avec le fait que la distance de p_1 à tous les K_{α_n} est supérieure à $\delta > 0$.

Le théorème est ainsi démontré.

Les deux théorèmes que nous venons de démontrer peuvent être réunis de manière que l'on dise: dans les conditions 1°, 2°, 3°, 4° la décomposition $\{K\}$ est *partout localement continue*, si l'on définie ce mot par les deux propriétés spécifiées dans les thèses.

A Theorem on Arbitrary Functions of Two Variables with Applications.

By

Henry Blumberg (Columbus, Ohio, U. S. A.).

The object of the present paper is to enunciate a theorem — on unrestricted real functions of two variables — which, somewhat unexpectedly, appears as a common source of a number of theorems — on arbitrary real functions of a single variable — that have appeared, from time to time, in the literature without in themselves readily suggesting a common origin [1]). It turns out that these special theorems are obtainable by considering *particular* real interval functions [2]) associated with a given function; whereas the theorem of the present paper yields an analogous result for *every* interval function.

Let $f(x, y)$, then, be an arbitrary, real, one-valued [3]) function of two real variables defined in an entire plane π. Let s be a straight line of π, and d, a given direction in π. If P is a point of s, we denote by l_{Pd}, u_{Pd} the *lim inf*, *lim sup* of f as (x, y) approaches P along the direction d; and by $I_{Pd} = (l_{Pd}, u_{Pd})$ the interval of approach of f at P along d [4]). We may now state the

[1]) See Examples 1, 2, 4 and 5 below.

[2]) The argument of an interval function is a linear interval, or the pair of its end points, their order being disregarded. If the interval function is real, the dependent variable is a real number.

[3]) The condition of one-valuedness is inserted for simplicity of statement. The argument holds essentially for many-valued functions also, the requisite modifications in statement for the latter case being evident.

[4]) We permit l_{Pd} and u_{Pd} to be $\pm\infty$.

Theorem. *If $f(x, y)$ is an arbitrary real function, defined in a plane π; s, a straight line in π; and d_1, d_2 two directions of approach to s on the same side of it, then I_{Pd_1} overlaps or abuts I_{Pd_2} for every point P of s with the possible exception of $\aleph_0$ points. In other words $u_{Pd_1} \geqq l_{Pd_2}$ for all except possibly $\aleph_0$ points of s.*

Proof. Let E_k be the set of points of s where $l_{Pd_1} \geqq k$, a given real number. If P is a point of E_k, and n a positive integer, there exists an interval $P_n P$ of length $< \frac{1}{n}$ and having the direction d_1, such that $f(x, y) > k - \frac{1}{n}$ for every point (x, y) of $P_n P$ — except possibly P. Let J_{Pn} be the projection of the closed interval $P_n P$ upon s in the direction d_2, and D_{kn} the set of points of s that are end points of one or more of the J_{Pn} (n fixed) as P ranges over E_k, but are interior to none of these J_{Pn}. It follows that D_{kn} ist at most denumerable. Hence $D_k = \sum_{n=1}^{\infty} D_{kn}$ is at most denumerable. If P is a point of E_k not belonging to D_k, it must be interior to some J_{Pn} for every n. It follows that $u_{Pd_2} \geqq k$. Let k now take the values r, r ranging over the set R of rational numbers. Then $D = \Sigma D_r$ (r ranging over R) is a denumerable set (at most). If P is a point of s such that $l_{Pd_1} > u_{Pd_2}$ there is a value r of R such that $l_{Pd_1} > r > u_{Pd_2}$ and therefore P belongs to D_r. Consequently every point P of s for which $l_{Pd_1} > u_{Pd_2}$ belongs to $\Sigma D_r = D$, that is, to a denumerable set.

