

**Topological entropy
of nonautonomous piecewise monotone
dynamical systems on the interval**

by

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Dedicated to the memory of Wiesław Szlenk

Abstract. The topological entropy of a nonautonomous dynamical system given by a sequence of compact metric spaces $(X_i)_{i=1}^{\infty}$ and a sequence of continuous maps $(f_i)_{i=1}^{\infty}$, $f_i : X_i \rightarrow X_{i+1}$, is defined. If all the spaces are compact real intervals and all the maps are piecewise monotone then, under some additional assumptions, a formula for the entropy of the system is obtained in terms of the number of pieces of monotonicity of $f_n \circ \dots \circ f_2 \circ f_1$. As an application we construct a large class of smooth triangular maps of the square of type 2^{∞} and positive topological entropy.

1. Introduction and main results. Let $X_{1,\infty} := (X_i)_{i=1}^{\infty}$ be a sequence of compact metric spaces and $f_{1,\infty} := (f_i)_{i=1}^{\infty}$ a sequence of continuous maps, where f_i is a map from X_i to X_{i+1} . For any positive integers i, n set $f_i^n = f_{i+(n-1)} \circ \dots \circ f_{i+1} \circ f_i$ and additionally $f_i^0 = \text{id}_{X_i}$. We also write $f_i^{-n} = (f_i^n)^{-1}$ (this notation will be applied to sets; we do not assume that the maps f_i are invertible). We call $(X_{1,\infty}; f_{1,\infty})$ a *nonautonomous discrete dynamical system*. The *trajectory* of a point $x \in X_1$ is the sequence $(f_1^n(x))_{n=0}^{\infty}$. We will speak of the *n*th iterate $(X_{1,\infty}^{[n]}; f_{1,\infty}^{[n]})$ of the system, where $X_i^{[n]} = X_{(i-1)n+1}$ and $f_i^{[n]} = f_{(i-1)n+1}^n$.

The topological entropy $h(f)$ of an autonomous dynamical system $(X; f)$ was introduced by Adler, Konheim and McAndrew [AKM] and equivalent definitions were given by Bowen [B] and Dinaburg [D]. The topological entropy $h(f_{1,\infty})$ of a nonautonomous dynamical system $(X_{1,\infty}; f_{1,\infty})$ was studied in [KS] under the additional assumption that all the spaces X_i co-

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incide. In the present paper we generalize these definitions to the system $(X_{1,\infty}; f_{1,\infty})$.

The main aim of the paper is to prove analogues of the following result.

THEOREM 1.1 ([MS]). *If f is a piecewise monotone continuous selfmap of a compact interval and c_n denotes the number of pieces of monotonicity of f^n then*

$$(1.1) \quad h(f) = \lim_{n \rightarrow \infty} \frac{1}{n} \log c_n.$$

We will consider a dynamical system $(I_{1,\infty}; f_{1,\infty})$, where I_i is a compact real interval for any i . Moreover, we assume that every f_i is piecewise monotone. By this we mean that there is a finite partition of I_i into intervals such that f_i is monotone (not necessarily strictly) on each element of this partition. Then f_1^n is also piecewise monotone. A *lap* of a piecewise monotone map is any maximal (with respect to inclusion) interval on which it is monotone. We denote the number of laps of f_1^n by $c_{1,n}$.

We want to find conditions under which the formula

$$(1.2) \quad h(f_{1,\infty}) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log c_{1,n},$$

analogous to (1.1), holds. We present three sets of such conditions. In Theorem A, the assumptions are the weakest and easy to state. However, they are difficult to verify directly. Therefore we strengthen them to get weaker Theorems B and C, whose assumptions are often easy to verify.

Since the application in Section 5 uses Theorem B, this theorem can be perceived as the main result of the paper.

In the autonomous case compactness of the space plays an important role in the theory. In order not to lose this advantage, we have to put some restrictions on the behavior of our sequence of spaces.

DEFINITION 1.2. A dynamical system $(I_{1,\infty}; f_{1,\infty})$ is said to be *bounded* if the lengths of the intervals are uniformly bounded from above.

To state Theorem B, we need the following definitions.

DEFINITION 1.3. Let $(I_{1,\infty}; f_{1,\infty})$ be a dynamical system. It is said to have the *Markov property* if there exists $\delta > 0$ and a sequence $\mathcal{C}_{1,\infty}$ of finite subsets $\mathcal{C}_i \subset I_i$ such that for every $i \geq 1$,

- (a) the endpoints of I_i belong to \mathcal{C}_i ,
- (b) either f_i is monotone on the whole interval I_i or the length of every component of $I_i \setminus \mathcal{C}_i$ is at least δ ,
- (c) for every component J of $I_i \setminus \mathcal{C}_i$ the map f_i is monotone on J ,
- (d) $f_i(\mathcal{C}_i) \subset \mathcal{C}_{i+1}$.

We use the name “Markov” because of the properties (a), (c) and (d). However, property (b) is also very important. While we do not want the elements of the Markov partitions to be too short (this holds automatically in the autonomous case), we admit exceptions if f_i is globally monotone.

DEFINITION 1.4. A bounded dynamical system $(I_{1,\infty}; f_{1,\infty})$ is said to be a *Markov system* if it has the Markov property and the maps f_i , $i = 1, 2, \dots$, are equicontinuous.

THEOREM B. *If $(I_{1,\infty}; f_{1,\infty})$ is a Markov dynamical system then (1.2) holds.*

Another important case when (1.2) holds is the following.

DEFINITION 1.5. A bounded dynamical system $(I_{1,\infty}; f_{1,\infty})$ is said to be a *finite piecewise monotone system* if the set $\{f_i : i \in \mathbb{N}\}$ of maps has only finitely many distinct elements and each of them is piecewise monotone.

THEOREM C. *If $(I_{1,\infty}; f_{1,\infty})$ is a finite piecewise monotone system then (1.2) holds.*

Note that Theorem 1.1 is a special case of Theorem C.

We deduce Theorems B and C from a more general, but more technical result.

DEFINITION 1.6. A system $(I_{1,\infty}; f_{1,\infty})$, where all f_i are piecewise monotone, is called *long-lapped* if there exists $\delta > 0$ such that for every i either f_i is monotone or every lap of f_i has length at least δ . It will be called *totally long-lapped* if its n th iterate is long-lapped for every $n \geq 1$.

THEOREM A. *If $(I_{1,\infty}; f_{1,\infty})$ is a bounded totally long-lapped dynamical system then (1.2) holds.*

Theorem A will be proved in Section 3. In Section 4 we show that Markov and finite systems are totally long-lapped, so Theorems B and C follow from Theorem A. We also present some related examples.

In Section 5 we apply Theorem B to triangular maps. A continuous map F of the square $I \times I$ into itself is called *triangular* (see e.g. [Kl], [Ko]) if it is of the form

$$F(x, y) = (f(x), g_x(y)).$$

It is said to be *of type 2^∞* if it has periodic points of period 2^n for every $n \geq 0$ and of no other periods. Similarly to the case of continuous interval maps, if any other period is present, the topological entropy is positive, while if not all periods 2^n are present, the topological entropy is zero. However, the case of maps of type 2^∞ is different. While for the interval case they have zero entropy (see e.g. [ALM], [BC]), it is not necessarily the case for triangular maps (see [Ko]).

As an application of Theorem B we construct a large class of triangular maps of the square of type 2^∞ with positive topological entropy. Existence of such maps has been established in [Ko], but here we give a shorter proof based on Theorem B. In [BEL] an example of such a map of class C^r was given for any finite r . We present a much more general construction, leading to a whole class of C^r triangular maps of type 2^∞ with positive entropy (Theorem 5.7). Then we strengthen this result by showing that such maps exist even in the class C^∞ (Theorem 5.8).

As we show after Theorem 5.8, for any map F considered in Theorems 5.7 and 5.8 we have

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \text{card Fix}(F^n) = 0,$$

where $\text{Fix}(F^n)$ is the set of fixed points of F^n . Moreover, there are no homoclinic points or horseshoes of any type. This is in contrast to C^0 interval maps (see e.g. [ALM], [BC]) and two-dimensional diffeomorphisms of class $C^{1+\varepsilon}$ ([Ka]), where

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \text{card Fix}(F^n) \geq h(F),$$

and if $h(F) > 0$ then F has homoclinic points and horseshoes (at least for some iterate).

