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Bohr compactifications of discrete structures

by

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Abstract. The Bohr compactification and the Bohr topology are well known for groups, but they can easily be generalized to arbitrary structures. We prove a number of theorems about Bohr topologies in this general setting. Some of these results are new even for groups; for example, the weight of the Bohr compactification of a countable structure is either countable or continuum. In some cases, theorems about Bohr topologies are special cases of more general results in C_{p} theory. We also present applications of these generalities to the Bohr compactifications of lattices, semilattices, and loops.

1. INTRODUCTION

The Bohr topology and Bohr compactification for groups date back to the 1940 manuscript of Weil [35], and are well known in harmonic analysis. In fact these notions generalize to arbitrary algebraic structures, as pointed out by Holm [14] in 1964. For example, if $\mathfrak{A}$ is ring (with no topology on it yet), its *Bohr–Holm compactification* is a compact ring $b\mathfrak{A}$, together with a ring homomorphism Φ from $\mathfrak{A}$ into a dense subring of $b\mathfrak{A}$. The pair $(b\mathfrak{A}, \Phi)$ is characterized as the *maximal* compactification of $\mathfrak{A}$. Then $\mathfrak{A}^{\#}$ denotes the ring $\mathfrak{A}$ together with its *Bohr topology*—that is, the topology induced by the map $\Phi : \mathfrak{A} \to b\mathfrak{A}$. So, $\mathfrak{A}^{\#}$ is a topological ring.

These notions are made precise by Definition 2.3.6. Actually, unlike in Holm [14], we define $b\mathfrak{A}$ without reference to any algebraic axioms which $\mathfrak{A}$ may satisfy. As we point out though, $b\mathfrak{A}$ satisfies all the positive logical sentences satisfied by $\mathfrak{A}$ (see Lemma 2.3.9). In particular, if $\mathfrak{A}$ is a ring, then $b\mathfrak{A}$ will also be a ring, and if every element of $\mathfrak{A}$ has a square root, the same will be true in $b\mathfrak{A}$; it is not necessary to decide ahead of time whether to

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view $\mathfrak{A}$ as a member of the category of rings or of the category of rings all of whose elements have square roots.

Once the definitions are given, Section 2 develops some general theorems about $b\mathfrak{A}$ and $\mathfrak{A}^{\#}$, and Section 3 applies these to some specific classes of structures—primarily groups, quasigroups, semilattices, and lattices—where one sometimes has a fairly simple description of $b\mathfrak{A}$.

Some of the general results in this paper are new even in the case of groups. For example (see Section 2.8), given a structure $\mathfrak{B}$, one can always find a countable $\mathfrak{A} \subseteq \mathfrak{B}$ such that $b\mathfrak{A}$ is just the restriction of $b\mathfrak{B}$ to $\mathfrak{A}$. One cannot in general let $\mathfrak{A}$ be an arbitrary countable substructure of $\mathfrak{B}$ here; one can in the cases of semilattices (by Theorem 3.4.26) and abelian groups (as is well known), but one cannot in the cases of distributive lattices (see Section 3.5) or non-abelian groups.

Also, by Corollary 2.10.20, the weight of the closure of every countable subset of $\mathfrak{B}$ in $b\mathfrak{B}$ is either countable or $2^{\aleph_0}$. For groups, both these values are possible (see Section 3.3). For abelian groups, however, only $2^{\aleph_0}$ is possible, by arguments described in [11] and [21]. Actually, Corollary 2.10.20 is a special case of a more general result in C_{p} theory, as we explain in Section 2.10.

A general question, for infinite structures $\mathfrak{A}, \mathfrak{B}$, is whether $\mathfrak{A}^{\#}, \mathfrak{B}^{\#}$ can be homeomorphic topological spaces when $\mathfrak{A}, \mathfrak{B}$ are not isomorphic structures. For most varieties of structures, it is easy to give many such examples, but in the case of abelian groups, this is a long-standing question of van Douwen, and has generated a fairly large body of literature. Some references are given in Section 3.3, together with our proof that $\mathfrak{A}^{\#}, \mathfrak{B}^{\#}$ are always homeomorphic whenever $\mathfrak{A}$ is a subgroup of $\mathfrak{B}$ of finite index.

Other of our theorems show how some results which are known in the case of groups can be extended to more general classes of structures. For example, in Section 2.9, we discuss conditions which imply that $b(\mathfrak{A} \times \mathfrak{B}) = b\mathfrak{A} \times b\mathfrak{B}$. This equality holds for semigroups with an identity element 1 [16, 14, 15], but not for semigroups in general. The use of the 1 has an obvious generalization (Lemma 2.9.3) to other structures, but we also show $b(\mathfrak{A} \times \mathfrak{B}) = b\mathfrak{A} \times b\mathfrak{B}$ for some structures which lack the 1, such as semilattices, lattices, and quasigroups. Proving equality for this more general case involves our study of substructures in Section 2.7 (in some cases, it is "harmless" to extend the structure to add a 1), as well as conditions under which the basic functions in the structure may be modified; see Section 2.8. For example, for groups, $b(G; \cdot) = b(G; \cdot, i)$; that is, it does not matter whether or not we consider the variety to include the unary inverse function. A similar result holds for some (but not all) varieties of loops (see Section 3.2). For distributive lattices, one cannot in general identify $b(A; \vee, \wedge)$ with $b(A; \vee)$, even though $\wedge$ is first-order definable from $\vee$; one can drop the $\wedge$ in the case of total orders (see Sections 3.4 and 3.5).

For groups $\mathfrak{A}$, one may compute $b\mathfrak{A}$ by using the homomorphisms into various $U(n)$ (the group of all $n \times n$ unitary matrices); we say that $\{U(n) : 1 \leq n < \omega\}$ is *adequate* for groups. For abelian groups, the circle group $U(1)$ alone gives us the adequate set $\{U(1)\}$. The situation for general $\mathfrak{A}$ is discussed in Sections 2.6 and 2.10. The collection of all second countable compact structures is always adequate (Theorem 2.6.4), but this collection is uncountable. If there is a countable adequate family for $\mathfrak{A}$, then $\mathfrak{A}^{\#}$ is an Eberlein–Grothendieck space (a notion from C_{p} theory; see Definition 2.10.7). A countable $\mathfrak{A}$ for which $\mathfrak{A}^{\#}$ is not an Eberlein–Grothendieck space (because $\mathfrak{A}^{\#}$ is the Fréchet–Urysohn fan) is described in Example 3.6.7.

A number of other definitions of $b\mathfrak{A}$ and $\mathfrak{A}^{\#}$ are known to be equivalent in the case of groups. For example, $G^{\#}$ is the finest totally bounded topological group topology on G; the correct generalization of this (described in Section 2.4 and by Holm [14]) derives $\mathfrak{A}^{\#}$ from the finest totally bounded uniformity; it just happens that in the group case, the uniformity is obtained directly from the topology. For groups, bG can also be defined via almost periodic functions. In fact, the name of Harald Bohr is attached to bG and $G^{\#}$ in recognition of his work [4] on almost periodic functions. This approach does not seem to generalize to arbitrary structures; see also Remark 2.3.12.

Up to now, we have assumed that $\mathfrak{A}$ is just an abstract (discrete) structure. However, all the basic definitions easily generalize to topological structures, where $\mathfrak{A}$ already has a topology $\mathcal{T}$ on it, in which case $\mathfrak{A}^{\#}$ will be coarser than $\mathcal{T}$. Although the emphasis of this paper is on compactifications of discrete structures, we shall point out where the general theory also works for arbitrary topological structures.

We have tried to provide counter-examples to possible extensions or generalizations of our results. Where we could, we have chosen these examples from naturally occurring mathematical structures. In some cases, we did not see how to do this, so Section 3.6 collects a number of artificial counter-examples.

2. GENERALITIES

We prove some general results here which are common to all structures.

2.1. Topological structures. In discussing structures, we shall employ the standard terminology of first-order logic. Throughout, $\mathcal{L}$ denotes a (possibly empty) set consisting of constant symbols and function symbols; each function symbol has an arity ≥ 1. Using the symbols of $\mathcal{L}$ plus the predicate "=", one may build logical formulas in the usual way; we never consider predicates other than equality here (see Remark 2.3.13). A *structure* $\mathfrak{A}$ for $\mathcal{L}$ is a non-empty set A (the *domain*), together with actual elements of and functions on A corresponding to the constant and function symbols of $\mathcal{L}$. For example, when discussing groups, we could take $\mathcal{L} = \{\cdot, i, 1\}$ (the symbols for product, inverse, and identity). If $s \in \mathcal{L}$, we use $s_{\mathfrak{A}}$ for the corresponding constant or function on A. Then we frequently drop the subscript "$\mathfrak{A}$" when it is clear from context. So, for example, we display groups as $\mathfrak{A} = (A; \cdot, i, 1)$.

DEFINITION 2.1.1. Suppose $\mathfrak{A}$ is a structure for $\mathcal{L}$ and $\varphi : A \to X$. If $f \in \mathcal{L}$ is an n-ary function symbol, then $\varphi(f_{\mathfrak{A}})$ denotes $\{(\varphi(a_1), \ldots, \varphi(a_n), \varphi(b)) : (a_1, \ldots, a_n, b) \in f_{\mathfrak{A}}\}$.

Here, we identify $f_{\mathfrak{A}}$ with its graph, a subset of A^{n+1}. Note that $\varphi(f_{\mathfrak{A}}) \subseteq X^{n+1}$, but need not be the graph of an n-ary function.

DEFINITION 2.1.2. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two structures for $\mathcal{L}$, and let $\varphi : A \to B$. Then φ is a *homomorphism* from $\mathfrak{A}$ to $\mathfrak{B}$ iff $\varphi(f_{\mathfrak{A}}) \subseteq f_{\mathfrak{B}}$ for each function symbol f of $\mathcal{L}$, and $\varphi(c_{\mathfrak{A}}) = c_{\mathfrak{B}}$ for each constant symbol c of $\mathcal{L}$.

This is equivalent to the standard definition of homomorphism in universal algebra. The notation involving graphs introduced here will be useful in discussing compactifications, where we frequently use the fact that a function between compact Hausdorff spaces is continuous iff its graph is closed.

DEFINITION 2.1.3. A *topological structure* for $\mathcal{L}$ is a pair $(\mathfrak{A}, \mathcal{T})$, where $\mathfrak{A}$ is a structure for $\mathcal{L}$, and $\mathcal{T}$ is a topology on A which makes all the functions of $\mathfrak{A}$ continuous. We often write $\mathfrak{A}$ for $(\mathfrak{A}, \mathcal{T})$ if the topology is understood.

A special case is a *discrete structure*, where $\mathcal{T}$ is the discrete topology. At the other extreme, $\mathcal{L}$ could be $\emptyset$, in which case a structure is just a set and a topological structure is just a topological space.

DEFINITION 2.1.4. A *compact structure* for $\mathcal{L}$ is any topological structure $(\mathfrak{A}, \mathcal{T})$ in which $\mathcal{T}$ is a compact Hausdorff topology.

Note that compact structures are Hausdorff by definition, but topological structures in general have no separation axioms assumed about them.

Many of the common classes of structures are specified by sets of equations. The following table lists some equational classes which we use later in this paper. Also listed are the appropriate $\mathcal{L}$ and the arities of the symbols; symbols of arity 0 are constants.

Theory	$\mathcal{L}$	Arities
Groups	$(\cdot, i, 1)$	$(2, 1, 0)$
Quasigroups	$(\cdot, \backslash, /)$	$(2, 2, 2)$
Loops	$(\cdot, \backslash, /, 1)$	$(2, 2, 2, 0)$
Rings	$(+, \cdot, -, 0)$	$(2, 2, 1, 0)$
Semilattices	$(\vee)$ or $(\wedge)$	(2)
Lattices	$(\vee, \wedge)$	$(2, 2)$
Boolean algebras	$(\vee, \wedge, {}', 0, 1)$	$(2, 2, 1, 0, 0)$
Homogeneities	(f, g)	$(3, 3)$
Pairings	(p, L, R)	$(2, 1, 1)$

Of course, many modifications are possible. For example, for bounded lattices (with a largest and smallest element), take $\mathcal{L} = \{\vee, \wedge, 0, 1\}$. Note that since our languages do not use predicates, we consider theories such as lattices and boolean algebras to be presented only with functions, not with a $\leq$ predicate as is frequently done. Allowing predicates in $\mathcal{L}$ would be possible, but it makes the general theory ugly; see Remark 2.3.13.

We comment briefly on the theories listed which are not well known from elementary algebra. The axioms for *quasigroups* are

$$x \cdot (x \backslash y) = y, \quad (y/x) \cdot x = y, \quad x \backslash (x \cdot u) = u, \quad (u \cdot x)/x = u.$$

In terms of $\cdot$ alone, this is the same as postulating $\forall xy \exists! z (xz = y)$ and $\forall xy \exists! z (zx = y)$, but it is often convenient to express the axioms in a purely equational way by replacing the z in these two axioms by the functions $x \backslash y$ and y/x of x and y, as in the above equations. In combinatorics, quasigroups are identified with Latin squares. Every associative quasigroup is a group. A *loop* is a quasigroup with an identity element 1 (satisfying $x \cdot 1 = 1 \cdot x = x$). The texts [5], [7], [27] give further information on quasigroups and loops.

Some results for quasigroups and loops hold more generally for homogeneities; see Sections 3.1 and 3.2.

DEFINITION 2.1.5. A *homogeneity* is a structure $(A; f, g)$ satisfying

$$\begin{aligned} f(x, y, x) = g(x, y, x) &= y, \\ g(x, y, f(y, x, z)) &= z, \\ f(x, y, g(y, x, z)) &= z. \end{aligned}$$

That is, for each x, y, the maps $f(x, y, \_\_)$ and $g(x, y, \_\_)$ are both permutations of A taking x to y, and $f(x, y, \_\_)$ and $g(y, x, \_\_)$ are inverses of each other. Recall that a topological space is called *homogeneous* iff for all x, y, there is a homeomorphism moving x to y. So, a topological homogeneity is one way of expressing the informal notion that these homeomorphisms can be selected in a continuous way. Note that this puts a large restriction on the homogeneous space. For example, let A be an infinite compact Hausdorff space which supports a homogeneity. Then the weight of A equals its character (see Corollary 3.1.2), which is not true for many homogeneous A. In fact, A must be dyadic (this is easy to see from Theorem 1 of Uspenskiĭ [33]).

Definition 2.1.6. A *pairing* is a structure $(A; p, L, R)$ which satisfies: $x = L(p(x, y))$, $y = R(p(x, y))$, and $p(L(x), R(x)) = x$.

Thus, p provides a bijection from $A \times A$ onto A. These pairings will form a useful collection of examples and counter-examples.

We shall make use of the following elementary notions from logic: A *sentence* is a formula with no free variables. A *positive formula* is one which is logically equivalent to one expressed using quantifiers and only the propositional connectives AND and OR. A *theory* is a set of sentences and a *positive theory* is a set of positive sentences. A structure $\mathfrak{A}$ is a *model* of a theory Σ (i.e., $\mathfrak{A} \models \Sigma$) iff all the sentences of Σ are true in $\mathfrak{A}$. The theory Σ is *consistent* iff it has some model. An *equational theory* is a theory all of whose sentences are universally quantified equations. So, every equational theory is a positive theory. The theory of groups expressed in the language $\{\cdot, i, 1\}$ is equational, but if this theory is expressed in the language $\{\cdot\}$, it becomes positive but no longer equational; for example, one must say $\exists y \forall x (x \cdot y = x)$.

Every positive theory Σ is consistent, since it has a 1-element model. It is possible that Σ has only the 1-element model (e.g. $\Sigma = \{x = y\}$). It is also possible that Σ has infinite models but, as is the case for pairings, the 1-element model is the only finite model. Pairings also have infinite compact models, since there are infinite compact X homeomorphic to X^2, yet there are equational theories such as lattice ordered abelian groups, with infinite models, but only the 1-element compact model. For any equational theory with a compact model of size greater than one, infinite products will generate infinite compact models. In some cases, every compact model is a product of finite models. For example, by Strauss [32], every compact boolean algebra is of the form $\{0, 1\}^\kappa$.

2.2. Compactifications of sets. We make some remarks here on compactifications, since our definitions differ somewhat from the standard ones (see Kelley [19]) in general topology.

Definition 2.2.1. Let A be any non-empty set. A *compactification* of A is a pair (X, φ), where X is a compact Hausdorff space, $\varphi : A \to X$, and $\varphi(A)$ is dense in X. If (X, φ) and (Y, ψ) are two compactifications of A, then $(X, \varphi) \le_\Gamma (Y, \psi)$ means that $\Gamma : Y \to X$ is a continuous function and $\Gamma \circ \psi = \varphi$. $(X, \varphi) \le (Y, \psi)$ means that $(X, \varphi) \le_\Gamma (Y, \psi)$ for some Γ.

Since compactifications are Hausdorff, if $(X, \varphi) \le (Y, \psi)$ then the Γ such that $(X, \varphi) \le_\Gamma (Y, \psi)$ is uniquely determined, and $\Gamma(Y) = X$.

Definition 2.2.2. Two compactifications of A, (X,φ) and (Y,ψ), are *equivalent* iff $(X,\varphi) \le (Y,\psi) \le (X,\varphi)$.

In this case, we have $(X,\varphi) \le_\Gamma (Y,\psi) \le_\Delta (X,\varphi)$, where Γ, Δ are inverses of each other, so that X, Y are homeomorphic.

Definition 2.2.3. $\mathbb{K}(A)$ is the set of all equivalence classes of compactifications of A.

This is a set, not a proper class, since each compactification of A has size no more than $2^{2^{|A|}}$. Note that $\mathbb{K}(A)$ inherits the order $\le$. Actually, each equivalence class, $[(X,\varphi)]$, is a proper class, but that does not cause foundational problems (one can either take a set of representatives, or deal with the associated uniformities (see Section 2.4) instead). In the following, we frequently say (X,φ) when we really mean $[(X,\varphi)]$. Each (X,φ) induces a topology, $\mathcal{T}_\varphi$, on A:

Definition 2.2.4. If (X,φ) is a compactification of A, then $\mathcal{T}_\varphi = \{\varphi^{-1}U : U \text{ is open in } X\}$.

Lemma 2.2.5. *$\mathcal{T}_\varphi$ is Hausdorff iff φ is* 1-1.

Lemma 2.2.6. *If $(X,\varphi) \le (Y,\psi)$, then $\mathcal{T}_\varphi \subseteq \mathcal{T}_\psi$.*

Three simple examples: The maximal element of $\mathbb{K}(A)$ is $(\beta A, \varphi_1)$, where φ_1 is the usual inclusion of A into the Čech compactification of the set A with the discrete topology. The minimal element of $\mathbb{K}(A)$ is the *one-element* compactification, $(\{x\}, \varphi_2)$, where $\varphi_2(a) = x$ for all $a \in A$, and hence $\mathcal{T}_{\varphi_2}$ is indiscrete. This should not be confused with the *one-point* compactification, $(A \cup \{\infty\}, \varphi_3)$, which is neither minimal nor maximal in $\mathbb{K}(A)$, although it is the minimal compactification whose induced topology is discrete. The converse to Lemma 2.2.6 is false; for example $\mathcal{T}_{\varphi_1} = \mathcal{T}_{\varphi_3}$ but $(\beta A, \varphi_1) \not\le (A \cup \{\infty\}, \varphi_3)$ unless A is finite.

To get an "iff" in Lemma 2.2.6, one would have to use the induced uniformity, not the induced topology; see Section 2.4. A converse to Lemma 2.2.6 holds in some cases, when the spaces are endowed with sufficient algebraic structure to be able to read the uniformity from the topology; see Lemma 3.1.5.

Lemma 2.2.7. *$\mathbb{K}(A)$ is a complete lattice.*

Proof. We compute $\bigvee\{(X_i,\varphi_i) : i \in I\}$ to be (X,φ), where φ is the natural product map from A into $\prod_i X_i$, and X is the closure of the range of φ. Then $\bigwedge\{(X_i,\varphi_i) : i \in I\}$ is just $\bigvee\{(Y,\psi) : \forall i[(Y,\psi) \le (X_i,\varphi_i)]\}$. ∎

The $\bigwedge$ of compactifications seems a bit intractable, and is not widely discussed in the literature.

When the target space is clear from context, we frequently say φ when we mean (X,φ), as we already did in Definition 2.2.4, where T_φ should really be $T_{(X,\varphi)}$. Likewise, we might say that $\beta A = \bigvee\{\varphi : \varphi \in [0,1]^A\}$. Each φ here really designates the pair, $(\varphi, \mathrm{cl}(\mathrm{ran}(\varphi)))$. On the other hand, when the map is clear from context, we frequently say X when we mean (X,φ); for example, we use βA for the pair consisting of βA and the embedding of A into βA.

2.3. Compactifications of spaces and structures. Now that we have $\mathbb{K}(A)$, we may restrict our attention to those compactifications (X,φ) which are compatible with some topology on A, or with some structure $\mathfrak{A}$ on A, or both.

DEFINITION 2.3.1. If $(\mathfrak{A},\mathcal{T})$ is a topological structure and (X,φ) is a compactification of the set A, then (X,φ) is *compatible* with $(\mathfrak{A},\mathcal{T})$ iff φ is continuous and there is a topological structure $\mathfrak{X}$ built on the set X such that φ is a homomorphism.

LEMMA 2.3.2. *With the notation of Definition* 2.3.1, (X,φ) *is compatible with* $(\mathfrak{A},\mathcal{T})$ *iff* φ *is continuous and* $\mathrm{cl}(\varphi(f_{\mathfrak{A}}))$ (see Definition 2.1.1) *is the graph of a function on* X *for each function symbol* f *of* $\mathcal{L}$.

In this case, the topological structure $\mathfrak{X}$ built on X must have $c_{\mathfrak{X}} = \varphi(c_{\mathfrak{A}})$ (for constants c) and $f_{\mathfrak{X}} = \mathrm{cl}(\varphi(f_{\mathfrak{A}}))$ (for functions f).

LEMMA 2.3.3. *If* (X,φ) *and* (Y,ψ) *are both compatible compactifications of* $\mathfrak{A}$, *with associated topological structures* $\mathfrak{X}$ *and* $\mathfrak{Y}$, *and* $(X,\varphi) \le_\Gamma (Y,\psi)$ *as in Definition* 2.2.1, *then* $\Gamma : \mathfrak{Y} \to \mathfrak{X}$ *is a homomorphism.*

The point of this lemma is that, when dealing with structures rather than abstract sets, we do not have to re-define the ordering on compactifications. It might seem more natural to require Γ to be a homomorphism, but we get this for free anyway.

LEMMA 2.3.4. *If* (X,φ) *is compatible with* $(\mathfrak{A},\mathcal{T})$, *then* $\mathcal{T}_\varphi$ *is coarser than* $\mathcal{T}$, *and* $(\mathfrak{A},\mathcal{T}_\varphi)$ *is a topological structure.*

The following lemma is clear from the construction of the $\bigvee$ in the proof of Lemma 2.2.7.

