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On Whitney pairs

by

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Abstract. A simple arc ϕ is said to be a Whitney arc if there exists a non-constant function f such that

$$\lim_{x\to x_0} \frac{|f(x)-f(x_0)|}{|\phi(x)-\phi(x_0)|}=0$$

for every x_0. G. Petruska raised the question whether there exists a simple arc ϕ for which every subarc is a Whitney arc, but for which there is no parametrization satisfying

$$\lim_{t\to t_0} \frac{|t-t_0|}{|\phi(t)-\phi(t_0)|}=0.$$

We answer this question partially, and study the structural properties of possible monotone, strictly monotone and VBG_* functions f and associated Whitney arcs.

1. Introduction. A simple arc $\phi:[0,1]\to\mathbb{R}^n$ is said to be a *Whitney arc* if there exists a non-constant function f such that

$$(*)\qquad \lim_{x\to x_0} \frac{|f(x)-f(x_0)|}{|\phi(x)-\phi(x_0)|}=0$$

for every $x_0\in[0,1]$.

Several constructions of Whitney arcs are known ([5]). In particular, in [3] the authors construct a Whitney arc $\phi:[0,1]\to[0,1]^2$ such that

$$(**)\qquad \lim_{t\to t_0} \frac{|t-t_0|}{|\phi(t)-\phi(t_0)|}=0$$

for every $t_0\in[0,1]$. G. Petruska raised the question whether there exists a simple arc for which every subarc is a Whitney arc, but for which there is no parametrization satisfying $(**)$. We show that for every $n>2$ there exists a simple arc $\phi:[0,1]\to[0,1]^n$ with the required property, however,

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the planar case remains open. We also study the structural properties of possible Whitney arcs.

Definition 1. A simple arc $\phi : [0,1] \to \mathbb{R}^n$ and a non-constant function $f : [0,1] \to \mathbb{R}$ are said to form an (n-*dimensional*) *Whitney pair* (ϕ, f) if they satisfy $(*)$. That is, ϕ is a Whitney arc iff there exists an f for which (ϕ, f) is a Whitney pair. Analogously, we say that f is an (n-*dimensional*) *Whitney function* if there exists a $\phi : [0,1] \to \mathbb{R}^n$ for which (ϕ, f) is a Whitney pair.

It is immediate from the definition that any n-dimensional Whitney pair is an m-dimensional Whitney pair for every $m \geq n$. Therefore the existence of a simple arc $\phi : [0,1] \to \mathbb{R}^2$ satisfying $(**)$ means that $\mathrm{id}_{[0,1]}$ is an n-dimensional Whitney function for every $n \geq 2$. This implies immediately that every continuous function f is an n-dimensional Whitney function for $n \geq 3$. Indeed, let $(\phi_0, \mathrm{id}_{[0,1]})$ be a 2-dimensional Whitney pair, and for a given continuous function f define $\phi : [0,1] \to \mathbb{R}^3 \subset \mathbb{R}^n$ as follows:

$$\phi(x) = (\phi_0(f(x)), x) \in \mathbb{R}^{2+1},$$

where the third coordinate of the three-dimensional point $\phi(x)$ is indicated following the two-dimensional point $\phi_0(f(x))$. Now, ϕ is a simple arc, and $|\phi(x) - \phi(x_0)| \geq |\phi_0(f(x)) - \phi_0(f(x_0))|$. If f is constant in a neighbourhood of x_0 then $(*)$ obviously holds; in the other case we have

$$\begin{aligned}\lim_{x\to x_0} \frac{|f(x) - f(x_0)|}{|\phi(x) - \phi(x_0)|} &\leq \lim_{x\to x_0,\, f(x)\neq f(x_0)} \frac{|f(x) - f(x_0)|}{|\phi_0(f(x)) - \phi_0(f(x_0))|} \\ &= \lim_{t\to t_0 = f(x_0)} \frac{|t - t_0|}{|\phi_0(t) - \phi_0(t_0)|} = 0.\end{aligned}$$

The construction of injective arcs on the plane is much more difficult, and the characterization of the 2-dimensional Whitney functions is only partly solved. It is easy to see that if f is a Whitney function and $h : [0,1] \to [0,1]$ is a homeomorphism, then $f \circ h$ is a Whitney function as well (since the composition with h means only a re-parametrization of the curve ϕ). Thus, since $\mathrm{id}_{[0,1]}$ is a 2-dimensional Whitney function, every strictly monotone continuous function $f : [0,1] \to \mathbb{R}$ is a 2-dimensional Whitney function. Now, it is almost immediate that every monotone continuous function is a 2-dimensional Whitney function, and hence piecewise monotone continuous functions are 2-dimensional Whitney functions as well.

The Whitney property easily extends to continuous functions f of bounded variation as follows. Let $f = g_1 - g_2$, where the functions g_1 and g_2 are increasing, and let $h = g_1 + g_2$. Since h is monotone, we have a simple arc $\phi : [0,1] \to \mathbb{R}^2$ such that $(*)$ holds for h and ϕ. Now, for every $x, y \in [0,1]$ we have $|f(x) - f(y)| \leq |h(x) - h(y)|$, thus $(*)$ holds for f and ϕ as well.

As a generalization of this remark we will show that every continuous VBG_* function is a 2-dimensional Whitney function. (Property VBG_* is discussed in detail in [4] or [2]. The only fact that is needed here is a theorem of R. Fleissner and J. Foran, saying that every continuous VBG_* function can be transformed into a differentiable function by an inner homeomorphism (see [1], [2])).

DEFINITION 2. A Whitney pair (ϕ, f) is said to be *monotone*, *strictly monotone*, or VBG_* if f is monotone, strictly monotone, or VBG_*, respectively. A Whitney arc ϕ is called *monotone*, *strictly monotone*, or VBG_* if there exists a monotone, strictly monotone, or VBG_* function f for which (ϕ, f) is a Whitney pair.

REMARKS. (i) We remark that the "monotone", "strictly monotone" and "VBG_*" attributes refer to the Whitney function and not to the Whitney arc. Therefore, for example, a VBG_* Whitney arc $\phi : [0,1] \to \mathbb{R}^n$ is not necessarily a VBG_* arc in $\mathbb{R}^n$. On the other hand, "n-dimensional" refers to the arc, all the n-dimensional Whitney functions are from $\mathbb{R}$ to $\mathbb{R}$.

(ii) Every strictly monotone Whitney arc is obviously a monotone Whitney arc, and likewise, every monotone Whitney arc is VBG_*. We will study the possible reverse implications under suitable assumptions.