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2. Topological entropy. For $i = 1, 2, \dots$ let (X_i, ϱ_i) be a compact metric space and let $f_i : X_i \rightarrow X_{i+1}$ be a continuous map. We are going to define the topological entropy of the system $(X_{1,\infty}; f_{1,\infty})$.

For each $n \geq 1$ the function

$$\varrho_{1,n}(x, y) = \max_{0 \leq j \leq n-1} \varrho_{j+1}(f_1^j(x), f_1^j(y))$$

is a metric on X_1 , equivalent to ϱ_1 . A subset E of X_1 is called $(n, f_{1,\infty}, \varepsilon)$ -separated if for any two distinct points $x, y \in E$, $\varrho_{1,n}(x, y) > \varepsilon$. A set $F \subset X_1$ $(n, f_{1,\infty}, \varepsilon)$ -spans another set $K \subset X_1$ if for each $x \in K$ there is $y \in F$ for which $\varrho_{1,n}(x, y) \leq \varepsilon$.

We define $s_n(f_{1,\infty}, \varepsilon)$ as the maximal cardinality of an $(n, f_{1,\infty}, \varepsilon)$ -separated set and $r_n(f_{1,\infty}, \varepsilon)$ as the minimal cardinality of a set which $(n, f_{1,\infty}, \varepsilon)$ -spans X_1 .

Similarly to [KS] where all spaces were the same, we define the *topological entropy* $h(f_{1,\infty})$ of the system $(X_{1,\infty}; f_{1,\infty})$ by

$$h(f_{1,\infty}) = \lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \log s_n(f_{1,\infty}, \varepsilon) = \lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \log r_n(f_{1,\infty}, \varepsilon).$$

The equality of the two limits can be proved very similarly to less general situations [B], [KS], so we omit the proof.

It is also possible to provide an equivalent definition analogous to the original one from [AKM], using open covers. The only difference is that one has to use sequences of open covers of the spaces X_i with Lebesgue numbers bounded away from 0. However, in the present paper we are not going to use this definition, so we leave this easy task to the reader.

Although we keep using the traditional name “*topological entropy*,” its value can depend on the metrics $\varrho_{1,\infty}$. This is similar to the situation of one map on a noncompact space: if we change the metric to another one which is equivalent but not uniformly equivalent then the entropy can change. In our case a kind of noncompactness enters the scene when we let $i \rightarrow \infty$.

Note that the following three lemmas hold. The first two can be proved similarly to the corresponding results in [KS] dealing with the case when all the spaces X_i coincide, so we omit their proofs. The third one is simply taken from [KS].

LEMMA 2.1. *Let $(X_{1,\infty}; f_{1,\infty})$ be a dynamical system, where the spaces X_i are compact metric. Then $h(f_{1,\infty}^{[n]}) \leq nh(f_{1,\infty})$ for every $n \geq 1$.*

LEMMA 2.2. *Let $(X_{1,\infty}; f_{1,\infty})$ be a dynamical system, where the spaces X_i are compact metric and the maps f_i , $i = 1, 2, \dots$, are equicontinuous. Then $h(f_{1,\infty}^{[n]}) = nh(f_{1,\infty})$ for every $n \geq 1$.*

LEMMA 2.3 ([KS]). *Let $f_{1,\infty}$ be a sequence of continuous selfmaps of a compact metric space X converging uniformly to f . Then $h(f_{1,\infty}) \leq h(f)$.*

3. Piecewise monotone sequences of interval maps. In this section we prove the main technical result of the paper, Theorem A. Some of the main ideas of the proof from [MS] (cf. [ALM], [BC]) will be preserved.

From now on we consider a dynamical system $(I_{1,\infty}; f_{1,\infty})$, where I_i is a compact real interval for any i . Moreover, we assume that every f_i is piecewise monotone.

First of all we emphasize that the assumption that the system is bounded plays an important role in the theorem. For instance, for the system $(I_{1,\infty}; f_{1,\infty})$ where $f_i(x) = 2x$ and $I_i = [0, 2^i]$ the equality from the theorem does not hold. In Section 4 we show that the assumption that the system is totally long-lapped cannot be replaced by the assumption that it is long-lapped (Examples 4.1, 4.4).

By a *mono-partition* of I_i we will understand a finite partition of I_i into intervals (of any type, possibly degenerate) such that f_i is monotone on every element of this partition. Let $\mathcal{P}_{1,\infty}$ be a sequence of mono-partitions of $I_{1,\infty}$, i.e., \mathcal{P}_i is a mono-partition of I_i into intervals $P_{i,1}, \dots, P_{i,n_i}$, $i \geq 1$. We say that $(P_{1,j_1}, \dots, P_{n,j_n})$ is a *chain of length n* if the set

$$(3.1) \quad \bigcap_{i=1}^n f_1^{-(i-1)}(P_{i,j_i})$$

is nonempty. We denote the family of all such sets by \mathcal{P}_1^n . Thus, (3.1) gives us a one-to-one correspondence between chains and elements of \mathcal{P}_1^n . Clearly, \mathcal{P}_1^n is a mono-partition for $(I_{1,\infty}^{[n]}; f_{1,\infty}^{[n]})$. For a partition \mathcal{A} of an interval into subintervals we denote by $\text{esscard } \mathcal{A}$ the number of nondegenerate elements of \mathcal{A} . Note that

$$\text{esscard } \mathcal{A} \leq \text{card } \mathcal{A} \leq 2 \text{esscard } \mathcal{A} + 1.$$

Therefore we can set

$$H(f_{1,\infty}, \mathcal{P}_{1,\infty}) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log \text{esscard } \mathcal{P}_1^n = \limsup_{n \rightarrow \infty} \frac{1}{n} \log \text{card } \mathcal{P}_1^n.$$

For an interval J we denote its length by $|J|$. A sequence $\mathcal{P}_{1,\infty}$ of mono-partitions of $I_{1,\infty}$ is said to be a \mathcal{U} -sequence if there is $\delta > 0$ such that for any $k \geq 1$, the length of any nondegenerate element of \mathcal{P}_k is at least $\min(\delta, |I_k|)$.

Clearly, if $\mathcal{P}_{1,\infty}$ is a \mathcal{U} -sequence for a bounded system then the cardinalities of the partitions \mathcal{P}_k are uniformly bounded.

LEMMA 3.1. *Let $(I_{1,\infty}; f_{1,\infty})$ be a bounded dynamical system and let $\mathcal{P}_{1,\infty}$ be a \mathcal{U} -sequence. Then*

$$h(f_{1,\infty}) \leq H(f_{1,\infty}, \mathcal{P}_{1,\infty}) \leq h(f_{1,\infty}) + \log 2.$$

PROOF. By the definition of a \mathcal{U} -sequence, there is $\varepsilon > 0$ such that for any $k \in \mathbb{N}$, either the length of any nondegenerate element of \mathcal{P}_k is larger than $\min(2\varepsilon, |I_k|)$ or \mathcal{P}_k contains only one nondegenerate element. Let $E \subset I_1$ be an $(n, f_{1,\infty}, \varepsilon)$ -spanning set of minimal cardinality. Fix $x \in E$ and look at the points $y \in I_1$ for which $\varrho_{1,n}(x, y) \leq \varepsilon$. Then $\varrho_{j+1}(f_1^j(x), f_1^j(y)) \leq \varepsilon$ for $j = 0, \dots, n - 1$. The ε -neighborhood of $f_1^j(x)$ intersects at most 2 nondegenerate elements of \mathcal{P}_{j+1} , so the number of nondegenerate elements of \mathcal{P}_1^n to which y may belong is at most 2^n . Since for every $y \in I_1$ there is $x \in E$ with $\varrho_{1,n}(x, y) \leq \varepsilon$, we get $\text{esscard } \mathcal{P}_1^n \leq 2^n \text{card } E$, that is,

$$\frac{1}{n} \log \text{esscard } \mathcal{P}_1^n \leq \frac{1}{n} \log r_n(f_{1,\infty}, \varepsilon) + \log 2.$$

