & \text{if } \mathcal{C}(x, x') \neq \emptyset, \\ 1 & \text{otherwise,} \end{cases}$$

where

$$\mathcal{C}(x, x') = \{C \subset X \mid C \text{ is connected, } x, x' \in C \text{ and } \mathrm{diam}\, C < 1\}.$$

As is easily observed, if X is uniformly locally connected, then d_{c} is uniformly equivalent to d, hence d_{c} induces the same topology on $2^{X\times\mathbb{R}}$ as d. Then, by replacing d with d_{c}, we can assume that

(2) Recall that a map is identified with its graph.

(∗) each pair of points $x, x' \in X$ with $d(x, x') < \varepsilon < 1$ are contained in a connected set C in X with $\operatorname{diam} C < \varepsilon$.

2.1. Lemma. *Condition* (∗) *implies the following condition*:

(♯) $N_\varrho(\varphi, \varepsilon)(x)$ *is connected for each* $\varphi \in \mathrm{USCC_B}(X)$, $0 < \varepsilon < 1$ *and* $x \in X$.

Proof. Let $t_1, t_2 \in N_\varrho(\varphi, \varepsilon)(x)$ and $t_1 < t < t_2$. Then there are $x_1, x_2 \in X$ and $s_i \in \varphi(x_i)$ $(i = 1, 2)$ such that $d(x_i, x) < \varepsilon$ and $|s_i - t_i| < \varepsilon$. Let

$$s = \frac{t_2 - t}{t_2 - t_1} s_1 + \frac{t - t_1}{t_2 - t_1} s_2.$$

By (∗), X has connected subsets C_1 and C_2 such that $x_i, x \in C_i$ and $\operatorname{diam} C_i < \varepsilon$. Since $C = C_1 \cup C_2$ is connected, $s \in \varphi(x_0)$ for some $x_0 \in C$ by Lemma 1.1. Then $d(x_0, x) < \varepsilon$. Observe that

$$t = \frac{t_2 - t}{t_2 - t_1} t_1 + \frac{t - t_1}{t_2 - t_1} t_2.$$

It then follows that

$$|s - t| \le \frac{t_2 - t}{t_2 - t_1} |s_1 - t_1| + \frac{t - t_1}{t_2 - t_1} |s_2 - t_2| < \varepsilon.$$

So $(x, t) \in N_\varrho(\varphi, \varepsilon)$, i.e., $t \in N_\varrho(\varphi, \varepsilon)(x)$. Thus, $N_\varrho(\varphi, \varepsilon)(x)$ is connected. ∎

We denote by Δ^{n-1} the standard $(n-1)$-simplex in $\mathbb{R}^n$, that is,

$$\Delta^{n-1} = \Big\{(t_1, \ldots, t_n) \in \mathbb{R}^n \ \Big|\ t_i \ge 0,\ \sum_{i=1}^n t_i = 1\Big\}.$$

A space Y is called *hyper-connected* if there are functions $h_n : Y^n \times \Delta^{n-1} \to Y$ $(n \in \mathbb{N})$ which satisfy the following conditions:

(i) if $t_i = 0$ then

$$h_n(y_1, \ldots, y_n; t_1, \ldots, t_n) = h_{n-1}(y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n; t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n);$$

(ii) $\Delta^{n-1} \ni (t_1, \ldots, t_n) \mapsto h_n(y_1, \ldots, y_n; t_1, \ldots, t_n) \in Y$ is continuous for each $(y_1, \ldots, y_n) \in Y^n$;

(iii) each neighborhood U of $y \in Y$ contains a neighborhood V of y such that $h_n(V^n \times \Delta^{n-1}) \subset U$ for every $n \in \mathbb{N}$.

Notice that h_n need not be continuous. It was proved by C. R. Borges [Bo] that a metrizable space X is an AR if and only if X is hyper-connected ([3]). We apply this characterization to prove the following:

([3]) R. Cauty [Ca] introduced a local hyper-connectedness different from the one of [Bo] and showed that a metrizable space X is an ANR if and only if X is locally hyper-connected. The results of [Bo] and [Ca] hold for stratifiable spaces.

2.2. Theorem. *For any uniformly locally connected metric space* $X = (X, d)$, $\mathrm{USCC}_{\mathrm{B}}(X)$ *and* $\mathrm{USCC}(X, \mathbf{I})$ *are AR's.*

Proof. Since $\mathrm{USCC}(X, \mathbf{I})$ is a retract of $\mathrm{USCC}_{\mathrm{B}}(X)$, it suffices to show that $\mathrm{USCC}_{\mathrm{B}}(X)$ is an AR.

By replacing the metric d with d_{c}, we can assume condition $(*)$. Each point of $\Delta^{n-1} \setminus \{b_{n-1}\}$ can be uniquely represented as follows:

$$(1-t)b_{n-1} + z,\ z \in \partial\Delta^{n-1}, \quad 0 < t \le 1,$$

where b_{n-1} is the barycenter of Δ^{n-1} and $\partial\Delta^{n-1}$ is the boundary of Δ^{n-1}. We inductively define

$$h_n : \mathrm{USCC}_{\mathrm{B}}(X)^n \times \Delta^{n-1} \to \mathrm{USCC}_{\mathrm{B}}(X) \quad (n \in \mathbb{N}).$$

First, let $h_1(\varphi, 1) = \varphi$ for every $\varphi \in \mathrm{USCC}_{\mathrm{B}}(X)$. Assume that $h_1, \ldots, h_{n-1}$ have been defined, and define h_n as follows:

$$h_n(\varphi_1, \ldots, \varphi_n; b_{n-1})(x) = \Big[\min \bigcup_{i=1}^{n} \varphi_i(x), \max \bigcup_{i=1}^{n} \varphi_i(x)\Big]$$

and, for $z \in \partial\Delta^{n-1}$ and $0 < t \le 1$,

$$\begin{aligned} &h_n(\varphi_1, \ldots, \varphi_n; (1-t)b_{n-1} + tz)(x) \\ &\qquad = (1-t)h_n(\varphi_1, \ldots, \varphi_n; b_{n-1})(x) + th_n(\varphi_1, \ldots, \varphi_n; z)(x), \end{aligned}$$

where $h_n(\varphi_1, \ldots, \varphi_n; z)$ is defined by condition (i). Then conditions (i) and (ii) are clearly satisfied. We show that

$$h_n(B_{\varrho_{\mathrm{H}}}(\varphi, \varepsilon)^n \times \Delta^{n-1}) \subset B_{\varrho_{\mathrm{H}}}(\varphi, \varepsilon)$$

for each $\varphi \in \mathrm{USCC}_{\mathrm{B}}(X)$ and $0 < \varepsilon < 1$. For $\varphi_1, \ldots, \varphi_n \in B_{\varrho_{\mathrm{H}}}(\varphi, \varepsilon)$ and $z \in \Delta^{n-1}$, since $\varphi_1, \ldots, \varphi_n \subset N_\varrho(\varphi, \varepsilon)$, it follows from Lemma 2.1 and the definition of h_n that

$$h_n(\varphi_1, \ldots, \varphi_n; z) \subset h_n(\varphi_1, \ldots, \varphi_n; b_{n-1}) \subset N_\varrho(\varphi, \varepsilon).$$

On the other hand, since $h_n(\varphi_1, \ldots, \varphi_n; z)$ contains some φ_i and since $\varphi \subset N_\varrho(\varphi_i, \varepsilon)$, we have $\varphi \subset N_\varrho(h_n(\varphi_1, \ldots, \varphi_n; z), \varepsilon)$. Therefore,

$$\varrho_{\mathrm{H}}(h_n(\varphi_1, \ldots, \varphi_n; z), \varphi) < \varepsilon \quad (\text{i.e., } h_n(\varphi_1, \ldots, \varphi_n; z) \in B_{\varrho_{\mathrm{H}}}(\varphi, \varepsilon)).$$

Thus (iii) also holds. Consequently, $\mathrm{USCC}_{\mathrm{B}}(X)$ is hyper-connected, hence it is an AR. ■

3. Proof of Main Theorem. We use the following variant of Toruńczyk's characterization of Hilbert space [To$_3$] (cf. [To$_4$]):

3.1. Lemma. *Let* A *be a discrete space and* $H = (H, d)$ *a complete AR with weight* $w(H) = \operatorname{card} A$. *Then* $H \approx \ell_2(A)$ *if and only if the following condition is satisfied*:

(∗∗) *for any open cover* $\mathcal{U}$ *of* H, *there exists a map* $f : H \times A \to H$ *such that* $\{f_a(H) \mid a \in A\}$ *is discrete in* H *and each* f_a *is* $\mathcal{U}$*-close to* id,

where $f_a : H \to H$ *is defined by* $f_a(x) = f(x, a)$.