LEMMA 2.3.5. *If* (X_i,φ_i) *are compactifications of* $(A,\mathcal{T})$ *(for* $i \in I$*), and each* (X_i,φ_i) *is compatible with* $(\mathfrak{A},\mathcal{T})$ *then so is* $\bigvee\{(X_i,\varphi_i) : i \in I\}$.

In particular, there is a maximal compatible compactification, since there is at least one compatible compactification (namely, the 1-element compactification).

DEFINITION 2.3.6. The *Bohr–Holm compactification*, $(b(\mathfrak{A},\mathcal{T}),\Phi_{(\mathfrak{A},\mathcal{T})})$, of a given topological structure $(\mathfrak{A},\mathcal{T})$, is the maximal compatible compactification. The $\mathcal{T}$ is omitted when it is clear from context. $(\mathfrak{A},\mathcal{T})^{\#}$ denotes the structure $\mathfrak{A}$ with the *Bohr topology*—that is, the topology $\mathcal{T}_{\Phi}$ induced by the map $\Phi = \Phi_{(\mathfrak{A},\mathcal{T})}$.

Then, as in Theorem 8 of Holm [14], we get:

LEMMA 2.3.7. $\mathfrak{A}^{\#}$ *is a topological structure, and is Hausdorff iff* $\Phi_{(\mathfrak{A},\mathcal{T})}$ *is* 1-1. *The topology of* $\mathfrak{A}^{\#}$ *is coarser than the original topology*, $\mathcal{T}$.

As in Theorem 8 of [14] (and as usual in general topology), "coarser" need not imply "strictly coarser".

It may happen that the only compatible compactification is the one-element compactification ($\mathfrak{A}$ is "minimally almost periodic"), in which case $b\mathfrak{A}$ will be a singleton and $\mathfrak{A}^{\#}$ will be indiscrete. In the case where Φ is 1-1 ($\mathfrak{A}$ is "maximally almost periodic"), we may simply identify $\mathfrak{A}^{\#}$ as a subset of $b\mathfrak{A}$, with the subspace topology.

An important special case in general topology is where $\mathcal{L}$ is empty, so we just have a topological space $(A,\mathcal{T})$. If $\mathcal{T}$ is a completely regular Hausdorff topology, then $(b(A,\mathcal{T}),\Phi_{(A,\mathcal{T})})$ is just the natural embedding of A into its Čech compactification. In this (and only this) case, if we identify A as a subset of bA, then the subspace topology agrees with $\mathcal{T}$. There are regular Hausdorff spaces $(A,\mathcal{T})$ all of whose maps into compact Hausdorff spaces (equivalently, into $[0,1]$) are constant; for these, $b(A,\mathcal{T})$ is a singleton and $A^{\#}$ is indiscrete.

On the other hand, we may consider examples where $\mathfrak{A}$ is just an abstract structure, given the discrete topology. If $\mathfrak{A}$ is finite, then $b\mathfrak{A} = \mathfrak{A}$, Φ is the identity map, and $\mathfrak{A}^{\#}$ is discrete. It is possible for $\mathfrak{A}^{\#}$ to be discrete for infinite $\mathfrak{A}$ as well; for example, if $\mathcal{L}$ contains only constants and unary functions, then $bA = \beta A$, the Čech compactification of the discrete space A. If $\mathfrak{A}$ is an infinite group (or just a homogeneity), then $\mathfrak{A}^{\#}$ cannot be discrete (since it is dense in $b\mathfrak{A}$, which is dense in itself by homogeneity), but it might well be indiscrete, since by von Neumann and Wigner [25], [26], there are groups $\mathfrak{A}$ of all infinite cardinalities such that $b\mathfrak{A}$ is a singleton. If $\mathfrak{A}$ is an abelian group or a boolean algebra, then Φ is 1-1, so $\mathfrak{A}^{\#}$ is Hausdorff. However, if $\mathfrak{A} = (\mathbb{Z};+,-,0,\vee,\wedge)$, then $b\mathfrak{A}$ is a singleton, since the only compact lattice ordered abelian group is a singleton. Even if $\mathfrak{A} = (A;\vee,\wedge)$ is just a lattice, $\mathfrak{A}^{\#}$ may be indiscrete, although $\mathfrak{A}^{\#}$ is Hausdorff for distributive lattices (see Section 3.5). If $\mathfrak{A}$ contains functions of arity greater than one, then $b\mathfrak{A}$ will be βA only in trivial cases, but there are many examples where $\mathfrak{A}^{\#}$ is discrete. For example, if $\mathfrak{A} = (A;\vee,\wedge)$ is any total order, then $\mathfrak{A}^{\#}$ is discrete (see Section 3.4), but $b\mathfrak{A}$ is a compact LOTS, and hence will not be βA unless A is finite. The fact that $b\mathfrak{A}$ is indeed a LOTS in this case is a special case of Lemma 2.3.9 below.

LEMMA 2.3.8. *If $\psi(v_1,\ldots,v_n)$ is a positive logical formula, and $\mathfrak{X}$ is a compact structure, then $\{(x_1,\ldots,x_n)\in X^n : \mathfrak{X}\models\psi(x_1,\ldots,x_n)\}$ is closed in X^n.*

Proof. Induct on ψ. For the quantifier step, use the fact that the projection maps are both open and closed. ∎

LEMMA 2.3.9. *If $\psi(v_1,\ldots,v_n)$ is a positive logical formula, (X,φ) is a compactification of $\mathfrak{A}$, and $\mathfrak{A}\models\psi(a_1,\ldots,a_n)$, then $\mathfrak{X}\models\psi(\varphi(a_1),\ldots,\varphi(a_n))$.*

Proof. Induct on ψ. In the step for $\forall$, use Lemma 2.3.8 and the fact that $\mathrm{ran}(\varphi)$ is dense. ∎

For example, if $\mathfrak{A}=(A;\vee,\wedge)$ is a totally ordered lattice (total order is expressed by $\forall xy(x\vee y=x \text{ OR } x\vee y=y)$), then $b\mathfrak{A}$ must be totally ordered as well. Or, if $\mathfrak{A}=(A;\cdot,i,1)$ is a group, then, as expected, $b\mathfrak{A}$ is a group also. The ψ in Lemma 2.3.9 may include existential quantifiers. For example, suppose that $\mathfrak{A}=(A;\cdot)$ is a group (now, $\mathcal{L}=\{\cdot\}$). Then $b\mathfrak{A}$ is still a group, since the group axioms expressed using $\cdot$ (e.g., $\forall xy\exists z(xz=y)$) are all positive. In fact, one can identify $b(A;\cdot)$ with $b(A;\cdot,i,1)$ (see Section 2.8).

As with groups, homomorphisms are continuous with respect to the Bohr topology; this is easy to prove directly from the definition of $\mathfrak{A}^{\#}$:

LEMMA 2.3.10. *If $\mathfrak{A},\mathfrak{B}$ are topological structures and $\psi:\mathfrak{A}\to\mathfrak{B}$ is a homomorphism which is continuous with respect to the given topologies on $\mathfrak{A},\mathfrak{B}$, then ψ is also continuous as a map $\mathfrak{A}^{\#}\to\mathfrak{B}^{\#}$.*

The following lemma lets us prove general results about $b\mathfrak{A}$ by considering only the case where $\mathfrak{A}^{\#}$ is Hausdorff (equivalently, Φ is 1-1).

LEMMA 2.3.11. *Let $\mathfrak{A}$ be any topological structure, and let $\Phi=\Phi_{\mathfrak{A}}:A\to X=b\mathfrak{A}$. On A, define $a\sim b$ iff $\Phi(a)=\Phi(b)$. This defines a quotient map $\Phi/{\sim}: A/{\sim}\to X$. Then $\Phi/{\sim}$ is 1-1, and $b(\mathfrak{A}/{\sim})=(X,\Phi/{\sim})$.*

REMARK 2.3.12. One may define a function $f:A\to\mathbb{C}$ to be *almost periodic* iff $f=g\circ\Phi_{\mathfrak{A}}$ for some continuous $g:b\mathfrak{A}\to\mathbb{C}$. Then, trivially, $\mathfrak{A}^{\#}$ is the coarsest topology which makes all almost periodic functions continuous. However, we do not see how to define "almost periodic" directly (e.g., in terms of the translates of f having compact closure in $C(A)$), without reference to $b\mathfrak{A}$, and thereby use this as an independent way of defining the Bohr compactification, as one can for groups (or for some varieties of semigroups; see [16]).

REMARK 2.3.13. One might allow $\mathcal{L}$ to have predicate symbols as well as function symbols, but the theory is a little messier that way. The usual definition of topological structure requires that the interpretation of each predicate be closed (but not necessarily open). There are two definitions of "homomorphism" in the literature. Say $\varphi : \mathfrak{A} \to \mathfrak{B}$ and p is a binary predicate. One definition requires only that $p_{\mathfrak{A}}(a_1, a_2) \Rightarrow p_{\mathfrak{B}}(\varphi(a_1), \varphi(a_2))$, but in that case, the requirement of compatibility in Definition 2.3.1 would be trivial (we could always take $p_{\mathfrak{X}}$ to be $\mathrm{cl}(\varphi(p_{\mathfrak{A}})))$, so that the Bohr topology and Bohr compactification would be computed by ignoring the predicates. Another definition requires that $p_{\mathfrak{A}}(a_1, a_2) \Leftrightarrow p_{\mathfrak{B}}(\varphi(a_1), \varphi(a_2))$. But then we lose the fact that every structure has at least one compactification (the one-element structure), so that $b\mathfrak{A}$ would not always be defined.

We next consider the possibility that a given compact structure may be its own Bohr compactification:

DEFINITION 2.3.14. Let $\mathfrak{X}$ be a compact structure and let $\mathfrak{X}_{\mathrm{d}}$ be $\mathfrak{X}$ with the discrete topology. $\mathfrak{X}$ is *self-Bohrifying* iff $b(\mathfrak{X}_{\mathrm{d}})$ is just the identity map into $\mathfrak{X}$. $\mathfrak{X}$ is *self-compactifying* iff there is no other compact Hausdorff topology on the structure $\mathfrak{X}$ making all the functions of $\mathfrak{X}$ continuous.

Note that $\mathfrak{X}$ is self-Bohrifying iff every homomorphism from $\mathfrak{X}$ into any compact structure $\mathfrak{Y}$ is continuous iff $(\mathfrak{X}_{\mathrm{d}})^{\#}$ is the original compact topology on $\mathfrak{X}$. Clearly, self-Bohrifying implies self-compactifying, but the reverse implication can fail. For example, every compact total order $\mathfrak{X}$ (viewed either as a lattice or as a semilattice) is self-compactifying, since the only possible compact topology is the usual LOTS topology, but $\mathfrak{X}$ cannot be self-Bohrifying if it is infinite, since $(\mathfrak{X}_{\mathrm{d}})^{\#}$ is discrete (by Lemma 3.4.8). In fact, by Lawson [23], every compact semilattice and every compact lattice is self-compactifying.

Every finite structure is self-Bohrifying, but the infinite ones are a bit unusual. No infinite abelian group is self-Bohrifying (since $|b(\mathfrak{X}_{\mathrm{d}})| = 2^{2^{|X|}}$) (see Theorem 3.3.1), but there are infinite non-abelian examples. A finite-dimensional compact connected group is self-Bohrifying iff it is a semisimple Lie group (for example, $SO(3)$) (Anderson–Hunter [2]; see also van der Waerden [34]). There are also zero-dimensional self-Bohrifying groups (see Moran [24]). Observe that any compact group of the form X^{ω} with $|X| > 1$ cannot be self-Bohrifying because it will have discontinuous homomorphisms into itself. However, such a product can be self-compactifying. For example, $SO(3)^{\omega}$ is self-compactifying, since by Stewart [31], every compact connected group with a totally disconnected center is self-compactifying.

Semilattices and distributive lattices are self-Bohrifying iff they have no infinite chains. There is a large class of semilattice examples satisfying this condition, but for distributive lattices, this condition holds iff the lattice is finite (see Theorems 3.4.23 and 3.5.12).

It is also possible to consider just 0-dimensional compactifications, so we define:

DEFINITION 2.3.15. The 0-*dimensional Bohr compactification* of a topological structure $\mathfrak{A}$ is the maximal 0-dimensional compatible compactification, $(b_0\mathfrak{A}, \Phi_{0,\mathfrak{A}})$. $\mathfrak{A}^{\#_0}$ is the topology on $\mathfrak{A}$ induced by $\Phi_{0,\mathfrak{A}}$.

There is such a maximal compactification, since if one computes the $\bigvee$ in the lattice of compactifications (as in Lemma 2.2.7), the $\bigvee$ of 0-dimensional compactifications is 0-dimensional. The following lemma identifies the relationship between $b_0\mathfrak{A}$ and $b\mathfrak{A}$:

LEMMA 2.3.16. *Let $\mathfrak{A}$ be any topological structure, and define $\Gamma : b\mathfrak{A} \to b_0\mathfrak{A}$ so that $b_0\mathfrak{A} \leq_\Gamma b\mathfrak{A}$* (see Definition 2.2.1). *Then $\Gamma^{-1}(\Gamma(x))$ is the connected component of x for all $x \in b\mathfrak{A}$.*

Proof. Obtain $\Delta : b\mathfrak{A} \to \mathfrak{X}$ by collapsing each connected component in $b\mathfrak{A}$ to a point, and let $\varphi = \Delta \circ \Phi_{\mathfrak{A}} : \mathfrak{A} \to \mathfrak{X}$. Since $b_0\mathfrak{A}$ is 0-dimensional, Γ must be constant on each component of $b\mathfrak{A}$, so there is a $\Gamma' : \mathfrak{X} \to b_0\mathfrak{A}$ with $\Gamma = \Gamma' \circ \Delta$. Then $(b_0\mathfrak{A}, \Phi_0) \leq_{\Gamma'} (\mathfrak{X}, \varphi)$. But since $\mathfrak{X}$ is zero-dimensional and $b_0\mathfrak{A}$ is maximal, Γ' is a bijection, so that Γ also is the map which collapses components to points. ∎

In some cases, $b\mathfrak{A}$ turns out to be 0-dimensional, in which case $b_0\mathfrak{A} = b\mathfrak{A}$. This happens, for example, when $\mathfrak{A}$ is a discrete abelian group of finite exponent n (satisfying $\forall n(x^n = 1)$), in which case every compactification is 0-dimensional. If $\mathfrak{A}$ is just an infinite discrete set A, then not every compactification of $\mathfrak{A}$ is 0-dimensional, but the maximal compactification is 0-dimensional; here, $b_0\mathfrak{A} = b\mathfrak{A} = \beta A$.

For semilattices and distributive lattices (see Sections 3.4 and 3.5), it is useful to study $b_0\mathfrak{A}$, which always has a fairly simple description, and then to investigate conditions which imply $b_0\mathfrak{A} = b\mathfrak{A}$.

If $\mathfrak{A}$ is a discrete group, then $b_0\mathfrak{A}$ is obtained as the $\bigvee$ (in the lattice of compactifications) of homomorphisms into finite groups, and $\mathfrak{A}^{\#_0}$ is the coarsest topology which makes all cosets of normal subgroups of finite index clopen. For example, $\mathbb{Q}^{\#_0}$ is indiscrete.

2.4. Uniformities. One can also define the Bohr topology on a structure using uniformities. This may seem more elegant, as the whole construction resides just on the set A, and we do not need to deal with arbitrary representatives of equivalence classes of compactifications. Since the two presentations turn out to be equivalent in a fairly straightforward way, studying uniformities provides no new information, so we shall keep our remarks brief here.

Let A be any non-empty set and let $\Delta = \Delta(A) = \{(x, x) : x \in A\}$. If $U \subseteq A \times A$, let $U_x = \{y : (x, y) \in U\}$. A *uniformity* on A is a non-empty family $\mathcal{U} \subseteq \mathcal{P}(A \times A)$ satisfying the properties described in Kelley [19]. Let $\mathbb{T}(\mathcal{U})$ be the topology generated by the uniformity $\mathcal{U}$: $W \in \mathbb{T}(\mathcal{U})$ iff $\forall x \in W \exists U \in \mathcal{U}(U_x \subseteq W)$. In general, there may be many uniformities which generate the same topology. However, every compact Hausdorff space $(X, \mathcal{T})$ has a unique uniformity $\mathcal{U}$ such that $\mathbb{T}(\mathcal{U}) = \mathcal{T}$; namely, $\mathcal{U} = \{U \subseteq X \times X : \Delta \subseteq U^\circ\}$. This uniformity is always intended when discussing compact Hausdorff spaces.

Now, if (X, φ) is a compactification of the set A, then it induces a uniformity $\mathcal{U}$ on A, generated by sets of the form $\{(a, b) : (\varphi(a), \varphi(b)) \in V\}$, where V is a neighborhood of Δ in $X \times X$. This $\mathcal{U}$ is *totally bounded* (see [19], p. 198). Conversely, given any totally bounded uniformity on A, one may, by the standard completion process, construct a compactification which induces it. Thus, if we let $\mathbb{L}(A)$ be the lattice of all totally bounded uniformities on A, then $\mathbb{L}(A)$ and $\mathbb{K}(A)$ (see Definition 2.2.3) are isomorphic lattices.

Now, it is also easy to prove directly that $\mathbb{L}(A)$ is a complete lattice; then, as an alternative presentation, one could work directly with uniformities. Let $(\mathfrak{A}, \mathcal{T})$ be a topological structure. We may say that a uniformity $\mathcal{U}$ on A is *compatible* with $(\mathfrak{A}, \mathcal{T})$ iff every function of $\mathfrak{A}$ is uniformly continuous with respect to $\mathcal{U}$ and $\mathbb{T}(\mathcal{U})$ is coarser than $\mathcal{T}$. One may then define the *Bohr uniformity* as the finest (i.e., take the $\bigvee$ in $\mathbb{L}(A)$) uniformity compatible with $(\mathfrak{A}, \mathcal{T})$. Equivalently, the Bohr uniformity is the uniformity induced by the Bohr compactification.

REMARK 2.4.1. In the case of groups, the construction of the Bohr topology by constructing the Bohr uniformity first is due to Alfsen and Holm [1], and was the approach later emphasized by Holm [14] (see Theorem 8) for arbitrary structures. For groups, this approach seems very natural, since one may read the uniformity directly from the topology (via translations of neighborhoods of the identity), and then retrieve the usual definition of the Bohr topology as the finest totally bounded topological group topology. We do not know if this is possible for more general classes of structures; see also Remark 3.1.6.

2.5. Cardinal functions. We recall some basic results on weight and character in compact Hausdorff spaces (see Juhász [17]). If $\mathcal{T}$ is a topology on X, then $w(X, \mathcal{T})$, or just $w(X)$, denotes the *weight* of the topology (the least size of a basis). If F is a closed subset of X, then $\chi(F, X)$ denotes the *character* of F; that is, the least size of a local base of neighborhoods of F in X. Then $\chi(X)$ denotes $\sup\{\chi(\{x\}, X) : x \in X\}$.

Lemma 2.5.1. *If X is any infinite compact Hausdorff space, then*

$$(1)\qquad \chi(X) \le w(X) = w(X \times X) = \chi(\Delta, X \times X) \le |X| \le 2^{\chi(X)},$$

where $\Delta = \{(x,x) : x \in X\}$. If X is homogeneous, then

$$(2)\qquad \chi(X) \le w(X) = w(X \times X) = \chi(\Delta, X \times X) \le |X| = 2^{\chi(X)}.$$

We remark that $\chi(\Delta, X \times X)$ is the weight of the natural uniformity on X (see Section 2.4). The last "$\le$" in (1) is by Arkhangel'skiĭ's Theorem, and then the last "$=$" in (2) follows by applying the Čech–Pospíšil Theorem.

The two "$\le$"s in (2) may or may not be "$=$"s, depending on the homogeneous X. For example, if X is the double arrow space ($[0,1] \times \{0,1\}$, ordered lexically) with the endpoints deleted (to make it homogeneous), then $\chi(X) = \aleph_0$, while $w(X) = |X| = 2^{\aleph_0}$. However, if X supports a group operation (or in fact, a quasigroup, or just a homogeneity), then $\chi(X) = w(X)$; see Corollary 3.1.2. Obviously, the two "$\le$"s cannot both be "$=$"s, and under GCH, one of them must be an "$=$". However, if κ is any cardinal with $\aleph_0 < \kappa < 2^{\aleph_0}$, there is a separable compact homogeneous X with $\chi(X) = \aleph_0 < w(X) = \kappa < |X| = 2^{\aleph_0}$; just modify the double arrow construction to double only the points in $K \cap (0,1)$, where K is a subfield of $\mathbb{R}$ of size κ.

Now, in (1), assume that $X = b\mathfrak{A}$, where $\mathfrak{A}$ is a discrete structure, $|A| = \aleph_0$, and $|\mathcal{L}| \le \aleph_0$. One may then say more about the relevant cardinal functions. First, either $w(X) \le \aleph_0$ or $w(X) = 2^{\aleph_0}$ (see Corollary 2.10.20). If $\mathfrak{A}$ is an abelian group, then $w(X) = 2^{\aleph_0}$ (see Theorem 3.3.1), but this is not true for groups in general (see Section 3.3). It is true for groups (and, in fact, homogeneities; see Lemma 3.1.3) that whenever $b\mathfrak{A}$ is infinite, $w(\mathfrak{A}^{\#}) = \chi(\mathfrak{A}^{\#}) = w(b\mathfrak{A}) = \chi(b\mathfrak{A})$. These equalities do not hold for arbitrary structures. For example, if $\mathcal{L} = \emptyset$, then $\mathfrak{A}^{\#}$ is discrete, so its weight and character are countable, while $b\mathfrak{A} = \beta A$, so $w(b\mathfrak{A}) = \chi(b\mathfrak{A}) = 2^{\aleph_0}$. If $\mathfrak{A}$ is a total order, then again $\mathfrak{A}^{\#}$ is discrete, but $\chi(b\mathfrak{A}) = \aleph_0$; $w(b\mathfrak{A})$ can be either $\aleph_0$ or $2^{\aleph_0}$, depending on the order type (see Corollary 3.4.19). Furthermore, it is possible to have A countable and $\mathcal{L}$ finite and $\aleph_0 < w(\mathfrak{A}^{\#}) = \chi(\mathfrak{A}^{\#}) < 2^{\aleph_0}$; see Section 3.6.

2.6. Maps into standard structures. If G is an abelian group, then $G^{\#}$ and bG can be computed by considering only homomorphisms into the circle group, not arbitrary compact structures. We consider the extent to which this is possible for general structures.