(iii) If (ϕ, f) is a Whitney pair and h is a homeomorphism of $[0,1]$ onto itself, then $(\phi \circ h, f \circ h)$ is again a Whitney pair. By the theorem of R. Fleissner and J. Foran mentioned above, every continuous VBG_* function can be transformed into a differentiable function by an inner homeomorphism. Therefore in order to prove that every continuous VBG_* function is a 2-dimensional Whitney function it is enough to show that every differentiable function is a 2-dimensional Whitney function.

(iv) ϕ is a strictly monotone Whitney arc if and only if it admits a parametrization satisfying $(**)$.

2. Monotone, strictly monotone and VBG_* Whitney pairs

THEOREM 1. *Every continuous VBG_* function f is an n-dimensional Whitney function for every $n \geq 2$. Moreover, for every given continuous VBG_* function f and strictly monotone Whitney arc ϕ there is a homeomorphism h of* $[0,1]$ *for which* $(\phi \circ h, f)$ *is a Whitney pair.*

Proof. Let $\phi : [0,1] \to \mathbb{R}^n$ be a strictly monotone Whitney arc. We can suppose that $(**)$ holds and we choose a homeomorphism h such that $f \circ h^{-1}$ is differentiable (see remark (iii) above). Now, for every x_0 we have

$$\lim_{x\to x_0} \frac{|f(h^{-1}(x)) - f(h^{-1}(x_0))|}{|\phi(x)-\phi(x_0)|}$$
$$= \lim_{x\to x_0} \frac{|f(h^{-1}(x)) - f(h^{-1}(x_0))|}{|x-x_0|} \cdot \frac{|x-x_0|}{|\phi(x)-\phi(x_0)|} = |(f\circ h^{-1})'(x_0)|\cdot 0 = 0.$$

Putting $x = h(y)$ we have

$$\lim_{y\to y_0} \frac{|f(y)-f(y_0)|}{|\phi(h(y)) - \phi(h(y_0))|} = 0,$$

thus $(\phi\circ h, f)$ is a Whitney pair, as required. ■

We shall prove that a 2-dimensional Whitney function is not necessarily VBG_*. But the following problems remain open:

Problem 1. *Does there exist a continuous function which is not a* 2-*dimensional Whitney function?*

Problem 2. *Does there exist a non-*VBG_* *Whitney arc?*

Problem 3. *Does there exist an* n-*dimensional Whitney arc* ϕ *such that for every* n-*dimensional Whitney function there is a homeomorphism* h *of* $[0,1]$ *for which* $(\phi\circ h, f)$ *is a Whitney pair?*

Theorem 2. *There exists a non-*VBG_* 2-*dimensional Whitney function.*

Proof. In [3] the authors construct a simple arc $\phi:[0,1]\to[0,1]^2$ for which $3/7^n \le |t_1-t_2| \le 3/7^{n-1}$ implies $|\phi(t_1)-\phi(t_2)| \ge 1/5^n$. We choose an arbitrary number $1 < c < 7/5$, and a small number ε_m for every m. Let $f_m(t):[0,1]\to[0,\varepsilon_m]$ be defined as the distance of the number $c^m t$ and the sequence $\{0, 2\varepsilon_m, 4\varepsilon_m, \ldots\}$. It is clear that if the numbers ε_m are small enough then the function $f=\sum_m f_m$ exists and f is not of bounded variation on any subinterval of $[0,1]$, hence f is not VBG_*. We can also assume that $\varepsilon_m < 3/7^m$ for every m. Now, $3/7^n \le |t_1-t_2| \le 3/7^{n-1}$ implies

$$\frac{|f(t_2)-f(t_1)|}{|\phi(t_2)-\phi(t_1)|} \le \frac{\sum_m |f_m(t_2)-f_m(t_1)|}{|t_2-t_1|}\cdot\frac{|t_2-t_1|}{|\phi(t_2)-\phi(t_1)|},$$

where

$$\frac{\sum_m |f_m(t_2)-f_m(t_1)|}{|t_2-t_1|} \le \frac{\sum_{m=1}^n |f_m(t_2)-f_m(t_1)|}{|t_2-t_1|} + \frac{\varepsilon_{n+1}+\varepsilon_{n+2}+\ldots}{|t_2-t_1|}$$
$$\le c + c^2 + c^3 + \ldots + c^n + \frac{\frac{3}{7^{n+1}} + \frac{3}{7^{n+2}} + \ldots}{\frac{3}{7^n}} < nc^n$$

if n is large enough, and

$$\frac{|t_2-t_1|}{|\phi(t_2)-\phi(t_1)|} \le \frac{3}{7^{n-1}}\cdot 5^n = 21\left(\frac{5}{7}\right)^n.$$

Thus

$$\frac{|f(t_2)-f(t_1)|}{|\phi(t_2)-\phi(t_1)|} \le 21n\left(\frac{5c}{7}\right)^n.$$

Since $c < 7/5$, we have

$$\lim_{t_2\to t_1}\frac{|f(t_2)-f(t_1)|}{|\phi(t_2)-\phi(t_1)|} = 0,$$

as required. ∎

It will be useful to study functions defined only on a closed subset of the interval $[0,1]$. Clearly, Theorem 1 remains true if rather than property $(*)$,

$$(***) \qquad \lim_{x\in A,\, x\to x_0}\frac{|f(x)-f(x_0)|}{|\phi(x)-\phi(x_0)|} = 0 \quad \forall x_0 \in A$$

is to be satisfied, where $\phi : [0,1] \to \mathbb{R}^n$ is a strictly monotone Whitney arc and f is defined on a given closed subset $A \subset [0,1]$.

For functions defined only on the closed subset $A \subset [0,1]$ it makes sense to study also the 1-dimensional "arcs". We characterize the "1-dimensional Whitney functions on A".

THEOREM 3. *For every function $f : A \to \mathbb{R}$ the following two properties are equivalent*:

(i) *there exists a homeomorphism $\phi : [0,1] \to [0,1]$ such that* $(***)$ *holds*;
(ii) *f is continuous*, VBG_* *and $\lambda(f(A)) = 0$.*

Proof. Suppose f is continuous, VBG_* and $\lambda(f(A)) = 0$.

Since f is continuous and VBG_*, there is a homeomorphism h such that $f \circ h^{-1}$ is differentiable on $h(A)$. Since $f(A) = (f\circ h^{-1})(h(A))$ we can suppose that f is differentiable.