Proof. Obviously, (∗∗) implies conditions (∗1) and (∗2) in [To$_3$, Theorem 3.1] (cf. [To$_4$]), hence we have the "if" part. The "only if" part easily follows from the fact that the projection $\mathrm{pr}_1 : H \times H \to H$ onto the first factor is a near homeomorphism (cf. [Sc]). ∎

3.2. Lemma. *Assume condition* (∗) *of* §2 *is satisfied*, X *has no isolated points*, *and there exist* $D \subset X$ *and* $\delta, \varepsilon \in (0,1)$ *such that* $d(a, a') \geq \varepsilon$ *for* $a \neq a' \in D$ *and each* $a \in D$ *has a connected neighborhood with diameter* $> \delta$. *Then*, *for any open cover* $\mathcal{U}$ *of* $\mathrm{USCC_B}(X)$, *there exists a map* $h : \mathrm{USCC_B}(X) \times 2^D \to \mathrm{USCC_B}(X)$ *such that* $\{h_F(\mathrm{USCC_B}(X)) \mid F \in 2^D\}$ *is discrete in* $\mathrm{USCC_B}(X)$ *and each* h_F *is* $\mathcal{U}$*-close to* id, *where* $h_F : \mathrm{USCC_B}(X) \to \mathrm{USCC_B}(X)$ *is defined by* $h_F(\varphi) = h(\varphi, F)$.

Proof. Let $\mathcal{V}$ be an open star-refinement of $\mathcal{U}$. Since $\mathrm{USCC_B}(X)$ is an AR (Theorem 2.2), we have a simplicial complex K with maps

$$p : \mathrm{USCC_B}(X) \to |K| \quad \text{and} \quad q : |K| \to \mathrm{USCC_B}(X)$$

such that qp is $\mathcal{V}$-close to id. Let $\alpha : \mathrm{USCC_B}(X) \to (0,1)$ be a map such that $\alpha(\varphi) < \min\{\delta, \varepsilon\}$ for each $\varphi \in \mathrm{USCC_B}(X)$ and

$$\{\overline{B}_{\varrho_{\mathrm{H}}}(\varphi, 2\alpha(\varphi)) \mid \varphi \in \mathrm{USCC_B}(X)\} \prec \mathcal{V}.$$

By subdividing K, we can assume the following two conditions:

(1) $\mathrm{diam}_{\varrho_{\mathrm{H}}} q(\sigma) < \frac{1}{8}\alpha q(y)$ if $y \in \sigma \in K$;
(2) $\alpha q(y) < 2\alpha q(y')$ if $y, y' \in \sigma \in K$.

In fact, for each $\varphi \in \mathrm{USCC_B}(X)$, let

$$W_\varphi = B_{\varrho_{\mathrm{H}}}\big(\varphi, \tfrac{1}{24}\alpha(\varphi)\big) \cap \big\{\psi \in \mathrm{USCC_B}(X) \;\big|\; \tfrac{2}{3}\alpha(\varphi) < \alpha(\psi) < \tfrac{4}{3}\alpha(\varphi)\big\},$$

and subdivide K so that each simplex is contained in some $q^{-1}(W_\varphi)$.

For each $v \in K^{(0)}$, we define $f(v) \in \mathrm{USCC_B}(X)$ as follows:

$$f(v) = q(v) \cup \bigcup_{a \in D} \overline{B}\big(a, \tfrac{1}{8}\alpha q(v)\big) \times [b(v,a), t(v,a)],$$

where

$$b(v,a) = \inf q(v)\big(\overline{B}\big(a, \tfrac{1}{8}\alpha q(v)\big)\big), \qquad t(v,a) = \sup q(v)\big(\overline{B}\big(a, \tfrac{1}{8}\alpha q(v)\big)\big).$$

Obviously $\varrho_{\mathrm{H}}(f(v), q(v)) \leq \frac{1}{8}\alpha q(v)$. If u and v are vertices of the same simplex of K, then

$$\begin{aligned}\varrho_{\mathrm{H}}(f(u), f(v)) &\leq \varrho_{\mathrm{H}}(f(u), q(u)) + \varrho_{\mathrm{H}}(q(u), q(v)) + \varrho_{\mathrm{H}}(f(v), q(v)) \\ &< \tfrac{1}{8}\alpha q(u) + \tfrac{1}{8}\alpha q(v) + \tfrac{1}{8}\alpha q(v) < \tfrac{1}{4}\alpha q(v) + \tfrac{1}{4}\alpha q(v) = \tfrac{1}{2}\alpha q(v).\end{aligned}$$

For the barycenter $\widehat{\sigma}$ of each $\sigma \in K$, we define $f(\widehat{\sigma}) \in \mathrm{USCC_B}(X)$ by

$$f(\widehat{\sigma})(x) = \Big[\min \bigcup_{v\in\sigma^{(0)}} f(v)(x), \max \bigcup_{v\in\sigma^{(0)}} f(v)(x)\Big].$$

Then, by Lemma 2.1, $f(\widehat{\sigma}) \subset N_\varrho\big(f(v), \frac{1}{2}\alpha q(v)\big)$ for each $v \in \sigma^{(0)}$. Observe that if $0 < r \le \min_{v\in\sigma^{(0)}} \frac{1}{8}\alpha q(v)$, then

$$f(\widehat{\sigma})|\overline{B}(a,r) = \overline{B}(a,r) \times [b(\widehat{\sigma},a), t(\widehat{\sigma},a)] \quad \text{for each } a \in D,$$

where $b(\widehat{\sigma},a) = \min_{v\in\sigma^{(0)}} b(v,a)$ and $t(\widehat{\sigma},a) = \max_{v\in\sigma^{(0)}} t(v,a)$.

We define a map $f : |K| \to \mathrm{USCC_B}(X)$ as follows:

$$f(y)(x) = \sum_{i=1}^{k} s_i f(\widehat{\sigma}_i)(x) = \Big[\sum_{i=1}^{k} s_i \min f(\widehat{\sigma}_i)(x), \sum_{i=1}^{k} s_i \max f(\widehat{\sigma}_i)(x)\Big],$$

where $y = \sum_{i=1}^{k} s_i\widehat{\sigma}_i$, $\sigma_1 < \ldots < \sigma_k \in K$, $s_i \ge 0$ and $\sum_{i=1}^{k} s_i = 1$. In the above, note that $\frac{1}{2}\alpha q(y) < \alpha q(v)$ for each $v \in \sigma_k^{(0)}$. Then, for each $a \in D$,

$$f(y)|\overline{B}\big(a, \tfrac{1}{16}\alpha q(y)\big) = \overline{B}\big(a, \tfrac{1}{16}\alpha q(y)\big) \times [\min f(y)(a), \max f(y)(a)].$$

