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A partition theorem for α-large sets

by

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Abstract. Working with Hardy hierarchy and the notion of largeness determined by it, we define the notion of a partition of a finite set of natural numbers $A = \cup_{i<m} A_i$ being α-large and show that for ordinals $\alpha, \beta < \varepsilon_0$ satisfying suitable assumptions, if A is $(\omega^\beta \cdot \alpha)$-large and is partitioned as above and the partition itself is not α-large, then at least one A_i is ω^β-large.

The goal of this paper is to work out a combinatorial result which generalizes one of the results of Ketonen–Solovay [5]. Working below the ordinal ε_0 we define the notion of a partition $A = \cup_{i<m} A_i$ (where $A \subseteq \omega$) being α-large and show that (under suitable assumptions on α and β), if A is $(\omega^\beta \cdot \alpha)$-large and the partition itself is not α-large then there exists an ω^β-large homogeneous set. Of course, our paper heavily depends on the work of Ketonen–Solovay [5]. Indeed, from a point of view we generalize one of their results ([5], Theorem 4.7) from ω^2 to ε_0. We would like to point out that when working with the so-called Hardy hierarchy we are highly influenced by the work of Z. Ratajczyk (see [9], [6], [7] and his final [10]). It should be noticed that the idea of Hardy hierarchy was developed by several schools (see, e.g., [3] and [2]).

Let h be a finite increasing function (in the usual sense of the word, that is, $\forall x, y \in \mathrm{Dom}(h)\ [x < y \Rightarrow f(x) < f(y)]$). Assume moreover that $\forall x\ x < h(x)$. For every $\alpha < \varepsilon_0$ we define a function h_α, by induction on α. We put $h_0(x) = x$ and $h_{\alpha+1}(x) = h_\alpha(h(x))$.

Before defining the limit step we need to define, for each limit $\lambda < \varepsilon_0$, a sequence $\{\lambda\}(n)$ of ordinals convergent to λ from below. We put $\{\omega\}(n) = n$, and, more generally, $\{\omega^{\alpha+1}\}(n) = \omega^\alpha \cdot n$. For limit γ we put $\{\omega^\gamma\}(n) = \omega^{\{\gamma\}(n)}$. Finally,

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$$\{\omega^{\alpha_0}\cdot m_0+\ldots+\omega^{\alpha_s}\cdot m_s\}(n)=\omega^{\alpha_0}\cdot m_0+\ldots+\omega^{\alpha_s}\cdot(m_s-1)+\{\omega^{\alpha_s}\}(n),$$

where $\lambda=\omega^{\alpha_0}\cdot m_0+\ldots+\omega^{\alpha_s}\cdot m_s$ is the Cantor normal form expansion of λ, i.e., $\alpha_0>\ldots>\alpha_s$. It is easy to see that these conditions determine exactly one sequence $\{\{\lambda\}(n):n\}$, for each $\lambda<\varepsilon_0$. Observe also that Ketonen and Solovay [5] use a slightly different notion of $\{\lambda\}(n)$. We shall call the sequence $\{\lambda\}(n)$ the *fundamental sequence* for λ. It is possible to extend fundamental sequences to larger ordinals (cf. e.g. [8]). For example let $\omega_0=\omega$ and $\omega_{n+1}=\omega^{\omega_n}$. Then $\{\varepsilon_0\}(n)=\omega_n$ is a fundamental sequence for ε_0.

Now we are ready to define h_λ for λ limit. We simply put $h_\lambda(x)=h_{\{\lambda\}(x)}(x)$. The sequence $h_\alpha:\alpha<\varepsilon_0$ is called the *Hardy hierarchy based on h.*

This notion allows us to define a set A of natural numbers to be α*-large*. Namely, A is α-large iff $(h^A)_\alpha(a)$ is defined, where h^A denotes the successor in the sense of A (i.e., the function with domain $A\setminus\{\max A\}$ which associates with every b in its domain the next element of A) and $a=\min A$. We shall write just h if the meaning of A is clear from the context. One can restate this definition of largeness in the following manner. A set A is 0-large iff it is nonempty. A is $(\alpha+1)$-large iff $A\setminus\{\min A\}$ is α-large. A is λ-large, λ limit, iff it is $\{\lambda\}(\min A)$-large. Observe that Ketonen and Solovay [5] use a slightly different notion of largeness.

Let A be a finite subset of ω. We say that the partition $A=\cup_{0\le i\le e}B_i$ of A is α*-large* if the set $E=\{\min B_0,\ldots,\min B_e\}$ is α-large. A set or partition which is not α-large will be called α*-small*.

For ordinals $\alpha,\beta,\gamma<\varepsilon_0$ we write $\alpha\to(\beta)^1_\gamma$ if for every α-large set A with $\min A>0$ and every partition $A=\cup_{0\le i\le e}B_i$ of A which is γ-small, there exists $i\le e$ such that B_i is β-large. We keep the superscript 1 in the above notation just to follow the usual notation in Ramsey theory (cf. [4]).

For every $\alpha<\varepsilon_0$ let $\mathrm{LM}(\alpha)$ denote the greatest (i.e., leftmost) exponent in the Cantor normal form expansion of α. By $\varrho(\alpha)$ we mean the smallest (i.e., rightmost) exponent of α. We write $\beta\gg\alpha$ if either $\alpha=0$ or $\beta=0$ or all the exponents in the Cantor normal form of β are $\ge$ all the exponents in the normal form of α, i.e., $\varrho(\beta)\ge\mathrm{LM}(\alpha)$. Observe that $\beta\gg\alpha$ does not imply $\beta\ge\alpha$, indeed, $0\gg\alpha$ for each α and $\omega^4\gg\omega^4\cdot 3$. We should remark that if the relation $\beta\gg\alpha$ holds then the Cantor normal form of $\beta+\alpha$ is just the concatenation of the Cantor normal forms of β and α.