If f is differentiable and $\lambda(f(A)) = 0$, then $f'(x) = 0$ for a.e. $x \in A$. Let $B = \{x \in A : f'(x) \ne 0\}$. Since $\lambda(B) = 0$, there are open intervals $I_1, I_2, \ldots$ covering the interval $[0,1]$ such that every $x \in B$ is contained in infinitely many intervals and $\sum_n |I_n| < \infty$. Let $m(x)$ be the number of intervals covering x, and let $M = \int_0^1 m(x)\,dx$. It is immediate that $M = \sum_n |I_n \cap [0,1]| < \infty$. We define $\phi(x) = M^{-1}\int_0^x m(t)\,dt$.

For every $x, y \in A$ we have $|\phi(y)-\phi(x)| \ge M^{-1}|y-x|$ (because $m(x) \ge 1$ for every $x \in [0,1]$), thus

$$\left|\frac{f(y)-f(x)}{y-x}\right| \ge \frac{1}{M}\cdot\left|\frac{f(y)-f(x)}{\phi(y)-\phi(x)}\right|.$$

From this it is immediate that $(***)$ holds for every $x_0 \in A \setminus B$. Consider a point $x_0 \in B$ with $|f'(x_0)| = c > 0$. The intervals I_n are open, and x_0 is contained in infinitely many intervals, thus for a fixed N there is a positive

d such that $m(x) > N$ for every $x \in [x_0 - d, x_0 + d]$. We can suppose that d is so small that $x \in [x_0 - d, x_0 + d]$ implies

$$\left|\frac{f(x) - f(x_0)}{x - x_0}\right| \le 2c.$$

Now,

$$|\phi(x) - \phi(x_0)| \ge \frac{N}{M}|(x - x_0)| \ge \frac{N}{M} \cdot \frac{|f(x) - f(x_0)|}{2c},$$

and hence

$$\left|\frac{f(x) - f(x_0)}{\phi(x) - \phi(x_0)}\right| \le \frac{2Mc}{N}$$

for every N if $|x - x_0|$ is small enough.

The other direction is clear. Indeed, $f \circ \phi^{-1}$ is differentiable and we have $(f \circ \phi^{-1})'(x) = 0$ on $\phi(A)$, thus $f \circ \phi^{-1}$ is continuous, VBG_*, and $0 = \lambda(f \circ \phi^{-1}(\phi(A))) = \lambda(f(A))$. Finally, since ϕ^{-1} is a homeomorphism, the function f is continuous and VBG_*, as required. ∎

We know that every monotone Whitney arc is VBG_*. Now we prove the reverse implication.

Theorem 4. *Every VBG_* Whitney arc is also a monotone Whitney arc.*

Proof. Let $\phi : [0,1] \to \mathbb{R}^n$ be a VBG_* Whitney arc with the VBG_* function f. Without loss of generality we suppose that f is differentiable.

Let $Z = \{x \in [0,1] : f'(x) = 0\}$. Since f is a non-constant continuous function on $[0,1]$, we have $\lambda(Z) < 1$. Take an open covering set $Z \subset G$ with $\lambda(G) < 1$. Let $\chi(x)$ be the characteristic function of $[0,1] \setminus G$, and defining $g(x) = \int_0^x \chi(t)\,dt$, for the upper derivative we have $\overline{g}'(x) \le 1$ for every x.

Now we show that (ϕ, g) is a Whitney pair, i.e. $(*)$ holds for g and ϕ. If $x_0 \in G$, then there is a neighbourhood of x_0 where g is a constant, thus $(*)$ is trivial. If $x_0 \notin G$, then as $0 \le \overline{g}'(x_0) \le 1$ and $f'(x_0) \ne 0$ we obtain

$$\begin{aligned}\lim_{x\to x_0} \frac{|g(x) - g(x_0)|}{|\phi(x) - \phi(x_0)|} &= \lim_{x\to x_0} \frac{|g(x) - g(x_0)|}{|x - x_0|} \cdot \frac{|x - x_0|}{|f(x) - f(x_0)|} \cdot \frac{|f(x) - f(x_0)|}{|\phi(x) - \phi(x_0)|}\\ &\le 1 \cdot \frac{1}{|f'(x_0)|} \cdot 0 = 0,\end{aligned}$$

as stated. It is obvious that g is continuous, monotone, and since $\lambda(G) < 1$ it is not a constant. ∎

3. Structural properties. In this section we study the structure of the possible monotone but not strictly monotone Whitney arcs.

Lemma 1. *Let $\phi : [0,1] \to \mathbb{R}^n$ be a simple arc, and suppose that for every interval $I \subset [0,1]$ there is a function $f = f_I$ such that*

(i) *the derivative f' exists and is non-zero on a subset of I of positive measure*: $\lambda(\{x \in I : \exists f'(x) \neq 0\}) > 0$, *and*

(ii) $(\phi|_I, f_I)$ *is a Whitney pair.*

Let

$$R = \left\{x_0 \in [0,1] : \limsup_{x \to x_0} \frac{|x - x_0|}{|\phi(x) - \phi(x_0)|} > 0\right\}.$$

If R is of first category, then ϕ is a strictly monotone Whitney arc.

Proof. Let $R = \bigcup_n A_n$, where the sets A_n are nowhere dense.

Since $(\phi|_I, f_I)$ is a Whitney pair, the sets $\{x \in I : \exists f_I'(x) \neq 0\}$ and R are disjoint. Thus, by (i) we have $\lambda(I \setminus R) > 0$ for every subinterval I. We define a series of monotone functions f_n such that the sum of the series is a strictly monotone function satisfying $(*)$.

We arrange the rational intervals (i.e. the intervals $[a,b]$ where $a, b \in \mathbb{Q}$) in a sequence $I_1 = [0,1], I_2, \ldots,$ and put $f_0 = 0$. Suppose that the functions $f_0, f_1, \ldots, f_{n-1}$ have been defined. In the nth step we consider the nth rational interval I_n.