For each $y \in |K|$, choose $v \in \sigma^{(0)}$ so that $y \in |\mathrm{St}(v, \mathrm{Sd}\,K)|$. Since $f(v) \subset f(y) \subset f(\widehat{\sigma}) \subset N_\varrho\big(f(v), \frac{1}{2}\alpha q(v)\big)$, we have $\varrho_\mathrm{H}(f(y), f(v)) < \frac{1}{2}\alpha q(v)$, hence

$$\begin{aligned}\varrho_\mathrm{H}(f(y), q(y)) &\le \varrho_\mathrm{H}(f(y), f(v)) + \varrho_\mathrm{H}(f(v), q(v)) + \varrho_\mathrm{H}(q(v), q(y)) \\ &< \tfrac{1}{2}\alpha q(v) + \tfrac{1}{8}\alpha q(v) + \tfrac{1}{8}\alpha q(v) < \tfrac{3}{4}\alpha q(v) < \tfrac{3}{2}\alpha q(y).\end{aligned}$$

Now, for any $F \in 2^D$, we define $h_F : \mathrm{USCC_B}(X) \to \mathrm{USCC_B}(X)$ by

$$h_F(\varphi) = fp(\varphi) \cup \bigcup_{a\in F} \{a\} \times \big[\max fp(\varphi)(a), \max fp(\varphi)(a) + \tfrac{1}{2}\alpha qp(\varphi)\big].$$

Then h_F is $\mathcal{U}$-close to id. In fact, h_F is $\mathcal{V}$-close to qp because

$$\begin{aligned}\varrho_\mathrm{H}(h_F(\varphi), qp(\varphi)) &\le \varrho_\mathrm{H}(h_F(\varphi), fp(\varphi)) + \varrho_\mathrm{H}(fp(\varphi), qp(\varphi)) \\ &< \tfrac{1}{2}\alpha qp(\varphi) + \tfrac{3}{2}\alpha qp(\varphi) = 2\alpha qp(\varphi).\end{aligned}$$

To show the continuity of h_F, let $\varphi_n \to \varphi$ in $\mathrm{USCC_B}(X)$ as $n \to \infty$. Let $0 < r < \frac{1}{16}\alpha qp(\varphi)$. Since αqp is continuous, $r < \frac{1}{16}\alpha qp(\varphi_n)$ for sufficiently large n, whence for each $a \in F$,

$$\begin{aligned}fp(\varphi_n)|\overline{B}(a,r) &= \overline{B}(a,r) \times [\min fp(\varphi_n)(a), \max fp(\varphi_n)(a)] \quad \text{and} \\ fp(\varphi)|\overline{B}(a,r) &= \overline{B}(a,r) \times [\min fp(\varphi)(a), \max fp(\varphi)(a)].\end{aligned}$$

On the other hand, $fp(\varphi_n) \to fp(\varphi)$ because fp is continuous. Then, as is easily observed, $\max fp(\varphi_n)(a) \to \max fp(\varphi)(a)$ for each $a \in F$. From the definition, it follows that $h_F(\varphi_n) \to h_F(\varphi)$.

We show that $\{h_F(\mathrm{USCC_B}(X)) \mid F \in 2^D\}$ is discrete in $\mathrm{USCC_B}(X)$. Suppose that, on the contrary, there exist $\varphi, \varphi_i \in \mathrm{USCC_B}(X)$ and $F_i \in 2^D$ ($i \in \mathbb{N}$) such that $h_{F_i}(\varphi_i) \to \varphi$ as $i \to \infty$ and $F_i \ne F_j$ if $i \ne j$. Then

$\inf_{i\in\mathbb{N}} \alpha qp(\varphi_i) > 0$. Otherwise, $\lim_{n\to\infty} \alpha qp(\varphi_{i_n}) \to 0$ for some $i_1 < i_2 < \ldots$ As seen above, $\varrho_{\mathrm{H}}(h_{F_{i_n}}(\varphi_{i_n}), qp(\varphi_{i_n})) < 2\alpha qp(\varphi_{i_n})$. Then it follows that $qp(\varphi_{i_n}) \to \varphi$, hence $\alpha qp(\varphi) = \lim_{n\to\infty} \alpha qp(\varphi_{i_n}) = 0$, which is a contradiction.

Let $\varepsilon_0 = \inf_{i\in\mathbb{N}} \frac{1}{16}\alpha qp(\varphi_i) > 0$. For any $i \neq j \in \mathbb{N}$, there exists $a \in D$ such that $a \in F_i\setminus F_j$ or $a \in F_j\setminus F_i$. Without loss of generality, we may assume that $a \in F_j \setminus F_i$. For simplicity, we write $b_i = b(p(\varphi_i), a)$, $t_i = t(p(\varphi_i), a)$, $b_j = b(p(\varphi_j), a)$ and $t_j = t(p(\varphi_j), a)$. Then

$$h_{F_i}(\varphi_i)|\overline{B}(a,\varepsilon_0) = \overline{B}(a,\varepsilon_0) \times [b_i, t_i] \quad \text{and}$$
$$h_{F_j}(\varphi_j)|\overline{B}(a,\varepsilon_0) = \overline{B}(a,\varepsilon_0) \times [b_j, t_j] \cup \{a\} \times [t_j, t_j + \alpha qp(\varphi_j)].$$

In case $t_i \le t_j + \frac{1}{2}\alpha qp(\varphi_j)$, we have

$$\varrho_{\mathrm{H}}(h_{F_i}, h_{F_j}) \ge \varrho((a, t_j + \alpha qp(\varphi_j)), h_{F_i}) \ge \min\{\varepsilon_0, \tfrac{1}{2}\alpha qp(\varphi_j)\} = \varepsilon_0.$$

Recall that a has a connected neighborhood with diameter $> \delta$. Since $\varepsilon_0 < \frac{1}{16}\delta$, there is $c \in X$ so that $d(a, c) = \varepsilon_0/2$. In case $t_i \ge t_j + \frac{1}{2}\alpha qp(\varphi_j)$, it follows that

$$\varrho_{\mathrm{H}}(h_{F_i}, h_{F_j}) \ge \varrho((c, t_i), h_{F_j}) \ge \min\{\varepsilon_0/2, \tfrac{1}{2}\alpha qp(\varphi_j)\} = \varepsilon_0/2.$$

Consequently, $\varrho_{\mathrm{H}}(h_{F_i}(\varphi_i), h_{F_j}(\varphi_j)) \ge \varepsilon_0/2$ if $i \neq j$, whence $h_{F_i}(\varphi_i)$ is not convergent. This is a contradiction. ∎

3.3. LEMMA. *Assume that X is not totally bounded. For each $n \in \mathbb{N}$, let D_n be a maximal subset of X such that $d(x, y) \ge 2^{-n}$ for any distinct points $x, y \in D_n$* ([4]). *Then $w(\mathrm{USCC_B}(X)) = \sup_{n\in\mathbb{N}} 2^{\operatorname{card} D_n}$. In case X is separable, $w(\mathrm{USCC_B}(X)) = 2^{\aleph_0}$* ([5]).

Proof. For each $n \in \mathbb{N}$, let $\mathbb{Q}_n = \{2^{-n}m \mid m \in \mathbb{N}\} \subset \mathbb{R}$. Then $D_n \times \mathbb{Q}_n$ is discrete in $X \times \mathbb{R}$. Since X is not totally bounded, each D_n is infinite, hence $\operatorname{card}(D_n \times \mathbb{Q}_n) = \operatorname{card} D_n$. By the maximality, $d(x, D_n) < 2^{-n}$ for every $x \in X$, hence $\varrho(z, D_n \times \mathbb{Q}_n) < 2^{-n}$ for every $z \in X \times \mathbb{R}$. For each $E \in 2^{X\times\mathbb{R}}$ and $n \in \mathbb{N}$, let

$$F = \{z \in D_n \times \mathbb{Q}_n \mid \varrho(z, E) < 2^{-n}\} \in 2^{D_n\times\mathbb{Q}_n} \subset 2^{X\times\mathbb{R}}.$$

Then $\varrho_{\mathrm{H}}(E, F) \le 2^{-n}$. Hence, $\bigcup_{n\in\mathbb{N}} 2^{D_n\times\mathbb{Q}_n}$ is dense in $2^{X\times\mathbb{R}}$. Since the weight $w(2^{X\times\mathbb{R}})$ is equal to the density of $2^{X\times\mathbb{R}}$, it follows that

$$w(2^{X\times\mathbb{R}}) \le \operatorname{card} \bigcup_{n\in\mathbb{N}} 2^{D_n\times\mathbb{Q}_n}$$
$$\le \sup_{n\in\mathbb{N}} \operatorname{card} 2^{D_n\times\mathbb{Q}_n} = \sup_{n\in\mathbb{N}} 2^{\operatorname{card}(D_n\times\mathbb{Q}_n)} = \sup_{n\in\mathbb{N}} 2^{\operatorname{card} D_n},$$

which implies $w(\mathrm{USCC_B}(X)) \le \sup_{n\in\mathbb{N}} 2^{\operatorname{card} D_n}$.