The main result of this paper is as follows (we needed it as the main combinatorial lemma in [1]).

THEOREM 1. *If* $\alpha,\beta<\varepsilon_0$, $\alpha\ge 1$ *and* $\beta\gg\mathrm{LM}(\alpha)$ *then* $\omega^\beta\cdot\alpha\to(\omega^\beta)^1_\alpha$.

We shall need several other notions. We extend the notion of a fundamental sequence to nonlimit ordinals by putting $\{0\}(n)=0$ and $\{\alpha+1\}(n)=\alpha$.

For $\beta, \alpha < \varepsilon_0$ we write $\beta \to_n \alpha$ iff there exists a finite sequence $\alpha_0, \ldots, \alpha_k$ of ordinals such that $\alpha_0 = \beta$, $\alpha_k = \alpha$ and for every $m < k$ there exists $j_m \le n$ such that $\alpha_{k+1} = \{\alpha_k\}(j_m)$. We write $\beta \Rightarrow_n \alpha$ if there exists a sequence as above, but with each $j_m = n$. Observe that both relations $\to_n$, $\Rightarrow_n$ are transitive and imply $\beta \ge \alpha$.

LEMMA 2. (i) *For every α, b, $\alpha \Rightarrow_b 0$.*

(ii) *If $\beta \gg \alpha$ and $\alpha \Rightarrow_n \gamma$ then $\beta + \alpha \Rightarrow_n \beta + \gamma$.*

(iii) *If $k < l$ and $n > 0$ then $\omega^\alpha \cdot l \Rightarrow_n \omega^\alpha \cdot k$.*

(iv) *If $\beta \Rightarrow_n \alpha$ and $n > 0$ then $\omega^\beta \Rightarrow_n \omega^\alpha$.*

(v) *$\alpha \Rightarrow_n \{\alpha\}(j)$ and $\{\alpha\}(n) \Rightarrow_n \{\alpha\}(j)$ for $j \le n$.*

(vi) *$\{\alpha\}(n) \Rightarrow_1 \{\alpha\}(j)$ for $0 < j \le n$.*

(vii) *If $n \le b$ and $\alpha \Rightarrow_n \beta$ then $\alpha \Rightarrow_b \beta$.*

(viii) *$\beta \Rightarrow_n \alpha$ iff $\beta \to_n \alpha$.*

(ix) *If $\alpha < \beta$ then there exists b such that $\beta \Rightarrow_b \alpha$.*

Proof. See Ketonen–Solovay [5]. As pointed out above, they work with slightly different fundamental sequences, but their proofs work in our case as well. In fact, (i)–(viii) are not very difficult to prove (in the order as stated); the proof of the last claim (by induction on β) uses (vii). ■

LEMMA 3. *Let λ be a limit ordinal smaller than ε_0. Then if $\beta \gg \mathrm{LM}(\lambda)$ then for every $n \in \omega$, $\{\omega^\beta \cdot \lambda\}(n) = \omega^\beta \cdot \{\lambda\}(n)$.*

Proof. Let λ be limit and let ϱ be the smallest exponent in the Cantor normal form expansion of λ. Then $\lambda = \delta + \omega^\varrho$ for some $\delta \gg \omega^\varrho$. Let $\beta \gg \mathrm{LM}(\lambda)$ and $n \in \omega$. We have

$$\{\omega^\beta \cdot \lambda\}(n) = \{\omega^\beta(\delta + \omega^\varrho)\}(n) = \{\omega^\beta \cdot \delta + \omega^{\beta+\varrho}\}(n) = \omega^\beta \cdot \delta + \{\omega^{\beta+\varrho}\}(n).$$

The last equality holds because $\omega^\beta \cdot \delta \gg \omega^{\beta+\varrho}$. Obviously $\beta \gg \varrho$, hence if $\varrho = \alpha + 1$ for some α then

$$\begin{aligned} \omega^\beta \cdot \delta + \{\omega^{\beta+\varrho}\}(n) &= \omega^\beta \cdot \delta + \{\omega^{\beta+\alpha+1}\}(n) = \omega^\beta \cdot \delta + \omega^{\beta+\alpha} \cdot n \\ &= \omega^\beta \cdot (\delta + \omega^\alpha \cdot n) = \omega^\beta \cdot (\delta + \{\omega^{\alpha+1}\}(n)) \\ &= \omega^\beta \cdot \{\delta + \omega^\varrho\}(n) = \omega^\beta \cdot \{\lambda\}(n). \end{aligned}$$

Let ϱ be limit. By the assumption $\beta \gg \varrho$ we get

$$\begin{aligned} \omega^\beta \cdot \delta + \{\omega^{\beta+\varrho}\}(n) &= \omega^\beta \cdot \delta + \omega^{\{\beta+\varrho\}(n)} = \omega^\beta \cdot \delta + \omega^{\beta+\{\varrho\}(n)} \\ &= \omega^\beta \cdot (\delta + \omega^{\{\varrho\}(n)}) = \omega^\beta \cdot (\delta + \{\omega^\varrho\}(n)) \\ &= \omega^\beta \cdot \{\lambda\}(n). \ ■ \end{aligned}$$

Lemma 4. *Let h be a function as above. Then for every $\alpha < \varepsilon_0$:*

(i) *h_α is increasing.*

(ii) *For every β, b if $\alpha \Rightarrow_b \beta$ then if $h_\alpha(b)$ exists then $h_\beta(b)$ exists and $h_\alpha(b) \geq h_\beta(b)$.*

Proof. By simultaneous induction on α, left to the reader. ∎

Below if we write $A = \{a_0, \ldots, a_{\mathrm{card}\, A-1}\}$ we assume that this enumeration is the natural one, i.e., in increasing order.