Therefore

$$H(f_{1,\infty}, \mathcal{P}_{1,\infty}) \leq h(f_{1,\infty}) + \log 2.$$

Now we take an arbitrary ε and choose an $(n, f_{1,\infty}, \varepsilon)$ -separated set $F \subset I_1$ of maximal cardinality. Let us estimate how many elements of F can belong to one element of \mathcal{P}_1^n . An element A of \mathcal{P}_1^n corresponds to a chain $(P_{1,j_1}, \dots, P_{n,j_n})$. If x, y are two adjacent elements of $A \cap F$ then there is j with $0 \leq j \leq n - 1$ such that $\varrho_{j+1}(f_1^j(x), f_1^j(y)) > \varepsilon$ and $f_1^j(x), f_1^j(y)$ are two adjacent elements of $f_1^j(A \cap F)$. Since the system $(I_{1,\infty}; f_{1,\infty})$ is bounded, there is M such that $|I_i| < M\varepsilon$ for every $i \in \mathbb{N}$. Hence, for each j there are at most M pairs of adjacent elements of $f_1^j(A \cap F)$ which are more than ε apart. Hence, there are at most nM pairs of adjacent elements of $A \cap F$, and therefore $\text{card } A \cap F \leq nM + 1$. In such a way we get $\text{card } F \leq (nM + 1) \text{card } \mathcal{P}_1^n$, that is,

$$\frac{1}{n} \log s_n(f_{1,\infty}, \varepsilon) \leq \frac{1}{n} \log \text{card } \mathcal{P}_1^n + \frac{1}{n} \log(nM + 1).$$

Since $\lim_{n \rightarrow \infty} (1/n) \log(nM + 1) = 0$, we get $h(f_{1,\infty}) \leq H(f_{1,\infty}, \mathcal{P}_{1,\infty})$. ■

We need the following technical lemma (see [MS], [ALM]).

LEMMA 3.2. *Let $(a_n)_{n=1}^\infty$ be a sequence of real numbers and let $b, u \in \mathbb{R}$, $p \in \mathbb{N}$ be such that*

- (a) $u > 0$,
- (b) $a_{n+1} \leq a_n + b$ for all n ,
- (c) if $n \geq p$ and $a_n/n \geq u$ then $a_{n+1} \leq a_n + u$.

Then $\limsup_{n \rightarrow \infty} a_n/n \leq u$.

We call an element A of \mathcal{P}_1^n and the corresponding chain $(P_{1,j_1}, \dots, P_{n,j_n})$ proper if $P_{n,j_n} \subset f_1^n(A)$. Denote the set of all proper elements of \mathcal{P}_1^n by \mathcal{R}_n .

LEMMA 3.3. *Let $(I_{1,\infty}; f_{1,\infty})$ be a bounded dynamical system and let $\mathcal{P}_{1,\infty}$ be a \mathcal{U} -sequence. Assume that $H(f_{1,\infty}, \mathcal{P}_{1,\infty}) > \log 3$. Then*

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \text{card } \mathcal{R}_n = H(f_{1,\infty}, \mathcal{P}_{1,\infty}).$$

PROOF. Since the system $(I_{1,\infty}; f_{1,\infty})$ is bounded and $\mathcal{P}_{1,\infty}$ is a \mathcal{U} -sequence, there is $M > 2$ such that $\text{card } \mathcal{P}_i \leq M$ for all i . If we can apply Lemma 3.2 to $a_n = \log \text{card } \mathcal{P}_1^n$, $b = \log M$ and a number u such that $\log 3 < u < H(f_{1,\infty}, \mathcal{P}_{1,\infty})$ then $\limsup_{n \rightarrow \infty} a_n/n \leq u < H(f_{1,\infty}, \mathcal{P}_{1,\infty})$, a contradiction. Since conditions (a) and (b) of Lemma 3.2 are satisfied, (c) cannot be satisfied. Since $H(f_{1,\infty}, \mathcal{P}_{1,\infty}) > u$, we can find an arbitrarily large n such that $a_n/n \geq u$. Therefore, we can find an arbitrarily large n such that both $a_n/n \geq u$ and $a_{n+1} > a_n + u > a_n + \log 3$. In other words, we can find an arbitrarily large n such that both $(1/n) \log \text{card } \mathcal{P}_1^n \geq u$ and $\text{card } \mathcal{P}_1^{n+1} > 3 \text{card } \mathcal{P}_1^n$.

For any i , denote by l_i the number of chains of length i that can be completed to a chain of length $i + 1$ (by attaching an element of \mathcal{P}_{i+1}

at the end) in at least 3 ways. Then for arbitrarily large n we have both $(1/n) \log \text{card } \mathcal{P}_1^n \geq u$ and

$$3 \text{card } \mathcal{P}_1^n < \text{card } \mathcal{P}_1^{n+1} \leq Ml_n + 2(\text{card } \mathcal{P}_1^n - l_n) = 2 \text{card } \mathcal{P}_1^n + (M - 2)l_n.$$

Thus, we get $\exp(nu) \leq \text{card } \mathcal{P}_1^n < (M - 2)l_n$. Therefore, since u can be chosen as close to $H(f_{1,\infty}, \mathcal{P}_{1,\infty})$ as we wish,

$$\limsup_{i \rightarrow \infty} \frac{1}{i} \log l_i \geq H(f_{1,\infty}, \mathcal{P}_{1,\infty}).$$

When we complete a chain of length i to a chain of length $i + 1$ and we can do it in at least 3 ways, then at least one of these chains of length $i + 1$ is proper. Thus, $\text{card } \mathcal{R}_{i+1} \geq l_i$. Hence

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \text{card } \mathcal{R}_n = \limsup_{n \rightarrow \infty} \frac{1}{n-1} \log \text{card } \mathcal{R}_n \geq H(f_{1,\infty}, \mathcal{P}_{1,\infty}).$$

The opposite inequality follows from the fact that $\mathcal{R}_n \subset \mathcal{P}_1^n$. ■

LEMMA 3.4. *If \mathcal{Q} and \mathcal{S} are partitions of an interval I into intervals (possibly degenerate) and \mathcal{P} is their common refinement then $\text{card } \mathcal{P} \leq \text{card } \mathcal{Q} + \text{card } \mathcal{S} - 1$.*

We omit a simple proof, consisting of counting the number of places at which we have to cut the interval to obtain corresponding partitions.

LEMMA 3.5. *Let $\mathcal{Q}_{1,\infty}$ and $\mathcal{S}_{1,\infty}$ be sequences of mono-partitions for a system $(I_{1,\infty}; f_{1,\infty})$. Assume that there is a constant N such that $\text{card } \mathcal{Q}_n \leq N$ and $\text{card } \mathcal{S}_n \leq N$ for every n . Then $H(f_{1,\infty}, \mathcal{S}_{1,\infty}) = H(f_{1,\infty}, \mathcal{Q}_{1,\infty})$.*

PROOF. For each i , let \mathcal{P}_i be the common refinement of \mathcal{Q}_i and \mathcal{S}_i . Then by Lemma 3.4, $\text{card } \mathcal{P}_i < 2N$. Let $(Q_{1,j_1}, \dots, Q_{n,j_n})$ be the chain corresponding to an element J of \mathcal{Q}_1^n . Each Q_{i,j_i} is partitioned into less than $2N$ elements of \mathcal{P}_i . Since f_1^i is monotone on J , the partition of J into elements of $f_1^{-i}(\mathcal{P}_i)$ consists of less than $2N$ elements, all of them intervals. The partition of J into elements of \mathcal{P}_1^n is the common refinement of n such partitions. Hence, by Lemma 3.4, the cardinality of \mathcal{P}_1^n restricted to J is less than $2nN$. Thus, $\text{card } \mathcal{P}_1^n \leq 2nN \text{card } \mathcal{Q}_1^n$, and therefore $H(f_{1,\infty}, \mathcal{P}_{1,\infty}) \leq H(f_{1,\infty}, \mathcal{Q}_{1,\infty})$. Since $\mathcal{P}_{1,\infty}$ is finer than $\mathcal{Q}_{1,\infty}$, we get $H(f_{1,\infty}, \mathcal{P}_{1,\infty}) = H(f_{1,\infty}, \mathcal{Q}_{1,\infty})$. Similarly, $H(f_{1,\infty}, \mathcal{P}_{1,\infty}) = H(f_{1,\infty}, \mathcal{S}_{1,\infty})$. Therefore $H(f_{1,\infty}, \mathcal{S}_{1,\infty}) = H(f_{1,\infty}, \mathcal{Q}_{1,\infty})$. ■

Let f be piecewise monotone on I . Note that two adjacent laps of f always overlap. We will call their intersection a *turning interval* (it can be a degenerate interval, consisting of one point).