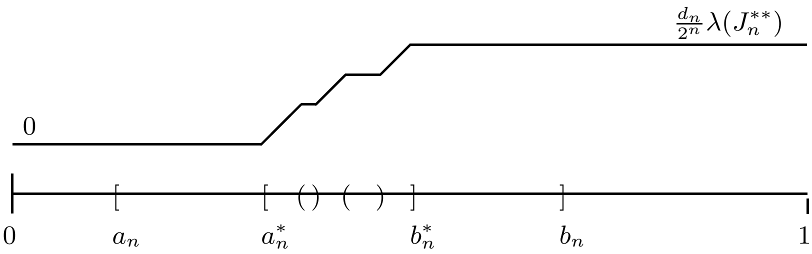

Fig. 1. Graph of f_n

We choose a subinterval $J_n = [a_n, b_n] \subset I_n$ disjoint from $A_1, \ldots, A_n$. Denote the middle 1/3 of J_n by $J_n^* = [a_n^*, b_n^*]$, and choose an open cover $G \supset R \cap J_n^*$ such that $\lambda(G) < |J_n^*|$. Let $J_n^{**} = J_n^* \setminus G$. Let d_n denote the minimum of the distances between the subarcs $\phi([0, a_n]), \phi([a_n^*, 1])$ and $\phi([0, b_n^*]), \phi([b_n, 1])$, and define

$$f_n(x) = \frac{d_n}{2^n}\lambda([0,x] \cap J_n^{**}) \quad (x \in [0,1])$$

(see Figure 1 above). Now, f_n is a non-constant monotone function such that for every x, y we have

$$\frac{|f_n(x) - f_n(y)|}{|x - y|} \leq \frac{d_n}{2^n} \leq \frac{\operatorname{diam}\phi([0,1])}{2^n}. \tag{1}$$

Thus f_n satisfies $(*)$, because for $x_0 \in R$ there is a neighbourhood where f_n

is constant, and for $x_0 \notin R$ we have

$$\limsup_{x\to x_0} \frac{|f_n(x)-f_n(x_0)|}{|\phi(x)-\phi(x_0)|} = \limsup_{x\to x_0} \frac{|f_n(x)-f_n(x_0)|}{|x-x_0|} \cdot \frac{|x-x_0|}{|\phi(x)-\phi(x_0)|} \le \frac{\operatorname{diam}\phi([0,1])}{2^n}\cdot 0 = 0.$$

Now we put $f=\sum_n f_n$. It is clear that f is strictly monotone. In order to verify $(*)$ we consider first $x_0 \in R$. For such points there is an N such that $n>N$ implies $x_0 \notin J_n$. Put $m \ge N$. If x_0 does not belong to J_n then $x_0 \in [0,a_n]$ or $x_0 \in [b_n,1]$. It is easy to see from the definition that f_n is constant on the intervals $[0,a_n^*]$ and $[b_n^*,1]$, thus $f_n(x)=f_n(x_0)$ or $x_0\in[0,a_n]$, $x\in[a_n^*,1]$, or $x_0\in[b_n,1]$, $x\in[0,b_n^*]$. In the last two cases, by the choice of d_n we have

$$\frac{|f_n(x)-f_n(x_0)|}{|\phi(x)-\phi(x_0)|} = \frac{1}{2^n}\cdot\lambda([x_0,x]\cap J_n^{**})\cdot\frac{d_n}{|\phi(x)-\phi(x_0)|} \le \frac{1}{2^n}\cdot 1\cdot 1,$$

and hence

$$\frac{|\sum_{k=m+1}^\infty f_k(x)-\sum_{k=m+1}^\infty f_k(x_0)|}{|\phi(x)-\phi(x_0)|} \le \sum_{k=m+1}^\infty \frac{|f_k(x)-f_k(x_0)|}{|\phi(x)-\phi(x_0)|} \le \frac{1}{2^m}.$$

Since $\sum_{k=1}^m f_k$ is a Whitney function, $(*)$ follows.

On the other hand, at a point $x_0 \notin R$, that is, if

$$\limsup_{x\to x_0}\frac{|x-x_0|}{|\phi(x)-\phi(x_0)|}=0,$$

then the finite upper derivative of f (see (1)) guarantees $(*)$. ■

Corollary 1. *If ϕ is a Whitney arc and for a dense sequence of pairwise disjoint closed subintervals $I_\alpha \subset [0,1]$, $\phi|_{I_\alpha}$ is a strictly monotone Whitney arc, then so is ϕ itself.*

Proof. Indeed, let the union $\bigcup_n I_n$ of pairwise disjoint closed intervals be everywhere dense, and assume that there exists a strictly monotone function $f_n : [0,1]\to\mathbb{R}$ satisfying $(*)$ on I_n. Re-parametrizing the intervals $I_n=[a_n,b_n]$ we can suppose that every f_n is linear. Indeed, let T_n be the linear bijection

$$[a_n,b_n] \xrightarrow{T_n} [f_n(a_n), f_n(b_n)],$$

and put $\phi^*(t)=\phi(f_n^{-1}(T_n t))$ for every $t\in[a_n,b_n]$ and $\phi^*(t)=\phi(t)$ for every $t\notin\bigcup_n I_n$. Then $(*)$ obviously holds for every $x_0\in\operatorname{int} I_n$, ϕ^* and $f=\mathrm{id}_{[0,1]}$:

putting $u = f_n^{-1}(T_n x)$ we have

$$\lim_{x\to x_0}\frac{|x-x_0|}{|\phi^*(x)-\phi^*(x_0)|} = \lim_{x\to x_0}\frac{|x-x_0|}{|\phi(f_n^{-1}(T_n x))-\phi(f_n^{-1}(T_n x_0))|}$$
$$= \lim_{u\to u_0}\frac{|T_n^{-1}f_n(u)-T_n^{-1}f_n(u_0)|}{|\phi(u)-\phi(u_0)|}$$
$$= \lim_{u\to u_0}\frac{b_n-a_n}{f(b_n)-f(a_n)}\cdot\frac{|f_n(u)-f_n(u_0)|}{|\phi(u)-\phi(u_0)|} = 0.$$

Since $f'(x) = 1$, the set

$$R = \left\{x_0\in[0,1] : \limsup_{x\to x_0}\frac{|x-x_0|}{|\phi^*(x)-\phi^*(x_0)|} > 0\right\}$$

is disjoint from the interior of our intervals I_n, thus it is nowhere dense. Finally, for every subinterval $I=[a,b]\subset[0,1]$ there exists an interval $I_n = [a_n,b_n]$ for which $\lambda(I_n\cap I)>0$, and for the function f_I defined by $f_I(x)=a_n$ if $x\in[0,a_n]$, $f_I(x)=x$ if $x\in[a_n,b_n]=I_n$, $f_I(x)=b_n$ if $x\in[b_n,1]$ we find that $(\phi^*|_I, f_I)$ is a Whitney pair and $f_I'(x)=1$ for every $x\in \operatorname{int}(I_n\cap I)$. Lemma 1 shows that ϕ^* is strictly monotone, and then ϕ is strictly monotone, as stated. ∎

This means that if a simple arc $\phi:[0,1]\to\mathbb{R}^n$ is not a strictly monotone Whitney arc, then it has a subarc which has no strictly monotone subarc. Thus, in order to study monotone but not strictly monotone Whitney arcs it is enough to consider Whitney arcs having no strictly monotone subarc.