Lemma 5. (i) *For every α if A, B are finite sets of the same cardinality and such that for every $i < \mathrm{card}\, A$, $b_i \leq a_i$ then for every $i < \mathrm{card}\, A$ if $(h^A)_\alpha(a_i)$ exists then $(h^B)_\alpha(b_i)$ exists and $(h^A)_\alpha(a_i) \geq (h^B)_\alpha(b_i)$.*

(ii) *If A, B are finite sets, A is α-large, $\mathrm{card}\, A = \mathrm{card}\, B$ and for every $i < \mathrm{card}\, A$, $b_i \leq a_i$ then B is α-large.*

(iii) *If $A \subseteq B$ and A is α-large then B is α-large.*

Proof. The first part is immediate by induction on α, the second is a direct consequence of the first one. The third part follows from the observation that if $A \subseteq B$ then B has an initial segment of cardinality $\mathrm{card}\, A$. But obviously, if a set has an α-large initial segment then it is α-large itself, so the second part may be applied. ∎

The following is a minor variant of Lemma 5 in which we speak of sets of different cardinalities. We write $h_\alpha(x)\downarrow$ rather than "$h_\alpha(x)$ exists".

Lemma 6. *For every α and every D, E, if $D \subseteq E$, $x \in D$ and $(h^D)_\alpha(x)\downarrow$ then $(h^E)_\alpha(x)\downarrow$ and $(h^E)_\alpha(x) \leq (h^D)_\alpha(x)$.*

Proof (by induction on α). If $\alpha = 0$ the conclusion is obvious. Assume the conclusion for α; we derive it for $\alpha + 1$. So let D, E satisfy the assumption. Let $x \in D$ be such that $(h^D)_{\alpha+1}(x)$ exists. Then $(h^D)_{\alpha+1}(x) = (h^D)_\alpha((h^D)(x))$. Let $y = (h^D)(x)$. We apply the inductive assumption to y. Thus we infer $(h^E)_\alpha(y)\downarrow$ and $(h^E)_\alpha(y) \leq (h^D)_\alpha(y)$. But $(h^E)(x) \leq (h^D)(x) = y$, hence $(h^E)_{\alpha+1}(x) = (h^E)_\alpha((h^E)(x)) \leq (h^E)_\alpha((h^D)(x)) \leq (h^D)_\alpha((h^D)(x)) = (h^D)_{\alpha+1}(x)$ because $(h^E)_\alpha$ is increasing by Lemma 4. We leave the limit step to the reader. ∎

Lemma 7. *Let h be as above. Then for every α and every $\beta \gg \alpha$, $h_{\beta+\alpha} = h_\beta \circ h_\alpha$.*

Proof. By induction on α. ∎

Let us restate this fact in the following manner.

Lemma 8. *Let A be a finite set and let $\beta \gg \alpha$. Then A is $(\beta+\alpha)$-large iff there exists $u \in A$ such that $\{x \in A : x \leq u\}$ is α-large and $\{x \in A : u \leq x\}$ is β-large.* ∎

We shall need one more idea (once again, known from Ketonen–Solovay [5]). For every $\alpha < \varepsilon_0$ we define the *norm* of α, $\|\alpha\|$, in the following manner. We let $\|0\| = 0$. If $\alpha > 0$ we write $\alpha = \omega^{\alpha_0} \cdot a_0 + \ldots + \omega^{\alpha_r} \cdot a_r$ in the Cantor normal form and let $\|\alpha\| = \sum_{i=0}^{r} a_i \cdot (1 + \|\alpha_i\|)$. The following fact strengthens Lemma 2(ix) so that b may be chosen to depend only on α.

LEMMA 9. (i) *For every $\alpha < \varepsilon_0$ if $a \geq \|\alpha\|$ then for every $\beta > \alpha$ we have $\beta \Rightarrow_a \alpha$ and hence $\{\beta\}(a) \geq \alpha$.*

(ii) *For every α and every β, if $\varrho(\beta) > \alpha$ and $a = \|\alpha\|$ then $\{\beta\}(a) \gg \omega^\alpha$ and $\{\beta\}(a) + \omega^\alpha < \beta$.*

Proof. For (i) see Ketonen–Solovay [5]. We prove (ii). We write $\beta = \delta + \omega^{\beta'}$ where $\delta \gg \omega^{\beta'}$. By the assumption, $\beta' > \alpha$. If $\beta' = \beta'' + 1$ then $\{\beta\}(a) = \delta + \omega^{\beta''} \cdot a$ with $\beta'' \geq \alpha$, so the first conclusion is immediate. The second one follows from the fact that in the decisive step the exponent β' was changed to the smaller one, i.e. β''. If β' is limit then $\{\delta + \omega^{\beta'}\}(a) = \delta + \omega^{\{\beta'\}(a)}$. By (i), $\{\beta'\}(a) \geq \alpha$, so the first conclusion holds. The second does as well because in the decisive step the exponent β' was lowered to $\{\beta'\}(a)$. ∎

We shall need an additional lemma.

LEMMA 10. (i) $\forall \gamma > 0\ \forall b > 0\ \gamma \Rightarrow_b 1$.

(ii) $\forall \alpha \gg \omega\ \forall u > b > 1\ \{\alpha\}(u) \Rightarrow_b \{\alpha\}(b) + 1$.

(iii) $\forall \alpha \gg \omega\ \forall \delta \gg \omega^\alpha\ \forall u > b > 1\ \delta + \omega^{\{\alpha\}(u)} \Rightarrow_b \delta + \omega^{\{\alpha\}(b)} \cdot b$.

