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avec le fait que la distance de $p_1$ à tous les $K_n$ est supérieure à $\delta > 0$.

Le théorème est ainsi démontré.

Les deux théorèmes que nous venons de démontrer peuvent être réunis de manière que l'on dise: dans les conditions 1°, 2°, 3°, 4° la décomposition $\{K\}$ est partout localement continue, si l'on définit ce mot par les deux propriétés spécifiées dans les thèses.

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A Theorem on Arbitrary Functions of Two Variables with Applications.

By

Henry Blumberg (Columbus, Ohio, U. S. A.).

The object of the present paper is to enunciate a theorem — on unrestricted real functions of two variables — which, somewhat unexpectedly, appears as a common source of a number of theorems — on arbitrary real functions of a single variable — that have appeared, from time to time, in the literature without in themselves readily suggesting a common origin). It turns out that these special theorems are obtainable by considering particular real interval functions) associated with a given function; whereas the theorem of the present paper yields an analogous result for every interval function.

Let $f(x, y)$, then, be an arbitrary, real, one-valued function of two real variables defined in an entire plane $\pi$. Let $s$ be a straight line of $\pi$, and $d$, a given direction in $\pi$. If $P$ is a point of $s$, we denote by $l_P$, $u_P$ the $\lim \inf$, $\lim \sup$ of $f$ as $(x, y)$ approaches $P$ along the direction $d$; and by $I_P = (l_P, u_P)$ the interval of approach of $f$ at $P$ along $d$). We may now state the

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1) See Examples 1, 2, 4 and 5 below.
2) The argument of an interval function is a linear interval, or the pair of its end points, their order being disregarded. If the interval function is real, the dependent variable is a real number.
3) The condition of one-valuedness is inserted for simplicity of statement. The argument holds essentially for many-valued functions also, the requisite modifications in statement for the latter case being evident.
4) We permit $l_P$ and $u_P$ to be $\pm \infty$.

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(2, 1), we set \( f(x, y) = -1 \). If \( P \) is in \( M_1 \), we have \( l_{p_n} = 0 \), \( u_{p_n} = 1 \); \( l_{p_n} = -1 \), \( u_{p_n} = 0 \). If \( P \) is in \( M_1 \), we have \( l_{p_n} = -1 \), \( u_{p_n} = 0 \), \( l_{p_n} = 1 \), \( u_{p_n} = 1 \). Therefore, the intervals \( I_{p_n}, I_{p_n} \) abut at every point, but overlap at no point \( P \) of the \( x \)-axis.

If \( \langle d_n \rangle \) is a sequence of directions of approach to \( s \) on one and the same side of it, then \( D_{m_n} \{ m, n = 1, 2, \ldots \} \), the set of points \( P \) of \( s \) where \( u_{p_n} < l_{p_n} \) is at most denumerable, and therefore \( D = \sum_{n=1}^{\infty} D_{m_n} \) is at most denumerable. Hence we have the following

Corollary 1. If \( f(x, y) \) is an arbitrary real function; \( s \), a given straight line; and \( \langle d_n \rangle, n = 1, 2, \ldots, \infty \), a sequence of directions of approach to \( s \) on the same side of it, then there exists a denumerable set \( D \), such that if \( P \) is a point of \( s \) not in \( D \), we have \( u_{p_n} \geq l_{p_n} \) for every pair of integers \( m, n \). In other words, for a point \( P \) not in \( D \), the intervals \( I_{p_n}, \{ n = 1, 2, \ldots \} \) overlap or abut one another.

It is easy to see that this result is exhaustive, in a sense presently to be made more precise. For suppose \( \pi \) is the \( xy \)-plane, and \( s \), for simplicity, assumed to be the \( x \)-axis. Let \( D = \{ P_n \} \) be any given denumerable subset of \( s \) and \( \langle d_n \rangle \) a given sequence of directions of approach to \( s \) from the upper half-plane of \( \pi \), which we denote by \( \pi^+ \). It is possible to draw, for each \( P_n \), a circle \( C_n \) tangent to \( s \) at \( P_n \) and with center in \( s \) in such a way that no two \( C_n \) have points in common. Let \( K_{m_n} \) \{ \( m, n = 1, 2, \ldots \} \) be the chord of the circle \( C_n \) having the direction \( d_n \) and terminating in \( P_n \). Since no pair of the chords \( K_{m_n} \) intersect in \( \pi^+ \), the intervals of approach \( I_{m_n} \) can be made whatever we please, and quite independently of one another, by means of an appropriate definition of \( f(x, y) \) on these chords. We are still free to define \( f(x, y) \) elsewhere as we please. Denote by \( K \) the set of points on the \( K_{m_n} \), and by \( \overline{K} \), the complement of \( K \) with respect to \( \pi^+ \). Regarding then, \( f(x, y) \) as defined on \( K \) so as to make \( I_{m_n} = J_{m_n} \) where \( \{ J_{m_n} \} \) \{ \( m, n = 1, 2, \ldots, \infty \} \), is an arbitrarily given set of real intervals, we define \( f(x, y) = -1 \) in \( \overline{K} \). If \( P \) is a point of \( s \) not belonging to \( D \), and \( d \) is a direction of approach to \( P \) from \( \pi^+ \), it

\^1 \( D \), of course, has a different meaning from that above.