([4]) The existence of such $D_n \subset X$ is guaranteed by Zorn's Lemma.

([5]) In general, $\sup_{n\in\mathbb{N}} 2^{\operatorname{card} D_n} \neq 2^{\sup_{n\in\mathbb{N}} \operatorname{card} D_n} = 2^{w(X)}$.

On the other hand, for each $n \in \mathbb{N}$ and $F \in 2^{D_n}$, let

$$\varphi_F = F \times \mathbf{I} \cup X \times \{0\} \in \mathrm{USCC_B}(X).$$

Since $\varrho_{\mathrm{H}}(\varphi_F, \varphi_{F'}) \geq 2^{-n}$ for each $F \neq F' \in 2^{D_n}$, $\{B_{\varrho_{\mathrm{H}}}(\varphi_F, 2^{-n-1}) \mid F \in 2^{D_n}\}$ is pairwise disjoint. Therefore, $w(\mathrm{USCC_B}(X)) \geq \operatorname{card} 2^{D_n} = 2^{\operatorname{card} D_n}$, hence $w(\mathrm{USCC_B}(X)) \geq \sup_{n\in\mathbb{N}} 2^{\operatorname{card} D_n}$. ■

Proof of Main Theorem. We apply Lemma 3.1 to show that $\mathrm{USCC_B}(X) \approx \ell_2(A)$, where $\operatorname{card} A = w(\mathrm{USCC_B}(X))$. We have proved that $\mathrm{USCC_B}(X)$ is a completely metrizable AR (Corollary 1.3 and Theorem 2.2). It remains to construct a map $f : \mathrm{USCC_B}(X) \times A \to \mathrm{USCC_B}(X)$ such as in Lemma 3.1. Let $\mathfrak{C}$ be the collection of all components of X and take D_n ($n \in \mathbb{N}$) as in Lemma 3.3. Then observe that

$$\operatorname{card} \mathfrak{C} \leq w(X) = \operatorname{card} \bigcup_{n\in\mathbb{N}} D_n = \sup_{n\in\mathbb{N}} \operatorname{card} D_n.$$

Case (1): $\operatorname{card} \mathfrak{C} = w(X)$. Since X is uniformly locally connected, $\operatorname{card} D_n \geq \operatorname{card} \mathfrak{C} = w(X)$ for sufficiently large $n \in \mathbb{N}$. On the other hand, $\operatorname{card} D_n \leq w(X)$ for all $n \in \mathbb{N}$ by definition. Then $\sup_{n\in\mathbb{N}} 2^{\operatorname{card} D_n} = 2^{w(X)}$, hence Lemma 3.3 yields $\operatorname{card} A = w(\mathrm{USCC_B}(X)) = 2^{w(X)}$.

We can write $\mathfrak{C} = \bigcup_{i\in\mathbb{N}} \mathfrak{C}_i$, where $\mathfrak{C}_i \cap \mathfrak{C}_j = \emptyset$ if $i \neq j$ and $\operatorname{card} \mathfrak{C}_i = w(X)$ for each $i \in \mathbb{N}$. For each $i \in \mathbb{N}$, let $r_i : \mathrm{USCC_B}(X) \to m(\mathfrak{C}_i)$ be the map defined by $r_i(\varphi)(C) = \sup \varphi(C)$ ($\leq \sup \varphi(X)$) for each $C \in \mathfrak{C}_i$. Since $m(\mathfrak{C}_i) \approx \ell_2(2^{\mathfrak{C}_i})$ ([BP, Ch. VII, Theorem 6.1]) and $w(\mathrm{USCC_B}(X) \times A) = 2^{w(X)} = \operatorname{card} 2^{\mathfrak{C}_i}$, there is a closed embedding $g_i : \mathrm{USCC_B}(X) \times A \to m(\mathfrak{C}_i)$ such that $\|g_i(\varphi, a) - r_i(\varphi)\| < 2^{-i}$ for each $(\varphi, a) \in \mathrm{USCC_B}(X) \times A$. Note that $\{(g_i)_a(\mathrm{USCC_B}(X)) \mid a \in A\}$ is discrete in $\mathrm{USCC_B}(X)$.

For any open cover $\mathcal{U}$ of $\mathrm{USCC_B}(X)$, let $\alpha : \mathrm{USCC_B}(X) \to (0,1)$ be a map such that $\{B_{\varrho_{\mathrm{H}}}(\varphi, \alpha(\varphi)) \mid \varphi \in \mathrm{USCC_B}(X)\} \prec \mathcal{U}$. Now, we define a map $f : \mathrm{USCC_B}(X) \times A \to \mathrm{USCC_B}(X)$ as follows:

$$f(\varphi, a)(x) = \begin{cases} \varphi(x) + g_i(\varphi, a)(C) - r_i(\varphi)(C) \\ \qquad \text{for } x \in C \in \mathfrak{C}_i \text{ and } 2^{-i+1} < \alpha(\varphi), \\ \varphi(x) + 2^i(\alpha(\varphi) - 2^{-i})(g_i(\varphi, a)(C) - r_i(\varphi)(C)) \\ \qquad \text{for } x \in C \in \mathfrak{C}_i \text{ and } 2^{-i} \leq \alpha(\varphi) \leq 2^{-i+1}, \\ \varphi(x) \quad \text{otherwise.} \end{cases}$$

Then f_a is $\mathcal{U}$-close to id. In fact, for every $C \in \mathfrak{C}_i$,

$$|g_i(\varphi, a)(C) - r_i(\varphi)(C)| \leq \|g_i(\varphi, a) - r_i(\varphi)\| < 2^{-i},$$

hence $\varrho_{\mathrm{H}}(f_a(\varphi), \varphi) < \alpha(\varphi)$.

We prove that $\{f_a(\mathrm{USCC_B}(X)) \mid a \in A\}$ is discrete in $\mathrm{USCC_B}(X)$. Suppose that, on the contrary, there is a sequence $(\varphi_k, a_k) \in \mathrm{USCC_B}(X) \times A$ ($k \in \mathbb{N}$) such that $a_k \neq a_{k'}$ if $k \neq k'$, and $f_{a_k}(\varphi_k)$ converges to some

$\varphi_0 \in \mathrm{USCC_B}(X)$. Then there is some $i_0 \in \mathbb{N}$ such that $2^{-i_0+1} < \alpha(\varphi_k)$ for all $k \in \mathbb{N}$. Otherwise, $\lim_{j\to\infty} \alpha(\varphi_{k(j)}) = 0$ for some $k(1) < k(2) < \ldots$, whence $\lim_{j\to\infty} \varrho_{\mathrm{H}}(f_{a_{k(j)}}(\varphi_{k(j)}), \varphi_{k(j)}) = 0$. It follows that $\varphi_{k(j)}$ converges to φ_0, so $\alpha(\varphi_0) = \lim_{j\to\infty} \alpha(\varphi_{k(j)}) = 0$, which is a contradiction. For each $C \in \mathfrak{C}_{i_0}$,

$$\begin{aligned} r_{i_0}(f_{a_k}(\varphi_k))(C) &= \sup f(\varphi_k, a_k)(C) \\ &= \sup \varphi_k(C) + g_{i_0}(\varphi_k, a_k)(C) - r_{i_0}(\varphi_k)(C) \\ &= g_{i_0}(\varphi_k, a_k)(C) = (g_{i_0})_{a_k}(\varphi_k). \end{aligned}$$

Since r_{i_0} is continuous, $(g_{i_0})_{a_k}(\varphi_k) = r_{i_0}(f_{a_k}(\varphi_k))$ converges to $r_{i_0}(\varphi_0)$, which contradicts the fact that $\{(g_{i_0})_a(\mathrm{USCC_B}(X)) \mid a \in A\}$ is discrete in $\mathrm{USCC_B}(X)$. Therefore, $\{f_a(\mathrm{USCC_B}(X)) \mid a \in A\}$ is discrete in $\mathrm{USCC_B}(X)$.