(iv) *If a set D is $(\delta + \omega^{\{\alpha\}(u)})$-large and $b = \min D$ satisfies $u > b > 1$ then D is $(\delta + \omega^{\{\alpha\}(b)} \cdot b)$-large.*

Proof. (i) is immediate by induction on γ. (ii) is proved by induction on α, the cases $\alpha = \omega$ and $\alpha \to \alpha + \omega$ being immediate, so we show only the step $\alpha \gg \omega^2$. Write $\alpha = \delta + \omega^\tau$, where $\delta \gg \omega^\tau$. Thus, $\tau > 1$. If $\tau = \varrho + 1$ then

$$\{\alpha\}(u) = \{\delta + \omega^{\varrho+1}\}(u) = \delta + \omega^\varrho \cdot u = \delta + \omega^\varrho \cdot b + \omega^\varrho \cdot (u - b).$$

We use (i) to infer $\{\alpha\}(u) \Rightarrow_b \{\alpha\}(b) + 1$ as required. So let τ be limit. Then $\{\tau\}(u) \Rightarrow_b \{\tau\}(b) + 1$ by the inductive assumption, so by Lemma 2(iii),

$$\{\alpha\}(u) = \{\delta + \omega^\tau\}(u) = \delta + \omega^{\{\tau\}(u)} \Rightarrow_b \delta + \omega^{\{\tau\}(b)+1}.$$

Moreover, $\omega^{\{\tau\}(b)+1} \Rightarrow_b \omega^{\{\tau\}(b)} \cdot b = \omega^{\{\tau\}(b)} + \omega^{\{\tau\}(b)} \cdot (b-1)$ and the same argument as above works.

(iii) follows from (ii) and Lemma 2(iii).

In order to prove part (iv), let D, u, b satisfy the assumption. That is, we have $h_{\delta + \{\omega^\alpha\}(u)}(\min D)\downarrow$. By (iii) and Lemma 4, $h_{\delta + \{\omega^\alpha\}(b) \cdot b}(\min D)\downarrow$ as required. ∎

The main lemma needed for the proof of Theorem 1 is as follows.

LEMMA 11. *For every* α, *every* $\beta \gg \omega^\alpha$ *and every* A, B, *if* $\min A > 0$ *and* A *is* $(\beta + \omega^\alpha)$*-large*, $B \subseteq A$ *and* B *is* ω^α*-small then* $A \setminus B$ *is* β*-large.*

Proof. Let $T(\beta, \alpha)$ be the following property:

for every A, B, if A is $(\beta + \omega^\alpha)$-large and $B \subseteq A$ is ω^α-small then $A \setminus B$ is β-large

and we shall prove the statement $\forall\alpha\ \forall\beta \gg \omega^\alpha\ T(\beta, \alpha)$ by induction on α.

CASE $\alpha = 0$. Then A is $(\beta + \omega^0)$-large, i.e. $(\beta + 1)$-large, and B is 1-small. If $\beta = 0$ then A is 1-large, i.e., has at least two elements, but B being 1-small has at most one element, so $A \setminus B$ is nonempty, so 0-large. If $\beta > 0$ then A is $(\beta + 1)$-large so $A \setminus \{a_0\}$ is β-large. Also, B being 1-small has at most one element. It follows that $A \setminus \{a_0\}$ and $A \setminus B$ satisfy the assumption of Lemma 5 (these sets have the same cardinality and the ith element of $A \setminus B$ is $\le$ the ith element of $A \setminus \{a_0\}$), hence $A \setminus B$ is β-large.

CASE $\alpha = 1$. Exactly as above, the case $\beta = 0$ is obvious. For other cases we proceed by induction on β.

Let $\beta = \omega$. So let A be $(\omega + \omega)$-large and let $B \subseteq A$ be ω-small. Let $u = (h^A)_\omega(a_0)$.

Case 1: $b_0 > u$. Then $\{x \in A : x \le u\} \subseteq A \setminus B$. The first of these sets is ω-large, so the second is as well by Lemma 5.

Case 2: $b_0 = u$. Then there exists $z \in A \setminus B$ with $z > u$ (otherwise $\{x \in A : u \le x\} \subseteq B$, so B is ω-large by Lemma 5). It follows that $\{x \in A : x < u\} \cup \{z\} \subseteq A \setminus B$, so this set is ω-large, again by Lemma 5.

Case 3: $a_0 < b_0 < u$. In order to show that $A \setminus B$ is ω-large it suffices to show that it has more than a_0 elements, indeed, $\min(A \setminus B) = a_0$. But A has more than $a_0 + u$ elements and B, being ω-small, has at most $b_0 < u$ elements.

Case 4: $a_0 = b_0$. Then B has at most a_0 elements, so $A \setminus B$ has more than u elements. If $\min(A \setminus B) \le u$ then we are done. Otherwise $\{x \in A : x \le u\} \subseteq B$, so this set is ω-large, contrary to assumption.

Assume $T(\beta, 1)$; we prove $T(\beta + \omega, 1)$. So let A be $(\beta + \omega + \omega)$-large and let $B \subseteq A$ be ω-small. Let $u = (h^A)_\omega(a_0)$ and $w = (h^A)_\omega(u)$.

Case 1: $b_0 > u$. Let $A' = A \setminus \{x \in A : x < u\}$. Thus $B \subseteq A'$. By $T(\beta, 1)$, $A' \setminus B$ is β-large, hence $\{x \in A : x < u\} \cup (A' \setminus B) = A \setminus B$ is $(\beta + \omega)$-large by Lemma 8.

Case 2: $b_0 = u$. Let A' be $\{x \in A : u \le x\}$. By $T(\beta, 1)$, the set $C = A' \setminus B$ is β-large. Let c_0 be, as usual, the smallest element of C. Then $A \setminus B = \{x \in A : x < u\} \cup C = (\{x \in A : x < u\} \cup \{c_0\}) \cup C$ is $(\beta + \omega)$-large by Lemma 8.