Definition 3. Given a simple arc $\phi:[0,1]\to\mathbb{R}^n$ and a nowhere dense closed subset $H\subset[0,1]$ we say that a non-constant monotone function f is *associated with* H if

(i) (ϕ,f) is a Whitney pair;
(ii) f is constant on every interval contiguous to H;
(iii) f is *ϕ-Lipschitz*, that is, the set

$$\left\{\frac{|f(x)-f(y)|}{|\phi(x)-\phi(y)|} : x,y\in[0,1],\ x\neq y\right\}$$

is bounded.

Lemma 2. *Given a simple arc ϕ, every nowhere dense closed set $P\subset[0,1]$ admits a disjoint decomposition $P = A\cup\bigcup_n B_n$ such that*

(i) *A is a closed set, and for every portion $A\cap I$ there is a function f associated with* $\mathrm{cl}(A\cap I)$;
(ii) *$B_n\subset[a_n,b_n]$ and the intervals $[a_n,b_n]$ are non-overlapping*;
(iii) *there is no monotone function f associated with the closure of B_n.*

Proof. Let B^* be the set of points $x \in P$ such that for every function f associated with P, f is constant on a neighbourhood of x. Let B be the relative interior of B^* in P, and let B_n denote the relative components of B. (Thus every B_n is a portion of P, $B = \bigcup_n B_n$, and (ii) holds.)

By the definition of B^*, a function f associated with P could only have countably many different values on B^*. Thus if it is constant on the intervals contiguous to the closure of some $B_n \subset B^*$ as well, then since B_n is a portion of P we see that it is a continuous function of countable range, thus it must be constant on the whole interval $[0,1]$. So (iii) is proved.

Let $A = P \setminus B$. If for an interval $I = (a,b)$ the intersection $A \cap I$ is a portion of A, then there is a point $x \in I \setminus B$, and then in a small neighbourhood there is a point $y \in I \setminus B^*$. Thus, there exists an f associated with P which is not constant on I. We can suppose that f is constant on the intervals of $[0,1] \setminus I$. It is enough to show that f is constant on the intervals contiguous to $\operatorname{cl} A$ in I.

Since f is monotone, we have $f(t) = f(a) + \lambda(f([a,t] \cap A)) + \sum_n f_n(t)$ for every $t \in I$, where $f_n(t) = \lambda(f([a,t] \cap B_n))$. Since (ϕ, f) is a Whitney pair and f is ϕ-Lipschitz, so is f_n, and it is immediate that f_n is constant on the intervals contiguous to $\operatorname{cl} B_n$. Thus (iii) implies that $f_n \equiv 0$ for every n, hence $f(t) = f(a) + \lambda(f([a,t] \cap A))$, f is constant on the intervals contiguous to $\operatorname{cl} A$, and the proof is complete. ∎

The main result of this section is the following theorem.

THEOREM 5. *Let $\phi : [0,1] \to \mathbb{R}^n$ be a Whitney arc for which every subarc is a monotone Whitney arc. Then the following two properties are equivalent*:

(1) *ϕ has no strictly monotone Whitney subarc*;

(2) *ϕ can be parametrized so that $[0,1] = R \cup P \cup A$, $P \cap A = \emptyset$, and*

  (2.1) *the set $R = \{x_0 : \limsup_{x \to x_0} |x - x_0|/|\phi(x) - \phi(x_0)| > 0\}$ is residual*;

  (2.2) *$P = \bigcup_n P_n$ is dense in $[0,1]$, and the sets P_n are closed and nowhere dense*; *$P_n \subset [a_n, b_n]$ where $b_n - a_n \to 0$ and for every $m \neq n$ we have $P_m \cap (a_n, b_n) = \emptyset$ or $P_n \cap (a_m, b_m) = \emptyset$*;

  (2.3) *no monotone function is associated with P_n*;

  (2.4) *A is a nowhere dense set.*

Proof. (2)⇒(1). Suppose that a subarc of ϕ is strictly monotone, i.e. there is a function f which is strictly monotone on $I = [a,b] \subset [0,1]$, and for which $(*)$ holds. By (2.4) we can suppose that $I \cap A = \emptyset$. It is clear that we can suppose that $f(a) = 0$ and f is constant on the components of $[0,1] \setminus I$.

For $x \in R$ clearly $f'(x) = 0$, and since f is monotone we have $\lambda(f(R)) = 0$. Thus, on I we have $f = \sum_n f_n$, where $f_n(x) = \lambda(f([a,x] \cap P_n))$.

Since f has property $(*)$, so does f_n. Moreover, f_n is clearly monotone, continuous and constant on the intervals contiguous to P_n. Hence, according to (2.3), either $f_n \equiv 0$ or f_n is not ϕ-Lipschitz; in the second case we can find points $x, y \in [a_n, b_n]$ for which

$$\frac{|f_n(y) - f_n(x)|}{|\phi(y) - \phi(x)|} > 1.$$

Since f_n and ϕ are continuous, the same inequality holds in a small neighbourhood of x and y. Now, (2.2) and the fact that f is strictly monotone and f_n is constant on the intervals contiguous to P_n imply that every subinterval of $[0,1]$ contains an interval $[a_n, b_n]$ for some n where f_n is not ϕ-Lipschitz, thus we can find intervals $I \supset [a_{n_1}, b_{n_1}] \supset [a_{n_2}, b_{n_2}] \supset \ldots$ and points $y_1, y_2, \ldots$ such that $y_i \in [a_{n_i}, b_{n_i}] \setminus [a_{n_{i+1}}, b_{n_{i+1}}]$, $b_{n_i} - a_{n_i} \to 0$, and for every $x_i \in [a_{n_{i+1}}, b_{n_{i+1}}]$ we have

$$\frac{|f_{n_i}(y_i) - f_{n_i}(x_i)|}{|\phi(y_i) - \phi(x_i)|} > 1.$$

This means that for $\{x\} = \bigcap_i [a_{n_i}, b_{n_i}]$ there is a sequence $y_i \to x$ and a sequence $n_1, n_2, \ldots$ of integers such that

$$\frac{|f_{n_i}(y_i) - f_{n_i}(x)|}{|\phi(y_i) - \phi(x)|} > 1.$$

Since the functions f_{n_i} are monotone we have

$$\frac{|f(y_i) - f(x)|}{|\phi(y_i) - \phi(x)|} > 1,$$

a contradiction to $(*)$.