Remark. The set of points P such that I_{Pd_1} abuts I_{Pd_2} need not be denumerable and, indeed, every point of s may have this character, as the following example shows: Let s be the x-axis and let $X = M_1 + M_2$ be a subdivision of X, the set of points on the x-axis, into two everywhere dense components. Confining ourselves to the half-plane above the x-axis, we term a line l inclined at a given angle α_1, to the x-axis, and passing through a point of M_1 or M_2 as of type (1, 1) or (1, 2) respectively; similarly, if a line l inclined at another given angle α_2 passes through a point of M_1 or M_2, we term it of type (2, 1) or (2, 2) respectively. If $P = (x, y)$, a point above the x-axis, lies on two lines of type (1, 1), (2, 2), we set $f(x, y) = 1$; if on two lines of type (1, 1), (2, 1) or (1, 2), (2, 2), we set $f(x, y) = 0$; finally, if on two lines of type (1, 2) (2, 1), we set $f(x, y) = -1$. If P is in M_1, we have $l_{Pd_1} = 0$, $u_{Pd_1} = 1$; $l_{Pd_2} = -1$, $u_{Pd_2} = 0$. If P is in M_2, we have $l_{Pd_1} = -1$, $u_{Pd_2} = 0$, $l_{Pd_2} = 0$, $u_{Pd_2} = 1$. Therefore, the intervals I_{Pd_1}, I_{Pd_2} abut at every point, but overlap at no point P of the x-axis.

If $\{d_n\}$ is a sequence of directions of approach to s on one and the same side of it, then $D_{mn}\{m, n = 1, 2, \ldots \infty\}$, the set of points P of s where $u_{Pd_m} < l_{Pd_n}$ is at most denumerable, and therefore $D = \sum_{m,n=1}^{\infty} D_{mn}$ is at most denumerable[5]). Hence we have the following

Corollary I. *If $f(x, y)$ is an arbitrary real function; s, a given straight line; and $\{d_n\}$, $n = 1, 2 \ldots \infty$, a sequence of directions of approach to s on the same side of it, then there exists a denumerable set D, such that if P is a point of s not in D, we have $u_{Pd_m} \geqq l_{Pd_n}$ for every pair of integers m, n. In other words, for a point P not in D, the intervals I_{Pd_n} $\{n = 1, 2, \ldots \infty\}$ overlap or abut one another.*

It is easy to see that this result is exhaustive in a sense presently to be made more precise. For suppose π is the xy-plane, and s, for simplicity, assumed to be the x-axis. Let $D = \{P_n\}$ be any given denumerable subset of s, and $\{d_n\}$ a given sequence of directions of approach to s from the upper half-plane of π, which we denote by π'[6]). It is possible to draw, for each P_n, a circle C_n tangent to s at P_n and with center in π, in such a way that no two C_n have points in common. Let $K_{mn}\{m, n = 1, 2, \ldots \infty\}$ be the chord of the circle C_m having the direction d_n and terminating in P_m. Since no pair of the chords K_{mn} intersect in π', the intervals of approach $I_{P_m d_n}$ can be made whatever we please, and quite independently of one another, by means of an appropriate definition of $f(x, y)$ on these chords. We are still free to define $f(x, y)$ elsewhere as we please. Denote by K the set of points on the K_{mn}, and by $\overline{K}$, the complement of K with respect to π'. Regarding, then, $f(x, y)$ as defined on K so as to make $I_{P_m d_n} = J_{mn}$ where $\{J_{mn}\}$, $\{m, n = 1, 2, \ldots \infty\}$, is an arbitrarily given set of real intervals, we define $f(x, y) = -\frac{1}{y}$ in $\overline{K}$. If P is a point of s not belonging to D, and d is a direction of approach to P from π', it

[5]) D, of course, has a different meaning from that above.