Definition 2.6.1. A class $\mathcal{K}$ of compact (Hausdorff) structures is *adequate* for a compact structure $\mathfrak{X}$ iff for each $x, y \in X$ with $x \ne y$, there is a $\mathfrak{Y} \in \mathcal{K}$ and a continuous homomorphism φ from $\mathfrak{X}$ to $\mathfrak{Y}$ with $\varphi(x) \ne \varphi(y)$.

DEFINITION 2.6.2. A class $\mathcal{K}$ of compact (Hausdorff) structures is *adequate* for a topological structure $\mathfrak{A}$ iff $\mathcal{K}$ is adequate for $b\mathfrak{A}$.

Often, to verify that $\mathcal{K}$ is adequate for $\mathfrak{A}$, we do not compute $b\mathfrak{A}$ explicitly, but rather verify that $\mathcal{K}$ is adequate for every compact model for some positive logical sentences true in $\mathfrak{A}$. For example, using the standard theory of representations for compact groups, we see that $\{U(n) : 1 \le n < \omega\}$ is adequate for every topological group; likewise, $U(1) = T$ alone is adequate for every abelian group. The two-element algebra is adequate for every boolean algebra (see Corollary 3.5.17). Given an adequate $\mathcal{K}$, one may use maps into structures in $\mathcal{K}$ to compute the Bohr compactification:

LEMMA 2.6.3. *Assume that $\mathcal{K}$ is a set of compact structures adequate for the topological structure $\mathfrak{A}$. Let $\varphi_\alpha : \mathfrak{A} \to \mathfrak{X}_\alpha$ (for $\alpha \in \kappa$) list all continuous homomorphisms which take $\mathfrak{A}$ into a structure in $\mathcal{K}$. Then $\mathfrak{A}^\#$ is the coarsest topology on A which makes all the φ_α continuous. Let Φ be the natural product map from $\mathfrak{A}$ into $\prod_\alpha \mathfrak{X}_\alpha$, and let X be the closure of the range of Φ. Then $(\mathfrak{X}, \Phi) = (b\mathfrak{A}, \Phi_{\mathfrak{A}})$.*

For example, the Bohr compactification of every abelian group is a subgroup of some power of the circle group. If $\mathcal{L} = \emptyset$, then $\{[0,1]\}$ is adequate for all compact Hausdorff spaces; then, if A is a completely regular space, Lemma 2.6.3 just expresses Tikhonov's embedding of A into a cube.

We do not have a simple description of classes adequate for an arbitrary structure, but one can bound the size of such classes by a Löwenheim–Skolem argument; the following theorem implies that the class can always be taken to be a *set*, of size no more than $2^{\max(\aleph_0, |\mathcal{L}|)}$.

THEOREM 2.6.4. *The class of all compact structures of weight no more than* $\max(\aleph_0, |\mathcal{L}|)$ *is adequate for every topological $\mathcal{L}$-structure.*

Proof. Fix a compact structure $\mathfrak{X}$, and let $\kappa = \max(\aleph_0, |\mathcal{L}|)$. Fix distinct $a, b \in X$. We shall produce a $\mathfrak{Y}$ of weight $\le \kappa$ and a continuous homomorphism $\varphi : \mathfrak{X} \to \mathfrak{Y}$ with $\varphi(a) \ne \varphi(b)$.

Since X is compact Hausdorff, we may assume that $X \subseteq [0,1]^P$ for some P, and then, by extending all the functions of $\mathfrak{X}$ arbitrarily, we may assume that $X = [0,1]^P$. Our Y will be $[0,1]^Q$ for some $Q \subseteq P$ with $|Q| \le \kappa$, and φ will be the projection $\pi_Q : [0,1]^P \to [0,1]^Q$. It is thus sufficient to find such a Q with π_Q compatible with $\mathfrak{X}$ and $\pi_Q(a) \ne \pi_Q(b)$.

In general, if $g : X^n \to M$ and $R \subseteq P$, say R is *big enough* for g iff $g(x_1, \ldots, x_n) = g(y_1, \ldots, y_n)$ whenever each $\pi_R(x_i) = \pi_R(y_i)$. Observe that if M is compact metric and g is continuous, then there is a countable R which is big enough for g. It follows that we may find a Q with $|Q| \le \kappa$ such that $\pi_Q(a) \ne \pi_Q(b)$ and Q is big enough for $\pi_q \circ f_{\mathfrak{X}}$ for each $q \in Q$ and each

$f \in \mathcal{L}$. Hence, Q is big enough for $\pi_Q \circ f_{\mathfrak{X}}$ for each f, which implies that π_Q is compatible with $\mathfrak{X}$. ∎

DEFINITION 2.6.5. A discrete $\mathcal{L}$-structure $\mathfrak{A}$ is *nice* iff $\mathcal{L}$ is countable and there is a single compact second countable $\mathcal{L}$-structure $\mathfrak{X}$ with $\{\mathfrak{X}\}$ adequate for $\mathfrak{A}$.

Even when $\mathcal{L}$ and A are countable, $\mathfrak{A}$ could fail to be nice (see Example 3.6.7). One may apply Theorem 2.6.4 and take the product of all second countable $\mathcal{L}$-structures to produce a single $\mathfrak{X}$ with $\{\mathfrak{X}\}$ adequate for $\mathfrak{A}$. However, this $\mathfrak{X}$ might well have weight $2^{\aleph_0}$.

Many of the structures commonly studied turn out to be nice; one usually proves this by producing one $\mathfrak{X}$ which is adequate for a whole variety. For example, groups are nice, taking $\mathfrak{X}$ to be $\prod_{0<n<\omega} U(n)$, and boolean algebras are nice, taking $\mathfrak{X}$ to be the two-element algebra (see Corollary 3.5.17).

2.7. Substructures. As usual, $\mathfrak{A} \subseteq \mathfrak{B}$ means not only that $A \subseteq B$, but also that all the functions of $\mathfrak{A}$ are the restrictions of the corresponding functions of $\mathfrak{B}$; in the case of topological structures, $\mathfrak{A} \subseteq \mathfrak{B}$ means also that the topology on A is the relative topology inherited from B. If we are given a structure $\mathfrak{B}$, and a subset A of B which is closed under the functions of $\mathfrak{B}$, then $\mathfrak{B}{\restriction}A$ denotes the structure $\mathfrak{A}$ on A obtained by restricting all these functions to A (and relativizing the topology in the case of topological structures). Using $\Phi_{\mathfrak{B}}$ to denote the Bohr compactification, $(b\mathfrak{B}, \Phi_{\mathfrak{B}})$, we see that $\Phi_{\mathfrak{B}}{\restriction}\mathfrak{A}$ (i.e., $(\mathrm{cl}(\Phi_{\mathfrak{B}}(A)), \Phi_{\mathfrak{B}}{\restriction}\mathfrak{A})$) is some compactification of $\mathfrak{A}$, but it need not be the maximal compactification, $\Phi_{\mathfrak{A}}$. We investigate conditions which imply that $\Phi_{\mathfrak{B}}{\restriction}\mathfrak{A}$ does equal $\Phi_{\mathfrak{A}}$, and in particular, we prove a "Löwenheim–Skolem" result (Corollary 2.7.4), saying that given $\mathfrak{B}$, we can find a countable $\mathfrak{A} \subseteq \mathfrak{B}$ with $\Phi_{\mathfrak{B}}{\restriction}\mathfrak{A} = \Phi_{\mathfrak{A}}$.

LEMMA 2.7.1. *Suppose that* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are topological structures*, $\mathfrak{A} \subseteq \mathfrak{B}$, *and* $\mathcal{K}$ *is a class of compact structures adequate for* $\mathfrak{A}$. *Suppose further that whenever* $\varphi : \mathfrak{A} \to \mathfrak{X}$ *is a continuous homomorphism from* $\mathfrak{A}$ *to some* $\mathfrak{X} \in \mathcal{K}$, *there is an extension* ψ *of* φ *to a continuous homomorphism from* $\mathfrak{B}$ *to some compact structure* $\mathfrak{Y}$ (*not necessarily in* $\mathcal{K}$) *with* $\mathfrak{X} \subseteq \mathfrak{Y}$. *Then* $\Phi_{\mathfrak{B}}{\restriction}\mathfrak{A} = \Phi_{\mathfrak{A}}$.

This lemma is easily proved from the computation of $b\mathfrak{A}$ as a product (see Lemma 2.6.3). For discrete abelian groups, $\mathcal{K} = \{T\}$ and we can always let $\mathfrak{Y} = \mathfrak{X} = T$. An application with $\mathfrak{Y} \neq \mathfrak{X}$ occurs naturally in semilattices (see Theorem 3.4.26).

For discrete structures, the proof of Theorem 2.7.3 will verify the hypothesis of Lemma 2.7.1 whenever $\mathfrak{A}$ is algebraically closed in $\mathfrak{B}$.

DEFINITION 2.7.2. A *system of equations* over a structure $\mathfrak{A}$ is a finite set, $\sigma(\vec{x})$, of equations formed by using the symbols of $\mathcal{L}$, together with the elements of A as constants, together with some variables $\vec{x} = (x_1, \ldots, x_n)$. A *solution* of $\sigma(\vec{x})$ in $\mathfrak{A}$ is an n-tuple $\vec{a} = (a_1, \ldots, a_n)$ such that $\sigma(\vec{a})$ is true in $\mathfrak{A}$. If $\mathfrak{A} \subseteq \mathfrak{B}$, then $\mathfrak{A}$ is *algebraically closed* in $\mathfrak{B}$ iff every system of equations over $\mathfrak{A}$ which has a solution in $\mathfrak{B}$ also has a solution in $\mathfrak{A}$.

THEOREM 2.7.3. *If* $\mathfrak{A} \subseteq \mathfrak{B}$, *where* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are discrete structures*, *and* $\mathfrak{A}$ *is algebraically closed in* $\mathfrak{B}$, *then* $\Phi_{\mathfrak{B}} \restriction \mathfrak{A} = \Phi_{\mathfrak{A}}$.

Proof. By Lemma 2.7.1, it is sufficient to show that whenever $\varphi : \mathfrak{A} \to \mathfrak{X}$ is a homomorphism from $\mathfrak{A}$ to some compact structure $\mathfrak{X}$, there is an extension ψ of φ to a homomorphism from $\mathfrak{B}$ to $\mathfrak{X}$. We shall obtain ψ by using an ultrafilter to take a limit.

Let Δ be the set of all equations true in $\mathfrak{B}$. So, elements of Δ can be written as $\tau_1(\vec{b}) = \tau_2(\vec{b})$, where $\vec{b}$ is a tuple (possibly empty) of elements of $B \setminus A$ and $\tau_1(\vec{x}) = \tau_2(\vec{x})$ is an equation over A with $\tau_1(\vec{b}) = \tau_2(\vec{b})$ true in $\mathfrak{B}$.

Let I be the set of all finite subsets of Δ, and let $\mathcal{U}$ be an ultrafilter over I such that $\{i \in I : j \subseteq i\} \in \mathcal{U}$ for all $j \in I$.

Each $i \in I$ can be written as $i = \sigma(\vec{b})$, where $\sigma(\vec{x})$ is a system of equations over $\mathfrak{A}$ and $\vec{b} = (b_1, \ldots, b_n)$ is a tuple of elements of $B \setminus A$ which is a solution of $\sigma(\vec{x})$. For each such $i \in I$, let $C(i)$ be some tuple $\vec{a}$ of elements of A which is also a solution of $\sigma(\vec{x})$. Always choose $\vec{a}$ such that whenever $b_p = b_q$, we also have $a_p = a_q$; this is possible because we could always add $x_p = x_q$ to σ.

For $b \in B \setminus A$, let $F_b : I \to X$ be such that whenever $i = \sigma(\vec{b})$ with $b = b_q$ and $\vec{a} = C(i)$, we have $F_b(i) = \varphi(a_q)$. If b is not among the b_q, then choose $F_b(i)$ arbitrarily, but note that the set of i for which this happens is not in $\mathcal{U}$. If $a \in A$, set $F_b(i) = \varphi(a)$ for all i.

Now, define $\psi(b)$ to be the $\mathcal{U}$-limit of F_b; that is, $\psi(b)$ is the unique $x \in X$ such that $\{i : F_b(i) \in V\} \in \mathcal{U}$ for every neighborhood V of x. It is easy to verify that ψ is a homomorphism and extends φ. ∎

One may also view this proof as embedding $\mathfrak{B}$ into an ultrapower, $\mathfrak{A}^I/\mathcal{U}$, and then using $\mathcal{U}$-limits to extend a homomorphism $\varphi : \mathfrak{A} \to \mathfrak{X}$ to $\mathfrak{A}^I/\mathcal{U}$.

COROLLARY 2.7.4. *If* $\mathcal{L}$ *is countable and* $\mathfrak{B}$ *is a discrete* $\mathcal{L}$*-structure*, *then there is a countable* $\mathfrak{A} \subseteq \mathfrak{B}$ *such that* $\Phi_{\mathfrak{B}} \restriction \mathfrak{A} = \Phi_{\mathfrak{A}}$, *and hence* $\mathfrak{A}^{\#}$ *is the same as the topology of* $\mathfrak{B}^{\#}$ *restricted to* A.

In the cases of semilattices (Theorem 3.4.26) and discrete abelian groups, $\Phi_{\mathfrak{B}} \restriction \mathfrak{A} = \Phi_{\mathfrak{A}}$ holds for all $\mathfrak{A} \subseteq \mathfrak{B}$, and furthermore A is closed in $\mathfrak{B}^{\#}$, but these facts do not hold in general, even for groups. For example, $\mathfrak{B}$ could be one of the groups described by von Neumann and Wigner [25], [26], where $\mathfrak{B}^{\#}$ is indiscrete. If $\mathfrak{A}$ is an infinite abelian subgroup, then $\mathfrak{A}^{\#}$ is Hausdorff, and hence strictly finer than the topology of $\mathfrak{B}^{\#}$ restricted to $\mathfrak{A}$. Also, if

$\mathfrak{A}$ is any proper subgroup of $\mathfrak{B}$, then A is not closed in $\mathfrak{B}^{\#}$, so we do not expect to get A closed in Corollary 2.7.4.

2.8. Reducts. If $\mathcal{L}_0 \subseteq \mathcal{L}$, and $\mathfrak{A}$ is an $\mathcal{L}$-structure, then one defines the *reduct*, $\mathfrak{A}\restriction\mathcal{L}_0$, to be the $\mathcal{L}_0$-structure obtained from $\mathfrak{A}$ by applying the forgetful functor; $\mathfrak{A}$ is called an *expansion* of $\mathfrak{A}\restriction\mathcal{L}_0$. Now, $(b\mathfrak{A})\restriction\mathcal{L}_0$ is some compactification of $\mathfrak{A}\restriction\mathcal{L}_0$, but is not necessarily maximal. So, we have

LEMMA 2.8.1. *Suppose that $\mathcal{L}_0 \subseteq \mathcal{L}$ and $\mathfrak{A}$ is a topological $\mathcal{L}$-structure. Then $(b\mathfrak{A})\restriction\mathcal{L}_0 \le b(\mathfrak{A}\restriction\mathcal{L}_0)$, and $(\mathfrak{A}\restriction\mathcal{L}_0)^{\#}$ is finer than $\mathfrak{A}^{\#}$.*

In some cases, $\mathfrak{A}$ will be an "inessential" expansion of $\mathfrak{A}\restriction\mathcal{L}_0$, in which case we can identify $b\mathfrak{A}$ with $b(\mathfrak{A}\restriction\mathcal{L}_0)$. The notion of "inessential" here differs somewhat from the usual notion from logic. Constants are always inessential, as are functions defined explicitly by terms (Lemma 2.8.3). But functions defined "implicitly" by logical formulas are only sometimes inessential; not always, as in first-order logic (see Theorem 2.8.5 and following discussion).

DEFINITION 2.8.2. If $\mathfrak{A}$ is an $\mathcal{L}$-structure, $F : A^n \to A$, and $\tau(x_1, \ldots, x_n, z_1, \ldots z_m)$ is a term of $\mathcal{L}$, then F is *definable by τ on* $\mathfrak{A}$ iff for some fixed $d_1, \ldots, d_m \in A$, we have $F(a_1, \ldots, a_n) = \tau_{\mathfrak{A}}(a_1, \ldots a_n, d_1, \ldots, d_m)$ for all elements $a_1, \ldots, a_n \in A$.

LEMMA 2.8.3. *Assume that $\mathcal{L} \subseteq \mathcal{L}'$, $\mathfrak{A}'$ is a topological $\mathcal{L}'$-structure, and $\mathfrak{A} = \mathfrak{A}'\restriction\mathcal{L}$. Assume also that every symbol of $\mathcal{L}' \setminus \mathcal{L}$ is either a constant symbol or denotes a function on A which is definable on $\mathfrak{A}$ by some term of $\mathcal{L}$. Then every compactification compatible with $\mathfrak{A}$ is compatible with $\mathfrak{A}'$ as well*; *hence*, $b\mathfrak{A} = (b\mathfrak{A}')\restriction\mathcal{L}$, *and the topologies $\mathfrak{A}^{\#}$ and $\mathfrak{A}'^{\#}$ are the same.*

For example, for groups, $b(A; \cdot, i, 1) = b(A; \cdot, i)$. But also, if we fix any $a \in A$ and define $\sigma(x) = a^{-1}xa$, then $b(A; \cdot, i, \sigma, 1) = b(A; \cdot, i)$.

Next, we consider functions defined implicitly by logical formulas.

DEFINITION 2.8.4. If $\mathfrak{A}$ is an $\mathcal{L}$-structure and $F : A^n \to A$, then F is *positively definable* on $\mathfrak{A}$ iff for some positive formula $\phi(x_1, \ldots, x_n, y, z_1, \ldots, z_m)$ of $\mathcal{L}$, and some fixed $d_1, \ldots, d_m \in A$, we have:

1. For all $a_1, \ldots, a_n, b \in A$, $\mathfrak{A} \models \phi(\vec{a}, b, \vec{d})$ iff $F(\vec{a}) = b$.
2. The logical sentence $\forall\vec{x}\,\exists! y\,\phi(\vec{x}, y, \vec{d})$ is provable from the positive logical sentences true of $\vec{d}$ in $\mathfrak{A}$.

A function defined by a term is a special case of a positively definable function, since here $\phi(x_1, \ldots, x_n, y, z_1, \ldots, z_m)$ is just $\tau(\vec{x}, \vec{z}) = y$.

Note that $\forall\vec{x}\,\exists! y\,\phi(\vec{x}, y, \vec{d})$ is not a positive sentence, so its truth in $\mathfrak{A}$ does not imply its truth in $b\mathfrak{A}$. However, if it happens to be provable from positive sentences true in $\mathfrak{A}$, then $\forall\vec{x}\,\exists! y\,\phi(\vec{x}, y, \Phi(\vec{d}))$ will be true in $b\mathfrak{A}$ by

Lemma 2.3.9, so that $\phi(\vec{x}, y, \Phi(\vec{d}))$ will define a function in $b\mathfrak{A}$. This function will be continuous, since its graph is closed by Lemma 2.3.8. Hence,

THEOREM 2.8.5. *Assume that $\mathcal{L} \subseteq \mathcal{L}'$, $\mathfrak{A}'$ is a topological $\mathcal{L}'$-structure, and $\mathfrak{A} = \mathfrak{A}'{\restriction}\mathcal{L}$. Assume also that every symbol of $\mathcal{L}' \setminus \mathcal{L}$ is either a constant symbol or denotes a function on A which is positively definable on $\mathfrak{A}$ by a formula of $\mathcal{L}$. Then every compactification compatible with $\mathfrak{A}$ is compatible with $\mathfrak{A}'$ as well; hence, $b\mathfrak{A} = (b\mathfrak{A}'){\restriction}\mathcal{L}$, and the topologies $\mathfrak{A}^{\#}$ and $\mathfrak{A}'^{\#}$ are the same.*

For example, for groups, $b(A;\cdot) = b(A;\cdot,i,1)$, since i is positively definable on $(A;\cdot)$ (using $y = i(x)$ iff $x\cdot y = 1$). Theorem 2.8.5 applies because the sentence $\forall x \exists! y(x\cdot y = 1)$ is provable from positive facts true about 1 in $(A;\cdot)$ (namely, the associative law, $\forall x(x \cdot 1 = x)$, and $\forall x\exists y(x\cdot y = 1)$). This argument also works for some (but not all) varieties of loops (see Lemma 3.2.1). A similar argument (see Lemma 3.4.1) shows that $b(A;\vee,\wedge) = b(A;\wedge) = b(A;\vee)$ whenever $(A;\vee,\wedge)$ is a total order; but this does not hold in general for distributive lattices, and in fact fails for boolean algebras, although for boolean algebras, it is true that $b(A;\vee,\wedge) = b(A;\vee,\wedge,')$; see Theorem 3.5.19.

One must take a bit of care in stating Theorem 2.8.5. Say $\mathcal{L}' = \{p, L, R\}$ and $\mathcal{L} = \{L, R\}$, and let $\mathfrak{A}'$ be a discrete infinite pairing (see Section 2.1). Then $b\mathfrak{A}'$ is also a pairing, so it is homeomorphic to its square, whereas in $\mathfrak{A} = (A; L, R)$, all functions are unary, so $b\mathfrak{A} = \beta A$, which is not homeomorphic to its square. On A, the function p is definable by a positive $\mathcal{L}$-formula, $\phi(x_1, x_2, y)$ (namely, $x_1 = L(y)$ & $x_2 = R(y)$), but $\forall x_1 x_2 \exists! y \phi(x_1, x_2, y)$ is not provable from the positive logical sentences true in $\mathfrak{A}$, and in fact is false in $b\mathfrak{A}$. For an example with loops, see Example 3.2.2.

Note that there is no Löwenheim–Skolem theorem for languages. That is, if $\mathfrak{A}$ is countable but $\mathcal{L}$ is uncountable, there need not be a countable $\mathcal{L}_0 \subset \mathcal{L}$ such that the topology $(\mathfrak{A}{\restriction}\mathcal{L}_0)^{\#}$ is the same as $\mathfrak{A}^{\#}$; see Example 3.6.4.

2.9. Products. The product of two topological structures is a topological structure in a natural way, and the product of two compactifications is a compactification of the product.

LEMMA 2.9.1. *Suppose that $\mathfrak{A}$ and $\mathfrak{B}$ are topological structures for $\mathcal{L}$. Then $b\mathfrak{A} \times b\mathfrak{B} \le b(\mathfrak{A} \times \mathfrak{B})$, and hence the product topology, $\mathfrak{A}^{\#} \times \mathfrak{B}^{\#}$, is coarser than $(\mathfrak{A} \times \mathfrak{B})^{\#}$.*

For semigroups with an identity element 1, we have $b\mathfrak{A} \times b\mathfrak{B} = b(\mathfrak{A} \times \mathfrak{B})$; deLeeuw and Glicksberg [16] does this in the commutative case, Holm [14] does it for groups, and Hušek and de Vries [15] has a common generalization to [16, 14]. The key idea is to use the 1 to build compactifications of the factors from compactifications of the product. In Lemma 2.9.3, we abstract what is needed for this idea to work in our current setting, and then we describe some cases where we can prove $b\mathfrak{A} \times b\mathfrak{B} = b(\mathfrak{A} \times \mathfrak{B})$ by using other results in this paper, even though Lemma 2.9.3 does not apply directly. One cannot assert in general that $b\mathfrak{A} \times b\mathfrak{B} = b(\mathfrak{A} \times \mathfrak{B})$; this fails for infinite sets ($\mathcal{L} = \emptyset$), since $\beta A \times \beta B \not\cong \beta(A \times B)$, and hence it fails for semigroups without a 1, since if $x \cdot y = 0$ for all x, y, then $b\mathfrak{A} = \beta A$.