(1)⇒(2). By assumption for every subarc there exists a non-constant monotone function satisfying $(*)$ on the corresponding subinterval. It is easy to see that there is a parametrization of $[0,1]$ such that for every subinterval there is a non-constant monotone differentiable function satisfying $(*)$. Since R is a Borel set which by Lemma 1 is not of first category in any subinterval of $[0,1]$, we obtain (2.1).

Thus, the complement of R can be covered by pairwise disjoint nowhere dense closed sets $F^1, F^2, \ldots$ Applying Lemma 2 to each of the sets F^n we obtain $F^n = A^n \cup \bigcup_k B_k^n$. Put $A = \bigcup_n A^n$ and $P = \bigcup_n \bigcup_k B_k^n$.

It is easy to see that the sets B_k^n can be further decomposed into the union of closed sets satisfying all of (2.2) but for the density of P. Since there is no monotone function f associated with B_k^n, it is automatic that there is no monotone function associated with a closed subset of B_k^n, i.e. (2.3) holds. For every subarc of ϕ there is a non-constant monotone function satisfying $(*)$, thus the set R cannot contain an interval. Hence it is enough to show that A is nowhere dense, which implies that P is also dense.

Suppose that A is dense in an interval I. By the definition of A we have monotone functions $f_1, f_2, \ldots$ satisfying $(*)$ such that the set

$$\left\{\frac{|f_i(x)-f_i(y)|}{|\phi(x)-\phi(y)|} : x, y \in [0,1],\ x \neq y\right\}$$

is bounded, say, by K_i, and on every subinterval $J \subset I$ some f_i is non-constant.

Now, let $K_i^* = \max|f_i(x)|$, and put

$$f(x) = \sum_i \frac{f_i(x)}{2^i \max(K_i, K_i^*)}.$$

It is easy to check that the series converges uniformly, f is strictly monotone on I, and $(*)$ holds. This contradicts (1) and completes our proof. ■

4. Construction of a Whitney arc Φ which is not strictly monotone but for which every subarc is Whitney

THEOREM 6. *There exists a simple arc $\phi : [0,1] \to [0,1]^3$, a Cantor set $C \subset [0,1]$ and a non-constant function f such that*

(i) *f is constant on the intervals contiguous to C;*

(ii) *(ϕ, f) is a monotone Whitney pair;*

(iii) *for every function f satisfying* (i) *and* (ii), *the set*

$$\left\{\frac{|f(y)-f(x)|}{|\phi(y)-\phi(x)|} : x, y \in [0,1],\ x \neq y\right\}$$

is unbounded; that is, there is no function associated with C.

Proof. Let $\phi_0 : [0,1] \to [0,1]^2$ be a strictly monotone Whitney arc with a strictly monotone function f_0, $\phi_0(0) = (0,0)$, $\phi_0(1) = (1,1)$, and let $C_0 \subset [0,1]$ be a Cantor set for which $\lambda(f_0(C_0)) > 0$.

Let $\mathcal{I}_0 = \emptyset$, and let

$$\mathcal{I}_n = \{I_0^n, I_1^n, \ldots, I_{m_n}^n\} = \{[a_0^n, b_0^n], [a_1^n, b_1^n], \ldots, [a_{m_n}^n, b_{m_n}^n]\}$$

be a finite set of pairwise disjoint intervals disjoint from C_0 and indexed by the real ordering such that

- $a_0^n = b_0^n = 0$, $a_{m_n}^n = b_{m_n}^n = 1$, and $a_k^n < b_k^n$ for every $0 < k < m_n$;
- $\mathcal{I}_n \subset \mathcal{I}_{n+1}$;
- $|\phi_0(a_k^n) - \phi_0(b_{k-1}^n)| < 1/(3 \cdot 2^n)$;
- $|\phi_0(b) - \phi_0(a)| < 1/(3 \cdot 2^n)$ for every interval $[a,b] \in \mathcal{I}_n \setminus \mathcal{I}_{n-1}$.

For every interval $[a,b] = [a_k^n, b_k^n] \in \mathcal{I}_n \setminus \mathcal{I}_{n-1}$ let $u = u_{a,b} : [(2a+b)/3, (a+2b)/3] \to [0,1]^2$ be a polygon connecting $\phi_0(a)$ and $\phi_0(b)$ such that

- $\max|u(t) - \phi_0(a)| < 1/2^n$;
- $\phi_0(a_l^n) \neq u(t) \neq \phi_0(b_l^n)$ $(a_l^n \neq t \neq b_l^n)$;
- the distance of the polygons $u_{a_k^n, b_k^n}$, $u_{a_{k+1}^n, b_{k+1}^n}$ is less than $1/(m_n \cdot 2^n)$.

We choose a sequence (say, an equidistant subdivision) $1/2^n < d_0^n < d_1^n < \ldots < d_{m_n}^n < 1/2^{n-1}$, and let

$$\phi(t) = \begin{cases} (\phi_0(t), 0) & \text{if } t \text{ is not covered by } \operatorname{int} \bigcup_n \bigcup_{I \in \mathcal{I}_n} I; \\ (u(t), d_k^n) & \text{if } t \in [(2a+b)/3, (a+2b)/3], \\ & [a,b] = [a_k^n, b_k^n] \in \mathcal{I}_n \setminus \mathcal{I}_{n-1}; \\ \text{linear} & \text{on } [a, (2a+b)/3] \text{ and } [(a+2b)/3, b], \text{ where } [a,b] \in \mathcal{I}_n, \end{cases}$$

where the third coordinate of the three-dimensional point $\phi(t)$ is indicated following a two-dimensional point $\phi_0(t)$ and $u(t)$, respectively.

Since $d_k^n \to 0$ and $\max |\phi_0(a_k^n) - u(t)| \to 0$, the function $\phi : [0,1] \to [0,1]^3$ is continuous, and it is easy to see that it is a simple arc as well. Let $C = [0,1] \setminus \operatorname{int} \bigcup_n \bigcup_{I \in \mathcal{I}_n} I$.

The function $f(x) = \lambda(f_0([0,x] \cap C))$ is monotone and non-constant, since $C_0 \subset C$ and $\lambda(f_0(C_0)) > 0$. It is clear that f is constant on the intervals contiguous to C, and $|f(x) - f(x_0)| \le |f_0(x) - f_0(x_0)|$. We claim that $(*)$ holds for f and ϕ.