[6]) It is understood here that the points of s do not belong to π'.

follows, from the denumerability of the K_{mn}, that $l_{Pd} = -\infty$; therefore I_{Pd_m} and I_{Pd_n} overlap or abut for every pair of positive integers m, n. We have thus proved the

Converse of Corollary I. *Let s be a given straight line in the xy-plane; $D = \{P_n\}$ a denumerable set on s; $\{d_n\}$, a denumerable set of directions of approach to s on one and the same side of it; and I_{mn} a given real interval associated with the pair of integers m, n $\{= 1, 2, \dots \infty\}$. Then there exists a function $f(x, y)$ such that $I_{P_m d_n} = I_{mn}$, and if P is a point of s not in D, all the intervals I_{Pd_n} $\{n = 1, 2, \dots\}$ overlap or abut. In particular, if $D = \{P_n\}$ and $\{d_n\}$ are given, there exists a function $f(x, y)$ such that on the one hand, for all n and $\mu \neq \nu$, $I_{P_n d_\mu}$ and $I_{P_n d_\nu}$ have no points in common; while on the other hand, for every P of s not in D, and for every μ, ν the intervals I_{Pd_μ} and I_{Pd_ν} overlap or abut.*

For the applications we have in mind to functions of one variable, we shall make use of the following corollary of our theorem:

Corollary II. *If $F(\xi, x)$ is a symmetric function in its arguments, then for all x, with the possible exception of a denumerable set, we have* $\lim\limits^{sup}_{inf} F(\xi, \xi - 0) \gtreqless \lim\limits^{inf}_{sup} F(\xi, \xi + 0)$.

This corollary is the particular form our theorem takes when π is the ξx-plane, of rectangular cartesian coordinates; s, the 45^0 line through the origin; and d_1, d_2 the positive y- and negative x-directions of approach. For, on account of the assumed symmetry, $F(\xi, \xi + 0) = F(\xi + 0, \xi)$.

We shall utilize Corollary II more particularly in the form it assumes when *$F(\xi, x)$ is taken to be a real interval function*, i. e., a variable real number associated with the variable interval $(\xi, x) = (x, \xi)$. Such an interval function may, for example, be defined in relation to a given set S or a given function $f(x)$.

Example 1. Let $F(\xi, x) = \dfrac{f(x) - f(\xi)}{x - \xi}$. Then we have the

(Theorem of G. C. Young) [7]. *The upper right (left) derivative of an arbitrary function is greater than or equal to the lower left (right) derivative, with the possible exception of a denumerable number of points.*

[7]) G. C. Young, Acta Mathematica, vol. 37 (1914), p. 147; W. Sierpiński, Bull. Acad. Sc. Cracovie 1912, p. 850; see Fund. Math., vol. IV, p. 809.

Example 2. Let $F(\xi, x) = u_0(f; \xi, x) = u_0(\xi, x)$, the ordinary upper boundary of a given function $f(x)$ in the closed [8]) interval (ξ, x); or let $F(\xi, x) = u_f(\xi, x)$, $u_d(\xi, x)$, $u_e(\xi, x)$ or $u_z(\xi, x)$ respectively [9]).

In this case, since $u_\lambda(\xi, y)$, $(\lambda = 0, f, d, e, z)$ is a monotone decreasing interval function — in the sense that $F(\xi', x') \leqq F(\xi, x)$ if (ξ', x') lies in (ξ, x) — $u_\lambda(\xi, \xi \pm 0) = u_\lambda^\pm(\xi)$, which represents the right (left) λ-upper-boundary of f at ξ, exists. Therefore, according to Corollary II, the right λ-upper-boundary $(\lambda = 0, f, d, e, z)$ of an arbitrary function at a point ξ equals the left λ-upper-boundary at ξ, with a possible denumerable number of exceptions. Likewise, of course, for the left and right λ-lower-boundaries $l_\lambda^\pm(\xi)$. The λ-upper-boundary $u_\lambda(\xi)$ of f at ξ, as distinguished from the right or left λ-upper-boundary at ξ, is the greater of these latter two numbers, and is therefore equal to them in case of their equality. Since $s_\lambda(\xi) = u_\lambda(\xi) - l_\lambda(\xi)$ and $s_\lambda^\pm(\xi) = u_\lambda^\pm(\xi) - l_\lambda^\pm(\xi)$, where $s_\lambda(\xi)$ means the λ-saltus of f at ξ, and $s_\lambda^\pm(\xi)$ means the right (left) λ-saltus of f at ξ, we conclude:

(Theorem of H. Blumberg) [10]). *The λ-saltus function $s_\lambda(x)$ of an arbitrary function $f(x)$ is identical with the right and left λ-saltus functions $s_\lambda^\pm(x)$ except possibly at the points of a denumerable set.*

Example 3. Let $p < q$ be any two real numbers, and $F(\xi, \eta) = u(f; \xi, \eta; p, q)$ the upper boundary of the values of $f(x)$ such that $p \leqq f(x) \leqq q$ and $\xi < x < \eta$; in case there is no x of this sort, we agree to set $u(f; \xi, \eta; p, q) = p$. Since the interval function of our present example is monotone decreasing, we conclude that

[8]) To change from closed to open intervals means simply to change $F(x, \xi)$.

[9]) The number $u_f(\xi, x)$ is the upper boundary (= least upper bound) of f in (ξ, x), on the understanding that an arbitrary finite set of points may be neglected. Likewise, $u_d(\xi, x)$, $u_e(\xi, x)$, $u_z(\xi, x)$ mean the upper boundary of f in (ξ, x) in case one may respectively neglect denumerable sets, exhaustible sets, or sets of measure 0. See H. Blumberg, „*Certain General Properties of Functions*“, Annals of Mathematics, vol. 18 (1917), p. 147.

[10]) *A Theorem on Semi-Continuous Functions*, Bull. Am. Math. Soc., 2nd ser., vol. 24 (1918), p. 381.

$u(f;\xi,\xi+0;p,q)=u(f;\xi,\xi-0;p,q)$, with the possibility of an exceptional set D_{pq} which is at most denumerable. Let $p<q$ range independently over the set of rational numbers, and let D be the sum of all these D_{pq}. Suppose ξ is a point such that there is a value y for which (ξ, y) is a limit point of the points of the curve $y=f(x)$ that lie to the right (left) of ξ but not a limit point of the curve points lying to the left (right) of ξ. It is then possible to select two rational numbers r_1, r_2 such that $r_1<y<r_2$ and no curve points $(x, f(x))$ lie in the strip from $y=r_1$ to $y=r_2$ for x sufficiently near and at the left (right) of ξ. Therefore, according to our convention $u(f;\xi,\xi\mp 0;r_1,r_2)=r_1$, whereas $u(f;\xi,\xi\pm 0;r_1,r_2)\geqq \geqq y>r_1$. ξ thus belongs to the, at most denumerable, set D. We therefore conclude:

(Theorem of W. H. Young) [11]. *If $f(x)$ in an arbitrary real function, and (x, y) a limit point of the points of $y=f(x)$ to the right (left) of x, then (x, y) is also a limit point of the points of $y=f(x)$ to the left (right) of x, there being at most a denumerble number of exceptional values of x.*

Example 4. Let C be a given class of linear point sets — such as the totality of denumerable sets, or that of non-denumerable sets, or that of sets of positive exterior measure. We define the C-upper-boundary of f relative to a given linear point set S, as the number u — which is allowably $\pm\infty$ — having the following properties: (a) if $u'>u$, the set $SE_{f>u'}$ — which signifies the set of points x of S such that $f(x)>u'$ — contains no subset that is an element of C; (b) for every $\varepsilon>0$, the set $SE_{f>u'-\varepsilon}$ contains at least one subset that is an element of C. If (ξ,η), (p, q) are two real intervals, we define $u_C(f;\xi,\eta;p,q)$ as the C-upper-boundary of f relative to the set of points x satisfying the inequalities $\xi<x<\eta$, $p<f(x)<q$; in case the subset of $E_{f>p}$ lying in the open interval (ξ,η) contains no subset that is an element of C, we agree to set $u_C(f;\xi,\eta;p,q)=p$, and in case, for every positive ε, there is a subset of $E_{f>q-\varepsilon}$ that is an element of C, we set $u_C(f;\xi,\eta;p,q)=q$. Suppose (ξ, y) is such that, for