LEMMA 3.6. *A system $(I_{1,\infty}; f_{1,\infty})$ is long-lapped if and only if there exists a \mathcal{U} -sequence of partitions for this system.*

PROOF. If there exists a \mathcal{U} -sequence, the system is obviously long-lapped. Conversely, assume that it is long-lapped. For a given n , let A be the set of the midpoints of turning intervals of f_n . As the elements of our partition of I_n we take the elements of A and the components of $I_n \setminus A$. The lengths of nondegenerate elements of this partition are at least $1/2$ of the lengths of the corresponding laps of f_n . Thus, since the system is long-lapped, the sequence of partitions which we constructed is a \mathcal{U} -sequence. ■

Recall that for a system $(I_{1,\infty}; f_{1,\infty})$ where all f_i are piecewise monotone, we denoted by $c_{1,n}$ the number of laps of f_1^n . The analogous number for the k th iterate of the system will be denoted by $c_{1,n}^{[k]}$. Thus, $c_{1,n}^{[k]} = c_{1,kn}$.

LEMMA 3.7. *If $(I_{1,\infty}; f_{1,\infty})$ is a bounded long-lapped dynamical system then*

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log c_{1,n}^{[k]} = k \limsup_{n \rightarrow \infty} \frac{1}{n} \log c_{1,n},$$

and if $\mathcal{P}_{1,\infty}$ is a \mathcal{U} -sequence of partitions then

$$H(f_{1,\infty}^{[k]}, \mathcal{P}_{1,\infty}^{[k]}) = kH(f_{1,\infty}, \mathcal{P}_{1,\infty}).$$

PROOF. Let $\mathcal{P}_{1,\infty}$ be a \mathcal{U} -sequence of partitions for $(I_{1,\infty}; f_{1,\infty})$. Then, since the system $(I_{1,\infty}; f_{1,\infty})$ is bounded, there is M such that $\text{card } \mathcal{P}_i \leq M$ for all i . Therefore if $0 \leq j \leq k - 1$ then $c_{1,kn+j} \leq M^k c_{1,kn}$. Thus,

$$\limsup_{i \rightarrow \infty} \frac{1}{i} \log c_{1,i} = \limsup_{n \rightarrow \infty} \frac{1}{kn} \log c_{1,kn} = \frac{1}{k} \limsup_{n \rightarrow \infty} \frac{1}{n} \log c_{1,n}^{[k]},$$

and we get the second equality in a similar way. ■

LEMMA 3.8. *Let $(I_{1,\infty}; f_{1,\infty})$ be a bounded dynamical system and let $\mathcal{P}_{1,\infty}$ be a \mathcal{U} -sequence. Assume that $H(f_{1,\infty}, \mathcal{P}_{1,\infty}) > \log 3$. Then*

$$(3.2) \quad \limsup_{n \rightarrow \infty} \frac{1}{n} \log c_{1,n} \geq H(f_{1,\infty}, \mathcal{P}_{1,\infty}).$$

PROOF. By Lemma 3.3 for every $\varepsilon > 0$ there exists an arbitrarily large n such that $\text{card } \mathcal{R}_n \geq \exp(n(H(f_{1,\infty}, \mathcal{P}_{1,\infty}) - \varepsilon))$. Thus, there exists an interval $B \in \mathcal{P}_n$ such that the number of elements $A \in \mathcal{P}_1^n$ for which $B \subset f_1^n(A)$ is at least $\exp(n(H(f_{1,\infty}, \mathcal{P}_{1,\infty}) - \varepsilon))/\text{card } \mathcal{P}_n$. These intervals A are disjoint and their images under f_1^n contain the same interval. Hence, the number of laps of f_1^n is larger than or equal to the number of these intervals. That is, we have

$$(3.3) \quad c_{1,n} \geq \exp(n(H(f_{1,\infty}, \mathcal{P}_{1,\infty}) - \varepsilon)) / \text{card } \mathcal{P}_n.$$

Since $(I_{1,\infty}; f_{1,\infty})$ is bounded, there is M such that $\text{card } \mathcal{P}_i \leq M$ for all i . Hence, we can replace in (3.3) $\text{card } \mathcal{P}_n$ by M . Thus,

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log c_{1,n} \geq H(f_{1,\infty}, \mathcal{P}_{1,\infty}) - \varepsilon.$$

Since ε was arbitrary, we get (3.2). ■

Now we are ready to prove Theorem A.

Proof of Theorem A. By Lemma 3.6 there exists a \mathcal{U} -sequence $\mathcal{P}_{1,\infty}$ of partitions for $(I_{1,\infty}; f_{1,\infty})$. Then f_1^n is monotone on every element of \mathcal{P}_1^n , and hence $\text{card } \mathcal{P}_1^n \geq c_{1,n}$. Thus, by Lemma 3.1, we get

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log c_{1,n} \leq h(f_{1,\infty}) + \log 2.$$

Replacing the system $(I_{1,\infty}; f_{1,\infty})$ by its k th iterate and using Lemmas 2.1 and 3.7, we get the same inequality with $\log 2$ replaced by $(1/k) \log 2$. Since k can be arbitrarily large, we get

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log c_{1,n} \leq h(f_{1,\infty}).$$

Now we prove the opposite inequality. If $h(f_{1,\infty}) = 0$ then clearly

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log c_{1,n} \geq h(f_{1,\infty})$$

and we are done. Assume that $h(f_{1,\infty}) > 0$. Then by Lemma 3.1, $H(f_{1,\infty}, \mathcal{P}_{1,\infty}) > 0$, so there is k such that $kH(f_{1,\infty}, \mathcal{P}_{1,\infty}) > \log 3$. By Lemma 3.7, $H(f_{1,\infty}^{[k]}, \mathcal{P}_{1,\infty}^{[k]}) > \log 3$. Although we do not know whether $\mathcal{P}_{1,\infty}^{[k]}$ is a \mathcal{U} -sequence, we can circumvent it. Since $(I_{1,\infty}; f_{1,\infty})$ is totally long-lapped, by Lemma 3.6 there exists a \mathcal{U} -sequence of partitions $\mathcal{Q}_{1,\infty}$ for $f_{1,\infty}^{[k]}$.

The system $(I_{1,\infty}; f_{1,\infty})$ (and therefore also its k th iterate) are bounded, so there is M such that $\text{card } \mathcal{P} \leq M$ and $\text{card } \mathcal{Q} \leq M$ for every n . Hence, for every n the cardinalities of $\mathcal{P}_n^{[k]}$ and \mathcal{Q}_n are bounded by $N = M^k$. Thus, by Lemma 3.5 we get $H(f_{1,\infty}^{[k]}, \mathcal{P}_{1,\infty}^{[k]}) = H(f_{1,\infty}^{[k]}, \mathcal{Q}_{1,\infty}) > \log 3$. Thus, by Lemma 3.8 we get

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log c_{1,n}^{[k]} \geq H(f_{1,\infty}^{[k]}, \mathcal{Q}_{1,\infty}) = H(f_{1,\infty}^{[k]}, \mathcal{P}_{1,\infty}^{[k]}).$$

In view of Lemmas 3.7 and 3.1, this yields

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log c_{1,n} \geq H(f_{1,\infty}, \mathcal{P}_{1,\infty}) \geq h(f_{1,\infty}). \quad \blacksquare$$

4. Markov systems and examples. The following example (and similarly Example 4.4) shows that in Theorem A the assumption that the system is totally long-lapped cannot be replaced by the assumption that it is long-lapped.

EXAMPLE 4.1. Let $I_n = I = [0, 1]$ for every n . Take a sequence of points $0 = x_1 < x_2 < \dots$ converging to $1/2$. Now consider the sequence $f_{1,\infty}$ of continuous selfmaps of I where for any n ,

$$f_n(x) = \begin{cases} 0 & \text{if } x \in \{0, 1\}, \\ x_n & \text{if } x \in \{x_n, 1 - x_n\}, \\ 1 - x_n & \text{if } x = 1/2, \\ \text{affine} & \text{in between.} \end{cases}$$

Since $f_{1,\infty}$ converges uniformly to the map

$$f(x) = \begin{cases} x & \text{if } x \in [0, 1/2], \\ 1 - x & \text{if } x \in [1/2, 1] \end{cases}$$

and $h(f) = 0$, by Lemma 2.3 we have $h(f_{1,\infty}) = 0$. Every f_n is a unimodal map and so $c_{1,n} \leq 2^n$. On the other hand, for every n we have $f_n(0) = f_n(1) = 0$ and $f_n(I) = [0, 1 - x_n] \supset [0, 1 - x_{n+1}]$ and so $c_{1,n} \geq 2^n$. Hence $c_{1,n} = 2^n$ and

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log c_{1,n} = \log 2.$$

Since sometimes it may be difficult to verify whether a system is totally long-lapped using only the definition, in Section 1 we specified two special cases: Markov systems and finite piecewise monotone systems. It turns out that they are totally long-lapped.