Definition 2.9.2. An element e of a structure $\mathfrak{A}$ is an *idempotent* iff for *all* functions f of arity ≥ 1, we have $e = f(e, e, \ldots, e)$. If $\circ$ is some binary function on A, then e is an *identity element* with respect to $\circ$ iff $e \circ x = x \circ e = x$ for all $x \in A$.

Lemma 2.9.3. *Suppose that $\mathfrak{A}$ and $\mathfrak{B}$ are any topological structures for $\mathcal{L}$, and there are binary operations $\circ_{\mathfrak{A}}$ and $\circ_{\mathfrak{B}}$ on $\mathfrak{A}$ and $\mathfrak{B}$ respectively, defined by the same term of $\mathcal{L}$ (in the sense of Definition 2.8.2). Suppose further that there are idempotents $0_{\mathfrak{A}} \in A$ and $0_{\mathfrak{B}} \in B$ such that $0_{\mathfrak{A}}, 0_{\mathfrak{B}}$ are identity elements in $\mathfrak{A}, \mathfrak{B}$ respectively, with respect to $\circ_{\mathfrak{A}}, \circ_{\mathfrak{B}}$ respectively. Then $b\mathfrak{A} \times b\mathfrak{B} = b(\mathfrak{A} \times \mathfrak{B})$, and hence the product topology, $\mathfrak{A}^{\#} \times \mathfrak{B}^{\#}$, is the same as $(\mathfrak{A} \times \mathfrak{B})^{\#}$.*

Proof. We suppress the subscripts $\mathfrak{A}$ and $\mathfrak{B}$. Since $\circ$ is defined by the same term in both $\mathfrak{A}$ and $\mathfrak{B}$, it is defined likewise on $\mathfrak{A} \times \mathfrak{B}$ and on any compactification of these structures. So, by Lemma 2.8.3, we may deal with $\circ$ and 0 as if they are symbols of $\mathcal{L}$; likewise, we may assume that $\mathcal{L}$ has no constant symbols other than 0. Let $\Phi = \Phi_{\mathfrak{A} \times \mathfrak{B}} : \mathfrak{A} \times \mathfrak{B} \to \mathfrak{X} = b(\mathfrak{A} \times \mathfrak{B})$. It is sufficient to produce compactifications $\psi : \mathfrak{A} \to Y$ and $\chi : \mathfrak{B} \to Z$ and prove that $\Phi \leq \psi \times \chi$. Let $\psi(a) = \Phi(a, 0)$ and $\chi(b) = \Phi(0, b)$; these are homomorphisms into $\mathfrak{X}$ because 0 is an idempotent and $\mathcal{L}$ has no constant symbols other than 0. Let $Y = \mathrm{cl}(\mathrm{ran}(\psi)) \subseteq X$ and $Z = \mathrm{cl}(\mathrm{ran}(\chi)) \subseteq X$. Now, by definition of "$\leq$", it is sufficient to produce a continuous $\Gamma : X \times X \to X$ such that $\Gamma((\psi \times \chi)(a, b)) = \Phi(a, b)$ holds for all $(a, b) \in A \times B$. So, let $\Gamma(y, z) = y \circ z$. Then, applying the fact that 0 is an identity element, we get $\Gamma((\psi \times \chi)(a, b)) = \Gamma(\psi(a), \chi(b)) = \Phi(a, 0) \circ \Phi(0, b) = \Phi((a, 0) \circ (0, b)) = \Phi((a \circ 0), (0 \circ b)) = \Phi(a, b)$. ∎

This lemma obviously applies to groups and loops. It also applies to rings (with $\circ$ as $+$), since 0 happens to be an idempotent with respect to $\cdot$, but in fact, by Lemma 3.1.4, it will apply to every structure which contains a group operation—or even just a homogeneity—plus arbitrary other functions. Using this, it will apply also to quasigroups; see Section 3.2.

In a lattice, every element is an idempotent, so that Lemma 2.9.3 applies immediately whenever the lattice has a least element 0 (which is an identity element with respect to $\vee$) or a greatest element 1 (which is an identity element with respect to $\wedge$). However, it turns out that $b\mathfrak{A} \times b\mathfrak{B} = b(\mathfrak{A} \times \mathfrak{B})$ does hold for all semilattices and all lattices; see Sections 3.4 and 3.5. In fact, our counter-examples to Lemma 2.9.3, if one drops the assumption on idempotent (Example 3.6.8), or if one drops the assumption on identity (Example 3.6.9), both seem a bit unnatural, so perhaps there is a better version of Lemma 2.9.3 which applies to a wider class of structures.

2.10. C_{p} theory. A number of facts about Bohr topologies may be proved using notions of C_{p} theory. We recall some relevant definitions; see Arkhangel'skiĭ [3] for further details.

DEFINITION 2.10.1. If H, K are topological spaces, then $C_{\mathrm{p}}(H, K)$ is the set of continuous functions from H to K, given the topology of pointwise convergence (i.e., regarding $C_{\mathrm{p}}(H, K)$ as a subset of K^H with the usual product topology).

All compactifications in the sense of Section 2.2 may be viewed in the context of C_{p} theory as follows:

DEFINITION 2.10.2. A compactification (X, ψ) of a set A is a $C_{\mathrm{p}}(H, K)$ *compactification* iff ψ is equivalent to some (Y, φ), where K is compact Hausdorff, $\varphi : A \to C_{\mathrm{p}}(H, K)$, and Y is the closure of $\mathrm{ran}(\varphi)$ in K^H.

Note that Y need not be a subset of $C_{\mathrm{p}}(H, K)$. Every compactification of A is a $C_{\mathrm{p}}(H, [0, 1])$ compactification for some (discrete) H, since every compact Hausdorff space can be embedded in a cube. We shall see that $C_{\mathrm{p}}(H, [0, 1])$ compactifications have additional properties when H has some additional properties, such as compactness. First, we note (Lemma 2.10.4) that we can often replace K by $[0, 1]$.

DEFINITION 2.10.3. Q is the Hilbert cube, $[0, 1]^\omega$.

LEMMA 2.10.4. *If K is compact and second countable, then each $C_{\mathrm{p}}(H, K)$ compactification is equivalent to a $C_{\mathrm{p}}(H \times (\omega + 1), [0, 1])$ compactification.*

Proof. We start with $\varphi : A \to C_{\mathrm{p}}(H, K)$. Since K can be embedded in Q, we may as well assume that $K = Q$. Now, define $\Gamma : Q^H \to [0, 1]^{H \times (\omega+1)}$ so that $\Gamma(x)(h, n) = (x(h))_n \cdot 2^{-n}$ and $\Gamma(x)(h, \omega) = 0$. Observe that Γ is 1-1 and continuous, and that Γ takes $C_{\mathrm{p}}(H, Q)$ into $C_{\mathrm{p}}(H \times (\omega + 1), [0, 1])$. Then φ is equivalent to the $C_{\mathrm{p}}(H \times (\omega + 1), [0, 1])$ compactification, $\Gamma \circ \varphi$. ∎

In the case of Bohr topologies, a natural H to use is $\mathrm{Hom}(\mathfrak{A}, \mathfrak{X})$:

DEFINITION 2.10.5. If $\mathfrak{A}$ and $\mathfrak{X}$ are $\mathcal{L}$-structures, then $\mathrm{Hom}(\mathfrak{A}, \mathfrak{X})$ is the set of homomorphisms from $\mathfrak{A}$ into $\mathfrak{X}$. If $\mathfrak{X}$ is given a topology, then $\mathrm{Hom}(\mathfrak{A}, \mathfrak{X}) \subseteq X^A$ is given the usual product topology.

Lemma 2.10.6. *If* $\mathfrak{X}$ *is a compact* $\mathcal{L}$*-structure and* $\mathfrak{A}$ *is a discrete* $\mathcal{L}$*-structure, then* $\mathrm{Hom}(\mathfrak{A}, \mathfrak{X})$ *is closed in* X^A, *and is hence a compact Hausdorff space.*

Definition 2.10.7. An *Eberlein–Grothendieck space* is any topological space homeomorphic to a subspace of $C_{\mathrm{p}}(H, [0,1])$ for some compact H.

Theorem 2.10.8. *If* $\mathcal{L}$ *is countable and* $\mathfrak{A}$ *is a nice* $\mathcal{L}$*-structure* (Definition 2.6.5), *then* $(b\mathfrak{A}, \Phi_{\mathfrak{A}})$ *is a* $C_{\mathrm{p}}(H, [0,1])$ *compactification for some compact* H, *where* $w(H) \le \max(|A|, \aleph_0)$.

Proof. Fix a compact second countable $\mathcal{L}$-structure $\mathfrak{X}$ with $\{\mathfrak{X}\}$ adequate for $\mathfrak{A}$. By Lemma 2.6.3, the Bohr compactification of $\mathfrak{A}$ is equivalent to the evaluation map $\Psi : \mathfrak{A} \to X^{\mathrm{Hom}(\mathfrak{A},\mathfrak{X})}$, where $\Psi(a)(\varphi) = \varphi(a)$. Note that $\mathrm{ran}(\Psi) \subseteq C_{\mathrm{p}}(\mathrm{Hom}(\mathfrak{A}, \mathfrak{X}), X)$, so that the Bohr compactification is a $C_{\mathrm{p}}(\mathrm{Hom}(\mathfrak{A}, \mathfrak{X}), X)$ compactification. Finally, let $H = \mathrm{Hom}(\mathfrak{A}, \mathfrak{X}) \times (\omega + 1)$, and apply Lemma 2.10.4. ∎

So, Theorem 2.10.8 implies that if $\mathfrak{A}$ is nice and $\mathfrak{A}^{\#}$ is Hausdorff, then $\mathfrak{A}^{\#}$ is an Eberlein–Grothendieck space. Note that we are not claiming that $\mathrm{cl}(\mathrm{ran}(\Phi_{\mathfrak{A}}))$ is a subset of $C_{\mathrm{p}}(H, [0,1])$; that would imply that $b\mathfrak{A}$ is an Eberlein compactum (see §III.3 of [3]); if $\mathfrak{A}$ is a group, then $b\mathfrak{A}$ is a compact group, which cannot be an Eberlein compactum unless it is second countable (see Theorem III.3.12 of [3]).

If $\mathfrak{A}$ is not nice, then we cannot just use one $\mathfrak{X}$, but as long as the language is countable, we can apply Theorem 2.6.4 to compute $b\mathfrak{A}$ using only $\mathrm{Hom}(\mathfrak{A}, \mathfrak{X})$ for various second countable $\mathfrak{X}$. Each such $\mathfrak{X}$ can be topologically embedded into Q, and then the functions of $\mathfrak{X}$ can be continuously extended to functions on all of Q, so that one can compute $b\mathfrak{A}$ by just considering topological structures with domain Q. This fact enables us to prove for $b\mathfrak{A}$ some (but not all) of the theorems true for nice structures. We can code all second countable compactifications of a discrete structure $\mathfrak{A}$ as follows:

Definition 2.10.9. For symbols $s \in \mathcal{L}$, let $F_s = Q$ when s is a constant, and let $F_s = C(Q^n, Q)$ whenever s is a function symbol of arity $n > 0$. Give $C(Q^n, Q)$ its usual metric topology. Let $P = P_{\mathcal{L}} = \prod_{s \in \mathcal{L}} F_s$, with the usual product topology.

Note that elements $p \in P$ are the compact $\mathcal{L}$-structures with domain Q. The co-ordinate p_s, for $s \in \mathcal{L}$, is what we were formerly calling s_p, the interpretation of the symbol s in the structure p.

Definition 2.10.10. If $\mathfrak{A}$ is any discrete $\mathcal{L}$-structure then $\mathrm{Homq}(\mathfrak{A}) \subseteq P_{\mathcal{L}} \times Q^A$ is the set of pairs (p, φ) such that $\varphi : \mathfrak{A} \to p$ is a homomorphism.

Here, "homomorphism" means that $\varphi(c_{\mathfrak{A}}) = p_c$ for each constant $c \in \mathcal{L}$, and that $\varphi(f_{\mathfrak{A}}(a_1, \ldots, a_n)) = p_f(\varphi(a_1), \ldots, \varphi(a_n))$ for each $f \in \mathcal{L}$ of arity

$n > 0$ and each $a_1, \ldots, a_n \in A$. Unlike the $\mathrm{Hom}(\mathfrak{A}, \mathfrak{X})$ of Definition 2.10.5, $\mathrm{Homq}(\mathfrak{A})$ is not compact, but it is still of a type useful for C_{p} theory. Recall that a *Polish space* is a separable complete metric space. A more general class of spaces are the ones of *type* $K_{\sigma\delta}$; these are spaces which are $F_{\sigma\delta}$ sets in some compact Hausdorff space. Every Polish space is $K_{\sigma\delta}$ because it can be embedded as a G_δ in Q. Both classes, Polish and $K_{\sigma\delta}$, are closed under countable products and closed subspaces.

LEMMA 2.10.11. $\mathrm{Homq}(\mathfrak{A})$ *is closed in* $P_{\mathcal{L}} \times Q^A$. *Hence*, $\mathrm{Homq}(\mathfrak{A})$ *is of type* $K_{\sigma\delta}$ *whenever* $\mathcal{L}$ *is countable*, *and is Polish whenever* A *is also countable.*

THEOREM 2.10.12. *If* $\mathcal{L}$ *is countable and* $\mathfrak{A}$ *is any discrete* $\mathcal{L}$*-structure*, *then* $(b\mathfrak{A}, \Phi_{\mathfrak{A}})$ *is a* $C_{\mathrm{p}}(H, [0,1])$ *compactification*, *where* H *is of type* $K_{\sigma\delta}$ *and* $w(H) \leq \max(|A|, \aleph_0)$. *If* A *is countable*, *then* H *is Polish.*

Proof. Let $D = Q^{\mathrm{Homq}(\mathfrak{A})}$. Make D into a compact $\mathcal{L}$-structure, $\mathfrak{D}$, by setting $c_{\mathfrak{D}}(p, \varphi) = p_c$ and $(f_{\mathfrak{D}}(d_1, \ldots, d_n))(p, \varphi) = p_f(d_1(p, \varphi), \ldots, d_n(p, \varphi))$. Define $\Phi : A \to D$ so that $(\Phi(a))(p, \varphi) = \varphi(a)$. Observe that $\Phi : \mathfrak{A} \to \mathfrak{D}$ is a homomorphism, and that $\mathrm{ran}(\Phi) \subseteq C_{\mathrm{p}}(\mathrm{Homq}(\mathfrak{A}), Q) \subseteq D$. So, Φ (that is $(\mathrm{cl}(\mathrm{ran}(\Phi)), \Phi)$) is a compactification of $\mathfrak{A}$. In fact, it is the maximal (Bohr) compactification, since it dominates every compactification φ of $\mathfrak{A}$ into a structure p with domain Q; that is, $(p, \varphi) \leq_\Gamma \Phi$, where $\Gamma : D \to Q$ is just projection; $\Gamma(d) = d(p, \varphi)$. This shows that $b\mathfrak{A}$ is a $C_{\mathrm{p}}(\mathrm{Homq}(\mathfrak{A}), Q)$ compactification. Now, let $H = \mathrm{Homq}(\mathfrak{A}) \times (\omega + 1)$. ∎

We now study some aspects of C_{p} theory which are relevant to Bohr topologies. We shall see that the nice and non-nice structures for a countable language share most of the same basic properties, but the following theorem is an exception, and can sometimes be used to prove that a structure is not nice. This type of argument was discovered twice: once by L. T. Ramsey [30] in the context of Bohr topologies on abelian groups, and once by Arkhangel'skiĭ (see [3], Theorem II.2.2) in the context of C_{p} theory. We give a proof, since neither reference has precisely the form we wish to quote.

THEOREM 2.10.13. *Assume that* A *is an Eberlein–Grothendieck space. Then for each* $E \subseteq A$, $p \in A$, *and* $n \in \omega$, *we may choose a finite subset*, $T(E, p, n) \subseteq E$, *with the following property*: *Whenever* $p \in \overline{E}_n$ *for each* n, *then* $p \in \mathrm{cl}(\bigcup_{n\in\omega} T(E_n, p, n))$.

Proof. We may assume that $A = C_{\mathrm{p}}(H, [0,1])$, where H is compact. For $p \in A$, finite n, $\varepsilon > 0$, and $u_1, \ldots, u_n \in H$, let $U(p; u_1, \ldots, u_n; \varepsilon)$ be the set of all $a \in A$ such that $|a(u_i) - p(u_i)| < \varepsilon$ for each $i = 1, \ldots, n$. Then the $U(p; u_1, \ldots, u_n; \varepsilon)$ form an open base at p.

Fix E, p, n. If $p \notin \overline{E}$, we set $T(E, p, n) = \emptyset$. If $p \in \overline{E}$ then for each $\vec{u} = (u_1, \ldots, u_n) \in H^n$, we may choose a point $a(\vec{u}) \in U(p; \vec{u}; 2^{-n}) \cap E$.

Since each $a \in A$ is continuous, $\{\vec{v} \in H^n : a(\vec{u}) \in U(p;\vec{v};2^{-n})\}$ is an open neighborhood of $\vec{u}$ in H^n. By compactness, finitely many of these open neighborhoods cover H^n, so we may choose $T(E,p,n)$ to be a finite subset of E such that $U(p;\vec{v};2^{-n}) \cap T(E,p,n) \neq \emptyset$ for all $\vec{v} \in H^n$.

Now, if $p \in \overline{E}_n$ for each n, then we have ensured that every neighborhood of p meets $\bigcup_{n\in\omega} T(E_n,p,n)$. ∎

In particular, as in [3], the Fréchet–Urysohn fan (or hedgehog), defined in Definition 3.6.5, is not an Eberlein–Grothendieck space, so that Example 3.6.7 will provide a countable structure for a finite language which is not nice.

However, we shall see now that two other consequences of this theorem turn out to apply also to non-nice $\mathfrak{A}$.

First, Theorem 2.10.13 implies that every Eberlein–Grothendieck space has countable tightness, but in fact we have:

Lemma 2.10.14. *$C_{\mathrm{p}}(H,[0,1])$ has countable tightness whenever H^n is Lindelöf for all n.*

This is Theorem II.1.1 of [3]; the proof is as above, but choose countable sets rather than finite sets, and apply the result with all $E_n = E$. Note that whenever H is $K_{\sigma\delta}$, all H^n are also $K_{\sigma\delta}$, and hence Lindelöf, so

Corollary 2.10.15. *$\mathfrak{A}^{\#}$ has countable tightness whenever $\mathcal{L}$ is countable.*

Second,

Definition 2.10.16. A topological space X has the *splitting property* iff whenever $E \subseteq X$ and $p \in \overline{E} \setminus E$, then there are disjoint $R, S \subset E$ such that $p \in \overline{R} \cap \overline{S}$.

One may apply Lemma 2.10.13 to prove that Eberlein–Grothendieck spaces have the splitting property: let $R = \bigcup_{n\in\omega} R_n$ and $S = \bigcup_{n\in\omega} S_n$, where $R_0 = S_0 = \emptyset$, each $R_{n+1} = R_n \cup T(E \setminus (R_n \cup S_n), p, n)$, and each $S_{n+1} = S_n \cup T(E \setminus (R_{n+1} \cup S_n), p, n)$. In the case of $A^{\#}$ for an abelian group A, this was exactly the argument of L. T. Ramsey [30]. However, the splitting property is weaker:

Lemma 2.10.17. *If H is a $K_{\sigma\delta}$ space, then $C_{\mathrm{p}}(H,[0,1])$ has the splitting property.*

Proof. Fix $p \in \overline{E} \setminus E$. Since all H^n are Lindelöf, we may assume that E is countable (by Corollary 2.10.14). Let $\mathcal{I} = \{R \subseteq E : p \notin \overline{R}\}$. Then $\mathcal{I}$ is an ideal on E. Also, using the fact that H is $K_{\sigma\delta}$, one may show that $\mathcal{I}$ is an analytic subset of $\mathcal{P}(E)$ (which we identify with 2^E). It follows that $\mathcal{I}$ cannot be a prime ideal, so we can find disjoint $R, S \notin \mathcal{I}$. ∎

COROLLARY 2.10.18. *If the language is countable and $\mathfrak{A}^{\#}$ is Hausdorff, then $\mathfrak{A}^{\#}$ has the splitting property.*

If $\mathfrak{A}$ is an infinite abelian group, then there is an infinite $D \subseteq A$ such that $\overline{D}$ (the closure of D in bA) is homeomorphic to βD [11]. This is also true of some (but not all) semilattices and distributive lattices (see Example 3.5.20). By Corollary 2.10.18, no such D can have any limit points in $A^{\#}$. The methods of [11] and [21] easily show that if E is any infinite subset of the discrete abelian group A, then there is an infinite $D \subseteq E$ such that $\overline{D}$ homeomorphic to βD; hence $w(\overline{E}) = 2^{\aleph_0}$ if E is countable. For various other structures, including some non-abelian groups (see Section 3.3), such closures can have countable weight. However, by Corollary 2.10.20 below, such closures cannot have weight strictly between $\aleph_0$ and $2^{\aleph_0}$.

THEOREM 2.10.19. *Let H be a Polish space, $E \subseteq C_{\mathrm{p}}(H, [0,1])$ with $|E| = \aleph_0$, and let X be the closure of E in the cube $[0,1]^H$. Then either $w(X) \le \aleph_0$ or $w(X) = 2^{\aleph_0}$.*

Proof. This proof is patterned after the proof that every uncountable analytic set has size $2^{\aleph_0}$. Let $\kappa = w(X)$; then $\kappa \le 2^{\aleph_0}$ because X is separable. Assume $\kappa > \aleph_0$, and we shall show that $\kappa = 2^{\aleph_0}$. We consider the Banach space $C(X) = C(X, \mathbb{R})$, with $\|\cdot\|$ the usual sup norm, and $d(\cdot,\cdot)$ the associated metric distance. Note that $\kappa = w(C(X))$. For each $u \in H$, let $\pi_u : X \to [0,1]$ be projection. We shall produce a perfect $P \subseteq H$ such that $\{\pi_u : u \in P\}$ is discrete in $C(X)$, proving $w(C(X)) \ge 2^{\aleph_0}$.