Case (2): $\operatorname{card} \mathfrak{C} < w(X)$. Since X is uniformly locally connected, we may assume the condition $(*)$ of §2. Let X_0 be the set of isolated points of X. Then $d(x, X \setminus \{x\}) \geq 1$ for every $x \in X_0$ by $(*)$. As is easily seen,

$$\mathrm{USCC_B}(X) \approx \mathrm{USCC_B}(X_0) \times \mathrm{USCC}(X \setminus X_0).$$

For each $n \in \mathbb{N}$, let $D'_n = D_n \setminus X_0$. Since $\operatorname{card} X_0 \leq \operatorname{card} \mathfrak{C} < w(X) = \sup_{n\in\mathbb{N}} \operatorname{card} D_n$, we have $\operatorname{card} X_0 < \operatorname{card} D_n$ for sufficiently large $n \in \mathbb{N}$, whence $\operatorname{card} D'_n = \operatorname{card} D_n$. By Lemma 3.3,

$$\begin{aligned} w(\mathrm{USCC_B}(X \setminus X_0)) &= \sup_{n\in\mathbb{N}} 2^{\operatorname{card} D'_n} = \sup_{n\in\mathbb{N}} 2^{\operatorname{card} D_n} \\ &= w(\mathrm{USCC_B}(X)). \end{aligned}$$

In case (1) above, we have shown that $\mathrm{USCC_B}(X_0)$ is homeomorphic to a Hilbert space, hence it is a completely metrizable AR with

$$w(\mathrm{USCC_B}(X_0)) \leq w(\mathrm{USCC_B}(X)).$$

By [To$_2$, Theorem 3.1], it suffices to show that $\mathrm{USCC_B}(X \setminus X_0)$ is homeomorphic to a Hilbert space with the same weight. Thus we can assume that X has no isolated points.

For each $\delta > 0$, let $\mathfrak{C}(\delta) = \{C \in \mathfrak{C} \mid \operatorname{diam} C < \delta\}$. Let

$$D^1_n = D_n \setminus \bigcup \mathfrak{C}(2^{-n}) \quad \text{for each } n \in \mathbb{N}.$$

Note that each point of D^1_n has a connected neighborhood in X with diam $\geq 2^{-n}$ because it is contained in a component of X with diam $\geq 2^{-n}$. Each member of $\mathfrak{C}(2^{-n})$ contains at most one point of D_n. Recall that $\operatorname{card} \mathfrak{C} < w(X) = \sup_{n\in\mathbb{N}} \operatorname{card} D_n$. Then, for sufficiently large $n \in \mathbb{N}$,

$$\operatorname{card}\Big(D_n \cap \bigcup \mathfrak{C}(2^{-n})\Big) \leq \operatorname{card} \mathfrak{C}(2^{-n}) \leq \operatorname{card} \mathfrak{C} < \operatorname{card} D_n,$$

whence $\operatorname{card} D_n = \operatorname{card} D_n^1$. Therefore, it follows from Lemma 3.3 that

$$\operatorname{card}\Big(\bigcup_{n\in\mathbb{N}} \{n\} \times 2^{D_n^1}\Big) = \sup_{n\in\mathbb{N}} \operatorname{card} 2^{D_n^1} = \sup_{n\in\mathbb{N}} 2^{\operatorname{card} D_n^1}$$
$$= \sup_{n\in\mathbb{N}} 2^{\operatorname{card} D_n} = w(\mathrm{USCC_B}(X)).$$

Thus we may assume that

$$A = \bigcup_{n\in\mathbb{N}} \{n\} \times 2^{D_n^1}.$$

For any open cover $\mathcal{U}$ of $\mathrm{USCC_B}(X)$, let $\mathcal{V}$ be an open star-refinement of $\mathcal{U}$. Since X is not totally bounded, we can apply Lemma 3.2 to obtain a map $g : \mathrm{USCC_B}(X) \times \mathbb{N} \to \mathrm{USCC_B}(X)$ such that $\{g_n(\mathrm{USCC_B}(X)) \mid n \in \mathbb{N}\}$ is discrete in $\mathrm{USCC_B}(X)$ and each g_n is $\mathcal{V}$-close to id. Choose an open refinement $\mathcal{W}$ of $\mathcal{V}$ so that the star $\mathrm{st}(W, \mathcal{W})$ of each $W \in \mathcal{W}$ meets at most one of $g_n(\mathrm{USCC_B}(X))$. Applying Lemma 3.2 again, we obtain maps $h_n : \mathrm{USCC_B}(X) \times 2^{D_n^1} \to \mathrm{USCC_B}(X)$ $(n \in \mathbb{N})$ such that $\{(h_n)_F(\mathrm{USCC_B}(X)) \mid F \in 2^{D_n^1}\}$ is discrete in $\mathrm{USCC_B}(X)$ and each $(h_n)_F$ is $\mathcal{W}$-close to id. Then we define a map $f : \mathrm{USCC_B}(X) \times A \to \mathrm{USCC_B}(X)$ by

$$f(\varphi, (n, F)) = h_n(g(\varphi, n), F) \quad (\text{i.e., } f_{(n,F)}(\varphi) = (h_n)_F \circ g_n(\varphi)).$$

Each $f_{(n,F)}$ is $\mathcal{U}$-close to id because it is $\mathcal{W}$-close to g_n.

We show that the collection $\{f_{(n,F)}(\mathrm{USCC_B}(X)) \mid (n, F) \in A\}$ is discrete in $\mathrm{USCC_B}(X)$. Each $\varphi \in \mathrm{USCC_B}(X)$ is contained in some $W \in \mathcal{W}$. Then this W meets at most one member of $\{f(\mathrm{USCC_B}(X) \times \{n\} \times 2^{D_n^1}) \mid n \in \mathbb{N}\}$. In fact, if $f_{(n,F)}(\psi), f_{(n',F')}(\psi') \in W$ for some $\psi, \psi' \in \mathrm{USCC_B}(X)$, $n \neq n' \in \mathbb{N}$, $F \in 2^{D_n^1}$ and $F' \in 2^{D_{n'}^1}$, then $g_n(\psi), g_{n'}(\psi') \in \mathrm{st}(W, \mathcal{W})$, which is a contradiction. In case

$$W \cap f(\mathrm{USCC_B}(X) \times \{n\} \times 2^{D_n^1}) \neq \emptyset,$$

we can choose a neighborhood W' of φ so that $W' \subset W$ and W' meets at most one of $(h_n)_F(\mathrm{USCC_B}(X))$. Since

$$f_{(n,F)}(\mathrm{USCC_B}(X)) = (h_n)_F \circ g_n(\mathrm{USCC_B}(X)) \subset (h_n)_F(\mathrm{USCC_B}(X)),$$

W' meets at most one of $f_{(n,F)}(\mathrm{USCC_B}(X))$. Thus $\{f_{(n,F)}(\mathrm{USCC_B}(X)) \mid (n, F) \in A\}$ is discrete in $\mathrm{USCC_B}(X)$.