LEMMA 4.2. *Markov systems are totally long-lapped.*

PROOF. Assume that the system $(I_{1,\infty}; f_{1,\infty})$ is Markov. Fix $k \geq 1$. We have to prove that the system $(I_{1,\infty}^{[k]}; f_{1,\infty}^{[k]})$ is long-lapped. By the definition of the Markov property, $(I_{1,\infty}; f_{1,\infty})$ is long-lapped, i.e., there is $\delta > 0$ such that for every n either f_n is monotone or every lap of f_n has length at least δ . Set $g_i = f_{(n-1)k+1}^i$ for $i \geq 0$. By equicontinuity, there is $\varepsilon > 0$ (independent of n) such that if $|a - b| < \varepsilon$ then $|g_i(a) - g_i(b)| < \delta$ for $i = 0, 1, \dots, k - 1$.

Let J be a lap of $f_n^{[k]}$. If f_{nk-j} is monotone for $j = 0, 1, \dots, k - 1$, then $f_n^{[k]}$ is monotone. If this is not the case, take the smallest j such that f_{nk-j} is not monotone. The lap J contains an interval (a, b) such that $g_i((a, b))$ is disjoint from $\mathcal{C}_{(n-1)k+1+i}$ for $i = 0, 1, \dots, k - j - 1$ and (a, b) is the maximal interval with this property. Then for at least one of $i = 0, \dots, k - j - 1$ we must have $g_i(a) \in \mathcal{C}_{(n-1)k+1+i}$. Therefore, by property (d) of Definition 1.3, $g_{k-j-1}(a) \in \mathcal{C}_{nk-j}$. Similarly, $g_{k-j-1}(b) \in \mathcal{C}_{nk-j}$. Since these two points are distinct, by property (b) of Definition 1.3 we have

$|g_{k-j-1}(a) - g_{k-j-1}(b)| \geq \delta$, and therefore $|a - b| \geq \varepsilon$. Hence, $|J| \geq \varepsilon$. This proves that $(I_{1,\infty}^{[k]}; f_{1,\infty}^{[k]})$ is long-lapped. ■

LEMMA 4.3. *Finite piecewise monotone systems are totally long-lapped.*

PROOF. Clearly, a finite piecewise monotone system is long-lapped. Every iterate of such a system is also finite piecewise monotone, and therefore long-lapped. Hence, the original system is totally long-lapped. ■

Now, Theorem B follows from Theorem A and Lemma 4.2, while Theorem C follows from Theorem A and Lemma 4.3.

A system $(I_{1,\infty}; f_{1,\infty})$ is Markov if it is bounded, has the Markov property and the maps f_i are equicontinuous. None of these can be removed from the definition of Markov systems if one wishes Theorem B to work for them. The system $(I_{1,\infty}; f_{1,\infty})$ where $f_i(x) = 2x$ and $I_i = [0, 2^i]$ shows that we cannot remove the assumption that the system is bounded. Example 4.1 serves as a counterexample if we remove the Markov property. Finally, the next example shows that equicontinuity cannot be omitted.

EXAMPLE 4.4. Let $I_n = [0, 1]$ for every n . Set $d_n = 1/(n + 2)$, $a_n = 1/2 - d_n$ and $b_n = 1/2 + d_n$. Define

$$f_n(x) = \begin{cases} 0 & \text{if } x \in \{0, 1\}, \\ 1 & \text{if } x = 1/2, \\ d_n & \text{if } x \in \{a_n, b_n\}, \\ \text{affine} & \text{in between.} \end{cases}$$

Then $c_{1,n} = 2^n$ for every n , and thus $\limsup_{n \rightarrow \infty} (1/n) \log \text{card } c_{1,n} = \log 2$. All conditions of Definition 1.4 are satisfied except equicontinuity. The system is long-lapped, but it is easy to see that it is not totally long-lapped (this will follow also from the fact that the conclusion of Theorem A does not hold for this system).

Let us show that $h(f_{1,\infty}) = 0$. Fix $\varepsilon > 0$ and an integer $m > 1/\varepsilon$. Divide $[0, 1]$ into m intervals A_1, \dots, A_m of length less than ε each. Take also an integer k such that if $n > k$ then $2d_n < \varepsilon$. Now we form the itinerary $(i_n(x))_{n=0}^\infty$ of a point $x \in [0, 1]$ in the following way. If $n \leq k$ then $i_n(x) = j$ if $f_1^n(x) \in A_j$. If $n > k$ then $i_n(x) = 0$ if $f_1^n(x) \in [a_n, b_n]$, $i_n(x) = m + 1$ if $f_1^n(x) \in [0, d_n]$, and $i_n(x) = j$ if $f_1^n(x) \in A_j \setminus ([a_n, b_n] \cup [0, d_n])$. Note that if $n > k$ and $i_n(x) > 0$ then $i_j(x) = m + 1$ for all $j > n$. Thus, if $n > k$ then the number of possible itinerary beginnings of length n is at most $m^k(n - k + 1)(m + 1)$. Moreover, if $i_j(x) = i_j(y)$ then $|f_1^j(x) - f_1^j(y)| < \varepsilon$. Therefore

$$r_n(f_{1,\infty}, \varepsilon) \leq m^k(n - k + 1)(m + 1).$$

Hence, $h(f_{1,\infty}) = 0$.

We conclude this section by noting that even if a system is bounded and totally long-lapped, the maps $f_i, i = 1, 2, \dots$, do not have to be equicontinuous. For instance, one can take $f_i = x^i$ on $[0, 1]$.

5. Application: triangular maps of type 2^∞ with positive topological entropy. As an application we are going to construct a large class of triangular maps of the square of type 2^∞ with positive topological entropy (on the interval, maps of type 2^∞ have zero topological entropy, see e.g. [ALM]). Existence of such maps has been established in [Ko], but here we give a shorter proof based on Theorem A. In [BEL] an example of such a map of class C^r was given for any finite r . We present a much more general construction, leading to a whole class of triangular maps of type 2^∞ , class C^r and with positive entropy. Then we strengthen this result by showing that such maps exist even in the class C^∞ (Theorem 5.8).

We start with a series of examples.

EXAMPLE 5.1. Denote by τ the full tent map, i.e., $\tau(x) = 1 - |1 - 2x|$. Set $\tau_k(x) = 2^{-k}\tau(x)$ for $k = 0, 1, 2, \dots$ (in particular, $\tau_0 = \tau$). Note that $\tau^k \circ \tau_k = \tau$ for every k and hence

$$(5.1) \quad \tau^r \circ \tau_k = \tau^{r-k+1} \quad \text{whenever} \quad r \geq k \geq 0.$$

Consider the sequence $\varphi_{1,\infty}$ of selfmaps of I starting with

$$\begin{aligned} &\tau_0, \tau_0, \tau_0, \underline{\tau_1}, \tau_0, \tau_0, \tau_0, \underline{\tau_2}, \tau_0, \tau_0, \tau_0, \underline{\tau_1}, \tau_0, \tau_0, \tau_0, \underline{\tau_3}, \\ &\tau_0, \tau_0, \tau_0, \underline{\tau_1}, \tau_0, \tau_0, \tau_0, \underline{\tau_2}, \tau_0, \tau_0, \tau_0, \underline{\tau_1}, \tau_0, \tau_0, \tau_0, \underline{\tau_4}, \dots \end{aligned}$$

(we have underlined all maps different from τ_0) and defined by $\varphi_n = \tau_{i(n)-1}$, where $i(n)$ is 1 if n is odd and is equal to the number of trailing zeros in the binary expansion of n if n is even.