This is clear on the intervals contiguous to C, since f is constant on these intervals. If $x_0 \in C$ then $\phi(x_0) = (\phi_0(x_0), 0)$, and there are two possibilities:

If there is no interval $[a,b] \in \bigcup_n \mathcal{I}_n$ for which $x \in [(2a+b)/3, (a+2b)/3]$, then $\phi(x) = (\phi_0(x), y)$ for a $y \in [0,1]$, thus $|\phi(x) - \phi(x_0)| \ge |\phi_0(x) - \phi_0(x_0)|$. If $x \in [(2a+b)/3, (a+2b)/3]$ for an interval $[a,b] = I_k^n \in \mathcal{I}_n \setminus \mathcal{I}_{n-1}$, then (using the trivial inequality $q^2/2 \le (q-r)^2 + s^2$ for every $0 \le q$ and $0 \le r \le s$) the inequality $d = d_k^n > 1/2^n > \max |u(t) - \phi_0(a)|$ implies

$$\begin{aligned} \frac{|\phi_0(a) - \phi_0(x_0)|^2}{2} &\le (|\phi_0(a) - \phi_0(x_0)| - |\phi_0(a) - u(x)|)^2 + d^2 \\ &\le |u(x) - \phi_0(x_0)|^2 + d^2 = |\phi(x) - \phi(x_0)|^2. \end{aligned}$$

Hence, in all cases we have

$$|\phi(x) - \phi(x_0)| \ge \frac{|\phi_0(x) - \phi_0(x_0)|}{\sqrt{2}} \quad \text{and} \quad |f(x) - f(x_0)| \le |f_0(x) - f_0(x_0)|,$$

thus

$$\limsup_{x \to x_0} \frac{|f(x) - f(x_0)|}{|\phi(x) - \phi(x_0)|} \le \sqrt{2} \cdot \limsup_{x \to x_0} \frac{|f_0(x) - f_0(x_0)|}{|\phi_0(x) - \phi_0(x_0)|} = 0.$$

Finally, we show (iii). Let f be an arbitrary monotone function satisfying $|f(y) - f(x)| \le K|\phi(y) - \phi(x)|$, and which is constant on the intervals contiguous to C. We show that f is constant on $[0,1]$.

For a given n, let c_k denote the value of f on $[a_k^n, b_k^n]$. Then

$$|f(1)-f(0)| \le \sum_{i=1}^{m_n} |c_i - c_{i-1}| \le K \sum_{i=1}^{m_n} \operatorname{dist}(\phi([a_i^n, b_i^n]), \phi([a_{i-1}^n, b_{i-1}^n]))$$
$$\le K \sum_{i=1}^{m_n} \operatorname{dist}(u_{a_i^n, b_i^n}, u_{a_{i-1}^n, b_{i-1}^n}) \le K \cdot m_n \cdot \frac{1}{m_n \cdot 2^n} = \frac{K}{2^n}$$

for every n. Hence f must indeed be constant on $[0,1]$. ∎

The main result of this paper is the following theorem:

THEOREM 7. *There exists a Whitney arc Φ which is not strictly monotone but for which every subarc is Whitney.*

Proof. We construct this arc by an iterative process.

STEP I. The notation below follows that of Theorem 6.

The iteration is started by applying Theorem 6 as it is. Then for every interval $[a,b] \in \bigcup_n \mathcal{I}_n$ we choose a Cantor set $T_{a,b} \subset [a,b]$ with appropriately short contiguous intervals and a set of small cubes $\mathcal{C}_{a,b}$ such that

(i) $\bigcup\{\mathcal{C}_{a,b} : [a,b] \in \bigcup_n \mathcal{I}_n\}$ is a pairwise disjoint system of subcubes of $[0,1]^3$;

(ii) all cubes have a base with diagonal lying on a side of the polygon $\phi([a,b])$, and the cube has no other point in common with $\phi([0,1])$;

(iii) every segment of $\phi([a,b])$ is covered by the cubes except for the Cantor set $\phi(T_{a,b})$;

(iv) the cubes are so small that if $[a,b]$ and $[c,d]$ are distinct intervals of $\bigcup_n \mathcal{I}_n$, then for any $x \in Q^1 \in \mathcal{C}_{a,b}$, $y \in Q^2 \in \mathcal{C}_{c,d}$, $x_1 \in Q^1 \cap \phi$ and $y_1 \in Q^2 \cap \phi$ we have $|x_1 - y_1| \le 2|x-y|$, and for all $x \in Q^1 \in \mathcal{C}_{a,b}$, $x_1 \in Q^1 \cap \phi$ and $z \in \phi(C)$ we have $|x_1 - z| \le 2|x - z|$.

It is important to note that for every simple arc ψ such that

$$\psi \subset \phi(C) \cup \bigcup_{a,b} \bigcup_{Q \in \mathcal{C}_{a,b}} Q,$$
$$\psi(t) = \phi(t) \qquad (\forall t \in C),$$
$$\psi(t) \in \phi([a,b]) \cup \bigcup_{Q \in \mathcal{C}_{a,b}} Q \qquad \Big(\forall t \in [a,b] \in \bigcup_n \mathcal{I}_n\Big),$$

the requirements (i)–(iii) of Theorem 6 hold with the same Cantor set C and monotone function f. Indeed, f is constant on the intervals $[a,b] \in \bigcup_n \mathcal{I}_n$, thus for given $u, v \in [0,1]$ either we have $f(u) = f(v)$ or there is no interval $[a,b] \in \bigcup_n \mathcal{I}_n$ for which $u, v \in [a,b]$. In the latter case by (iv) we have

$$|\phi(t) - \phi(u)| \le 2|\psi(t) - \psi(u)|,$$

thus
$$\frac{|f(u)-f(v)|}{|\phi(u)-\phi(v)|} \geq \frac{|f(u)-f(v)|}{2|\psi(u)-\psi(v)|} \quad \text{for every } u, v.$$

Now, we take subcubes in all our cubes in the sets $\mathcal{C}_{a,b}$, as follows. For every cube $Q \in \mathcal{C}_{a,b}$ we take a dense set of pairwise disjoint subintervals on its diagonal $Q \cap \phi([a,b]) = \phi([c,d])$ such that the measure of the union of the subintervals is less than $d-c$. Then we re-parametrize the segment $\phi([c,d])$ by the same interval $[c,d]$ so that all the subintervals considered have an arc length parametrization. Next we place a subcube on each subinterval as before (that is, each subinterval is the diagonal of a face of a new small subcube). Finally, we regularly divide the base of each subcube containing one of our subintervals as a diagonal into 9 smaller squares, the middle one again into 9 squares, etc., and we place subcubes on the squares on the diagonal. See Figure 2 below.