Let $M = \{\pi_u : u \in H\} \subseteq C(X)$. By the Stone–Weierstrass Theorem, the algebra generated by M is dense in $C(X)$, so that M cannot be separable. It follows that we can fix an $\varepsilon > 0$ such that whenever S is a countable subset of M, there is a $\pi_u \in M$ with $d(\pi_u, S) \ge \varepsilon$.

Call $Z \subseteq H$ *small* iff there is a countable $S \subseteq M$ such that $d(\pi_u, S) < \varepsilon$ for all $u \in Z$. So, H is not small, and the small subsets of H form a σ-ideal.

Fix an integer $N > 0$ such that $1/N < \varepsilon/2$. Let $I_i = [i/N, (i+1)/N]$, for $i < N$. Observe that if $Z \subseteq H$ is non-small, then for some $a \in E$ and some $i, j < N$, we have $i + 1 < j$ and the sets $\{u \in Z : a(u) \in I_i\}$ and $\{u \in Z : a(u) \in I_j\}$ are both non-small: If not, then for each $a \in E$, let P_a be the set of i such that $\{u \in Z : a(u) \in I_i\}$ is non-small. Note that P_a is either a singleton or a set of two adjacent integers, so if $J_a = \bigcup\{I_i : i \in P_a\}$, then J_a is an interval of length either $1/N$ or $2/N$. But then $Z = \{u \in Z : \forall a \in E[a(u) \in J_a]\} \cup \bigcup_{a \in E}\{u \in Z : a(u) \notin J_a\}$, which expresses Z as a countable union of small sets, a contradiction.

It follows that we may choose, for each $s \in 2^{<\omega}$, a closed $Z_s \subseteq H$ and an $a_s \in E$ such that:

(1) $Z_{()} = H$.

(2) $\mathrm{diam}(Z_s) < 1/n$ whenever s has length $n > 0$.

(3) Z_s is not small.

(4) For some i, j: $i + 1 < j$, and $Z_{s0} = \{u \in Z_s : a_s(u) \in I_i\}$ and $Z_{s1} = \{u \in Z_s : a_s(u) \in I_j\}$.

Here, diam refers to the diameter with respect to some (fixed) complete metric on H. By (2), for each $f \in 2^\omega$, there is a $u_f \in H$ with $\bigcap_{n\in\omega} Z_{f\restriction n} = \{u_f\}$. By (4), $\|\pi_{u_f} - \pi_{u_g}\| \geq 1/N$ whenever f, g are distinct. Thus, M has a discrete set of size $2^{\aleph_0}$, so $\kappa = 2^{\aleph_0}$. ∎

Corollary 2.10.20. *If $\mathfrak{A}$ is any structure for a countable language, $(b\mathfrak{A}, \Phi)$ is its Bohr compactification, and E is a countable subset of A, then the weight of the closure of $\Phi(E)$ in $b\mathfrak{A}$ is either $2^{\aleph_0}$ or countable.*

Proof. By Corollary 2.7.4, we may assume that A is countable. Now, apply Theorems 2.10.12 and 2.10.19. ∎

Question 2.10.21. *Suppose that $\mathfrak{A}$ is a nice discrete structure for a countable language, E is a countable subset of A, and $a \in E$ is not isolated in E (with the topology inherited from $\mathfrak{A}^\#$). Must $\chi(a, E)$ be either $2^{\aleph_0}$ or countable?*

Note that Example 3.6.7 provides a counter-example for non-nice structures. Question 2.10.21 seems to be open even for abelian groups, but observe that in that case we must have $\chi(a, E) \geq \mathfrak{p}$ (the least cardinal κ such that $MA(\kappa)$ fails for some σ-centered partial order; see, e.g., Fremlin [10]); otherwise, there would be a subsequence S of E which converges to a, which is impossible, since there must be a subsequence of S whose closure in bA is homeomorphic to $\beta\mathbb{N}$.

3. SPECIFIC STRUCTURES

We now consider these generalities for some specific structures. We consider two general types of structures. One is homogeneities (see Definition 2.1.5) and related structures, such as loops and groups. The other is semilattices and distributive lattices, and special varieties thereof, such as total orders and boolean algebras. In addition, in Section 3.6, we describe some special structures cooked up to provide counter-examples.

3.1. Homogeneities. These give us enough structure to prove that products and cardinal functions work out nicely.

Lemma 3.1.1. *Let $(X; f, g)$ be a compact homogeneity, and fix a "basepoint" 0. For each open neighborhood W of 0, let $U_W = \{(x, y) : f(x, 0, y)$*

$\in W\}$. *Then the set of all* U_W, *for* W *an open neighborhood of* 0 *in* X, *is a basis for* Δ *in* $X \times X$.

Proof. Fix any open V with $\Delta \subseteq V \subseteq X \times X$. By compactness, $W = X \setminus \{f(x, 0, y) : (x, y) \notin V\}$ is open. Then $0 \in W$ and $\Delta \subseteq U_W \subseteq V$. ∎

This plus Lemma 2.5.1 implies the following corollary, which also follows from the fact that X is dyadic:

COROLLARY 3.1.2. *Let* $(X; f, g)$ *be any infinite compact homogeneity. Then* $w(X) = \chi(X)$.

LEMMA 3.1.3. *Suppose* $\mathfrak{A} = (A; f, g, \ldots)$ *is any topological structure with* $b\mathfrak{A}$ *infinite and* $(A; f, g)$ *a homogeneity. Then* $w(\mathfrak{A}^{\#}) = \chi(\mathfrak{A}^{\#}) = w(b\mathfrak{A}) = \chi(b\mathfrak{A})$.

Proof. We may assume that $\mathfrak{A}^{\#}$ is Hausdorff (or else, pass to a quotient by Lemma 2.3.11), and now we may identify $\mathfrak{A}$ as a sub-structure of $b\mathfrak{A}$, with the induced topology. By homogeneity, all points of A have the same character in $\mathfrak{A}^{\#}$, and all points of $b\mathfrak{A}$ have the same character in $b\mathfrak{A}$. We already know (Corollary 3.1.2) that $w(b\mathfrak{A}) = \chi(b\mathfrak{A})$. Since A is dense in $b\mathfrak{A}$ and the spaces involved are all regular, $\chi(\mathfrak{A}^{\#}) = \chi(b\mathfrak{A}) = w(b\mathfrak{A})$. Finally, $\chi(\mathfrak{A}^{\#}) \le w(\mathfrak{A}^{\#}) \le w(b\mathfrak{A}) = \chi(\mathfrak{A}^{\#})$. ∎

Note that every element of a homogeneity is an idempotent with respect to f, g (that is, $f(x, x, x) = g(x, x, x) = x$). Using this, we obtain:

LEMMA 3.1.4. *Suppose that* $\mathcal{L}$ *contains* 3-*place function symbols* f, g, *and that* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are topological* $\mathcal{L}$-*structures*, *and are homogeneities* (*with respect to* f, g). *Then* $b(\mathfrak{A} \times \mathfrak{B}) = b\mathfrak{A} \times b\mathfrak{B}$, *and hence the topology* $(\mathfrak{A} \times \mathfrak{B})^{\#}$ *is the same as the product topology*, $\mathfrak{A}^{\#} \times \mathfrak{B}^{\#}$.

Proof. We may assume that $\mathcal{L}$ also contains a symbol 0 (interpreted arbitrarily in $\mathfrak{A}, \mathfrak{B}$). Set $x \circ y = f(0, x, g(0, 0, y))$, and note that $x \circ 0 = 0 \circ x = x$. Thus, the result *would* follow by Lemma 2.9.3, except that 0 may fail to be an idempotent with respect to the other functions besides f, g. To fix this problem, let $\mathcal{L}_1$ be obtained from $\mathcal{L}$ by replacing each function $h \in \mathcal{L} \setminus \{f, g\}$ with a new symbol h' of the same arity, and let $\mathcal{L}_2 = \mathcal{L} \cup \mathcal{L}_1$. Interpret h' in $\mathfrak{A}$ and $\mathfrak{B}$ by the formula $h'(x_1, \ldots, x_n) = f(h(0, \ldots, 0), 0, h(x_1, \ldots, x_n))$, so that 0 will be idempotent with respect to h'. We have now defined, in the obvious way, structures $\mathfrak{A}_1, \mathfrak{B}_1$ for $\mathcal{L}_1$ and $\mathfrak{A}_2, \mathfrak{B}_2$ for $\mathcal{L}_2$. Lemma 2.9.3 applies immediately to show $b(\mathfrak{A}_1 \times \mathfrak{B}_1) = b\mathfrak{A}_1 \times b\mathfrak{B}_1$. But also, we can retrieve h from h' by the formula $h(x_1, \ldots, x_n) = g(0, h(0, \ldots, 0), h'(x_1, \ldots, x_n))$, so applying Lemma 2.8.3, we get $b\mathfrak{A}_1 = b\mathfrak{A}_2 = b\mathfrak{A}$, and likewise for $\mathfrak{B}$ and $\mathfrak{A} \times \mathfrak{B}$. ∎

The following lemma, together with Lemma 2.2.6, implies that a compactification of $\mathfrak{A}$ is completely determined by its induced topology on A:

LEMMA 3.1.5. *Suppose that* $\mathfrak{A} = (A; f, g)$ *is a topological homogeneity, and that* (X, φ) *and* (Y, ψ) *are two compactifications of* A *compatible with* $\mathfrak{A}$. *If* $\mathcal{T}_\varphi \subseteq \mathcal{T}_\psi$, *then* $(X, \varphi) \le (Y, \psi)$.

Proof. Define $\Gamma \subseteq Y \times X$ by $\Gamma = \mathrm{cl}\{(\psi(a), \varphi(a)) : a \in A\}$. If we can show that in fact Γ is a function, we will have $(X, \psi) \le_\Gamma (Y, \varphi)$. So, fix $y \in Y$ and $x, z \in X$ with $(y, x) \in \Gamma$ and $(y, z) \in \Gamma$. We shall show that $x = z$.

Let D be a directed set, with the nets $\langle (\psi(a_\alpha), \varphi(a_\alpha)) : \alpha \in D\rangle$ converging to (y, x) and $\langle (\psi(b_\alpha), \varphi(b_\alpha)) : \alpha \in D\rangle$ converging to (y, z). Fix a "basepoint" $0 \in A$. In Y, we have $\psi(f(a_\alpha, 0, b_\alpha)) \to f(y, \psi(0), y) = \psi(0)$, so in A, $f(a_\alpha, 0, b_\alpha) \to 0$ in the topology $\mathcal{T}_\psi$ and hence in $\mathcal{T}_\varphi$, so that in X, $\varphi(f(a_\alpha, 0, b_\alpha)) \to \varphi(0)$. But also, $\varphi(f(a_\alpha, 0, b_\alpha)) \to f(x, \varphi(0), z)$, so $f(x, \varphi(0), z) = \varphi(0)$, so $x = z$. ∎

REMARK 3.1.6. Note that this lemma does not give us a simple criterion for deciding whether a given topology $\mathcal{T}$ on A is indeed of the form $\mathcal{T}_\varphi$ for some compactification (X, φ); the lemma only says that this compactification is unique if it exists. In the case of groups, the criterion is simply that $\mathcal{T}$ be *totally bounded*, in the sense that for each open set $U \subsetneq A$, we have all of A covered by a finite number of translates of U; then $\mathcal{T}$ corresponds to a totally bounded uniformity; see Remark 2.4.1. It is not clear to what extent this can be generalized to other varieties of loops.

The fact that homogeneities are homogeneous gives us:

LEMMA 3.1.7. *Suppose that* $\mathcal{L}$ *contains* 3-*place function symbols* f, g, *and that* $\mathfrak{A}$ *is a discrete* $\mathcal{L}$-*structure which is a homogeneity* (*with respect to* f, g). *Suppose also that there is a homomorphism* ψ *from* $\mathfrak{A}$ *onto the discrete* $\mathcal{L}$-*structure* $\mathfrak{B}$, *where* $|\mathfrak{B}| = \aleph_0$ *and* $\mathfrak{B}^\#$ *is Hausdorff. Then* $\mathfrak{A}^\#$ *is homeomorphic to* $\omega \times \mathfrak{A}^\#$ (*i.e. to a disjoint sum of* ω *copies of the space* $\mathfrak{A}^\#$).

Proof. Note that if X is any countably infinite homogeneous Hausdorff space, then X is homeomorphic to $\omega \times X$. If we examine the *proof* of this fact, applied in the case $X = \mathfrak{B}^\#$, we shall see that in this case, the homeomorphism lifts to $\mathfrak{A}^\#$ via the continuous (by Lemma 2.3.10) $\psi : \mathfrak{A}^\# \to \mathfrak{B}^\#$.

First, we observe that if U, V are any non-empty open subsets of $\mathfrak{B}^\#$, then $\psi^{-1}(U)$ and $\psi^{-1}(V)$ are homeomorphic open subsets of $\mathfrak{A}^\#$. To prove this, apply homogeneity of $\mathfrak{B}^\#$ (and the fact that there are no isolated points) to choose points $u_n, v_n \in B$ and clopen sets $U_n, V_n \subset B$ (for $n \in \omega$) so that $u_n \in U_n$, $v_n \in V_n$, U is the disjoint union of the U_n, V is the disjoint union of the V_n, and each U_n is homeomorphic to V_n via the map $x \mapsto f(u_n, v_n, x)$. Then choose $p_n, q_n \in A$ with $\psi(p_n) = u_n$ and $\psi(q_n) = v_n$. Now,

$\psi^{-1}(U_n)$ will be homeomorphic to $\psi^{-1}(V_n)$ via the map $y \mapsto f(p_n, q_n, y)$, so that $\psi^{-1}(U)$,$\psi^{-1}(V)$ are homeomorphic.

Finally, apply homogeneity again to partition B into non-empty clopen sets W_n (for $n \in \omega$). Then each $\psi^{-1}(W_n)$ and $\psi^{-1}(B) = A$ are homeomorphic, so that we have $\mathfrak{A}^\#$ partitioned into ω sets homeomorphic to $\mathfrak{A}^\#$. ∎

This lemma is most useful when $\mathfrak{A}$ is an abelian group, in which case we get $\psi, \mathfrak{B}$ for free (see Lemma 3.3.3). For non-abelian groups, we cannot delete the assumption about $\psi, \mathfrak{B}$. For example, let $A = SO(3)$, viewed as a discrete group. By van der Waerden [34], $bA = A^\#$ is just $SO(3)$ with its usual compact topology (i.e., $SO(3)$ is self-Bohrifying (Definition 2.3.14)). Thus, $A^\#$ is not homeomeorphic to $\omega \times A^\#$.

3.2. Quasigroups and loops. Every quasigroup $(A; \cdot, \backslash, /)$ has a loop operation, defined via isotopy, as in Bruck [5]. That is, fix any $a, b \in A$ and define

$$(*) \qquad x \circ y = (x/b) \cdot (a\backslash y);$$

then $\circ$ is a loop operation with identity $a \cdot b$. It follows (see Hofmann [12] or Chapter IX of [7]) that every topological space X which supports a continuous quasigroup operation $(X; \cdot, \backslash, /)$ also supports a continuous loop operation. However, it need not be the case that we can read the quasigroup operation back from the loop operation. For example, if A is infinite and $\circ$ is any loop operation on A, then there are $2^{|A|}$ different quasigroup operations which yield $\circ$ via $(*)$, at most $|A|$ of which can be first-order definable from $\circ$. So, we are not able to reduce the Bohr compactification of a quasigroup to the Bohr compactification of a loop.

Likewise, every quasigroup has a defined homogeneity: set $f(x, y, z) = z \cdot (x\backslash y)$ and $g(x, y, z) = z/(y\backslash x)$; in the case of groups, this reduces to $f(x, y, z) = g(x, y, z) = zx^{-1}y$. Hence, topological quasigroups are homogeneous. Furthermore, $b(A; \cdot, \backslash, /) = b(A; \cdot, \backslash, /, f, g)$ by Lemma 2.8.3, so applying Lemma 3.1.4 we have $b(\mathfrak{A} \times \mathfrak{B}) = b\mathfrak{A} \times b\mathfrak{B}$ for quasigroups. It is not clear whether $b(A; \cdot, \backslash, /) = b(A; f, g)$ in general, although this is certainly true for loops.

If $(X; \cdot, \backslash, /)$ is a quasigroup and X has a compact Hausdorff topology which makes the function $\cdot$ continuous, then $\backslash$ and $/$ must also be continuous, since their graphs are closed. However, one cannot in general identify $b(A; \cdot, \backslash, /)$ with $b(A; \cdot)$, since the $\cdot$ of $b(A; \cdot)$ may fail to be a quasigroup operation (see Example 3.2.2 below). In the case of IP-loops, one can make this identification. An *IP-loop*, or a *loop with the inverse property*, is a loop with a unary function i satisfying the equations $x/y = x \cdot i(y)$ and $y\backslash x = i(y) \cdot x$ (so that also $1/y = y\backslash 1 = i(y)$). Clearly, for these we have $b(A; \cdot, \backslash, /) = b(A; \cdot, i)$, but one can drop the i as well:

LEMMA 3.2.1. *Suppose that* $(A;\cdot,\backslash,/)$ *is an IP loop and* $\varphi : A \to X$ *is a compactification of the set* A *which is compatible with* $\cdot$. *Then* φ *is also compatible with* $\backslash$ *and* $/$. *Hence*, $b(A;\cdot) = b(A;\cdot,\backslash,/)$.

Proof. We apply Theorem 2.8.5; it is sufficient to show that the inverse operation, i, is positively definable from $\cdot$ and 1. Now, we can define i by $y = i(x) \Leftrightarrow x \cdot y = 1$. Furthermore, the statement $\forall x \exists! y[x \cdot y = 1]$ is provable from $\forall z[z1 = 1z = z]$ and $\forall x \exists z \forall y[z(xy) = (yx)z = y]$, which are positive statements about $\cdot$ and 1 which are true in all IP loops. ∎

This lemma does not hold for loops in general, as the next example shows. This example also shows that some care must be used in applying Theorem 2.8.5. In any loop, one may define $/$ and $\backslash$ by positive formulas involving just $\cdot$; that is, $y = x_1/x_2 \Leftrightarrow yx_2 = x_1$; but one may not be able to prove $\forall x_1 x_2 \exists! y[yx_2 = x_1]$ from positive statements true about $\cdot$ and 1 in the loop.

EXAMPLE 3.2.2. *Let* A *denote the rational points in the circle group* T, *and let* $\varphi : A \to T$ *be the identity map. Then there is a commutative loop operation* $\circ$ *on* A *such that* φ *is compatible with* $\circ$, *but not with its* $\backslash$ *and* $/$. *Furthermore*, *the* $\circ$ *of* $b(A;\circ)$ *is not a quasigroup operation.*

Proof. View T as $\mathbb{R}/\mathbb{Z}$, so that $A = \mathbb{Q}/\mathbb{Z}$. If $f : \mathbb{R} \to \mathbb{R}$ is continuous and periodic with period 1, we can define $\circ$ on T by

$$x \circ y = x + y + f(x)f(y) \pmod 1.$$

Fix an irrational $\gamma \in (0.4, 0.6)$. Then choose f together with rationals $0 = a_0 < a_1 < a_2 < \ldots < \gamma < \ldots < b_2 < b_1 < b_0 = 1$ so that $a_n \nearrow \gamma$ and $b_n \searrow \gamma$, and

(1) $0 \le f(x) \le 0.1$ for all x, and $f(\gamma) = 0.1$, but $f(x) < 0.1$ for all $x \in [0,1] \setminus \{\gamma\}$.

(2) On each $[a_n, a_{n+1}]$ and $[b_{n+1}, b_n]$, f is linear, with rational slope in $[-10, +10]$.

(3) $f(a_0) = f(a_1) = f(b_1) = f(b_0) = 0$.

(4) f has slope exactly -10 on $[b_2, b_1]$.

So, $\circ : T \times T \to T$ is a continuous function, and, by item (2), $\circ$ takes $A \times A$ to A. For any $c \in [0,1]$, define $L_c(x) = x + c + f(x)f(c)$. Then $L_c(0) = c$ and $L_c(1) = 1 + c$ by (3). If c is rational, then L_c has positive slope by (1) and (2), so L_c, viewed as a map on $\mathbb{Q}/\mathbb{Z}$, is invertible. Hence, $\circ$ is a quasigroup operation on A. It is a loop because 0 is an identity element. But also, by (1) and (4), $L_\gamma(x) = x + \gamma + f(x) \cdot 0.1$ is constant on $[b_2, b_1]$, so that $\circ$ is not a quasigroup operation on T. Hence, $\backslash$ and $/$ on A do not extend continuously to functions on T; that is, the closures of their graphs in T^3 fail to be functions.

Now, let $\Phi : (A; \circ) \to (X; \circ) = b(A; \circ)$, and let $\Gamma : X \to T$ with $\Gamma\Phi = \varphi$. In T, let d be the constant value of L_γ on $[b_2, b_1]$. In X, let $U = \Gamma^{-1}(b_2, b_1)$, and fix any δ with $\Gamma(\delta) = \gamma$. Let $V = \delta \circ U$. If $v = \delta \circ x \in V$, then $\Gamma(v) = \gamma \circ \Gamma(x) = d$. Hence, $\Gamma(V) = \{d\}$. If $(X; \circ)$ were a quasigroup, then V would be open, and we could cover X with translates of V: $X = x_1 \circ V \cup \ldots \cup x_n \circ V$. But then $T = \Gamma(X) = \{\Gamma(x_1) \circ d, \ldots, \Gamma(x_n) \circ d\}$. ∎

3.3. Groups. If G is any topological group, then it has a defined homogeneity, so that $w(G^\#) = \chi(G^\#) = w(bG) = \chi(bG)$ by Lemma 3.1.3. In some cases, we can compute this cardinal. Most notably, we have:

THEOREM 3.3.1. *If G is any discrete infinite abelian group, then $w(bG) = 2^{|G|}$ and $|bG| = 2^{2^{|G|}}$.*

Of course, the fact that $|bG| = 2^{w(bG)}$ follows immediately from homogeneity (see Lemma 2.5.1). This theorem is due to Kakutani [18], but was improved by Hartman and Ryll-Nardzewski [11], who showed that G contains an I_0-set, A, of size $|G|$. This set has the property that A is discrete in $G^\#$, and is C^*-embedded in bG (equivalently, its closure is homeomorphic to βA).

This theorem can fail for non-abelian groups, since by von Neumann and Wigner [26], there are discrete groups G in all infinite cardinalities such that $G^\#$ is indiscrete (so $w(bG) = 1$). Furthermore, by Moran [24], there is a countably infinite group G such that $G^\#$ is Hausdorff and $w(bG) = \aleph_0$. Note that for countable G, $w(bG)$ cannot be strictly between $\aleph_0$ and $2^{\aleph_0}$ by Corollary 2.10.20.