Case 3: $a_0 < b_0 < u$. Obviously, B has at most b_0 elements (otherwise it is ω-large), so B has less than u elements. Let $k = \mathrm{card}(\{x \in A : w < x\})$. Thus, $A \setminus B$ has at least $a_0 + k + 1$ elements. Let $c_0 = a_0, c_1, \ldots, c_{a_0}$ be the list of the first $a_0 + 1$ elements of $A \setminus B$ in increasing order. We claim that $c_{a_0} \le w$. For otherwise there are at least k elements of $A \setminus B$ which are $> c_{a_0} > w$. But this is impossible, as there are only $k-1$ such elements of A. Let $E = \{e_0, \ldots, e_k\}$ be the set of the $k+1$ consecutive elements of $A \setminus B$, beginning with $e_0 = c_{a_0}$. Then E is β-large, indeed, its cardinality is $k+1$ and its elements are $\le$ the corresponding elements of $\{x \in A : w \le x\}$. It follows that $A \setminus B$ contains $\{c_0, \ldots, c_{a_0}\} \cup E$, so it is $(\beta + \omega)$-large by Lemma 8.

Case 4: $b_0 = a_0$. Then B, being ω-small, has at most a_0 elements, hence $A \setminus B$ has more than $u + k$ elements. Let $E = \{e_0, \ldots, e_{u+k}\}$ be the set of the first $u+k+1$ of them. Then E is $(\beta+\omega)$-large because its elements are $\le$ the corresponding elements of $\{x \in A : u \le x\}$.

Assume $\forall \beta' < \beta\ T(\beta', 1)$ and $\varrho(\beta) > 1$; we check $T(\beta, 1)$. So let A be $(\beta + \omega)$-large. Let $u = (h^A)_\omega(a_0)$ as usual, so $A = \{x \in A : x \le u\} \cup \{x \in A : u \le x\}$. The first of these sets is ω-large and the second one is β-large, i.e., $\{\beta\}(u)$-large. As $\varrho(\beta) > 1$ we have (i) $\{\beta\}(u) \gg \omega$ and (ii) $\{\beta\}(u) + \omega < \beta$. Let B be an ω-small subset of A. The set A is $(\{\beta\}(u) + \omega)$-large and by $T(\{\beta\}(u), 1)$, $A \setminus B$ is $\{\beta\}(u)$-large. Observe that $\min(A \setminus B) = c_0 \le u$, for otherwise $\{x \in A : x \le u\} \subseteq B$, so B is ω-large contrary to assumption. If $c_0 = u$ then obviously $A \setminus B$ is β-large, so assume that $c_0 < u$. By Lemma 2(vi), $\{\beta\}(u) \Rightarrow_1 \{\beta\}(c_0)$, so $\{\beta\}(u) \Rightarrow_{c_0} \{\beta\}(c_0)$ by (vii) of the same lemma. By Lemma 4, $(h^{A\setminus B})_{\{\beta\}(c_0)}(c_0)$ exists, so $A \setminus B$ is β-large.

We show the nonlimit step in the proof of Lemma 11, i.e.,

$$\forall \alpha\ [(\forall \beta \gg \omega^\alpha\ T(\beta, \alpha)) \Rightarrow (\forall \beta \gg \omega^{\alpha+1}\ T(\beta, \alpha+1))].$$

Once again, the case $\beta = 0$ is obvious. Indeed, if A is $\omega^{\alpha+1}$-large and B is its $\omega^{\alpha+1}$-small subset, then $A \setminus B$ is nonempty, so 0-large.

Case $\beta = \omega^{\alpha+1}$. Let A be $(\omega^{\alpha+1} + \omega^{\alpha+1})$-large. Let $u = (h^A)_{\omega^{\alpha+1}}(a_0)$. Then $A = \{x \in A : x \le u\} \cup \{x \in A : u \le x\}$ and both of these sets are $\omega^{\alpha+1}$-large. Let B be an $\omega^{\alpha+1}$-small subset of A.

Case 1: $b_0 > u$. Then $\{x \in A : x \le u\}$ is contained in $A \setminus B$, so this set is $\omega^{\alpha+1}$-large.

Case 2: $b_0 = u$. Then there exists $z \in A \setminus B$ with $z > u$ (otherwise $\{x \in A : u \le x\} \subseteq B$ and hence B is $\omega^{\alpha+1}$-large, which contradicts the assumption), so $\{x \in A : x < u\} \cup \{z\}$ is contained in $A \setminus B$, so this set is $\omega^{\alpha+1}$-large.

Case 3: $a_0 < b_0 < u$. We let $c_0 = b_0 = \min B$ and $c_{i+1} = (h^B)_{\omega^\alpha}(c_i)$. This induction breaks after r steps, where $r \le b_0$, otherwise B is $\omega^{\alpha+1}$-large. That is, the last c_i is c_{r-1}. We let $A_0 = A$ and $A_{i+1} = A_i \setminus \{x \in B : c_i \le x < c_{i+1}\}$ and $A_r = A_{r-1} \setminus \{x \in B : c_{r-1} \le x\}$. Observe that A_0 is $(\omega^\alpha \cdot (a_0 + u))$-

large, and (by the inductive assumption), A_i is $(\omega^\alpha \cdot (a_0 + u - i))$-large. In particular, A_r is $(\omega^\alpha \cdot a_0)$-large, i.e., $\omega^{\alpha+1}$-large, indeed $a_0 = \min(A \setminus B)$.