\^2 It is understood here that the points of \( s \) do not belong to \( \pi^+ \).
of an arbitrary function is greater than or equal to the lower left (right) derivative, with the possible exception of a denumerable number of points.

Example 2. Let \( F(\xi, x) = u_0(f; \xi, x) = u_0(f; \xi, x) \), the ordinary upper boundary of a given function \( f(x) \) in the closed interval \((\xi, x)\); or let \( F(\xi, x) = u_0(f; \xi, x) \), \( u_0(f; \xi, x) \), \( u_0(f; \xi, x) \) or \( u_0(f; \xi, x) \) respectively.\(^7\)

In this case, since \( u_0(f; \xi, y) \), \( \lambda = 0, f, d, e, a, z \) is a monotone decreasing interval function — in the sense that \( F(\xi, x') \leq F(\xi, x) \) if \((\xi, x')\) lies in \((\xi, x)\) — \( u_0(f; \xi, \xi \pm 0) = u_0(f; \xi) \), which represents the right (left) \( \lambda \)-upper-boundary of \( f \) at \( \xi \) exists. Therefore, according to Corollary II, the right \( \lambda \)-upper-boundary \( \lambda = 0, f, d, e, a, z \) of an arbitrary function at a point \( \xi \) equals the left \( \lambda \)-upper-boundary at \( \xi \), with a possible denumerable number of exceptions. Likewise, of course, for the left and right \( \lambda \)-lower-boundaries \( \lambda \xi \) \( \xi \) of \( f \) at \( \xi \), as distinguished from the right or left \( \lambda \)-upper-boundary at \( \xi \), is the greater of these latter two numbers, and is therefore equal to them in case of their equality. Since \( u_0(f; \xi, y) = u_0(f; \xi) = \lambda \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \x
A theorem on arbitrary functions.

If \( f(x) \) is an arbitrary real function, and \( C \) an arbitrary class of linear point sets, then every point \( (x, y) \) with the possible exception of a denumerable number of \( x's \) that is a right (left) \( C \)-limit of the curve \( y = f(x) \) is also a left (right) \( C \)-limit of the curve.

**Example 5.** Let \( (\xi, \eta) \) and \( (p, q) \) be any two real intervals, and \( r \) a real number such that \( 0 < r < 1 \). By the number \( u(f; \xi, \eta; p, q, r) \), we understand the upper boundary of \( f(x) \) for the \( x's \) of \( (\xi, \eta) \) satisfying the inequality \( p < f(x) < q \), with the permissible neglect of any measurable subset \( M \) of \( (\xi, \eta) \) of relative measure \( r \); i.e., such that the Lebesgue measure of \( M \) is less than \( r(\eta - \xi) \).

Here \( u \) is, in general, not a monotone interval function, but at any rate, we conclude, according to Corollary II, that \( u(x; \xi, \eta; p, q, r) \geq u(x; \xi, \eta; p, q, r) = \limsup u(f; \xi, \eta; p, q, r) \), except possibly for the \( x's \) of an at most denumerable set \( D_{pq} \). Let \( r \) take a succession of values \( r_n \) — for example \( r_n = \frac{n}{n+1}, \quad n = 1, 2, \ldots \) — approaching 1 as a limit, and let \( D_{pq} = \bigcup_{n=1}^{\infty} D_{pq} \). Then \( u(x; \xi, \eta; p, q, r) \geq u(x; \xi, \eta; p, q, r_n) \) for all the numbers \( r_n \), and all points \( x \) not belonging to \( D_{pq} \). Since \( u(x; \xi, \eta; p, q, r_n) \) and \( u(x; \xi, \eta; p, q, r_n) \) decrease monotonically as \( r_n \to 1 \), they possess limits as \( r_n \to 1 \); we denote these respectively by \( u(x; \xi, \eta; p, q, 1) \), \( u(x; \xi, \eta; p, q, 1) \). Then \( u(x; \xi, \eta; p, q, 1) \) if \( \xi \) does not belong to \( D_{pq} \). In parti-