Finally, we show that $\mathrm{USCC}(X, [-1, 1]) \approx \ell_2(A)$ (i.e., $\mathrm{USCC}(X, \mathbf{I}) \approx \ell_2(A)$). Let

$$B = \{\varphi \in \mathrm{USCC}(X, [-1, 1]) \mid \inf \varphi(X) = -1 \text{ or } \sup \varphi(X) = 1\}.$$

Then B is clearly closed in $\mathrm{USCC}(X, [-1, 1])$ and

$$\mathrm{USCC}(X, [-1, 1]) \setminus B \approx \mathrm{USCC_B}(X) \approx \ell_2(A).$$

We show that B is a strong Z-set in $\mathrm{USCC}(X,[-1,1])$, whence we obtain $\mathrm{USCC}(X,[-1,1]) \approx \ell_2(A)$ by [To$_4$, Theorem B1] (cf. [To$_2$]). For any map $\alpha : \mathrm{USCC}(X,[-1,1]) \to (0,1)$, we define a map

$$h : \mathrm{USCC}(X,[-1,1]) \to \mathrm{USCC}(X,[-1,1])$$

by $h(\varphi)(x) = \left(1 - \frac{1}{2}\alpha(\varphi)\right) \cdot \varphi(x)$. Then $\varrho_{\mathrm{H}}(h(\varphi),\varphi) < \alpha(\varphi)$ for each $\varphi \in \mathrm{USCC}(X,[-1,1])$. For every $\varphi_0 \in \mathrm{cl}\, h(\mathrm{USCC}(X,[-1,1]))$, there is a sequence $\varphi_k \in \mathrm{USCC}(X,\mathbf{I})$ ($k \in \mathbb{N}$) such that $h(\varphi_k) \to \varphi_0$. Then $b = \inf_{k\in\mathbb{N}} \alpha(\varphi_k) > 0$. Otherwise, $\lim_{j\to\infty} \alpha(\varphi_{k_j}) = 0$ for some $k_1 < k_2 < \ldots$, hence φ_{k_j} converges to φ_0, so $\alpha(\varphi_0) = \lim_{j\to\infty} \alpha(\varphi_{k_j}) = 0$, which is a contradiction. For each $k \in \mathbb{N}$,

$$\sup \bigcup_{x\in X} h(\varphi_k)(x) = \left(1 - \frac{1}{2}\alpha(\varphi_k)\right) \cdot \sup \bigcup_{x\in X} \varphi_k(x) \le 1 - \frac{1}{2}b,$$

hence $\sup \bigcup_{x\in X} \varphi_0(x) \le 1 - \frac{1}{2}b < 1$. Similarly, we have $\inf \bigcup_{x\in X} \varphi_0(x) \ge -1 + \frac{1}{2}b > -1$. Therefore, $\varphi_0 \notin B$. This means that

$$B \cap \mathrm{cl}\, h(\mathrm{USCC}(X,[-1,1])) = \emptyset.$$

Thus B is a strong Z-set in $\mathrm{USCC}(X,[-1,1])$. ■

REMARK. Let P be the convex set in the Banach space $C_{\mathrm{B}}(X)^2 = C_{\mathrm{B}}(X) \times C_{\mathrm{B}}(X)$ defined as follows:

$$P = \{(f,g) \in C_{\mathrm{B}}(X)^2 \mid g(x) \ge 0 \text{ for all } x \in X\}.$$

Then it is easy to see that if $X = (X,d)$ is a discrete metric space (i.e., $\inf\{d(x,y) \mid x \ne y\} > 0$), then $\mathrm{USCC}_{\mathrm{B}}(X) \approx P$. In fact, for each $\varphi \in \mathrm{USCC}_{\mathrm{B}}(X)$, we define $m_\varphi, r_\varphi \in C_{\mathrm{B}}(X)$ by

$$m_\varphi(x) = \tfrac{1}{2}(\min \varphi(x) + \max \varphi(x)),$$
$$r_\varphi(x) = \tfrac{1}{2}(\max \varphi(x) - \min \varphi(x)).$$

Then the desired homeomorphism $\xi : \mathrm{USCC}_{\mathrm{B}}(X) \to P$ can be defined by $\xi(\varphi) = (m_\varphi, r_\varphi)$.

4. Remarks on topologies for $C_{\mathrm{B}}(X)$ and $C(X,\mathbf{I})$. Although the spaces $C_{\mathrm{B}}(X)$ and $C(X,\mathbf{I})$ with the sup-metric are AR's for an arbitrary metric space X, the example in the Introduction also shows that the spaces $C_{\mathrm{B}}(X)$ and $C(X,\mathbf{I})$ with the Hausdorff metric ϱ_{H} are not ANR's even if X is locally connected. One should also remark that $C_{\mathrm{B}}(X)$ is not a topological linear space in this topology. In fact, it can easily be derived from [FK, Remark 3.6] that the addition $C_{\mathrm{B}}(\mathbb{R})^2 \to C_{\mathrm{B}}(\mathbb{R})$ ($(f,g) \mapsto f+g$) is not continuous with respect to the Hausdorff metric. However, we can prove the following:

4.1. Theorem. *For any uniformly locally connected metric space* $X = (X,d)$, *the spaces* $C_{\mathrm{B}}(X)$ *and* $C(X,\mathbf{I})$ *with the Hausdorff metric are AR's.*

A subset Z of a space Y is said to be *homotopy dense* in Y if there exists a homotopy $h : Y \times \mathbf{I} \to Y$ such that $h_0 = \mathrm{id}$ and $h_t(Y) \subset Z$ for $t > 0$. As is easily observed, a homotopy dense subset of an AR (resp. ANR) is also an AR (resp. ANR). By Theorem 2.2, in case X has no isolated points, Theorem 4.1 is deduced from the following:

4.2. Theorem. *For any uniformly locally connected metric space* $X = (X,d)$ *with no isolated points*, $C_{\mathrm{B}}(X)$ (*resp.* $C(X,\mathbf{I})$) *is homotopy dense in* $\mathrm{USCC}_{\mathrm{B}}(X)$ (*resp.* $\mathrm{USCC}(X,\mathbf{I})$).

As a corollary of Theorem 4.2, we also have the following:

4.3. Corollary. *Let* $X = (X,d)$ *be an infinite* σ-*compact complete metric space*, *which is assumed to be uniformly locally connected in case* X *is non-compact. Then* $C_{\mathrm{B}}(X)$ *and* $C(X,\mathbf{I})$ *with the Hausdorff metric are homeomorphic to a Hilbert space.*

To prove Theorem 4.2, we need the following non-compact version of [SU, Lemma 2]:

4.4. Lemma. *Assume that condition* $(*)$ *of* §2 *holds and* X *has no isolated points. Let* $f_0 : K^{(0)} \to C_{\mathrm{B}}(X)$ *be a map of the* 0-*skeleton of a locally finite simplicial complex* K *such that* $\operatorname{diam}_{\varrho_{\mathrm{H}}} f_0(\sigma^{(0)}) < 1$ *for every* $\sigma \in K$, *where* $\sigma^{(0)} = \sigma \cap K^{(0)}$. *Then* f_0 *extends to a map* $f : |K| \to C_{\mathrm{B}}(X)$ *such that*

$$\operatorname{diam}_{\varrho_{\mathrm{H}}} f(\sigma) \le 4 \operatorname{diam}_{\varrho_{\mathrm{H}}} f_0(\sigma^{(0)}) \quad \text{for every } \sigma \in K,$$

where $C_{\mathrm{B}}(X)$ *has the topology induced by* ϱ_{H}. ■

Sketch of proof. By Lemma 2.1, we have property (♯). Then the proof is the same as that of [SU, Lemma 2], with $C(X,(-1,1))$ replaced by $C_{\mathrm{B}}(X)$. Now, since X is not compact, we cannot take $A_v \subset X$ as a finite set in the proof, but since K is locally finite and X has no isolated points, we can take $A_v \subset X$ as a discrete set with the same property, that is, $f(v) \subset N_\varrho(f(v)|A_v, \varepsilon_v)$ (in other words, $f(v)|A_v = f(v) \cap p^{-1}(A_v)$ is ε_v-dense in $f(v)$), and each A_v has an open neighborhood U_v in X with $U_v \cap U_{v'} = \emptyset$ if $v \ne v' \in \sigma^{(0)}$ and $\sigma \in K$. No other change is necessary. ■

Remark. In the above, if $\operatorname{card} \mathrm{St}(v_0,K) > \operatorname{card} X$ at some vertex $v_0 \in K^{(0)}$, it is impossible to obtain discrete sets $A_v \subset X$, $v \in K^{(0)}$, such that $A_v \cap A_{v_0} = \emptyset$ for every $v \in \mathrm{St}(v_0,K)^{(0)}$. Then the local finiteness of K is assumed.