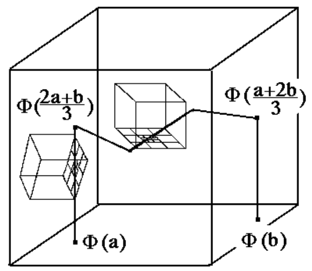

Fig 2. A typical step in the construction

Let $S_{c,d}$ be the set of those points $x \in [c,d]$ for which $\phi(x)$ is not inside a cube getting into this regular subdivision, that is, the points not belonging to the subintervals and the set of dividing points.

Step I concludes with the re-parametrized Whitney arc ϕ^*, a Cantor set C, a monotone Whitney function f, Cantor sets $T_{a,b}$, nowhere dense closed sets $S_{c,d}$, and an infinite system of new cubes lying on ϕ^* (say $Q^{(1)}, Q^{(2)}, \ldots$), where all the diagonals lying on ϕ^* are parametrized by arc length.

For each of these cubes we now apply Step II below.

Step II. If there is given a cube Q_α with a distinguished face (called the base) and a diagonal of the base with a given parametric interval $[u,v]$ where the length of the diagonal is $v-u$, then Step II consists of an application of Step I, such that the parametric interval $[0,1]$, the points $(0,0,0), (1,1,0)$, and the cube $[0,1]^3$ are replaced by the given parametric interval, the end-points of the diagonal of the base of the cube Q_α, and the cube itself.

In more detail, let $L : [u,v] \to [0,1]$ and $\Lambda : Q_\alpha \to [0,1]^3$ be linear bijections, where the Λ-image of the base of Q_α is $[0,1]^2 \times \{0\}$ and the

Λ-image of the distinguished diagonal is the segment with endpoints $(0,0,0)$ and $(1,1,0)$. With these notations and ϕ^*, C, f constructed in Step I, this step results in a Whitney curve $\phi_\alpha : [u,v] \to Q_\alpha$, a Cantor set $C_\alpha \subset [u,v]$, a non-constant monotone function $f_\alpha : [u,v] \to \mathbb{R}$, where $\phi_\alpha(t) = \Lambda^{-1}\phi(Lt)$, $C_\alpha = L^{-1}C$ and $f_\alpha(t) = f(Lt)$. It is clear that f_α is constant on the intervals contiguous to C_α, (ϕ_α, f_α) is a monotone Whitney pair, and there is no monotone function associated with C_α. At the end of Step II we also get Cantor sets of type $T_{a,b}$ and nowhere dense closed sets of type $S_{a,b}$ (these are the sets $T^\alpha_{a,b} = L^{-1}T_{a,b}$ and $S^\alpha_{a,b} = L^{-1}S_{a,b}$), and we get a new infinite system of subcubes, the Λ^{-1}-image of the cubes $Q^{(1)}, Q^{(2)}, \ldots$ It is easy to see that the diagonals are parametrized by arc length.

Then we apply Step II again for these new subcubes.

Our induction process yields a simple arc Φ. We show that Φ and a suitable decomposition $[0,1] = R \cup P$ (and $A = \emptyset$) satisfy requirement (2) of Theorem 5.

Let P denote the set of points whose image is not inside any cube from a certain step, i.e. P is the union of the Cantor sets C_α, the Cantor sets $T^\alpha_{a,b}$, and the nowhere dense closed sets $S^\alpha_{a,b}$.

It is clear by the very definition that P satisfies (2.2) of Theorem 5. It is also clear that the Cantor sets C_α satisfy (2.3). To verify (2.3) for the nowhere dense closed sets $T^\alpha_{a,b}$ and $S^\alpha_{a,b}$ we note that the Φ-image of each set is contained in a polygon, thus the measure of the image of these sets by a hypothetical associated function must be 0, as a consequence of a trivial application of Theorem 3. Thus (2.3) is trivial for these sets.

Finally, (2.1) is verified as follows. Let R^* be the set of points of $[0,1]$ belonging to a nested sequence of our cubes. It is clear that R^* is residual; we have to show that

$$R^* \subset R := \left\{ x_0 : \limsup_{x \to x_0} \frac{|x - x_0|}{|\phi(x) - \phi(x_0)|} > 0 \right\}.$$

For every point $x = \Phi(u)$ of a cube Q_α of our construction we can find another cube Q_β of the same size, namely, the one in a centrally symmetrical position on the diagonal of the previous base, such that for every point $y = \Phi(v)$ of Q_β we have $|u - v| \ge \operatorname{dist}(Q_\alpha, Q_\beta)$, and

$$|\Phi(u) - \Phi(v)| = |x - y| \le \operatorname{dist}(Q_\alpha, Q_\beta)\sqrt{(1+1+1)^2 + 1/2}$$

(see Figure 3 below).

Thus,

$$\frac{|u - v|}{|\Phi(u) - \Phi(v)|} \ge \frac{\operatorname{dist}(Q_\alpha, Q_\beta)}{\operatorname{dist}(Q_\alpha, Q_\beta)\sqrt{(1+1+1)^2 + 1/2}} = \sqrt{2/19}.$$

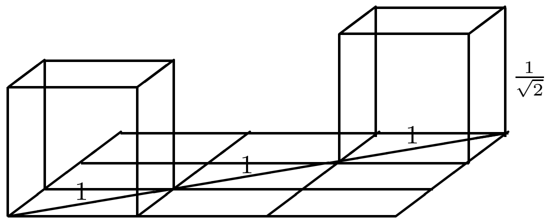

Fig. 3. Distances

Now, for every $x_0 \in R^*$ we have

$$\limsup_{x \to x_0} \frac{|x - x_0|}{|\Phi(x) - \Phi(x_0)|} \geq \sqrt{2/19} > 0,$$

which was to be proved.

That is, we constructed a curve Φ satisfying (2) of Theorem 5. We also have to show that every subarc is monotone. But this is immediate, since we know that for every Cantor set C_α there is a non-constant monotone function f_α satisfying $(*)$. ∎

The following two problems remain open.

PROBLEM 4. *Does there exist an everywhere monotone but nowhere strictly monotone Whitney arc on the plane?*

PROBLEM 5. *Does there exist a Whitney arc ϕ which is not a strictly monotone Whitney arc, but for which there is a VBG_* (or arbitrary) nowhere constant function f such that (ϕ, f) is a Whitney pair?*

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