For Moran's G, the topology of $G^\#$ is characterized as the unique countable regular space with no isolated points of weight $\aleph_0$. This raises the general question:

QUESTION 3.3.2. *If G, K are groups, $G^\#, K^\#$ are Hausdorff, and $|G| = |K|$, when are $G^\#, K^\#$ homeomorphic (just as topological spaces)?*

In the case of abelian groups, this is an old question of van Douwen, and some partial results are known: Say $|G| = |K| = \kappa \geq \aleph_0$. Then $G^\#, K^\#$ are Hausdorff and have all the same cardinal functions; for example, $w(G^\#) = w(K^\#) = 2^\kappa$. However, there are examples (for each κ) where $G^\#$ and $K^\#$ fail to be homeomorphic (see [20]). But there are also cases where G, K are non-isomorphic but $G^\#, K^\#$ are homeomorphic. Two classes of such examples are known. One class is provided by Comfort, Hernández, and Trigos [8]. Another is provided by:

LEMMA 3.3.3. *If G is a discrete infinite abelian group, then $G^\#$ is homeomorphic to $\omega \times G^\#$. Hence, if G is a subgroup of the abelian group K and $[K : G]$ is finite, then $G^\#$ and $K^\#$ are homeomorphic.*

Proof. Note that there must be a countably infinite B and a homomorphism ψ from G onto B. To see this, choose C, D such that $C \leq G$, $C \leq D$, D is divisible, and $|C| = |D| = \aleph_0$; then let $\psi : G \to D$ extend the identity map on C, and let $B = \operatorname{ran}(\psi)$. Thus, the result follows from Lemma 3.1.7. ∎

3.4. Semilattices. In this discussion, we emphasize $\wedge$-semilattices, which are structures of the form $(A; \wedge)$ where $\wedge$ is associative, commutative, and idempotent ($x \wedge x = x$). These define a partial order by $x \leq y \Leftrightarrow x \wedge y = x$. Of course, we may apply our results to $\vee$-semilattices, which have exactly the same axioms, but the binary function is called $\vee$ and the order is called $\geq$. So, in discussing lattices, the results apply both to the $\wedge$ and the $\vee$.

For total orders, it makes no difference whether we consider the structures to be lattices or semilattices:

LEMMA 3.4.1. *If A is totally ordered by $\leq$, and $\wedge$ and $\vee$ are the corresponding lattice operations*, *then* $b(A; \vee, \wedge) = b(A; \wedge) = b(A; \vee)$.

Proof. Apply Theorem 2.8.5. $\vee$ is positively definable from $\wedge$ by

$$x_1 \vee x_2 = y \Leftrightarrow \psi(x_1, x_2, y),$$

where ψ is

$$x_1 \wedge y = x_1 \text{ AND } x_2 \wedge y = x_2 \text{ AND } (y = x_1 \text{ OR } y = x_2).$$

The statement $\forall x_1 x_2 \exists! y\, \psi(x_1, x_2, y)$ is provable from positive statements (such as $\forall xy(x \wedge y = x \text{ OR } x \wedge y = y)$), which are true about $\wedge$ in A. ∎

In any lattice, the $\vee$ is first-order definable from the $\wedge$ (since $\leq$ is), but it may not be positively definable, and we need not have $b(A; \vee, \wedge) = b(A; \wedge)$; see Theorem 3.5.19.

Bohr topologies for abelian groups are handled via the Pontryagin Duality Theorem. There are similar duality theorems for a number of other algebraic varieties; see Davey [9]. Among these are the Priestly duality [28], [29] for bounded distributive lattices and the duality of Hofmann–Mislove–Stralka [13] for semilattices. In these dualities, the two-element lattice or semilattice, **2**, plays the role of the circle group for abelian groups. Using homomorphisms into **2**, one can prove that the Bohr topology for semilattices and distributive lattices is Hausdorff.

Now, the Pontryagin duality is with *all* compact abelian groups, and this enables us to prove that the circle group is adequate for abelian groups. However, the dualities for the lattice varieties are with compact *zero-dimensional* structures, so we do not get the analogous result that **2** is adequate for these varieties, since the Bohr topology is computed using homomorphisms into *all* compact structures. So, $\operatorname{Hom}(\mathfrak{A}, \mathbf{2})$ will give a fairly explicit description of $b_0\mathfrak{A}$ (see Definition 2.3.15) for all semilattices (Theorem 3.4.17) and for all distributive lattices (Theorem 3.5.14). Then we shall examine some cases in which $b\mathfrak{A} = b_0\mathfrak{A}$. In particular, this holds for boolean algebras and total orders, so that **2** is adequate for discrete boolean algebras and discrete total orders.

Of course, **2** is not adequate for *compact* total orders, since the order may be connected. It is easy to see that $[0,1]$ is adequate for compact total orders, and we shall show that if $\mathfrak{A}$ is any discrete lattice or semilattice, then $[0,1]$ is adequate for $\mathfrak{A}$ iff **2** is adequate for $\mathfrak{A}$.

Definition 3.4.2. A compact semilattice $(X;\wedge)$ has the *Lawson property* iff the semilattice $[0,1]$ is adequate for X.

Many equivalents are known for this property; see [6]. Not every compact semilattice has the Lawson property (by Lawson [22]), and this implies that there is a discrete semilattice $\mathfrak{A}$ for which $b_0\mathfrak{A} < b\mathfrak{A}$ (see [6] and Lemma 3.4.12).

We begin by listing some elementary properties of compact orders and semilattices; see, e.g., [6], [13] for proofs.

Definition 3.4.3. A *compact order* is a pair $(X;\le)$, where X is a compact Hausdorff space and $\le$ is a partial order on X which is closed in $X\times X$. If $S\subseteq X$, then $S{\downarrow} = \{x\in X : \exists y\in S[x\le y]\}$ and $S{\uparrow} = \{x\in X : \exists y\in S\ [x\ge y]\}$. If $x\in X$, then $x{\downarrow} = \{x\}{\downarrow}$ and $x{\uparrow} = \{x\}{\uparrow}$.

For example, the $\le$ induced by a compact semilattice is a compact order, by continuity of $\wedge$.

Lemma 3.4.4. *Suppose $(X;\le)$ is a compact order and F is a closed subset of X. Then $F{\downarrow}$ and $F{\uparrow}$ are closed.*

Lemma 3.4.5. *Suppose that $(X;\wedge)$ is a compact semilattice.*

(1) *If $U\subseteq X$ is open then $U{\uparrow}$ is open.*

(2) *If $K\subseteq X$ is closed, $E\subseteq X$, and $x\wedge y\in K$ for all distinct $x,y\in K$, then all limit points of E lie in K.*

(3) *If $E\subseteq X$ is infinite and $x\wedge y = c$ for all distinct $x,y\in E$, then c is the unique limit point of E.*

(4) *If $K\subseteq X$ is clopen, and M is the set of minimal elements of K, then M is finite, $K{\uparrow} = \bigcup\{m{\uparrow} : m\in M\}$, and $m{\uparrow}$ is clopen for each $m\in M$.*

(5) *Suppose that $\{x_\alpha : \alpha<\kappa\}\subseteq X$. Let D be $[\kappa]^{<\omega}$ ordered by subset, and for each $d\in D$, let $x_d = \bigwedge\{x_\alpha : \alpha\in d\}$. Then $\bigwedge\{x_\alpha : \alpha<\kappa\}$ exists in X, and the net $\langle x_d : d\in D\rangle$ converges to it.*

(6) *If $(Y;\wedge)$ is another compact semilattice and $\varphi : X\to Y$ is a continuous homomorphism, then for any $\{x_\alpha : \alpha<\kappa\}\subseteq X$, we have $\varphi(\bigwedge_{\alpha<\kappa} x_\alpha) = \bigwedge_{\alpha<\kappa}\varphi(x_\alpha)$.*

By (5)–(6), compact semilattices are complete, and homomorphisms between them are complete homomorphisms. (4) is essential to the duality results with compact 0-dimensional semilattices. We do not need the fact the details of this theory, but we do need the fact that homomorphisms into **2** separate points (Lemma 3.4.7).

Definition 3.4.6. If $(X; \wedge)$ is a semilattice and $b \in X$, define $\gamma_b : X \to \mathbf{2}$ so that $\gamma_b(x)$ is 1 if $x \geq b$ and 0 if $x \not\geq b$.

Lemma 3.4.7. *Each γ_b is a homomorphism. If $(X; \wedge)$ is a compact 0-dimensional semilattice, and $x, y \in X$ with $x \neq y$, then for some $b \in X$, γ_b is continuous and $\gamma_b(x) \neq \gamma_b(y)$.*

Proof. To get a continuous γ_b (so $b\uparrow$ is clopen), use Lemma 3.4.5(4). ∎

Now, just the fact that γ_b is a homomorphism lets us prove:

Lemma 3.4.8. *If $\mathfrak{A} = (A; \wedge)$ is a discrete semilattice, then $\mathfrak{A}^{\#}$ is Hausdorff. If $C \subseteq A$ is a chain, then C is relatively discrete in $\mathfrak{A}^{\#}$.*

Proof. $\mathfrak{A}^{\#}$ is Hausdorff because the γ_b separate points.

Now, let C be a chain, and fix $b \in C$. Then $\gamma_b^{-1}\{1\}$ is a clopen set containing b and no smaller element of C. Define another homomorphism $\psi(x) = \sup\{\gamma_c(x) : c \in C \;\&\; c > b\}$. Then $\psi^{-1}\{0\}$ is a clopen set containing b and no greater element of C. ∎

It is immediate from Lemma 3.4.7 that we may compute $b_0\mathfrak{A}$ by taking the $\bigvee$ (in the lattice of compactifications) of all homomorphisms into **2**. But in fact we get the same thing if we consider all homomorphisms into $[0, 1]$:

Lemma 3.4.9. *Suppose that $(A; \wedge)$ is a semilattice and $\varphi : A \to X$ is some compactification of A, where $(X; \wedge)$ is a compact semilattice which is totally ordered. Then $\varphi \leq b_0(A; \wedge)$.*

Proof. Let $Y = X \times 2$, ordered lexically. So, Y is obtained by doubling all the points of X. Let $\Gamma : Y \to X$ be the undoubling: $\Gamma(x, i) = x$. Then Γ is continuous. The map φ lifts to $\psi : A \to Y$ defined by $\psi(a) = \varphi(a, 0)$. Then ψ (as a map into $\mathrm{cl}(\mathrm{ran}(\psi))$) is a compactification of A and $\varphi \leq_\Gamma \psi$. Since Y is 0-dimensional, $\psi \leq b_0(A; \wedge)$. ∎

Corollary 3.4.10. $b_0\mathfrak{A} = \bigvee(\mathrm{Hom}(\mathfrak{A}, \mathbf{2})) = \bigvee(\mathrm{Hom}(\mathfrak{A}, [0, 1]))$ *for every semilattice* $\mathfrak{A}$.

Here, the $\bigvee$ is in the lattice of compactifications.

Corollary 3.4.11. *If $(A; \wedge)$ is a discrete semilattice and $b(A; \wedge)$ has the Lawson property, then $b(A; \wedge) = b_0(A; \wedge)$, and is hence 0-dimensional.*

Conversely, as in [6], pp. 22–23, we have:

LEMMA 3.4.12. *If $(A;\wedge)$ is a discrete semilattice and $b(A;\wedge) = b_0(A;\wedge)$, then every compactification of $(A;\wedge)$ has the Lawson property.*

P r o o f. Since $b(A;\wedge)$ is a compact 0-dimensional semilattice, it has the Lawson property (by Lemma 3.4.7). Now, just use the fact that the Lawson property is preserved under continuous homomorphic images; this fact is easily proved from the equivalents to the Lawson property given in [6]. ∎

COROLLARY 3.4.13. *There is a countable discrete semilattice $(A;\wedge)$ such that $b_0(A;\wedge) < b(A;\wedge)$.*

P r o o f. By Lawson [22], there is a compact second countable semilattice $(X;\wedge)$ which does not have the Lawson property. Let A be a countable dense sub-semilattice of X, and apply Lemma 3.4.12. ∎

Corollary 3.4.11 may be used to prove that $b(A;\wedge) = b_0(A;\wedge)$ when A has finite breadth:

DEFINITION 3.4.14. For n finite, a semilattice $(A;\wedge)$ has *breadth* $\leq n$ iff whenever $E \subseteq A$ and $n < |E| < \aleph_0$, there is an $F \subset E$ with $|F| = n$ such that $\bigwedge F = \bigwedge E$.

LEMMA 3.4.15. *If $(A;\wedge)$ has breadth $\leq n$, then $b(A;\wedge) = b_0(A;\wedge)$.*

P r o o f. $b(A;\wedge)$ also has breadth $\leq n$, since this property is expressed by a positive logical sentence. Thus, $b(A;\wedge)$ has the Lawson property by [6], Theorem 2.30. ∎

The clopen sets in a topological space form a semilattice (in which $\wedge$ is $\cap$). Every semilattice can be isomorphically embedded into such a semilattice. The standard way of doing this, as indicated in the proof of the next lemma, can be used to compute $b_0\mathfrak{A}$.

LEMMA 3.4.16. *Let $(B;\wedge)$ be a semilattice. Then there is a compact 0-dimensional Hausdorff space X and a sub-semilattice A of the clopen sets of X such that B is isomorphic to A, and for $K \in A$,*

$$(*) \qquad K \setminus \bigcup \{L \in A : L \not\supseteq K\} \neq \emptyset.$$

P r o o f. Let $X = \mathrm{Hom}(B, \mathbf{2})$, which we regard as a subset of 2^B, with the usual product topology. Define $\Psi(b) = \{\varphi \in X : \varphi(b) = 1\}$. Then Ψ is a 1-1 homomorphism from B into the semilattice of clopen subsets of X, so A is just the range of Ψ. $(*)$ holds because each $K = \Psi(b)$ in A contains the element γ_b and $\gamma_b \notin \Psi(a)$ for any $a \not\geq b$. ∎

Condition $(*)$ implies that Ψ is not a *lattice* homomorphism; that is, if $a \not\leq b$ and $b \not\leq a$ and $a \vee b$ happens to exist, then $\Psi(a) \cup \Psi(b)$ is a proper subset of $\Psi(a \vee b)$. For the analogous embedding to use for lattices in the computation of the Bohr compactification, see Section 3.5.

Theorem 3.4.17. *Suppose that* $(A;\wedge)$ *is a sub-semilattice of the semilattice of clopen subsets of the compact Hausdorff space* X, *and assume that* $(*)$ *above holds. Give* $\mathcal{P}(X)$ *the usual product topology by identifying it with* 2^X. A *is also a sub-semilattice of* $\mathcal{P}(X)$. *Let* Z *be the closure of* A *in* $\mathcal{P}(X)$. *Then* $Z = b_0(A;\wedge)$.

Proof. It is sufficient to show that whenever $\varphi : A \to \mathbf{2}$ is a semilattice homomorphism, φ extends to a continuous homomorphism on $\mathcal{P}(X)$, and hence on Z. Let $E = \bigcap\{K \in A : \varphi(K) = 1\} \setminus \bigcup\{L \in A : \varphi(L) = 0\}$. By $(*)$ and compactness, $E \neq \emptyset$, so fix $x \in E$. Define $\psi : \mathcal{P}(X) \to \mathbf{2}$ so that $\psi(S) = 1$ iff $x \in S$. Then ψ extends φ and is continuous with respect to the product topology on $\mathcal{P}(X)$. ■

In particular, for total orders, A is a chain of clopen sets, and we get its closure by taking unions and intersections. Note that an increasing union of clopen sets is open but not closed, while a decreasing intersection of clopen sets is closed but not open. This gives us the following simple description of bA, which may also easily be verified directly from the basic definitions:

Lemma 3.4.18. *If* A *is a total order*, *then* $A^\#$ *is discrete*, *and* $bA = b_0A$ *is computed as follows*: *First*, *let* $A \subseteq Y$, *where* Y *is the Dedekind completion. Then replace each element* $y \in Y\backslash A$ *by two points* $\{y^+, y^-\}$, *except in the cases where* y *is the first or the last element of* Y. *Finally*, *for each point* $z \in A$, *add a new point* z^+ *directly above* z *if* z *is a limit from above*, *and also add a new point* $z^- < z$ *if* z *is a limit from below. So*, *if* z *is a limit point from both sides*, *it becomes a triple of points*: $z^- < z < z^+$.

Then applying the standard Hausdorff analysis of total orders gives:

Corollary 3.4.19. *If* A *is a countably infinite total order*, *then* $\chi(bA) = \aleph_0$. *If* A *contains a copy of the rationals*, *then* $w(bA) = |bA| = 2^{\aleph_0}$. *If not*, *then* $w(bA) = |bA| = \aleph_0$.

One way to ensure $b_0A = bA$ is to bound the breadth (Lemma 3.4.15). In the opposite direction, we may bound the chain length. In fact, if all chains in A are finite, then we shall show (Theorem 3.4.23) that $b_0A = bA = A$; that is, there is a natural compact topology on A which makes the semilattice self-Bohrifying (see Definition 2.3.14).

Lemma 3.4.20. *Let* A *be any Hausdorff topological semilattice. Then each* $a\!\uparrow$ *is closed. If* $a < b$ *and there is no* c *with* $a < c < b$, *then* $a\!\uparrow \setminus b\!\uparrow$ *is closed.*

Proof. $a\!\uparrow = \{x : x \wedge a = a\}$ is closed by continuity of $\wedge$. If $a\!\uparrow \setminus b\!\uparrow$ fails to be closed, then there is an $x \in b\!\uparrow$ such that $x \in \mathrm{cl}(a\!\uparrow \setminus b\!\uparrow)$. Then, by continuity of $\wedge$, we have $b = b \wedge x \in \mathrm{cl}(b \wedge (a\!\uparrow \setminus b\!\uparrow))$. However, $b \wedge (a\!\uparrow \setminus b\!\uparrow) = \{x : a \le x < b\} = \{a\}$, so we would have $b \in \mathrm{cl}\{a\}$. ■

LEMMA 3.4.21. *Let $(A;\wedge)$ be a semilattice with no infinite chains. Let $\mathcal{T}$ be the coarsest topology which makes all $a{\uparrow}$ clopen. Then*

(1) *$\mathcal{T}$ is Hausdorff.*

(2) *Each $a \in A$ has a base consisting of sets of the form $a{\uparrow} \setminus (b_1{\uparrow} \cup \ldots \cup b_n{\uparrow})$, where $n \in \omega$ and each $b_i > a$.*

(3) *$\mathcal{T}$ is compact.*

(4) *$\mathcal{T}$ is coarser than every other Hausdorff topological lattice topology on A.*

Proof. (1) is clear, and holds in any partial order.

Next, note that since A has no decreasing ω-sequences, it is actually a complete semilattice. Hence, A must have a least element, 0. Also, if $a, c \in A$ have some upper bound, they must have a least upper bound, $a \vee c$.

Now, for (2), use the fact that the $a{\uparrow}$ and $A \setminus b{\uparrow}$ form a sub-base for $\mathcal{T}$, plus the fact that $a_1{\uparrow} \cap a_2{\uparrow}$ is either $\emptyset$ or $(a_1 \vee a_2){\uparrow}$.

Suppose (3) fails. Then $A = 0{\uparrow}$ is not compact in the topology $\mathcal{T}$. Since A has no increasing ω-sequences, there is some maximal $a \in A$ such that $a{\uparrow}$ is not compact in the topology $\mathcal{T}$. Let $\mathcal{U}$ be any open cover of $a{\uparrow}$ with no finite subcover, and let $a \in a{\uparrow} \setminus (b_1{\uparrow} \cup \ldots \cup b_n{\uparrow}) \subseteq U \in \mathcal{U}$, where each $b_i > a$. But then, by maximality, we can get a finite subcover for each $b_i{\uparrow}$, and hence for $a{\uparrow}$, yielding a contradiction.

For (4), suppose that $\mathcal{T}'$ is a topological lattice topology. It is sufficient to prove that each $b{\uparrow}$ is clopen in $\mathcal{T}'$. If not, then fix b such that $b{\uparrow}$ is not clopen, but $d{\uparrow}$ is clopen for all $d < b$, and then fix $a < b$ such that there is no c with $a < c < b$. We then get a contradiction by applying Lemma 3.4.20 to $\mathcal{T}'$. ■

The following *tree orders* provide a class of such semilattices: Let I be any set and let A be any sub-tree of $I^{<\omega}$ which is well-founded (that is, $\neg\exists f \in I^{\omega} \forall n (f{\restriction}n \in A)$). A has the usual tree order, with the empty sequence at the bottom. Of course, a semilattice with no infinite chains need not be a tree. Still, by Lemma 3.4.21, its intrinsic compact topology is uniquely determined by the order in a fairly simple way. The next lemma describes how such a semilattice looks when embedded in a larger compact semilattice.

LEMMA 3.4.22. *Let $(A;\wedge)$ be a sub-semilattice of the compact lattice $(X;\wedge)$, with A dense in X. Assume that A has no infinite chains. Then $b{\uparrow} = \{x \in X : b \le x\}$ is clopen in X for each $b \in A$.*

Proof. Since A has no infinite chains, it must have a 0, which is then the least element of X as well (since $\{x \in X : x \wedge 0 = 0\}$ is closed and contains A). Then $0{\uparrow} = X$ is clopen in X. Thus, if the lemma fails, then we can fix $a, b \in A$ such that $a < b$, $a{\uparrow}$ is clopen in X, $b{\uparrow}$ is not clopen in X, and there is no $c \in A$ with $a < c < b$. Then $a{\uparrow} \setminus b{\uparrow}$ fails to be closed, so there is an $x \in b\uparrow$ such that $x \in \mathrm{cl}(a\uparrow \setminus b\uparrow)$. Since $a\uparrow \setminus b\uparrow$ is open in X and A is dense, we have $x \in \mathrm{cl}((a\uparrow \setminus b\uparrow) \cap A)$. Then, as in the proof of Lemma 3.4.20, we have $b = b \wedge x \in \mathrm{cl}(b \wedge ((a\uparrow \setminus b\uparrow) \cap A)) = \mathrm{cl}\{a\}$. ■

THEOREM 3.4.23. *For a semilattice*, $(A;\wedge)$, *the following are equivalent*:

(1) *A has no infinite chains.*

(2) *A has a compact topology*, $\mathcal{T}$, *such that* $(A;\wedge,\mathcal{T})$ *is self-Bohrifying.*

Proof. (2)⇒(1). If C is a chain, then so is $\overline{C}$. If C is infinite, then $\overline{C}$ cannot be discrete in the topology $\mathcal{T}$, but it is relatively discrete in bA by Lemma 3.4.8.