[11]) See, for example, Bull. des Sci. Math., 2nd ser., vol. 52 (1928), p. 274. The portion of Young's Theorem, as given l. c., but not included in our statement, can be obtained by changing the inequality $\xi<x<\eta$ employed in the definition of $u(f;\xi,\eta;p,q)$ to $\xi\leqq x\leqq\eta$.

every positive ε, the set of points x satisfying the inequalities $\xi<x<\xi+\varepsilon$, $y-\varepsilon<f(x)<y+\varepsilon$ has at least one subset that is an element of C; we shall then say that (ξ,η) is a left C-limit of the curve $y=f(x)$; and similarly for a right C-limit. We can then show, by means of this interval function $u_C(f;\xi,\eta;p,q)$, precisely as in Example 3, that if a point (ξ,η) is a left (right) C-lmit of $y=f(x)$ without being a right (left) C-limit, the number ξ belongs to a set which is at most denumerable. We thus have the following theorem, which specializes into the Young Theorem of Example 3 in case C is the totality of sets consisting of a single point.

Theorem. *If $y=f(x)$ is an arbitrary real function, and C an arbitrary class of linear point sets, then every point (x, y) — with the possible exception of a denumerable number of x's — that is a right (left) C-limit of the curve $y=f(x)$ is also a left (right) C-limt of the curve.*

Example 5. Let (ξ,η) and (p, q) be any two real intervals, and r a real number such that $0<r<1$. By the number $u(f;\xi,\eta;p,q,r)$, we understand the upper boundary of $f(x)$ for the x's of (ξ,η) satisfying the inequality $p\leqq f(x)\leqq q$, with the permissible neglect of any measurable subset M of (ξ,η) of relative measure $<r$; i. e., such that the Lebesgue measure of M is less than $r(\eta-\xi)$ [12]. Here u is, in general, not a monotone interval function, but at any rate, we conclude, according to Corollary II, that $u^{\pm}(\xi;p,q;r)\geqq \geqq u_{\mp}(\xi;p,q;r)$ — here $u^{\pm}(\xi;p,q;r)=\limsup u(f;\xi,\xi\pm 0;p,q;r)$, $u_{\mp}(\xi;p,q;r)=\liminf u(f;\xi,\xi\mp 0;p,q;r)$ — except possibly for the ξ's of an at most denumerable set D_{pqr}. Let r take a succession of values r_n — for example $r_n=\frac{n}{n+1}$, $n=1,2,\ldots\infty$ — approaching 1 as a limit, and let $D_{pq}=\sum_{n=1}^{\infty} D_{pqr_n}$. Then $u^{\pm}(\xi;p,q,r_n)\geqq \geqq u_{\mp}(\xi;p,q;r_n)$ for all the numbers r_n, and all points ξ not belonging to D_{pq}. Since $u^{\pm}(\xi;p,q;r_n)$ and $u_{\pm}(\xi;p,q;r_n)$ decrease monotonically as $r_n\to 1$, they possess limits as $r_n\to 1$; we denote these respectively by $u^{\pm}(\xi;p,q;1-0)$, $u_{\pm}(\xi;p,q;1-0)$. Then $u^{\pm}(\xi;p,q;1-0)\geqq u_{\mp}(\xi;p,q;1-0)$ if ξ does not belong to D_{pq}. In parti-

[12]) We here admit $\pm\infty$ as real numbers.