Case 4: $b_0 = a_0$. Arguing as in case 3 we see that $A \setminus B$ is $(\omega^\alpha \cdot u)$-large. Thus if $\min(A \setminus B) = u$ this set is $\omega^{\alpha+1}$-large. If $d = \min(A \setminus B) < u$ then $\{\omega^{\alpha+1}\}(u) \Rightarrow_1 \{\omega^{\alpha+1}\}(d)$ by Lemma 2(vi), hence $\omega^\alpha \cdot u \Rightarrow_d \omega^\alpha \cdot d$ by (vii) of the same lemma. By Lemma 4, $(h^{A\setminus B})_{\omega^\alpha \cdot d}(d)$ exists (because $(h^{A\setminus B})_{\omega^\alpha \cdot u}(d)$ exists).

We prove the implication $T(\beta, \alpha+1) \Rightarrow T(\beta+\omega^{\alpha+1}, \alpha+1)$ for $\beta \gg \omega^{\alpha+1}$. So let A be $(\beta + \omega^{\alpha+1} \cdot 2)$-large and let B be its $\omega^{\alpha+1}$-small subset. Let $u = (h^A)_{\omega^{\alpha+1}}(a_0)$ and $w = (h^A)_{\omega^{\alpha+1}}(u)$.

Case 1: $b_0 \ge u$. Let $A' = \{x \in A : x \ge u\}$. Then $B \subseteq A'$. By $T(\beta, \alpha+1)$, $A' \setminus B$ is β-large, hence $A \setminus B = (\{x \in A : x < u\} \cup \{c_0\}) \cup (A' \setminus B)$, where $c_0 = \min(A' \setminus B)$, is $(\beta + \omega^{\alpha+1})$-large.

Case 2: $a_0 < b_0 < u$. We put $d_0 = b_0 = \min B$ and $d_{i+1} = (h^B)_{\omega^\alpha}(d_i)$. Let r be the greatest i such that d_i exists. We must have $r < b_0$ for otherwise B would be $\omega^{\alpha+1}$-large. Let $D_i = \{x \in B : d_i \le x < d_{i+1}\}$ and $D_r = \{x \in B : d_r \le x\}$. Observe that none of these sets is ω^α-large. On the other hand, A is $(\beta+\omega^\alpha(u+a_0))$-large. It follows that $A\setminus D_0$ is $(\beta+\omega^\alpha\cdot(u+a_0-1))$-large, etc., $A \setminus B = A \setminus \cup_{i\le r} D_i$ is $(\beta + \omega^\alpha \cdot (u + a_0 - r))$-large. But $r+1 \le u$, hence $A \setminus B$ is $(\beta + \omega^\alpha \cdot a_0)$-large, so it is $(\beta + \omega^{\alpha+1})$-large because its minimum is a_0.

Case 3: $b_0 = a_0$. Exactly as above, by subtracting B from A in parts which are not ω^α-large we derive that $A\setminus B$ is $(\beta+\omega^\alpha\cdot u)$-large. Indeed, there are only a_0 parts as above because $\min B = a_0$ and this set is $\omega^{\alpha+1}$-small. If $\min(A \setminus B) = u$ then we are done. Otherwise $e = \min(A \setminus B) < u$. But $\omega^\alpha \cdot u \Rightarrow_1 \omega^\alpha \cdot e$ by Lemma 2(vi), and hence $\omega^\alpha \cdot u \Rightarrow_e \omega^\alpha \cdot e$ by (vii) of the same lemma. By Lemma 4, $(h^{A\setminus B})_{\omega^\alpha \cdot e}(e)$ exists because $(h^{A\setminus B})_{\omega^\alpha \cdot u}(e)$ exists.

Thus in order to prove the nonlimit step $\alpha+1$ in the proof of Lemma 11 it remains to check the case $\varrho(\beta) > \alpha + 1$. So let $\varrho(\beta) > \alpha + 1$ and assume that for all $\beta' < \beta$, $T(\beta', \alpha + 1)$ holds. Let A be $(\beta + \omega^{\alpha+1})$-large and let B be its $\omega^{\alpha+1}$-small subset. As usual, we let $u = (h^A)_{\omega^{\alpha+1}}(a_0)$, so that $A = \{x \in A : x \le u\} \cup \{x \in A : u \le x\}$; the first of these sets is $\omega^{\alpha+1}$-large, the second being β-large. It follows that A is $(\{\beta\}(u) + \omega^{\alpha+1})$-large. Observe that $u = (h^A)_{\omega^{\alpha+1}}(a_0) \ge \|\alpha + 1\|$. By Lemma 9, $\{\beta\}(u) \gg \omega^{\alpha+1}$ and $\{\beta\}(u)+\omega^{\alpha+1} < \beta$. By $T(\{\beta\}(u), \alpha+1)$, $A\setminus B$ is $\{\beta\}(u)$-large. Observe that $\min(A\setminus B) = c \le u$, otherwise $\{x \in A : x \le u\} \subseteq B$, so B is $\omega^{\alpha+1}$-large, contrary to assumption. If $c = u$ then we are done, $A\setminus B$ is $\{\beta\}(\min(A\setminus B))$-large. So assume that $c < u$. Then $(h^{A\setminus B})_{\{\beta\}(u)}(c)\downarrow$. Also we have $\{\beta\}(u) \Rightarrow_1 \{\beta\}(c)$ by Lemma 2(vi), hence, by Lemma 2(vii), $\{\beta\}(u) \Rightarrow_c \{\beta\}(c)$. By Lemma 4, $(h^{A\setminus B})_{\{\beta\}(c)}(c)\downarrow$ and $A \setminus B$ is β-large.

Case α limit. So, by assumption we have $\forall\alpha' < \alpha\ \forall\beta \gg \omega^{\alpha'}\ T(\beta,\alpha')$; we want to prove $\forall\beta \gg \omega^{\alpha}\ T(\beta,\alpha)$. As usual, the case $\beta = 0$ is obvious.