(1)⇒(2). First we show that $A = b_0A$. We may identify $A \subseteq X = b_0A$, and we need to prove that $A = X$. Let $\mathcal{T}$ be the (unique) compact semilattice topology on A, let $\mathcal{T}'$ be the topology on X, and let $\mathcal{T}'\restriction A$ be the subspace topology $\mathcal{T}'$ induces on A. By Lemma 3.4.22, $\mathcal{T} \subseteq \mathcal{T}'\restriction A$. It is enough to prove that $\mathcal{T} = \mathcal{T}'\restriction A$, since then A will be compact in $\mathcal{T}'$ and hence $A = X$, because A is dense in X. Since $\mathcal{T}'$ is 0-dimensional, it is enough to fix a $\mathcal{T}'$-clopen $H \subseteq X$ and prove that $H \cap A$ is $\mathcal{T}$-closed. If not, fix $a \in A \setminus H$ such that a is a $\mathcal{T}$-limit point of $H \cap a$. Since $a\uparrow$ is clopen in both topologies, we may assume that $H \subseteq a\uparrow$. Let M be the set of minimal elements of $H \cap A$. Then M is infinite: otherwise, $\bigcup_{m\in M}(m\uparrow \cap A)$ would be a $\mathcal{T}$-clopen set containing $H \cap A$ but not a. Also, $m \wedge n \notin H$ whenever m, n are distinct elements of M. But this contradicts H being $\mathcal{T}'$-clopen in X: if $x \in X$ is a $\mathcal{T}'$-limit point of M, then $x \in H$ (since H is closed); but also x would be a limit of $\{m \wedge n : m, n \in M \ \&\ m \ne n\} \subseteq X \setminus H$, so $x \notin H$ (since $X \setminus H$ is closed).

Now we show that $A = bA$, so we identify $A \subseteq Y = bA$, and we need to prove that $A = Y$. In the following, all topological notions refer to the compact topology on Y. Applying Lemma 2.3.16, we know that each connected component of Y contains precisely one element of A. So, it is enough to fix $a \in A$, let K be the component of a in Y, and show that $K = \{a\}$. Since A has no infinite chains, we may assume that for all $b \in A$ with $b > a$, the component of b in Y is $\{b\}$, so that $a\uparrow \setminus A \subseteq K$. Note that $K \subseteq a\uparrow$, since $a\uparrow$ is clopen. Also, if $a < y < x$ and $x \in K$, then $y \in K$; otherwise, $y \in A$ and $y\uparrow$ would be a clopen set disconnecting K.

Now, assume $K \ne \{a\}$, and we derive a contradiction. Let K_0 be the set of $x \in K \setminus \{a\}$ such that $x \le b$ for some $b \in A$. $K_0 \ne \emptyset$: If $K_0 = \emptyset$, then $b \wedge x = a$ for all $b \in a\uparrow \cap A$ and all $x \in K \setminus \{a\}$. Fix $x \in K \setminus \{a\}$. Since $x \in \mathrm{cl}(a\uparrow \cap A)$, we have $x = x \wedge x \in \mathrm{cl}((a\uparrow \cap A) \wedge x) = \mathrm{cl}(\{a\})$, which is impossible.

For $y \in K_0$, let $y^+ \in A$ be the (unique) minimal element of $A \cap y\uparrow$. Note that if $a < x \le y \in K_0$, then $x \in K_0$ and $x^+ \le y^+$. Since A has no infinite chains, we can fix $y \in K_0$ so that whenever $a < x \le y \in K_0$, we have $x^+ = y^+$. Since A is dense in Y, we have $y \in \overline{P}$, where $P = A \cap (a\uparrow \setminus y^+\uparrow)$.

For $p \in P$, $a \le p \wedge y \le y$; but then $a = p \wedge y$; if not then $(p \wedge y)^+ = y^+$, so $p \in y^+\uparrow$. But then $P \wedge y = \{a\}$, and $y = y \wedge y \in \overline{P \wedge y} = \overline{\{a\}}$, which is impossible. ∎

The following lemma adds a few more remarks on semilattices with no infinite chains.

LEMMA 3.4.24. *Let $(A; \wedge)$ be a compact semilattice with no infinite chains. Then every sub-semilattice is closed. Hence, the closure of every countable set is countable, so that A is scattered. Furthermore, A has countable tightness.*

Proof. Let S be a sub-semilattice, and suppose $a \in \overline{S}$. Since there are no infinite chains, there is a least element, b, of $S \cap a\uparrow$. But then $a = b \in S$; otherwise, $a\uparrow \setminus b\uparrow$ would be a neighborhood of a disjoint from S.

Let $E \subseteq A$ and $p \in \overline{E}$. Let S be a countable sub-semilattice such that $p \in S$ and for all $n \in \omega$ and all $b_1, \ldots, b_n \in S$ with each $b_i > a$, we have $S \cap E \cap a\uparrow \setminus (b_1\uparrow \cup \ldots \cup b_n\uparrow) \neq \emptyset$. Then $p \in \overline{S \cap E}$. ∎

Note that the tree orders provide a class of examples which can be constructed to have arbitrary Cantor–Bendixson rank. Actually, Lemma 3.4.24 can be proved directly from more general facts. A has countable tightness because the topology is the same as $(A; \wedge)^{\#}$, which has countable tightness by Corollary 2.10.15, and to prove that sub-semilattices are closed, one could apply Theorem 3.4.26 below, since $bA = A$.

The following lemma says more about $a\uparrow$ and $a\downarrow$ in $b\mathfrak{A}$.

LEMMA 3.4.25. *Let $\mathfrak{A}$ be any semilattice, let X denote either $b\mathfrak{A}$ or $b_0\mathfrak{A}$, and identify $\mathfrak{A}$ as a sub-semilattice of X. For $a \in A$, we use $a\uparrow$ and $a\downarrow$ as computed in X. Then for each $a \in A$:*

(1) *$a\uparrow$ is clopen and $a\downarrow$ is closed in X.*
(2) *$A \cap a\uparrow$ is dense in $a\uparrow$ and $A \cap a\downarrow$ is dense in $a\downarrow$.*

Proof. For (1), we need only show that $a\uparrow$ is clopen. By maximality of X, it dominates γ_a, so there is a continuous homomorphism $\Gamma : X \to \mathbf{2}$ such that $\Gamma(b) = 1$ iff $b \ge a$, for each $b \in A$. Then $K = \{x : \Gamma(x) = 1\}$ is clopen; we show that $K = a\uparrow$. Now $x \ge a \Rightarrow x \in K$ because Γ is a homomorphism. Furthermore, $A \cap K$ is dense in K and $\forall b \in A \cap K [b \ge a]$, so $\forall x \in K [x \ge a]$.

For (2), $A \cap a\uparrow$ is dense because $a\uparrow$ is clopen. If $x \in a\downarrow$, then there is a net $\langle c_\alpha : \alpha \in D\rangle$ from A converging to x, and then $\langle c_\alpha \wedge a : \alpha \in D\rangle$ is a net from $A \cap a\downarrow$ converging to $x \wedge a = x$. ∎

THEOREM 3.4.26. *Let S be a sub-semilattice of A.*

(1) *S is closed in $A^{\#_0}$, and hence in $A^{\#}$.*
(2) *$b_0 S$ is the closure of S in $b_0 A$.*
(3) *bS is the closure of S in bA.*

Proof. For (1), suppose $a \notin S$ is a limit point of S. Since $a\uparrow$ is clopen, it is a limit point of $S \cap a\uparrow$. But that is impossible, since there is a homomorphism from A to $\mathbf{2}$ which takes $(S \cap a\uparrow)\uparrow$ to 1 and everything else (including a) to 0.

For (2), it is sufficient to show that every $\varphi \in \mathrm{Hom}(S, \mathbf{2})$ extends to some $\psi \in \mathrm{Hom}(A, \mathbf{2})$. So, let $\psi(a) = 1$ iff $a \geq s$ for some $s \in S$ with $\varphi(s) = 1$.

For (3), we apply Lemma 2.7.1, with $\mathcal{K}$ the class of all compact semilattices. Let $\varphi : S \to X$. Let X^* be the compact semilattice (in the Vietoris topology) of all closed non-empty $E \subseteq X$ such that $E = E\downarrow$. Define $\varphi^* : S \to X^*$ by $\varphi^*(s) = \varphi(s)\downarrow$. Then φ^* is equivalent to φ. We can extend φ^* to $\psi : A \to X^*$ by defining $\psi(a) = (\mathrm{cl}(\varphi(a\downarrow \cap S)))\downarrow$. ∎

Such use of the hyperspace is a standard trick in this subject; see, e.g., Theorem 1.2 of [22]. Note that in (3), we cannot always extend φ itself to some $\psi : A \to X$. For a counter-example, consider $S = X = \{0, a, b\}$, where a, b are incomparable, and let $A = \{0, a, b, 1\}$.

COROLLARY 3.4.27. *For semilattices*, $b(A\times B) = bA\times bB$ *and* $b_0(A\times B) = b_0A \times b_0B$.

Proof. If A and B both have a 1, this is immediate from Lemma 2.9.3 (the same proof works for b_0). If not, simply extend them to semilattices with a 1 by adding a new element on top, and apply Theorem 3.4.26. ∎

3.5. Distributive lattices. It is easy to find lattices $(A; \vee, \wedge)$ whose Bohr topology is indiscrete. For example, we may let $A = \{0, 1\} \cup \{a_n : n \in \omega\}$, where 0 is the smallest element, 1 is the largest element, and the a_i are incomparable. Suppose φ were a non-constant lattice homomorphism into a compact lattice. Then $\varphi(0) \neq \varphi(1)$, whence the $\varphi(a_n)$ must all be distinct, since $\varphi(a_m) = \varphi(a_n)$ for any $m \neq n$ would imply that $\varphi(1) = \varphi(a_m) \vee \varphi(a_n) = \varphi(a_m) = \varphi(a_m) \wedge \varphi(a_n) = \varphi(0)$. But then, by applying Lemma 3.4.5(3) (both with $\wedge$ and $\vee$), $\{\varphi(a_n) : n \in \omega\}$ has both $\varphi(0)$ and $\varphi(1)$ as its unique limit point, so that $\varphi(0) = \varphi(1)$, a contradiction.

However, for distributive lattices, we obtain a reasonable theory of $b\mathfrak{A}$ which closely parallels the theory for semilattices. Unfortunately, we know of no way of deriving one theory directly from the other, and in fact the parallel is not exact; for example, the analog of Theorem 3.4.26.3 turns out to be false (see Example 3.5.20). So, we shall just prove a sequence of results paralleling those of Section 3.4, abbreviating the proofs when they are similar. In particular, we obtain a simple description of $b_0\mathfrak{A}$, which, as for semilattices, is obtained by homomorphisms into $\mathbf{2}$ (which now denotes the two-element *lattice*). These homomorphisms separate points, so that $\mathfrak{A}^\#$ is always Hausdorff. In some cases, such as with total orders and boolean algebras, $b_0\mathfrak{A} = b\mathfrak{A}$, but this is not true in general, as we shall see using lattices which fail to have the Lawson property:

DEFINITION 3.5.1. A compact distributive lattice $(X; \vee, \wedge)$ has the *Lawson property* iff the lattice $[0, 1]$ is adequate for X.

As with semilattices, every compact distributive lattice without the Lawson property (these exist by Lawson [22]) yields a discrete distributive lattice $\mathfrak{A}$ for which $b_0\mathfrak{A} < b\mathfrak{A}$ (see Lemma 3.5.9).

For discrete distributive lattices, there is no analog to the simply defined homomorphisms γ_b of Definition 3.4.6, but lattice filters do give rise to point separating homomorphisms into $\mathbf{2}$. Ideals and filters are defined as usual; so, $\mathcal{I} \subseteq A$ is an *ideal* iff $\forall x, y \in \mathcal{I}\, [x \vee y \in \mathcal{I}]$ and $\forall x \in \mathcal{I}\, \forall y \in A [x \wedge y \in \mathcal{I}]$; *filter* is the dual notion. Note that $\emptyset$ and A are both ideals and filters in A.

DEFINITION 3.5.2. If $(A; \vee, \wedge)$ is a lattice, call $(\mathcal{I}, \mathcal{F})$ an *ideal-filter pair* of $(A; \vee, \wedge)$ iff $\mathcal{I}$ is an ideal, $\mathcal{F}$ is a filter, and $\mathcal{I} \cap \mathcal{F} = \emptyset$.

By Zorn's lemma, every ideal-filter pair $(\mathcal{I}_0, \mathcal{F}_0)$ in $(A; \vee, \wedge)$ can be extended to a maximal pair $(\mathcal{I}, \mathcal{F})$. If A is distributive, then $\mathcal{I} \cup \mathcal{F} = A$.

DEFINITION 3.5.3. If $(\mathcal{I}, \mathcal{F})$ is a maximal ideal-filter pair in the distributive lattice $(A; \vee, \wedge)$, then $\delta_{\mathcal{I},\mathcal{F}} : A \to \mathbf{2}$ is defined to be 0 on $\mathcal{I}$ and 1 on $\mathcal{F}$.

LEMMA 3.5.4. *$\delta_{\mathcal{I},\mathcal{F}}$ is a homomorphism.*

Note that allowing ideals and filters to be empty is consistent with the fact that constant maps into $\mathbf{2}$ are homomorphisms by our definition.

LEMMA 3.5.5. *If $(A; \vee, \wedge)$ is a distributive lattice, and $a, b \in A$ with $a \neq b$, then $\varphi(a) \neq \varphi(b)$ for some homomorphism $\varphi : A \to \mathbf{2}$.*

Proof. Assume $a \not\geq b$. Then $(a{\downarrow}, b{\uparrow})$ is an ideal-filter pair, so let $(\mathcal{I}, \mathcal{F})$ be a maximal ideal-filter pair with $a{\downarrow} \subseteq \mathcal{I}$ and $b{\uparrow} \subseteq \mathcal{F}$. Then $\delta_{\mathcal{I},\mathcal{F}}$ separates a and b. ∎

These homomorphisms enable us to prove, in analogy with Lemma 3.4.8:

LEMMA 3.5.6. *If $\mathfrak{A} = (A; \vee, \wedge)$ is a discrete distributive lattice, then $\mathfrak{A}^{\#}$ is Hausdorff. If $C \subseteq A$ is a chain, then C is relatively discrete in $\mathfrak{A}^{\#}$.*

Proof. $\mathfrak{A}^{\#}$ is Hausdorff by Lemma 3.5.5.

Let C be a chain, and fix $b \in C$. Let $\mathcal{I}_0 = b{\downarrow}$ and let $\mathcal{F}_0 = \bigcup\{c{\uparrow} : c \in C \ \&\ b < c\}$. Let $(\mathcal{I}, \mathcal{F})$ be a maximal ideal-filter pair with $\mathcal{I}_0 \subseteq \mathcal{I}$ and $\mathcal{F}_0 \subseteq \mathcal{F}$. Then $\delta_{\mathcal{I},\mathcal{F}}^{-1}\{0\} \cap C = \{a \in C : a \leq b\}$ is relatively clopen in C. Likewise, $\{a \in C : a \geq b\}$ is relatively clopen in C, so b is isolated in C. ∎

Since for total orders, semilattice and lattice homomorphisms are the same, Lemma 3.4.9 and its corollaries are essentially unchanged:

LEMMA 3.5.7. *Suppose that* $(A; \vee, \wedge)$ *is a distributive lattice and* $\varphi : A \to X$ *is some compactification of* A, *where* $(X; \vee, \wedge)$ *is a compact lattice which is totally ordered. Then* $\varphi \le b_0(A; \vee, \wedge)$.

COROLLARY 3.5.8. $b_0\mathfrak{A} = \bigvee(\mathrm{Hom}(\mathfrak{A}, \mathbf{2})) = \bigvee(\mathrm{Hom}(\mathfrak{A}, [0,1]))$ *for each distributive lattice* $\mathfrak{A}$. *If* $b\mathfrak{A}$ *has the property that the continuous lattice homomorphisms into* $[0,1]$ *separate the points of* $\mathfrak{A}$, *then* $b\mathfrak{A} = b_0\mathfrak{A}$, *and is hence* 0-*dimensional.*

Note that identifying $b_0\mathfrak{A}$ with $\bigvee(\mathrm{Hom}(\mathfrak{A}, \mathbf{2}))$ here requires the fact (see [28], [29]) that the *continuous* homomorphisms from a 0-dimensional compact distributive lattice into **2** separate points.

Analogously to Lemma 3.4.12 and its corollary, we have:

LEMMA 3.5.9. *If* $(A; \vee, \wedge)$ *is a discrete distributive lattice and* $b(A; \vee, \wedge) = b_0(A; \vee, \wedge)$, *then every compactification of* $(A; \vee, \wedge)$ *has the Lawson property.*

Proof. This follows (as in Lemma 3.4.12) from the fact that the Lawson property is preserved under continuous homomorphic images. To see this, use the fact (see Strauss [32] and Theorem 6 of Lawson [23]) that the Lawson property is equivalent to complete distributivity, which is preserved by continuous homomorphisms by Lemma 3.4.5(6). ∎

COROLLARY 3.5.10. *There is a countable discrete distributive lattice* $(A; \vee, \wedge)$ *such that* $b_0(A; \vee, \wedge) < b(A; \vee, \wedge)$.

For some distributive lattices $\mathfrak{A}$, we do have $b(A; \vee, \wedge) = b_0(A; \vee, \wedge)$, as in the lattice version of Lemma 3.4.15:

LEMMA 3.5.11. *Let* $(A; \vee, \wedge)$ *be a discrete distributive lattice. If* $(A; \wedge)$ *has breadth* $\le n$, *where* n *is finite*, *then* $b(A; \vee, \wedge) = b_0(A; \vee, \wedge)$.

Note that $(A; \wedge)$ has breadth $\le n$ iff $(A; \vee)$ has breadth $\le n$; this is true because in terms of the order, both are equivalent to the non-existence of $x_0, \ldots, x_n, y_0, \ldots, y_n$ such that $x_i \le y_j$ for $i \ne j$, but each $x_i \not\le y_i$. To prove Lemma 3.5.11, we apply the fact (see [23]) that compact distributive lattices with finite breadth have the Lawson property.

Analogously to Theorem 3.4.23, we may characterize the self-Bohrifying distributive lattices. Unfortunately, this characterization reduces to:

THEOREM 3.5.12. *A compact distributive lattice is self-Bohrifying iff it is finite.*

Proof. It is sufficient to show that if A is a distributive lattice with no infinite chains, then A must be finite. Let $M(x) = \{y > x : \neg\exists z(y > z > x)\}$. Each $M(x)$ is finite, since otherwise, by distributivity, we would have an infinite chain of the form $y_0 < y_0 \vee y_1 < y_0 \vee y_1 \vee y_2 < \ldots$, where the y_n are distinct elements of $M(x)$. Also, note that the lack of infinite chains implies that $x\uparrow = \{x\} \cup \bigcup\{y\uparrow : y \in M(x)\}$. But then, if A were infinite, we would inductively construct an infinite chain, $0 = x_0 < x_1 < x_2 < \ldots$, where each $x_n\uparrow$ is infinite and $x_{n+1} \in M(x_n)$. ∎

The computation of $b_0\mathfrak{A}$ for semilattices also works for distributive lattices. In fact, it becomes somewhat simpler, since now we can use an arbitrary representation of the lattice in a clopen algebra (see [28], [29]):

LEMMA 3.5.13. *Every distributive lattice is isomorphic to a sub-lattice of the clopen sets of some compact Hausdorff space.*

THEOREM 3.5.14. *Suppose that* $(A; \vee, \wedge)$ *is a sub-lattice of the lattice of clopen subsets of the compact Hausdorff space* X. *Give* $\mathcal{P}(X)$ *the usual product topology by identifying it with* 2^X. *Let* Z *be the closure of* A *in* $\mathcal{P}(X)$. *Then* $Z = b_0(A; \vee, \wedge)$.

Proof. Since Z is a zero-dimensional compactification of A, it suffices to show that $Z \geq b_0(A; \vee, \wedge)$. By Corollary 3.5.8, this will follow if we can show that each lattice homomorphism $\varphi : A \to \mathbf{2}$ extends to a continuous homomorphism on $\mathcal{P}(X)$, and hence on Z. Let $\mathcal{I}_0 = \{H \in Z : \varphi(H) = 0\}$ and $\mathcal{F}_0 = \{K \in Z : \varphi(K) = 1\}$. If $H \in \mathcal{I}_0$ and $K \in \mathcal{F}_0$, then $K \not\subseteq H$. Since elements of Z are clopen subsets of X, it follows, by compactness, that we may fix a point $x \in (\bigcap \mathcal{F}_0) \setminus (\bigcup \mathcal{I}_0)$. Let $(\mathcal{I}, \mathcal{F})$ be the principal ideal-filter pair in $\mathcal{P}(X)$ generated by x; that is, $\mathcal{I} = \{H \in \mathcal{P}(X) : x \notin H\}$ and $\mathcal{F} = \{K \in \mathcal{P}(X) : x \in K\}$. Then $\delta_{\mathcal{I},\mathcal{F}} : 2^X \to \mathbf{2}$ is a continuous homomorphism extending φ. ∎

In particular, we can apply this to boolean algebras. As defined in Section 2.1, these are structures of the form $(B; \vee, \wedge, ', 0, 1)$. However, one can consider them simply as lattices:

LEMMA 3.5.15. $b(B; \vee, \wedge, ', 0, 1) = b(B; \vee, \wedge)$ *whenever* $(B; \vee, \wedge, ', 0, 1)$ *is a discrete boolean algebra.*

Proof. By Lemma 2.8.3, one can always drop the constants $0, 1$. To drop the $'$, apply Theorem 2.8.5: One can define x' by $x' = y \Leftrightarrow \phi(x, y)$, where $\phi(x, y)$ is the formula $x \wedge y = 0$ & $x \vee y = 1$. Furthermore, the assertion $\forall x \exists! y \phi(x, y)$ is provable from positive logical sentences true in $(B; \vee, \wedge, 0, 1)$— namely, $\forall x \exists y \phi(x, y)$ and the axioms for distributive lattices. ∎

Theorem 3.5.19 below expands on this lemma. First, we identify $b\mathfrak{B}$ explicitly (Theorem 3.5.18).

THEOREM 3.5.16 (Strauss [32]). *Every compact boolean algebra is continuously isomorphic to* $\{0, 1\}^\kappa$ *for some* κ.

Corollary 3.5.17. *The two-element algebra is adequate for every boolean algebra.*

Then, applying Theorem 3.5.14, we get:

Theorem 3.5.18. *Let $\mathfrak{B}$ be the clopen algebra of the compact 0-dimensional Hausdorff space X. Then $b\mathfrak{B} = b_0\mathfrak{B} = \mathcal{P}(X)$, where we identify $\mathcal{P}(X)$ with $\{0,1\}^X$.*

Now, if $\mathfrak{B} = (B; +, \vee, \wedge, ')$ is a boolean algebra (where $+$ is symmetric difference), we obtain potentially 16 different Bohr compactifications by reducting the language to various subsets of $\{+, \vee, \wedge, '\}$. However, only 5 of these are distinct, by the following theorem. Note that constants, such as $0, 1$, are irrelevant for computing $b\mathfrak{B}$ (by Lemma 2.8.3).