Important moments when we look how the composition of our maps looks like are $2^n - 1$. Therefore we introduce the notation $\Phi_n = \varphi_1^{2^n - 1}$. By the definition, for all n we have $\Phi_{n+1} = \Phi_n \circ \tau_{n-1} \circ \Phi_n$. We show by induction that

$$(5.2) \quad \Phi_n = \tau^{a_n},$$

where $a_n = 2^{n-1} + n - 1$. For $n = 1$ we have $a_1 = 1$ and $\Phi_1 = \tau$, so (5.2) holds. Assume now that it holds for some n . Then $\Phi_{n+1} = \tau^{a_n} \circ \tau_{n-1} \circ \tau^{a_n}$. Since $a_n \geq n - 1$, we can use (5.1), and we get $\Phi_{n+1} = \tau^{2a_n+1-(n-1)} = \tau^{a_{n+1}}$. This completes the proof of (5.2).

Now we can consider the system $(I_{1,\infty}, f_{1,\infty})$ such that $I_1 = I, I_2 = \varphi_1(I), I_3 = \varphi_2(\varphi_1(I)), \dots$, and $f_1 = \varphi_1, f_2 = \varphi_2|_{I_2}, f_3 = \varphi_3|_{I_3}, \dots$. We know that then $h(f_{1,\infty}) = h(\varphi_{1,\infty})$. Moreover, the system $(I_{1,\infty}, f_{1,\infty})$ (unlike $(I_{1,\infty}, \varphi_{1,\infty})$) is evidently Markov. The number of pieces of monotonicity of

$\varphi_1^{2^n-1}$ (and therefore of $f_1^{2^n-1}$) is 2^{a_n} . Clearly,

$$\limsup_{n \rightarrow \infty} \frac{1}{2^n - 1} \log 2^{a_n} = \limsup_{n \rightarrow \infty} \frac{2^{n-1} + n - 1}{2^n - 1} \log 2 = \frac{1}{2} \log 2,$$

and therefore by Theorem B, $h(\varphi_{1,\infty}) \geq (1/2) \log 2$.

In [Ko] there is an example of a class of triangular maps $F = (f, g_x)$ of type 2^∞ with positive topological entropy. The entropy is positive due to the fact that $h(F; I_\alpha) > 0$ for some α . The entropy $h(F; I_\alpha)$ is in fact the entropy of the sequence of maps $g_\alpha, g_{f(\alpha)}, g_{f^n(\alpha)}, \dots$. For one of the above-mentioned triangular maps F this sequence coincides with the sequence $\varphi_{1,\infty}$ from Example 5.1. Thus the long computation of $h(F; I_\alpha)$ from [Ko] can be replaced by the one from Example 5.1, which is much shorter and simpler.

Now we modify Example 5.1 in order to make maps smooth. We introduce a class of maps \mathcal{T}^r consisting of concave unimodal maps $f : [0, 1] \rightarrow [0, 1]$ of class C^r such that $f(0) = f(1) = 0$ and $f(1/2) = 1$.

EXAMPLE 5.2. Choose maps $\gamma_n \in \mathcal{T}^r$. We set $\sigma_n = \lambda_n \gamma_n$ for some $\lambda_n \in (0, 1]$, which we will define in a moment. Then we set $\varphi_n = \sigma_{i(n)-1}$, where $i(n)$ is defined as in Example 5.1. The numbers a_n are also defined as in Example 5.1, and we set $\Phi_n = \varphi_1^{2^n-1}$. Then again, by the definition, for all n we have $\Phi_{n+1} = \Phi_n \circ \sigma_{n-1} \circ \Phi_n$.

Now our construction starts to differ from the one from Example 5.1. Namely, instead of iterating τ like we did there, we use some sequence of maps $\eta_{1,\infty}$ from \mathcal{T}^r . Instead of $\tau^k \circ \tau_k = \tau$ (and instead of (5.1)) we will have

$$(5.3) \quad \eta_1^{k-1} \circ \sigma_{k-1} = \eta_{a_k+1},$$

and instead of (5.2),

$$(5.4) \quad \Phi_n = \eta_1^{a_n}.$$

We define η_n together with λ_n by induction. We start with $\eta_1 = \gamma_0$ and $\lambda_0 = 1$. Now suppose that $\eta_n \in \mathcal{T}^r$ is defined for all $n \leq a_k$ (note that $a_1 = 1$) and λ_n is defined for all $n < k - 1$. We take as λ_{k-1} the leftmost preimage of 1 under η_1^{k-1} (since $k \leq a_k$, we use η_n 's that are already defined). Then $\eta_1^{k-1}|_{[0, \lambda_{k-1}]}$ is an increasing map from $[0, \lambda_{k-1}]$ onto $[0, 1]$. As a composition of concave increasing maps it is also concave. Since $\sigma_{k-1} = \lambda_{k-1} \gamma_{k-1}$, it is easy to check that $\eta_1^{k-1} \circ \sigma_{k-1} \in \mathcal{T}^r$. Thus, (5.3) defines $\eta_{a_k+1} \in \mathcal{T}^r$. For $i = 2, \dots, a_{k+1} - a_k$ we set $\eta_{a_k+i} = \eta_{k-2+i}$ (cf. (5.5) below). Here we also use η_n 's that are already defined, since $a_{k+1} - a_k = a_k - k + 2$. Hence, the induction step is complete.

Now we have to prove (5.4). Again we use induction. If $n = 1$ then $\Phi_1 = \sigma_0 = \eta_1^1$, so (5.4) holds. Assume now that it holds for some n . Then

$\Phi_{n+1} = \eta_1^{a_n} \circ \sigma_{n-1} \circ \eta_1^{a_n}$. By the definition of $\eta_{1,\infty}$ we have

$$(5.5) \quad \begin{aligned} \eta_1^{a_n} \circ \sigma_{n-1} &= (\eta_{a_n} \circ \eta_{a_n-1} \circ \dots \circ \eta_n) \circ (\eta_1^{n-1} \circ \sigma_{n-1}) \\ &= (\eta_{a_{n+1}} \circ \eta_{a_{n+1}-1} \circ \dots \circ \eta_{a_n+2}) \circ \eta_{a_n+1}. \end{aligned}$$

Therefore $\Phi_{n+1} = \eta_1^{a_{n+1}}$. This completes the induction step.

Now we get $h(\varphi_{1,\infty}) \geq (1/2) \log 2$ as in Example 5.1.

We will need some additional estimates connected with Example 5.2.

LEMMA 5.3. *Let α_n be the number of $i \in \{1, \dots, n\}$ for which $\eta_i = \gamma_0$ in Example 5.2. Then $\alpha_m/m \geq 1/4$ for all m .*

PROOF. It is easy to produce estimates of α_{a_n} by induction. First, by definition, we have $\eta_1 = \gamma_0$, so $\alpha_{a_1} = 1$. For $k = 1$, (5.3) reads $\sigma_0 = \eta_2$ (η_1^0 is the identity), so also $\eta_2 = \gamma_0$. Then $a_2 - a_1 = 3 - 1 = 2$, so $\eta_3 = \eta_{a_1+2} = \eta_{1-2+2} = \eta_1 = \gamma_0$. Thus, $\alpha_{a_2} = \alpha_3 = 3$. Now, if $k \geq 2$ then we get $\eta_{a_k+2}, \dots, \eta_{a_{k+1}}$ by repeating some η_i with $i \leq a_k$ (we take $a_{k+1} - a_k - 1$ distinct i 's). Among these η_i 's there may be at most $a_k - \alpha_{a_k}$ maps different from γ_0 , so at least $(a_{k+1} - a_k - 1) - (a_k - \alpha_{a_k})$ of them are equal to γ_0 . This gives us

$$\alpha_{a_{k+1}} - \alpha_{a_k} \geq (a_{k+1} - a_k - 1) - (a_k - \alpha_{a_k}),$$

so $\alpha_{a_{k+1}} - (k + 1) \geq 2(\alpha_{a_k} - k)$. Since $\alpha_{a_2} - 2 = 1$, by induction we get $\alpha_{a_n} \geq 2^{n-2} + n$ for all $n \geq 2$.

If $n \geq 2$ and $a_n \leq m < a_{n+1}$ then

$$\frac{\alpha_m}{m} \geq \frac{\alpha_{a_n}}{a_{n+1}} \geq \frac{2^{n-2} + n}{2^n + n} \geq \frac{1}{4}.$$

If $m = 1, 2$ then $\alpha_m/m = 1 > 1/4$. ■

In the next example we construct a triangular map using maps from Example 5.2.