Let $\beta = \omega^{\alpha}$. Let A be $(\omega^{\alpha}+\omega^{\alpha})$-large and let B be its ω^{α}-small subset. As usual, let $u = (h^A)_{\omega^{\alpha}}(a_0)$.

Case 1: $b_0 > u$. Then $\{x \in A : x \le u\} \subseteq A \setminus B$, so this set is ω^{α}-large as required.

Case 2: $b_0 = u$. Then there exists $z > u$ with $z \in A \setminus B$, for otherwise $\{x \in A : u \le x\} \subseteq B$, so B is ω^{α}-large contrary to assumption. Thus $\{x \in A : x < u\} \cup \{z\} \subseteq A \setminus B$ and $A \setminus B$ is ω^{α}-large.

Case 3: $a_0 < b_0 < u$ (the main case). Let $D = \{x \in A : b_0 \le x\}$ and let $E = \{x \in A : u \le x\}$. Then E is ω^{α}-large, i.e. it is $\omega^{\{\alpha\}(u)}$-large. It follows that D is $\omega^{\{\alpha\}(u)}$-large, indeed, it contains E. By Lemma 10(iv), D is $(\{\omega^{\alpha}\}(b_0)\cdot b_0)$-large, in particular, it is $(\{\omega^{\alpha}\}(b_0)+\{\omega^{\alpha}\}(b_0))$-large (reason: $a_0 < b_0$, hence $b_0 > 1$). We apply the inductive assumption $T(\{\omega^{\alpha}\}(b_0),\{\alpha\}(b_0))$ and infer that $D \setminus B$ is $\{\omega^{\alpha}\}(b_0)$-large. By Lemma 5(iii), $A \setminus B$ is $\{\omega^{\alpha}\}(b_0)$-large. We also have $\{\omega^{\alpha}\}(b_0) \Rightarrow_{a_0} \{\omega^{\alpha}\}(a_0)$ by Lemma 2(vii), hence $A \setminus B$ is $\{\omega^{\alpha}\}(a_0)$-large, i.e., ω^{α}-large.

Case 4: $b_0 = a_0$. In this case B is $\{\omega^{\alpha}\}(a_0)$-small. But A is $(\{\omega^{\alpha}\}(u)+\{\omega^{\alpha}\}(a_0))$-large. By the inductive assumption $T(\{\omega^{\alpha}\}(u),\{\alpha\}(a_0))$, $A \setminus B$ is $\{\omega^{\alpha}\}(u)$-large. Let $s = \min(A \setminus B)$. If $s = u$ then we are done. If $s < u$ then $\{\omega^{\alpha}\}(u) \Rightarrow_s \{\omega^{\alpha}\}(s)$, hence $A \setminus B$ is $\{\omega^{\alpha}\}(s)$-large, i.e., ω^{α}-large. The case $s > u$ cannot happen, for if it does then $\{x \in A : x \le u\} \subseteq B$, so B is ω^{α}-large, contrary to assumption.

Assume $T(\beta,\alpha)$, where $\beta \gg \omega^{\alpha}$; we prove $T(\beta+\omega^{\alpha},\alpha)$. So let a set A be $(\beta+\omega^{\alpha}+\omega^{\alpha})$-large and let B be its ω^{α}-small subset. Let u, w be as before, i.e., $u = (h^A)_{\omega^{\alpha}}(a_0)$ and $w = (h^A)_{\omega^{\alpha}}(u)$.

Case 1: $b_0 \ge u$. Then $B \subseteq \{x \in A : u \le x\}$ and by the inductive assumption $T(\beta,\alpha)$, $\{x \in A \setminus B : u \le x\}$ is β-large. It follows that $A \setminus B = \{x \in A : x \le u\} \cup \{x \in A \setminus B : u \le x\}$ is $(\beta+\omega^{\alpha})$-large.

Case 2: $a_0 < b_0 < u$. Let $E = \{x \in A : u \le x\}$ and $D = \{x \in A : b_0 \le x\}$. Then E is $(\beta+\omega^{\alpha})$-large, hence it is $(\beta+\{\omega^{\alpha}\}(u))$-large. It follows that D is $(\beta+\{\omega^{\alpha}\}(u))$-large as well. Exactly as above, it follows that D is $(\beta+\{\omega^{\alpha}\}(b_0)\cdot b_0)$-large, hence it is $(\beta+\{\omega^{\alpha}\}(b_0)\cdot 2)$-large. By the inductive assumption $T(\beta+\{\omega^{\alpha}\}(b_0),\{\alpha\}(b_0))$, $D \setminus B$ is $(\beta+\{\omega^{\alpha}\}(b_0))$-large, i.e. $(\beta+\omega^{\alpha})$-large. Hence $A \setminus B$ is $(\beta+\omega^{\alpha})$-large as a superset of $D \setminus B$.

Case 3: $b_0 = a_0$. Then B is $\{\omega^{\alpha}\}(a_0)$-small. By the inductive assumption $T(\beta+\omega^{\alpha}+\{\omega^{\alpha}\}(a_0),\{\alpha\}(a_0))$, $A \setminus B$ is $(\beta+\omega^{\alpha})$-large.

Finally, let $\varrho(\beta) > \alpha$. Let, as usual, A be $(\beta+\omega^{\alpha})$-large and let B be its ω^{α}-small subset. Let also $u = (h^A)_{\omega^{\alpha}}(a_0)$. Clearly $u \ge \|\alpha+1\|$, hence A is $(\{\beta\}(u)+\omega^{\alpha})$-large. By Lemma 9, $\{\beta\}(u) \gg \omega^{\alpha}$ and $\{\beta\}(u)+\omega^{\alpha} < \beta$. By the inductive assumption $T(\{\beta\}(u),\alpha)$, $A \setminus B$ is $\{\beta\}(u)$-large. Let $s = \min(A \setminus B)$.