We can apply Lemma 4.4 to prove the following result the same way as [SU, Lemma 3].

4.5. Lemma. *Let $X = (X, d)$ be a uniformly locally connected metric space with no isolated points and $f : Y \to \mathrm{USCC_B}(X)$ a map of a separable metrizable space Y. Then there exists a homotopy $h : Y \times \mathbf{I} \to \mathrm{USCC_B}(X)$ such that $h_0 = f$ and $h_t(Y) \subset C_\mathrm{B}(X)$ for $t > 0$.*

Proof. By replacing the metric d by d_c, we can assume condition $(*)$ of §2. For each $n \in \mathbb{N}$, let $\mathcal{U}_n$ be an open cover of $\mathrm{USCC_B}(X)$ with $\mathrm{mesh}_{\varrho_\mathrm{H}} \mathcal{U}_n < (n+1)^{-1}$. Since Y is separable metrizable, the open cover $f^{-1}(\mathcal{U}_n)$ of Y has a countable star-finite open refinement $\mathcal{V}_n$, whence the nerve of $\mathcal{V}_n$ is locally finite. We define

$$\mathcal{W}_1 = \{U \times (2^{-1}, 1] \mid U \in \mathcal{U}_1\},$$
$$\mathcal{W}_n = \{U \times ((n+1)^{-1}, (n-1)^{-1}) \mid U \in \mathcal{U}_n\} \quad \text{for } n > 1.$$

Thus we have a star-finite open cover $\mathcal{W} = \bigcup_{n\in\mathbb{N}} \mathcal{W}_n$ of $Y \times (0,1]$. Let K be the nerve of $\mathcal{W}$ and $g : Y \times (0,1] \to |K|$ a canonical map, that is, each $g(y,t)$ is contained in the simplex spanned by all vertices $W \in \mathcal{W}$ containing (y,t). Then K is locally finite. For each $n \in \mathbb{N}$, let K_n be the nerve of $\mathcal{W}_n \cup \mathcal{W}_{n+1}$. Then each K_n is a subcomplex of K and $K = \bigcup_{n\in\mathbb{N}} K_n$. Note that $K^{(0)} = \bigcup_{n\in\mathbb{N}} \mathcal{W}_n$. For each $W \in \mathcal{W}_n$, since $\mathrm{pr}_Y(W) \in \mathcal{V}_n \prec f^{-1}(\mathcal{U}_n)$, we can choose $\pi(W) \in \mathcal{U}_n$ so that $f\,\mathrm{pr}_Y(W) \subset \pi(W)$.

Since $C_\mathrm{B}(X)$ is dense in $\mathrm{USCC_B}(X)$ by Theorem 1.5, we can also choose $k_0(W) \in \pi(W) \cap C_\mathrm{B}(X)$, whence $\varrho_\mathrm{H}(k_0(W), f(y)) \le \mathrm{mesh}_{\varrho_\mathrm{H}} \mathcal{U}_n < (n+1)^{-1}$ for any $y \in \mathrm{pr}_Y(W)$. Thus we have a map $k_0 : K^{(0)} \to C_\mathrm{B}(X)$ such that $\varrho_\mathrm{H}(k_0(W), f(y)) < (n+1)^{-1}$ for any $W \in K_n^{(0)} = \mathcal{W}_n$ and $y \in \mathrm{pr}_Y(W)$, hence $\mathrm{diam}_{\varrho_\mathrm{H}} k_0(\sigma^{(0)}) < 2(n+1)^{-1}$ for each $\sigma \in K_n$. By using Lemma 4.4, we can extend k_0 to a map $k : |K| \to C_\mathrm{B}(X)$ such that $\mathrm{diam}_{\varrho_\mathrm{H}} k(\sigma) < 4\,\mathrm{diam}_{\varrho_\mathrm{H}} k_0(\sigma^{(0)})$. Thus we obtain the map

$$kg : Y \times (0,1] \to C_\mathrm{B}(X) \subset \mathrm{USCC_B}(X).$$

For each $(y,t) \in Y \times (0,1]$, choose $n \in \mathbb{N}$ and $W \in \mathcal{W}_n$ so that $(n+1)^{-1} < t \le n^{-1}$ and $(y,t) \in W$. Then there is $\sigma \in K_n$ such that $g(y,t) \in \sigma$ and $W \in \sigma^{(0)}$. Since $k(W), kg(y,t) \in k(\sigma)$ and $\mathrm{diam}_{\varrho_\mathrm{H}} k(\sigma) < 4\,\mathrm{diam}_{\varrho_\mathrm{H}} k(\sigma^{(0)}) < 8(n+1)^{-1}$, it follows that

$$\begin{aligned}\varrho_\mathrm{H}(kg(y,t), f(y)) &\le \varrho_\mathrm{H}(kg(y,t), k(W)) + \varrho_\mathrm{H}(k(W), f(y)) \\ &< 8(n+1)^{-1} + (n+1)^{-1} = 9(n+1)^{-1} < 9t.\end{aligned}$$

Then kg can be extended to the desired homotopy h by $h_0 = f$. ■

Remark. In the above lemma, the separability of Y is necessary because the local finiteness of K is assumed in Lemma 4.4. Note that $\mathrm{USCC_B}(X)$ is non-separable.

A subset $Z \subset Y$ is called *locally homotopy negligible* in Y if every neighborhood U of each point $x \in X$ contains a neighborhood V of x such that each map $f : (I^n, \partial I^n) \to (V, V \setminus Z)$, $n \in \mathbb{N}$, is homotopic in $(U, U \setminus Z)$ to a map g with $g(I^n) \subset U \setminus Z$ (cf. [To_1]). By using Lemma 4.5, it is easy to prove the following:

4.6. Corollary. *For any uniformly locally connected metric space* $X = (X, d)$ *with no isolated points*, $\mathrm{USCC_B}(X) \setminus C_\mathrm{B}(X)$ *is locally homotopy negligible in* $\mathrm{USCC_B}(X)$. ∎

Proof of Theorem 4.2. Since $\mathrm{USCC_B}(X)$ is an AR by Theorem 2.2, according to [To_1, Theorem 2.4], Corollary 4.6 implies that $C_\mathrm{B}(X)$ is homotopy dense in $\mathrm{USCC_B}(X)$.