EXAMPLE 5.4. The main idea of the construction is the same as in [Ko]. The map is going to be of the form $F(x, y) = (f(x), g_x(y))$, where on essential pieces g_x is equal to the maps σ_n from Example 5.2. As $f : I \rightarrow I$ we take a unimodal map of type 2^∞ without wandering intervals. In the construction we will use some well known combinatorial properties of such maps (see e.g. [CE], [dMvS], [Ko]). Let us list them.

For every n there are 2^n disjoint intervals whose union K_n is invariant for f . Each component of K_n contains 2 components of K_{n+1} and a periodic point of period 2^n between them. The intersection of all K_n 's is an invariant Cantor set K . The critical point c of f belongs to K .

Thus, the set K_1 has 2 components. Denote the one that does not contain c by L_1 . Now $K_2 \setminus L_1$ has 2 components and we denote the one that does not contain c by L_2 . We continue this construction and obtain a sequence

of intervals L_n that approach c and lie alternately to the left and right of c . We denote by M_n the gap between L_n and L_{n+2} . Note that for each n the periodic point p_n closest to c , of period 2^{n-1} , belongs to M_n . There is only one periodic orbit of period 2^{n-1} for every n .

The trajectory of c passes through L_1 every two units of time (starting with $f(c)$), through L_2 every four units of time (starting with $f^2(c)$), and in general through L_k every 2^k units of time, starting with $f^{2^{k-1}}(c)$. This means that $f^j(c) \in L_i$ if the number of trailing zeros in the binary expansion of j is $i - 1$.

Comparing this property with the description of $i(n)$ from Example 5.1, we see that we can apply Example 5.2 if we set $g_x = \sigma_0$ if $x \in L_1$ and $g_x = \sigma_{k-2}$ if $x \in L_k$ for $k > 1$. Then $h(F) \geq (1/2) \log 2$.

On the other hand, if $g_x \equiv 0$ at one point x of every periodic orbit of f , then the only periodic points of F are of the form $(p, 0)$, where p is a periodic point of f , and then F is of type 2^∞ .

We have not completed our construction yet. If there are points in I outside the smallest interval containing L_1 and L_2 , we set $g_x = \sigma_0$ there. More importantly, we have to define g_c and g_x for $x \in M_n$. The first part is very simple, namely we set $g_c \equiv 0$. The second part is slightly more complicated. We start by setting $g_{p_n} \equiv 0$ (so we get F of type 2^∞). Now we have to fill the gaps between p_n and the endpoints of M_n . Take an auxiliary function $\psi : \mathbb{R} \rightarrow [0, 1]$ of class C^∞ such that ψ is 0 on $(-\infty, 0]$ and 1 on $[1, \infty)$. For $J = [a, a + \alpha]$ define $\psi_J^+, \psi_J^- : J \rightarrow [0, 1]$ by

$$(5.6) \quad \psi_J^+(a + t) = \psi_J^-(a + \alpha - t) = \psi(t/\alpha).$$

Now on the gaps at both sides of p_n we define g_x as the corresponding $\psi_J^\pm(x) \cdot \sigma_k$, so that the maps agree at both endpoints of the gap with the previously defined ones.

In Example 5.2, λ_n was defined as the leftmost preimage of 1 under η_1^n . Since each η_i belongs to \mathcal{T}^r , we have

$$(5.7) \quad \lambda_n = (\eta_1|_{[0,1/2]})^{-1} \circ (\eta_2|_{[0,1/2]})^{-1} \circ \dots \circ (\eta_n|_{[0,1/2]})^{-1}(1).$$

Each $(\eta_i|_{[0,1/2]})^{-1}$ maps $[0, 1]$ onto $[0, 1/2]$ and it is convex since η_i is concave. Therefore

$$(5.8) \quad (\eta_i|_{[0,1/2]})^{-1}(x) \leq x/2$$

for each x . Hence, $\lambda_n \leq 2^{-n}$ for every n , that is, λ_n decreases exponentially with n . Hence, σ_n converges to 0 uniformly, and therefore g_x converges to $g_c \equiv 0$ uniformly as $x \rightarrow c$. This proves continuity of F .

In order to specify the triangular map F in Example 5.4 we have to choose the maps f and γ_n . There is enormous freedom in doing that, so we can try to narrow the choice in order to get better properties of F . For

instance, we can choose f and γ_n with negative Schwarzian derivative, and then F has maps with negative Schwarzian derivative both in the base and in each fiber.

Another problem is to make F as smooth as possible. First, choose f and γ_n of class C^r for some fixed finite r . We need some estimates. We use the notation from Example 5.4.

LEMMA 5.5. *Assume that f is of class C^3 and $f''(c) \neq 0$. Then there are constants $\delta, \zeta > 0$ such that for every n the distance of p_n from each endpoint of M_n is larger than $\zeta\delta^n$.*

PROOF. It is easy to check that our f belongs to the class \mathcal{U}^2 defined in Chapter VI of [dMvS], and therefore we can use the results of that chapter.

From Theorem 2.1 of Chapter VI of [dMvS] it follows that there exist $\delta, \zeta_1 > 0$ such that for every n the length of M_n is larger than $\zeta_1\delta^n$.

Denote the common endpoint of L_n and M_n by a_n and the common endpoint of L_{n+2} and M_n by b_n . Then denote by J_n the interval with endpoints c and b_n . Again by Theorem 2.1 of Chapter VI of [dMvS], the ratios $|M_n|/|J_n|$ and $|L_n|/|M_n|$ are bounded from above by a constant ζ_2 independent of n .

The map $f^{2^{n-1}}$ is decreasing on $M_n \cup J_n$. It fixes p_n , maps a_n to b_n and b_n to some point of L_n (that is, beyond a_n , if we look from p_n). By Lemma 2.1 of Chapter VI of [dMvS], on $M_n \cup J_n$ we have $f^{2^{n-1}} = \psi_1 \circ \psi_2 \circ \psi_3$, where ψ_1 and ψ_3 have distortion bounded by a constant ζ_3 independent of n and ψ_2 is the quadratic map $\psi_2(x) = x^2$. By the chain rule, the distortion of $f^{2^{n-1}}$ on M_n is bounded by ζ_3^2 multiplied by the distortion of ψ_2 on $\psi_3(M_n)$. Since the only critical point of $\psi_1 \circ \psi_2 \circ \psi_3$ is c , we have $\psi_3(c) = 0$. Thus, the distortion of ψ_2 on $\psi_3(M_n)$ is equal to $|\psi_3(M_n \cup J_n)|/|\psi_3(J_n)|$. This number is in turn bounded by $\zeta_3|M_n \cup J_n|/|J_n| \leq \zeta_3(\zeta_2 + 1)$. Hence, the distortion of $f^{2^{n-1}}$ on M_n is bounded by a constant ζ_4 independent of n .

The average of $|(f^{2^{n-1}})'|$ on M_n is equal to $|f^{2^{n-1}}(M_n)|/|M_n|$. Since $f^{2^{n-1}}(M_n)$ is contained in $M_n \cup L_n$, it is less than or equal to $1 + \zeta_2$. Therefore the maximum of $|(f^{2^{n-1}})'|$ on M_n is less than or equal to $\zeta_4(1 + \zeta_2)$. This gives us $|b_n - p_n| \leq \zeta_4(1 + \zeta_2)|a_n - p_n|$ and $|a_n - p_n| \leq \zeta_4(1 + \zeta_2)|b_n - p_n|$. Hence, both $|a_n - p_n|$ and $|b_n - p_n|$ are larger than or equal to $\zeta\delta^n$, where the constant $\zeta = \zeta_1/(1 + \zeta_4(1 + \zeta_2))$ is independent of n . ■

We denote the C^r -norm of a function g by $\|g\|_r$.

LEMMA 5.6. *Assume that in Example 5.4 the map f is of class C^r and $\gamma_n \in \mathcal{T}^r$, $n = 0, 1, 2, \dots$, the norms $\|\gamma_n\|_r$ are bounded uniformly in n , and*

$$(5.9) \quad \liminf_{n \rightarrow \infty} \frac{1}{n} \log d_n > -\frac{1}{4r} \log \gamma'_0(0),$$

where d_n is the minimum of the distances of p_n from the endpoints of M_n . Then F is of class C^r .