Exactly as above, $s \le u$ for otherwise $\{x \in A : x \le u\} \subseteq B$, so B is ω^α-large contrary to assumption. If $s = u$ then we are done. Otherwise, $s < u$, hence $\{\beta\}(u) \Rightarrow_s \{\beta\}(s)$, so $A \setminus B$ is $\{\beta\}(s)$-large, i.e. β-large. ■

It should be noticed that Lemma 11 admits a generalization in which we speak not only about ordinals of the form ω^α. It is as follows.

THEOREM 12. *For every α and $\beta \gg \alpha$ and every A, B, if A is $\beta+\alpha$-large, $B \subseteq A$ and B is α-small, then $A \setminus B$ is β-large.*

Proof. Let A be $(\beta + \alpha)$-large where $\beta \gg \alpha$, and let B be its α-small subset. Write $\alpha = \omega^{\alpha_s} + \ldots + \omega^{\alpha_0}$, where $\alpha_s \ge \ldots \ge \alpha_0$. Let $e = \max\{i \le s : B$ is $(\omega^{\alpha_i} + \ldots + \omega^{\alpha_0})$-large$\}$. Let h denote the successor in the sense of B. Let $B_0 = \{x \in B : x < h_{\omega^{\alpha_0}}(\min B)\}$, $B_{i+1} = \{x \in B : h_{\omega^{\alpha_i}}(\min B_i) \le x < h_{\omega^{\alpha_{i+1}}}(h_{\omega^{\alpha_i}}(\min B_i))\}$ for $i < e$. We let $B_{e+1} = B \setminus \cup_{0\le i\le e} B_i$. Then $B = \cup_{0\le i\le e+1} B_i$. Observe that no B_i, $i \le e+1$, is ω^{α_i}-large. By Lemma 11, by induction on i, we infer that $A \setminus (B_0 \cup \ldots \cup B_i)$ is $(\beta + \omega^{\alpha_s} + \ldots + \omega^{\alpha_{i+1}})$-large. It follows that $A \setminus B$ is β-large. ■

Proof of Theorem 1. By induction on α. The case $\alpha = 1$ is obvious, indeed, if a partition is 1-small then there is only one part.

Assume the conclusion holds for α; we derive it for $\alpha + 1$. Let A be an $(\omega^\beta \cdot (\alpha + 1))$-large subset of ω and let $A = \cup_{0\le i\le e} B_i$ be an $(\alpha + 1)$-small partition of A. Let $E = \{\min B_0, \ldots, \min B_e\}$, so E is $(\alpha + 1)$-small. We may assume that $\min E = \min B_0$. We put $C = A \setminus B_0$. If B_0 is ω^β-small then by Lemma 11, C is $(\omega^\beta \cdot \alpha)$-large. Consider the partition $C = \cup_{1\le i\le e} B_i$ of C. Let $E_1 = \{\min B_1, \ldots, \min B_e\}$. But the partition of A is $(\alpha+1)$-small, hence $h_{\alpha+1}(\min B_0)\uparrow$ (where h denotes the successor in the sense of E). It follows that $h_\alpha(h(\min B_0))\uparrow$. We have $h(\min B_0) = \min E_1$. Thus the above partition of C is α-small. We apply the inductive assumption to the set C and the above-mentioned partition. Summing up, B_0 or at least one of $B_i, 1 \le i \le e$, is ω^β-large.

Assume the conclusion for all ordinals smaller than λ, λ limit. Let A be an $(\omega^\beta \cdot \lambda)$-large set, where $\beta \gg \mathrm{LM}(\lambda)$. Let a partition $A = \cup_{0\le i\le e} B_i$ be given and λ-small. Exactly as above, let $E = \{\min B_0, \ldots, \min B_e\}$. Then A is $(\{\omega^\beta \cdot \lambda\}(\min A))$-large. Thus A is $(\omega^\beta \cdot \{\lambda\}(\min A))$-large by Lemma 3. Obviously, E is $\{\lambda\}(\min A)$-small and $\beta \gg \mathrm{LM}(\{\lambda\}(\min A))$. By the inductive assumption, at least one of B_i, $i \le e$, is ω^β-large. ■

We show that the result of Theorem 1 is the best possible. Let A be a finite subset of ω, let $\beta < \varepsilon_0$ and let $A = \cup_{0\le i\le e} B_i$ be the partition of A determined by the following conditions:

(i) Each B_i is of the form $A \cap [u, w]$ for some $u, w \in A$.
(ii) $\min B_0 = \min A$ and for all $i = 0, \ldots, e-1$, $\min B_{i+1} = h_{\omega^\beta}(\min B_i)$.
(iii) $h_{\omega^\beta}(\min B_e)\uparrow$.

Of course, h denotes the successor in the sense of A. Let, as usual, E denote the set $\{\min B_i : i \le e\}$. Let H denote the successor in the sense of E. Obviously, $H(n) = h_{\omega^\beta}(n)$ for $n \in E \setminus \{\max E\}$. We show that

(∗) if A is $(\omega^\beta \cdot \alpha)$-large, where $\beta \gg \mathrm{LM}(\alpha)$, then $H_\alpha(\min A) = h_{\omega^\beta \cdot \alpha}(\min A)$.

We prove (∗) by induction on α, the steps $\alpha = 0$ and $\alpha \to \alpha + 1$ being evident. In the limit step one uses Lemma 3.

If A is $(\omega^\beta \cdot (\alpha+1))$-small, where $\beta \gg \varrho(\alpha)$, then by (∗) we infer immediately that the partition of A determined by the above mentioned conditions is $(\alpha+1)$-small. But none of the sets B_i is ω^β-large.

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