By small adjustments, we can see that Lemmas 4.4 and 4.5 are valid for $\mathrm{USCC}(X, \mathbf{I})$. It follows that $C(X, \mathbf{I})$ is homotopy dense in $\mathrm{USCC}(X, \mathbf{I})$ for any uniformly locally connected metric space $X = (X, d)$ with no isolated points. ∎

Proof of Theorem 4.1. Let X_0 be the set of all isolated points of X. Since X is uniformly locally connected, there is $\delta > 0$ such that $d(a, X \setminus \{a\}) > \delta$ for every $a \in X_0$. It is easy to see that

$$C_\mathrm{B}(X) \approx C_\mathrm{B}(X_0) \times C_\mathrm{B}(X \setminus X_0),$$

where the topology of each space is induced by the Hausdorff metric ϱ_H. By Theorems 2.2 and 4.2, $C_\mathrm{B}(X \setminus X_0)$ with the Hausdorff metric is an AR. On the other hand, $C_\mathrm{B}(X_0)$ with the Hausdorff metric is also an AR because the Hausdorff metric on $C_\mathrm{B}(X_0)$ induces the same topology as the sup-norm. Therefore, $C_\mathrm{B}(X)$ with the Hausdorff metric is an AR. Moreover, $C(X, \mathbf{I})$ with the Hausdorff metric is also an AR because it is a retract of $C_\mathrm{B}(X)$ with the Hausdorff metric. ∎

Proof of Corollary 4.3. In case X is compact and infinite, the Hausdorff metric induces the same topology as the sup-metric. The separable Banach space $C(X) = C_\mathrm{B}(X)$ is homeomorphic to the separable Hilbert space ℓ_2 [BP, Ch. VI, Theorem 5.1]. The space $C(X, \mathbf{I})$ is homeomorphic to the closed unit ball $C(X, [-1, 1])$ of $C(X)$, hence $C(X, \mathbf{I}) \approx \ell_2$ [BP, Ch. VI, Theorem 5.1].

If X is non-compact, then as in Theorem 4.1, the corollary reduces to the case where X has no isolated points. It then suffices to show that $\mathrm{USCC_B}(X) \setminus C_\mathrm{B}(X)$ is an F_σ-set in $\mathrm{USCC_B}(X)$. In fact, $\mathrm{USCC_B}(X) \setminus C_\mathrm{B}(X)$ would be a Z_σ-set in $\mathrm{USCC_B}(X)$ by Theorem 4.2, hence $\mathrm{USCC_B}(X) \approx C_\mathrm{B}(X)$ by [Cu, Corollary 1]. Moreover, since $\mathrm{USCC}(X, \mathbf{I}) \setminus C(X, \mathbf{I})$ would also be an F_σ-set in $\mathrm{USCC}(X, \mathbf{I})$, it would similarly follow that $\mathrm{USCC}(X, \mathbf{I}) \approx C(X, \mathbf{I})$.

Since X is σ-compact, X has compact subsets $X_1 \subset X_2 \subset \ldots$ with $X = \bigcup_{n \in \mathbb{N}} X_n$. For each $n \in \mathbb{N}$, let

$$F_n = \{\varphi \in \mathrm{USCC_B}(X) \mid \text{there is } x \in X_n \text{ such that } \operatorname{diam}\varphi(x) \ge 1/n\}.$$

Then $\mathrm{USCC_B}(X) \setminus C_\mathrm{B}(X) = \bigcup_{n\in\mathbb{N}} F_n$. To see that each F_n is closed in $\mathrm{USCC_B}(X)$, let $\varphi_i \in F_n$, $i \in \mathbb{N}$, and assume $\varphi_i \to \varphi \in \mathrm{USCC_B}(X)$ as $i \to \infty$. Then all φ_i and φ are contained in some $X \times [-r, r]$. For each $i \in \mathbb{N}$, there is $x_i \in X_n$ such that $\operatorname{diam}\varphi(x_i) \ge 1/n$, whence there are $s_i, t_i \in \varphi(x_i)$ with $t_i - s_i \ge 1/n$. Since X_n and $[-r, r]$ are compact, we may assume that $x_i \to x$ in X_n, $s_i \to s$ and $t_i \to t$ in $[-r, r]$. Then $t - s \ge 1/n$ and $s, t \in \varphi(x)$. Thus we have $\operatorname{diam}\varphi(x) \ge 1/n$, hence $\varphi \in F_n$. Therefore, $\mathrm{USCC_B}(X) \setminus C_\mathrm{B}(X)$ is an F_σ-set in $\mathrm{USCC_B}(X)$. ■

Let $C^\mathrm{U}_\mathrm{B}(X)$ be the subspace of the Banach space $C_\mathrm{B}(X)$ consisting of the uniformly continuous functions, and $C^\mathrm{U}(X, \mathbf{I}) = C(X, \mathbf{I}) \cap C^\mathrm{U}_\mathrm{B}(X)$. In case X is compact, $C^\mathrm{U}_\mathrm{B}(X) = C(X)$ and $C^\mathrm{U}(X, \mathbf{I}) = C(X, \mathbf{I})$. As just seen, the Banach space $C_\mathrm{B}(X)$ is not a subspace of $\mathrm{USCC_B}(X)$, but $C^\mathrm{U}_\mathrm{B}(X)$ can be regarded as a subspace of $\mathrm{USCC_B}(X)$, that is, we have

4.7. Proposition. *The topology of $C^\mathrm{U}_\mathrm{B}(X)$ induced by the sup-norm $\|\cdot\|$ coincides with the one induced by the Hausdorff metric ϱ_H.*

Proof. Let $f \in C^\mathrm{U}_\mathrm{B}(X)$. By the uniform continuity of f, for each $\varepsilon > 0$, there exists $\delta > 0$ such that $d(x, y) < \delta$ implies $|f(x) - f(y)| < \varepsilon/2$. Let $g \in C_\mathrm{B}(X)$ be such that $\varrho_\mathrm{H}(f, g) < \min\{\varepsilon/2, \delta\}$. For each $x \in X$, since $\varrho((x, g(x)), f) < \min\{\varepsilon/2, \delta\}$, we can choose $y \in X$ so that

$$\varrho((x, g(x)), (y, f(y))) = \max\{d(x, y),\ |g(x) - f(y)|\} < \min\{\varepsilon/2, \delta\}.$$

Since $d(x, y) < \delta$, we have $|f(x) - f(y)| < \varepsilon/2$. Hence,

$$|f(x) - g(x)| \le |f(x) - f(y)| + |f(y) - g(x)| < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon.$$

Therefore, $\|f - g\| < \varepsilon$. Conversely, if $\|f - g\| < \varepsilon$ then

$$\begin{aligned} \varrho_\mathrm{H}(f, g) &= \max\{\sup_{x\in X} \varrho((x, f(x)), g),\ \sup_{x\in X} \varrho((x, g(x)), f)\} \\ &\le \sup_{x\in X} |f(x) - g(x)| = \|f - g\| < \varepsilon. \ ■ \end{aligned}$$

Comparing with the result of the previous paper [SU], one may want to replace $C(X, \mathbf{I})$ and $C_\mathrm{B}(X)$ in Theorem 1.5 (or Corollary 1.6) by $C^\mathrm{U}(X, \mathbf{I})$ and $C^\mathrm{U}_\mathrm{B}(X)$, respectively, since the latter are subspaces of $\mathrm{USCC}(X, \mathbf{I})$ and $\mathrm{USCC_B}(X)$, respectively. However, $C^\mathrm{U}(X, \mathbf{I})$ is not dense in $\mathrm{USCC}(X, \mathbf{I})$ even if X is locally connected and has no isolated point. In fact, let $X = \bigcup_{n\in\mathbb{N}} [n - n^{-1}, n] \subset \mathbb{R}$ and define $f \in C(X, \mathbf{I}) \subset \mathrm{USCC}(X, \mathbf{I})$ by $f(n - t) = nt$ if $0 \le t \le n^{-1}$. Then no $g \in C(X, \mathbf{I})$ with $\varrho_\mathrm{H}(f, g) < 1/4$ is uniformly continuous because $1/4, 3/4 \in g([n - n^{-1}, n])$ for all $n \in \mathbb{N}$. As another example, let $X = \mathbb{R} \setminus \{0\}$ and define $f \in C(X, \mathbf{I}) \subset \mathrm{USCC}(X, \mathbf{I})$ by $f(x) = 0$ for $x < 0$ and $f(x) = 1$ for $x > 0$. Then no $g \in C(X, \mathbf{I})$ with $\varrho_\mathrm{H}(f, g) < 1/4$ is uniformly continuous because $g(x) < 1/4$ if $x < 0$ and $g(x) > 3/4$ if $x > 0$.

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