cular, if $p=-\infty$ and $q=+\infty$, we set $u^{\pm}(\xi; -\infty, +\infty; 1-0)=u^{\pm}(\xi)$ and $u_{\pm}(\xi; -\infty, +\infty; 1-0)=u_{\pm}(\xi)$. These numbers $u^{\pm}(\xi)$, $u_{\pm}(\xi)$ — as we may readily see — are so related to the numbers $L^{\pm}(\xi)$, $l^{\pm}(\xi)$ of Kempisty[13]) that our result, that the validity of $u^{\pm}(\xi) \geqq u_{\mp}(\xi)$ for all ξ's not in a certain denumerable set, implies the theorem of Kempisty, l. c. Kempisty defines $L^{\pm}(\xi)$ as the lower boundary of all real numbers a such that $E[f(x) \leqq a, x \gtrless \xi]$ is of metric density 1 at ξ; and $l^{\pm}(\xi)$, as the upper boundary of all real numbers a such that $E[f(x) \geqq a, x \gtrless \xi]$ is of metric density 1 at ξ. If $y > L^{+}(\xi)$, the metric density of $E[f(x) > y, x > \xi]$ is 0 et ξ; therefore, if $0 < r < 1$ and $\eta > \xi$ is sufficiently near ξ, we have $u(f; \xi, \eta; -\infty, +\infty; r) \leqq y$ and hence $u^{+}(\xi) \leqq y$. We conclude that $u^{+}(\xi) \leqq L^{+}(\xi)$. If $y < l^{+}(\xi)$ the metric density of $E[f(x) < y, x > \xi]$ is 0 at ξ, therefore, if $0 < r < 1$, we have $u(f; \xi, y; -\infty, +\infty; r) \geqq y$; therefore $u_{+}(\xi) \geqq y$, and hence $u_{+}(\xi) \geqq l^{+}(\xi)$. Likewise, of course, $u^{-}(\xi) \leqq L^{-}(\xi)$ and $u_{-}(\xi) \geqq l^{-}(\xi)$. Thus, since $u^{\pm}(\xi) \geqq u_{\mp}(\xi)$ except possibly for $\aleph_0$ ξ's, we conclude:

(Theorem of Kempisty). *For every real function $f(x)$ $E[L^{+}(x) < l^{-}(x)]$ is at most denumerable.*

Since every interval function offers an application of Corollary II, it would be easy to cite other interesting implications for arbitrary real functions, or for that matter, for arbitrary planar sets.

[13]) *Sur les fonctions approximativement discontinues*, Fund. Math., vol. 6 (1924), p. 6.

Sur les suites des fonctions presque partout continues.

Par

L. Kantorovitch (Leningrad, U. R. S. S.).

Le but de cette Note est de démontrer la proposition suivante

Théorème. *Pour que la fonction $F(x)$ puisse être représentée par la relation de la forme*

$$F(x) = \lim_{n\to\infty} f_n(x), \tag{1}$$

où les fonctions $f_n(x)$ sont presque partout continues dans (a, b), il faut et il suffit qu'il existe deux fonctions: $\varphi(x)$ du type (g_2)[1]): au plus [dans la classification de M. Young] et $\psi(x)$ du type (G_2) au plus, remplissant partout la relation

$$\varphi(x) \leqslant F(x) \leqslant \psi(x) \tag{2}$$

et presque partout la relation

$$\varphi(x) = F(x) = \psi(x). \tag{3}$$

Les conditions sont nécessaires. Supposons que la relation (1) a lieu. Posons:

$$\bar{\varphi}_n(x) = \mathrm{Max}\,\{f_n(x), f_{n+1}(x), \ldots\}. \tag{4}$$

On voit que la fonction $\bar{\varphi}_n(x)$ est semicontinue inférieurement au moins dans tous les points, où toutes les fonctions $f_\nu(x)$ $[\nu = n, n+1, \ldots]$ sont continues, c'est-à-dire presque partout. D'après la définition de la fonction $\bar{\varphi}_n(x)$, nous aurons immédiatement

$$\bar{\varphi}_{n+1}(x) \leqslant \bar{\varphi}_n(x); \quad \lim_{n\to\infty} \bar{\varphi}_n(x) = F(x). \tag{5}$$

[1]) Nous utilisons ici [et dans la suite] les notations de M. H. Hahn, v. *Theorie der reellen Funktionen* [Berlin 1921], pp. 328, 334.