Proof. The whole base interval is divided into countably many intervals and over each of them the map F is of the form $(f(x), \varphi(x)\gamma_n(y))$. The maps f and γ_n are of class C^r and φ is of class C^∞ , so F is of class C^r there. In order to show that it is of class C^r everywhere, we have to check what happens where two such intervals meet (that is, at the points p_n) and where these intervals accumulate (that is, at c).

The map φ can be defined on the whole real line by setting $\varphi(x) = 0$ outside the interval mentioned in the preceding paragraph, and then it is of class C^∞ everywhere. Therefore over the points p_n all partial derivatives of the second component of F up to order r are 0 from both sides and they are continuous over p_n from both sides. This leaves the problem of what happens when we approach c .

If a function is of the form $\varphi(x)\gamma_n(y)$ then its partial derivatives are of the form $\varphi^{(k)}(x)\gamma_n^{(l)}(y)$. Let us look at both factors as x approaches c . On L_n the function φ is constant, so its derivatives are 0. On the gaps where the functions ψ_J^\pm were used in the construction, we have $\varphi = \lambda_n \cdot \psi_J^\pm$. By (5.6) we get

$$\sup_J |(\psi_J^+)^{(k)}| = \sup_J |(\psi_J^-)^{(k)}| = |J|^{-k} \sup_{[0,1]} |\psi^{(k)}|.$$

In this formula, J is the interval between p_i and one of the endpoints of M_i , where $i = n$ or $n + 2$. Hence, $|\varphi^{(k)}| \leq \lambda_n \cdot d_i^{-k} \sup_{[0,1]} |\psi^{(k)}|$. By Lemma 5.3, (5.7) and (5.8), we have $\lambda_n \leq (\gamma_0|_{[0,1/2]})^{-E(n/4)}(1)$, where $E(\cdot)$ denotes the integer part. Therefore

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \lambda_n \leq -\frac{1}{4} \log \gamma'_0(0).$$

By (5.9) and since $k \leq r$, we have

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log d_n^{-k} < \frac{1}{4} \log \gamma'_0(0).$$

Thus,

$$(5.10) \quad \lim_{x \rightarrow c} \varphi^{(k)}(x) = 0.$$

Note that (5.10) also holds for $k = 0$.

Since the norms $\|\gamma_n\|_r$ are bounded uniformly in n , from (5.10) it follows that all partial derivatives of the second component of F up to order r go to 0 uniformly in y as $x \rightarrow c$. Since we already know that they are continuous except possibly at the points of the fiber over c , it remains to prove that

they are all 0 if $x = c$. However, this follows immediately by induction with respect to the order of differentiation with respect to x and the mean value theorem. ■

The next theorem follows directly from Lemmas 5.5 and 5.6. It shows that for a very wide choice of maps f and γ_n the map F from Example 5.4 is of class C^r with finite r .

THEOREM 5.7. *Assume that in Example 5.4 the map f is of class C^r with $3 \leq r < \infty$ and $f''(c) \neq 0$. Then there exists a constant N such that if the functions $\gamma_n \in \mathcal{T}^r$, the norms $\|\gamma_n\|_r$ are bounded uniformly in n and $\gamma'_0(0) > N$ then the map F is of class C^r .*

If we want to get F of class C^∞ then the choice is not so wide, but the construction still works if we take the map from [M] on the base.

THEOREM 5.8. *Assume that in Example 5.4 the map f is the C^∞ map constructed in [M], the maps γ_n are in \mathcal{T}^∞ and for every finite r their norms $\|\gamma_n\|_r$ are bounded uniformly in n (although the bounds may depend on r). Then the map F is of class C^∞ .*

Proof. There are formulas in [M] for the distance of the point p_n to the endpoints of M_n . They give estimates of d_n from below by reciprocals of polynomials in n . Therefore

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log d_n = 0,$$

while by concavity $\gamma'_0(0) \geq 2$. Therefore (5.9) holds for all r , and by Lemma 5.6 the map F is of class C^r for every finite r . ■

Note that for any map F considered in Theorems 5.7 and 5.8 there is at most one periodic orbit of a given period (we may assume for simplicity that the base map has only one fixed point). Therefore

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \text{card Fix}(F^n) = 0.$$

Moreover, any periodic point x is a saddle. Its stable manifold is the fiber over this point. The image of the whole space intersects this fiber only at x , and therefore there are no homoclinic points. Similarly, there are no horseshoes of any type, even for iterates of F .

There is another way to prove that the topological entropy of C^∞ maps from Theorem 5.8 is positive. One can proceed as follows. If G is a C^1 function defined on I , denote the length of the graph Γ_G of G by $\text{Vol}(\Gamma_G)$. For a C^∞ triangular map F from Theorem 5.8, by the theorem of Yomdin

([Y], cf. [G]) we have

$$h(F; I_{f(c)}) \geq \limsup_{n \rightarrow \infty} \frac{1}{n} \log \text{Vol}(\Gamma_{g_{f(c)}^{[n]}}),$$

where $g_{f(c)}^{[n]} = g_{f^n(c)} \circ \dots \circ g_{f^2(c)} \circ g_{f(c)}$. We know (see Example 5.2) that

$$\text{Vol}(\Gamma_{g_{f(c)}^{[2^n-1]}}) = \text{Vol}(\Gamma_{\varphi_1^{2^n-1}}) = \text{Vol}(\Gamma_{\eta_1^{a_n}}) \geq 2^{a_n}.$$

Thus

$$h(F) \geq h(F; I_{f(c)}) \geq \limsup_{n \rightarrow \infty} \frac{1}{2^n - 1} \log 2^{a_n} = \frac{1}{2} \log 2 > 0.$$

References

- [AKM] R. L. Adler, A. G. Konheim and M. H. McAndrew, *Topological entropy*, Trans. Amer. Math. Soc. 114 (1965), 309–319.
- [ALM] L. Alsedà, J. Llibre and M. Misiurewicz, *Combinatorial Dynamics and Entropy in Dimension One*, World Sci., Singapore, 1993.
- [BEL] F. Balibrea, F. Esquembre and A. Linero, *Smooth triangular maps of type 2^∞ with positive topological entropy*, Internat. J. Bifur. Chaos 5 (1995), 1319–1324.
- [BC] L. S. Block and W. A. Coppel, *Dynamics in One Dimension*, Lecture Notes in Math. 1513, Springer, Berlin, 1992.
- [B] R. Bowen, *Entropy for group endomorphisms and homogeneous spaces*, Trans. Amer. Math. Soc. 153 (1971), 401–414.
- [CE] P. Collet and J.-P. Eckmann, *Iterated Maps on the Interval as Dynamical Systems*, Progr. in Phys. 1, Birkhäuser, Boston, 1980.
- [D] E. I. Dinaburg, *Connection between various entropy characterizations of dynamical systems*, Izv. Akad. Nauk SSSR 35 (1971), 324–366 (in Russian).
- [G] M. Gromov, *Entropy, homology and semialgebraic geometry (after Y. Yomdin)*, Astérisque (Séminaire Bourbaki, 1985–86, no. 663) 145–146 (1987), 225–240.
- [Ka] A. Katok, *Lyapunov exponents, entropy and periodic orbits for diffeomorphisms*, Publ. Math. IHES 51 (1980), 137–174.
- [Kl] P. E. Kloeden, *On Sharkovsky's cycle coexistence ordering*, Bull. Austral. Math. Soc. 20 (1979), 171–177.
- [Ko] S. F. Kolyada, *On dynamics of triangular maps of the square*, Ergodic Theory Dynam. Systems 12 (1992), 749–768.
- [KS] S. Kolyada and L. Snoha, *Topological entropy of nonautonomous dynamical systems*, Random Comput. Dynam. 4 (1996), 205–233.
- [dMvS] W. de Melo and S. van Strien, *One-Dimensional Dynamics*, Series of Modern Surveys in Math., Springer, Berlin, 1993.
- [M] M. Misiurewicz, *Attracting set of positive measure for a C^∞ map of an interval*, Ergodic Theory Dynam. Systems 2 (1982), 405–415.
- [MS] M. Misiurewicz and W. Szlenk, *Entropy of piecewise monotone mappings*, Studia Math. 67 (1980), 45–63.

[Y] Y. Yomdin, *Volume growth and entropy*, Israel J. Math. 57 (1987), 285–300.

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