Theorem 3.5.19. *If $\mathfrak{B} = (B; +, \vee, \wedge, ')$ is an infinite boolean algebra, and $\mathcal{L} \subseteq \{+, \vee, \wedge, '\}$, then*

(1) *If $\mathcal{L}$ is $\emptyset$ or $\{'\}$, then $b(\mathfrak{B}{\restriction}\mathcal{L}) = \beta B$.*
(2) *If $\{\vee, \wedge\} \subseteq \mathcal{L}$, then $b(\mathfrak{B}{\restriction}\mathcal{L}) = b\mathfrak{B}$.*
(3) *If $\{\vee, \wedge\} \cap \mathcal{L} \neq \emptyset$ and $\{+, '\} \cap \mathcal{L} \neq \emptyset$, then $b(\mathfrak{B}{\restriction}\mathcal{L}) = b\mathfrak{B}$.*
(4) *The compactifications, $b(B; \wedge)$, $b(B; \vee)$, and $b(B; +) = b(B; +, ')$, are all incomparable with each other, and hence lie strictly between $b\mathfrak{B}$ and βB.*

Proof. (1) just uses the fact that $'$ is unary, and (2) is immediate from Lemma 3.5.15. For (3), just use the fact that either of $\{\vee, \wedge\}$ together with 1 and either of $\{+, '\}$ is sufficient to express every propositional connective.

To prove (4), we first show that $b(B; +) \not\leq b(B; \wedge)$. Since B is infinite, there is a strictly decreasing ω-sequence of elements, $b_0 > b_1 > \ldots$ Then the b_n are independent as elements of the abelian group $(B; +)$, so the closure of $\{b_n : n \in \omega\}$ in $b(B; +)$ is homeomorphic to $\beta\mathbb{N}$ (see [11], [21]). However, by Theorem 3.4.17, the closure of $\{b_n : n \in \omega\}$ in $b(B; \wedge)$ is homeomorphic to $\omega + 1$; if we embed B into the clopen subsets of X, then the unique limit point of $\{b_n : n \in \omega\}$ is $\bigcap_n b_n \in \mathcal{P}(X)$. The same argument shows that $b(B; +) \not\leq b(B; \vee)$.

To show that $b(B; \vee) \not\leq b(B; +)$, we show that the Bohr topology, $(B; +)^\#$, is not finer than $(B; \wedge)^\#$. If b_n (for $n \in \omega$) are distinct elements of B, then they have some limit point $z \in b(B; +)$, so that $0 = z + z$ is a limit point of $\{b_n + b_k : k < n < \omega\}$ in $(B; +)^\#$. However, 0 cannot be a limit point of any set in $(B, \vee)^\#$, since $0{\downarrow} = \{0\}$ is clopen by Lemma 3.4.25(1) (replacing the $\wedge$ there with $\vee$). Likewise, using $1 + b_n + b_k$, with limit 1, we see that $b(B; \wedge) \not\leq b(B; +)$.

Finally, we show that $b(B; \wedge)$ and $b(B; \vee)$ are not comparable. Let b_n, for $n \in \omega$, be pairwise disjoint. Then in $(B; \wedge)^\#$, the sequence $\langle x_n : n \in \omega \rangle$ converges to 0 by Lemma 3.4.5(3), whereas, by Lemma 3.4.25(1), $\{0\}$ is clopen in $(B; \vee)^{\#}$ (note the $\uparrow$ and $\downarrow$ are reversed). Hence, $b(B; \vee) \not\leq b(B; \wedge)$, and the dual proof shows $b(B; \wedge) \not\leq b(B; \vee)$. ∎

EXAMPLE 3.5.20. Suppose that the boolean algebra B contains $\{d_n : n \in \omega\}$ which are independent in the boolean algebra sense; this is possible iff the Stone space of B is not scattered. Then the closure of $\{d_n : n \in \omega\}$ in $b(B; \vee, \wedge)$ (and hence also in $b(B; \vee)$ and $b(B; \wedge)$) is homeomorphic to $\beta\mathbb{N}$. Note that $b(B; \vee, \wedge)$ is 0-dimensional, while $b(B; \wedge)$ and $b(B; \vee)$ are not; to see this, let $(A; \wedge)$ be a countable discrete semilattice with $b_0(A; \wedge) < b(A; \wedge)$; by virtue of the d_n, we may assume $(A; \wedge) \subseteq (B; \wedge)$; then $b(A; \wedge)$ is not 0-dimensional, and is a closed subset of $b(B; \wedge)$ (by Theorem 3.4.26(3)). Note that if $(S; \vee, \wedge)$ is a countable distributive lattice with $b_0(S; \vee, \wedge) < b(S; \vee, \wedge)$, we may again assume that $(S; \vee, \wedge) \subseteq (B; \vee, \wedge)$, and again $b(S; \vee, \wedge)$ is not 0-dimensional. Now, since $b(B; \vee, \wedge)$ *is* 0-dimensional, we see that the analog of Theorem 3.4.26(3), that bS is the closure of S in bA, is false for distributive lattices.

We do have the analog of the rest of Theorem 3.4.26:

THEOREM 3.5.21. *Let S be a sub-lattice of the distributive lattice* $(A; \vee, \wedge)$.

(1) *S is closed in $A^{\#_0}$, and hence in $A^{\#}$.*

(2) *$b_0 S$ is the closure of S in $b_0 A$.*

Proof. For (1), fix any $a \in A \setminus S$. To show that a is not in the closure of S, it is sufficient to produce $\varphi, \psi \in \mathrm{Hom}(A, \mathbf{2})$ such that $\varphi(a) = 1$, $\psi(a) = 0$, and for all $s \in S$ we have either $\varphi(s) = 0$ or $\psi(s) = 1$. To do this, first choose φ such that $\varphi(a) = 1$ and $\varphi(s) = 0$ for all $s \in a{\downarrow} \cap S$; this is possible because the filter $a{\uparrow}$ is disjoint from the ideal $(a{\downarrow} \cap S){\downarrow}$. Then choose ψ such that $\psi(a) = 0$ and $\psi(s) = 1$ for all $s \in S$ such that $\varphi(s) = 1$; this is possible because the ideal $a{\downarrow}$ is disjoint from the filter $\{s \in S : \varphi(s) = 1\}{\uparrow}$.

For (2), it is sufficient to show that every $\varphi \in \mathrm{Hom}(S, \mathbf{2})$ extends to some $\psi \in \mathrm{Hom}(A, \mathbf{2})$. But this is trivial, by Zorn's Lemma. ∎

We also have the special case of 3.4.26(3) needed to prove the analog of Corollary 3.4.27, and in fact we do not even need distributivity here:

THEOREM 3.5.22. *For lattices*, $b(A \times B) = bA \times bB$ *and* $b_0(A \times B) = b_0 A \times b_0 B$.

Proof. Let $\widehat{A}$ be A if A has a 1, and let $\widehat{A}$ be A with a 1 added otherwise. Likewise, define $\widehat{B}$. Note that A is algebraically closed (Definition 2.7.2) in $\widehat{A}$: This is trivial unless $\widehat{A} \neq A$. Consider a system of equations over A with a solution in $\widehat{A}$. Since $\widehat{A} \setminus A = \{1\}$, we may assume our system is of the form $\sigma(x)$, in just one variable x, and that $\sigma(1)$ is true in $\widehat{A}$. Say $\sigma(x)$ mentions elements $a_1, \ldots, a_n \in A$ as constants. Fix $b \in A$ with $b > a_1 \vee \ldots \vee a_n$. Then it is easy to see that $\sigma(b)$ holds in $\mathfrak{A}$.

Likewise, B is algebraically closed in $\widehat{B}$. It follows that $A \times B$ is algebraically closed in $\widehat{A} \times \widehat{B}$. Now use Theorem 2.7.3; note that this theorem applies also to b_0, with the same proof. ∎

Observe that for semilattices, one could not claim that A is algebraically closed in $\widehat{A}$, since the equations $\{x \wedge a = a\ ,\ x \wedge b = b\}$ must have a solution in $\widehat{A}$, but need not have a solution in A.

3.6. Counter-examples. We collect here illustrations of some phenomena for which we were not able to find examples among the more well known classes of structures.

EXAMPLE 3.6.1. *Let* $\mathfrak{A} = (\omega+1; \vee, g)$, *where* $\vee$ *is the usual* max *operation on* $\omega+1$, *and* $g(m)$ *is* m *if* m *is finite and odd*, *and* ω *otherwise. Give* $\omega+1$ *the usual ordinal topology. Then* $\mathfrak{A}$ *is self-Bohrifying* (*see Definition* 2.3.14).

Proof. By Lemma 3.4.18, we know that $b(\omega+1; \vee) = \omega+1 \cup \{d\}$, where $m < d < \omega$ for all $m \in \omega$. Thus, the only possibility for $b\mathfrak{A}$, other than $\mathfrak{A}$ itself, is this $\omega+1 \cup \{d\}$. However, $\omega+1 \cup \{d\}$ is not possible here, as one cannot extend g continuously to it: since $\{x : g(x) = x\}$ is closed, we would have $g(d) = d$, but since $\{x : g(x) = \omega\}$ is closed, we would have $g(d) = \omega$. ∎

Now, we can make a similar construction with almost disjoint families. If P, Q are any sets, we say $P \perp Q$ iff $P \cap Q$ is finite, and $P \subseteq^* Q$ iff $P \setminus Q$ is finite. For us, an *almost disjoint family* will be a non-empty family $\mathcal{F}$ of countably infinite subsets of some index set I such that $P \perp Q$ whenever P, Q are distinct elements of $\mathcal{F}$. Note that I itself need not be countable.

DEFINITION 3.6.2. If $\mathcal{F} \subseteq \mathcal{P}(I)$ is an almost disjoint family, its *induced topology* on $I \cup \{\infty\}$ is defined by letting U be open iff either $\infty \notin U$ or $P \subseteq^* U$ for all $P \in I$.

Note that I is always open and discrete. If $\mathcal{F}$ is a maximal almost disjoint family (which implies that $\mathcal{F}$ is either finite or uncountable), then the induced topology is just the 1-point compactification. In particular, when $\mathcal{F} = \{I\}$, the construction described below reduces to the construction in Example 3.6.1. If $\mathcal{F}$ is not maximal, then the induced topology is not compact, but we can always get it to be contained in a Bohr topology:

EXAMPLE 3.6.3. *If* $\mathcal{F} \subseteq \mathcal{P}(I)$ *is an almost disjoint family*, *then there is an* $\mathcal{L}$ *with* $|\mathcal{L}| \le \max(|\mathcal{F}|, \aleph_0)$ *and a structure* $\mathfrak{A}$ *built on* $A = I \cup \{\infty\}$ *such that the topology of* $\mathfrak{A}^\#$ *is the topology induced by* $\mathcal{F}$.

Proof. $\mathcal{L}$ now has symbols $\vee_P$ and g_P for each $P \in \mathcal{F}$. For each P, choose a bijection π_P from ω onto P. Let $\pi_P(m) \vee_P \pi_P(n) = \pi_P(\max(m,n))$, and let $x \vee_P y = \infty$ unless $x, y \in P$. Let $g_P(x) = \infty$ unless $x = \pi_P(m)$ for some odd m, in which case $g_P(x) = x$.

We first describe what we are claiming to be $b\mathfrak{A}$. Let $\mathcal{B} = \{E \in \mathcal{P}(I) : \forall P \in \mathcal{F}[P \subseteq^* E] \text{ OR } \forall P \in \mathcal{F}[P \perp E]\}$. $\mathcal{B}$ is a boolean algebra. $\mathcal{B}$ is the finite-cofinite algebra when $\mathcal{F}$ is maximal, but is larger otherwise. Let X be the Stone space of $\mathcal{B}$; so, elements of X are ultrafilters on $\mathcal{B}$. We identify A with a subset of X by identifying ∞ with the (unique) ultrafilter containing all E such that $\forall P \in \mathcal{F}[P \subseteq^* E]$, and by identifying each $i \in I$ with the principal ultrafilter generated by $\{i\}$. With this identification, the identity map φ defines a compatible compactification; that is, all the g_P and $\vee_P$ extend naturally to X.

Note that the topology, $\mathcal{T}_\varphi$, is just the topology induced by $\mathcal{F}$. So, we are done if we verify that this compactification is maximal. Consider any larger compactification. We may assume it is also an inclusion, $A \subset Y$, where Y is compact, A is dense in Y, and all the g_P and $\vee_P$ extend to Y. The statement $X \le Y$ is expressed by a map $\Gamma : Y \to X$ with Γ the identity on A. We must show that Γ is 1-1. Each $i \in I$ is isolated in X and hence in Y, so Γ is 1-1 on $I = \Gamma^{-1}(I)$. Also, if $E \subseteq I$ and $\forall P \in \mathcal{F}[P \perp E]$, then in X, $\overline{E} \cong \beta E$, so that Γ is 1-1 on $\Gamma^{-1}(\overline{E})$. Thus, the only possibility for Γ *not* to be 1-1 is that there is some $y \in Y$ with $\Gamma(y) = \infty$ but $y \neq \infty$. In Y, let $y \in U$ with U open and $\infty \notin \overline{U}$; if $E = U \cap I$, then $\mathrm{cl}_Y(E)$ contains y but not ∞, whereas $\infty = \Gamma(y) \in \mathrm{cl}_X(E)$. In Y, each $P \cup \{\infty\} \cong \omega + 1$ (by the argument of Example 3.6.1, applied to the structure $(P \cup \{\infty\}; \vee_P, g_P)$); hence $P \perp E$ for each $P \in \mathcal{F}$. But then $E \in \mathcal{B}$ and $\infty \notin \mathrm{cl}_X(E)$. ∎

When $\mathcal{F}$ is uncountable, this produces a Bohr topology on A which is strictly finer than what one obtains by reducting it to some countable sub-language. In fact, we have:

Example 3.6.4. *There is a countable structure $\mathfrak{A}$ for a language $\mathcal{L}$ such that the topology $\mathfrak{A}^\#$, viewed as a subset of $2^A \equiv \mathcal{P}(A)$, is not analytic. Hence $\mathfrak{A} \restriction \mathcal{L}_0^\#$ is strictly finer than $\mathfrak{A}^\#$ whenever $\mathcal{L}_0$ is a countable sub-language of $\mathcal{L}$.*

Proof. Just use Example 3.6.3 with a countable I and a suitable $\mathcal{F}$. Note that there are only $2^{\aleph_0}$ analytic sets, but, as $\mathcal{F}$ varies, one gets $2^{2^{\aleph_0}}$ different topologies. ∎

As in the proof of Lemma 2.10.17, whenever $\mathcal{L}$ is countable, the Bohr topology on any countable subset of A is analytic. Thus, one cannot in general replace an uncountable language by a countable one on a larger set by coding it into a larger structure. In the case of Example 3.6.3, one might try to code all the $\vee_P$ by one ternary function, taking A now to be $I \cup \{\infty\} \cup \mathcal{F}$, but then the requirement that this new function extend continuously as a function of $P \in \mathcal{F}$ will change the Bohr compactification. One can sometimes use coding to replace a countable language by a finite one, as in Example 3.6.7 below.

Example 3.6.7 is of interest also for the following reason. We have seen (Corollary 2.10.20) that if $\mathfrak{A}$ is a countable discrete structure for a countable language, then $w(b\mathfrak{A})$ is either $2^{\aleph_0}$ or countable. However, this cannot be proved of the cardinal functions $w(\mathfrak{A}^\#)$ and $\chi(\mathfrak{A}^\#)$. Following van Douwen, we use $\mathfrak{d}$ to denote the least cardinality of a dominating family in ω^ω. We shall produce an $\mathfrak{A}$ with $\mathcal{L}$ finite and $w(\mathfrak{A}^\#) = \chi(\mathfrak{A}^\#) = \mathfrak{d}$; as is well known, $\mathfrak{d} \geq \aleph_1$, and it is consistent to have $\mathfrak{d} < \mathfrak{c}$. The topology on our $\mathfrak{A}^\#$ is just one of the standard hedgehog topologies:

Definition 3.6.5. H is the *hedgehog*, $(\mathbb{Z} \times \omega) \cup \{\infty\}$. The *hedgehog topology*, $\mathcal{H}$, on H is obtained by declaring all points in $\mathbb{Z} \times \omega$ to be isolated, and giving ∞ a base consisting of the sets $N_f = \{\infty\} \cup \{(m,n) \in \mathbb{Z} \times \omega : n > f(m)\}$, for all $f \in \omega^{\mathbb{Z}}$.

H with this topology is often called the Fréchet–Urysohn fan. If $\mathcal{F}$ is the set of spines, $\{m\} \times \omega$, then our topology is just the one induced by $\mathcal{F}$ as in Definition 3.6.2.

Lemma 3.6.6. *$w(H,\mathcal{H}) = \chi(H,\mathcal{H}) = \mathfrak{d}$.*

Since $\mathcal{H}$ has such a simple description, one can cook up a structure which has $\mathcal{H}$ as its Bohr topology:

Example 3.6.7. *There is a finite language $\mathcal{L}$, and a countable discrete structure, $\mathfrak{H}$, built on H, such that $\mathfrak{H}^\#$ is exactly the topology $\mathcal{H}$. This $\mathfrak{H}$ is not nice.*

Proof. If we wanted a countable language, this would just be Example 3.6.3, taking $\mathcal{F}$ to be the set of spines. Let $\mathcal{B}$ be the boolean algebra constructed from $\mathcal{F}$ as in the proof of Example 3.6.3. Let $\mathcal{L} = \{R, L, g, \vee\}$, where R, L, g are unary and $\vee$ is binary. We shall interpret $\mathcal{L}$ so that $b\mathfrak{H}$ is the Stone space X of $\mathcal{B}$. We again identify H as a subset of X, so that $\varphi : H \to X$ is inclusion. We must use a little care here. If our structure encodes too much information, the Bohr topology could wind up to be the coarser metric topology. We shall use $g, \vee$ to ensure that spine 0 (i.e., $\{0\} \times \omega \cup \{\infty\}$) is indeed homeomorphic to $\omega + 1$. Then R, L will ensure that all the spines look alike.

In $\mathfrak{H}$, interpret $R : H \to H$ and $L : H \to H$ as right and left shifts: $R(m,n) = (m+1,n)$; $L(m,n) = (m-1,n)$; $R(\infty) = L(\infty) = \infty$. So, R and L are bijections, and they define automorphisms R^* and L^* of $\mathcal{B}$, and then by Stone duality, they define homeomorphisms, R_* and L_*, of X onto X. So, if $\mathcal{U} \in X$, then $R_*(\mathcal{U}) = \{b \in \mathcal{B} : R^{-1}(b) \in \mathcal{U}\}$ and $L_*(\mathcal{U}) = \{b \in \mathcal{B} : L^{-1}(b) \in \mathcal{U}\}$. R_*, L_* extend R, L, so that φ is indeed compatible with R, L.

In $\mathfrak{H}$, interpret $\vee$ to be the lattice operation on spine 0, and trivial elsewhere; that is, $(0,m) \vee (0,k) = (0, \max(m,k))$ and $x \vee y = \infty$ if either x or y fails to be in $\{0\} \times \omega$. Define $g(0,m)$ to be $(0,m)$ if m is odd and

∞ if m is even; $g(x) = \infty$ if $x \notin \{0\} \times \omega$. Then both $\vee$ and g extend to continuous functions on X. Note that this could have failed to be true if we tried to make these functions non-trivial on all the spines. It is now easy to verify, as in Example 3.6.3, that X is indeed $b\mathfrak{H}$.

Finally, we show that $\mathfrak{H}$ is not nice. If it were, then $\mathfrak{H}^{\#} = \mathcal{H}$ would be an Eberlein–Grothendieck space by Theorem 2.10.8. However, this would contradict Theorem 2.10.13 (take E_n to be nth spine, $\{n\} \times \omega$). ∎

It is not clear what cardinals besides $\mathfrak{d}$ can be of the form $w(\mathfrak{A}^{\#})$. One cannot choose an arbitrary cardinal between $\aleph_0$ and $\mathfrak{c}$, however. For example, suppose the universe is of the form $V[G]$, where $V \models \mathrm{CH}$ and G is a random real extension. Then whenever $\mathfrak{A}$ is a countable discrete structure for a countable language, $w(\mathfrak{A}) = \chi(\mathfrak{A})$ is either countable or $\mathfrak{c}$ or $\mathfrak{d} = \aleph_1$.

Finally, we point out that in Lemma 2.9.3, one cannot drop either the assumption that 0 is an idempotent or that 0 is an identity element.

EXAMPLE 3.6.8. *For any $\mathcal{L}$, there is an $\mathcal{L}$-structure $\mathfrak{A}$ in which every element is an idempotent, such that $b(\mathfrak{A}\times\mathfrak{A})$ is strictly greater than $b\mathfrak{A}\times b\mathfrak{A}$.*

Proof. Let A be infinite, and define $f_{\mathfrak{A}}(x_1, \ldots, x_n) = x_1$ for each function symbol f of $\mathcal{L}$. Then $b\mathfrak{A} = \beta A$ and $b(\mathfrak{A} \times \mathfrak{A}) = \beta(A \times A) > \beta A \times \beta A$. ∎

EXAMPLE 3.6.9. *There is an $\mathcal{L}$ containing a binary operation $\vee$, and two $\mathcal{L}$-structures $\mathfrak{A}$ and $\mathfrak{B}$, such that $\mathfrak{A}, \mathfrak{B}$ both contain an identity element with respect to $\vee$, but $b(\mathfrak{A} \times \mathfrak{B})$ is strictly greater than $b\mathfrak{A} \times b\mathfrak{B}$.*

Proof. Let $\mathcal{L} = \{\vee, f\}$, where $\vee$ is binary and f is unary. Let $A = B = \omega$, and let $\mathfrak{A}$ and $\mathfrak{B}$ both interpret $\vee$ as the usual max operation. Let $f_{\mathfrak{A}}(n)$ be 0 if n is odd and n if n is even, so that $b\mathfrak{A}$ is a singleton and $\mathfrak{A}^{\#}$ is indiscrete. Let $f_{\mathfrak{B}}(n) = n + 1$, so that $b\mathfrak{B} = \omega + 1$. Thus, in $\mathfrak{A}^{\#} \times \mathfrak{B}^{\#}$, the closure of $\{(0, 0)\}$ is $\omega \times \{0\}$. However, $(0, 0)$ is isolated in $(\mathfrak{A} \times \mathfrak{B})^{\#}$; to see this, consider $\varphi : \omega \times \omega \to \{0, 1\}$ where $\varphi(0, 0) = 0$ and other points map to 1; on $\{0, 1\}$, we interpret $\vee$ as the usual max operation and let $f(0) = f(1) = 1$. ∎

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