\[ u(f; \xi, \eta; p, q) = u(f; \xi, \eta; p, q), \] with the possibility of an exceptional set \( D_{pq} \) which is at most denumerable. Let \( p < q \) range independently over the set of rational numbers, and let \( D \) be the sum of all these \( D_{pq} \). Suppose \( \xi \) is a point such that there is a value \( y \) for which \( (\xi, y) \) is a limit point of the points of the curve \( y = f(x) \) that lie to the right (left) of \( \xi \) but not a limit point of the curve points lying to the right (left) of \( \xi \). It is then possible to select two rational numbers \( r_1, r_2 \) such that \( r_1 < y < r_2 \) and no curve points \( (x, f(x)) \) lie in the strip from \( y = r_1 \) to \( y = r_2 \) for \( x \) sufficiently near and at the left (right) of \( \xi \). Therefore, according to our convention \( u(f; \xi, \eta; p, q, r_1) = r_1 \), whereas \( u(f; \xi, \eta; p, q, r_2) \geq y > r_1 \). \( \xi \) thus belongs to the, at most denumerable, set \( D \). We therefore conclude:

(1) If \( \xi \) is an arbitrary real function, and \( (\xi, y) \) a limit point of the points of \( y = f(x) \) to the right (left) of \( \xi \), then \( (\xi, y) \) is also a limit point of the points of \( y = f(x) \) to the right (left) of \( \xi \), there being at most a denumerable number of exceptional values of \( x \).

Example 4. Let \( C \) be a given class of linear point sets — such as the totality of denumerable sets, or that of non-denumerable sets, or that of sets of positive exterior measure. We define the \( C \)-upper-boundary of \( f \) relative to a given linear point set \( S \), as the number \( u \) which is allowably \( \pm \infty \) — having the following properties: (a) if \( u' > u \), the set \( SE_{p} \) — which signifies the set of points \( x \) of \( S \) such that \( f(x) > u' \) contains no subset that is an element of \( C \); (b) for every \( \epsilon > 0 \), the set \( SE_{p} \) contains at least one subset that is an element of \( C \). If \( (\xi, \eta) \) and \( (p, q) \) are two real intervals, we define \( u_{c}(f; \xi, \eta; p, q) \) as the \( C \)-upper-boundary of \( f \) relative to the set of points \( x \) satisfying the inequalities \( \xi < x < \eta \), \( p < f(x) < q \); in case the subset of \( E_{p} \) lying in the open interval \( (\xi, \eta) \) contains no subset that is an element of \( C \), we agree to set \( u_{c}(f; \xi, \eta; p, q) = p \), and in case, for every positive \( \epsilon \), there is a subset of \( E_{p} \) that is an element of \( C \), we set \( u_{c}(f; \xi, \eta; p, q) = q \). Suppose \( (\xi, \eta) \) is such that, for

\[ 1) \) See, for example, Bull. des Sci. Math., 2nd ser., vol. 52 (1928), p. 274. The portion of Young's Theorem, as given in this article, but not included in our statement, can be obtained by changing the inequality \( \xi < x < \eta \) employed in the definition of \( u(f; \xi, \eta; p, q) \) to \( \xi \leq x \leq \eta \).
Sur les suites des fonctions presque partout continues.

Par
L. Kantorovitch (Leningrad, U. R. S. S.).

Le but de cette Note est de démontrer la proposition suivante:

Théorème. Pour que la fonction \( F(x) \) puisse être représentée par la relation de la forme

\[
F(x) = \lim_{n \to \infty} f_n(x),
\]

où les fonctions \( f_n(x) \) sont presque partout continues dans \((a, b)\), il faut et il suffit qu'il existe deux fonctions : \( \varphi(x) \) du type \((G_3)\); au plus [dans la classification de M. Young] et \( \psi(x) \) du type \((G_4)\) au plus, remplissant partout la relation

\[
(2) \quad \varphi(x) \leq F(x) \leq \psi(x)
\]
et presque partout la relation

\[
(3) \quad \varphi(x) = F(x) = \psi(x).
\]

Les conditions sont nécessaires. Supposons que la relation \((1)\) a lieu. Posons:

\[
(4) \quad \Phi_n(x) = \max \{ f_n(x), f_{n+1}(x), \ldots \}.
\]

On voit que la fonction \( \Phi_n(x) \) est semicontinue inférieurement au moins dans tous les points, où toutes les fonctions \( f_n(x) \) \([n = n, n + 1, \ldots]\) sont continues, c'est-à-dire presque partout. D'après la définition de la fonction \( \Phi_n(x) \), nous aurons immédiatement

\[
(5) \quad \Phi_{n+1}(x) \leq \Phi_n(x) ; \lim_{n \to \infty} \Phi_n(x) = F(x).
\]