Some Generalizations of the Scherrer Fixed-Point Theorem [1]).

By

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1. Following Rosenthal and Kuratowski, we shall call a class L of Fréchet which is the continuous image of an interval a *Peano continuum* (or space). It follows that a Peano continuum is compact and metrisable [2]), so that we may assume distance as already defined. W. Scherrer has shown that every homeomorphism of an acyclic Peano continuum into a subset of itself has a fixed-point [3]). In this note we shall give several generalizations of the Scherrer theorem based on the cyclic structure of a Peano continuum. It is assumed that the reader is familiar with the terminology and properties of the cyclic structure as introduced by G. T. Whyburn. Reference may be made to the following papers of Whyburn and the author: (I) Proc. Nat. Acad. Sci., vol 13 (1927), pp. 31—38; (II) Amer. Jour. of Math., vol 50 (1928), pp. 167—194; (III) ibid., vol 51 (1929), pp. 577—594; (IV) Trans. Amer. Math. Soc., vol 30 (1928), pp. 567—578; (V) ibid., vol 31 (1929), pp. 595—612. In this note we shall use the term maximal cyclic set in place of maximal cyclic curve and arc-set in place of arc-curve.

[1]) Presented to the American Mathematical Society April 18, 1930. The principal results of this paper were read before the Polish Math. Soc. in Warsaw March. 26, 1929.

[2]) W. L. Ayres, *On continuous images of a compact metric space*, Fund. Math., vol. 14 (1928), pp. 334–8.

[3]) *Ueber ungeschlossene stetige Kurven*, Math. Zeit., vol. 24 (1925), pp. 125—130.

The first theorem gives a general result on the cyclic structure of Peano continua. From this we are able to derive three generalizations of the Scherrer theorem as corollaries.

2. **Theorem.** *If T is a homeomorphism of the Peano continuum M such that $T(M)\subset M$, then there is a cyclic element C of M such that $T(C)\subset C$.*

Let X be a cyclic element of M. If $X'=T(X)\subset X$, our proof is complete. If not, there is a cyclic element Y such that $X'\subset Y$. If $Y'\subset Y$, our proof is complete. If not, there is a cyclic element Z such that $Y'\subset Z$. In case X' is a cut point of M we will choose $Y=X'$ (if there is a choice), and similarly in case of Z. We distinguish several cases according to the position of Z. If X is a single point, let $x=X$; and if X is a maximal cyclic set of M, let x be a point of X which is a non-cut point of M. Let $y=x'$ and $z=y'$. From the choice of x and Y we have $X+Y\subset$ arc-set $M(x+y)$. Among the points of the arc-set $M(x+y)$ we may define order as follows: A point p precedes a point $q (p\neq q)$ if either (a) $p=x$, or (b) p separates x and q in $M(x+y)$, or (c) there is a cut point t of $M(x+y)$ such that t separates x and q in $M(x+y)$ but not x and p. By this definition order is defined unless p and q are both non-cut points of $M(x+y)$ belonging to the same maximal cyclic set.

Case I. Suppose $z\in$ arc-set $M(x+y)$. Then $T(M(x+y))\subset M(x+y)$. Let K denote the set of cut points of $M(x+y)$. Let N denote the set consisting of the point x together with all points p of K such that $p'=p$ or p precedes p'. Evidently if $p\in N$ then every point of K preceding p belongs to N. Due to the linear order of K, if $\overline{N}\neq N$ then $\overline{N}-N$ is a single point q and every point of N precedes q. As $K+x+y$ is closed and y is not a limit point of N, $q\in K$. Let $[p_i]$ be a sequence of elements of N such that $\lim\limits_{i\to\infty} p_i=q$. Then $\lim\limits_{i\to\infty} p_i'=q_i'$. As q non-$\in N$, q' precedes q. Then for i sufficiently large p_i' precedes q. Thus there is some i for which p_i' precedes p_i, a contradiction. Then $\overline{N}=N$ and there is a last point s of N.

If $s'=s$, our case is complete. If not, then s is not a limit point of points of K following s and thus s belongs to a maximal cyclic set A such that s precedes every point of $A-s$. For suppose s is a limit point of points of K following s. Then there is a cut point t of K between s and s'. There is a number δ such that all points within a distance δ of s precede t and all such points are carried by T into points following t. Then s cannot be the last point of N for all the points of K within a distance δ of s belong to N. Thus s a limit point of points of K following s gives a contradiction.

Let r denote the other point of K belonging to A. As r non-ϵN, r' belongs to the component of $M(x+y)-r$ that contains x, denoted by $C(M(x+y)-r, x)$. And $s' \epsilon C(M(x+y)-s, y)$. Then

$$A' \subset \overline{C(M(x+y)-r, x)} \cdot \overline{C(M(x+y)-s, y)} = A,$$

which proves Case I.

Case II. Arc-set $M(x+y)+$arc-set $M(y+z)=$arc-set $M(x+z)$. In this case arc-set $M(x+y)$ and arc-set $M(y+z)$ have in common only Y or a cyclic element containing Y (this last may happen if $Y=y$), and have no point in common which is a cut point of both $M(x+y)$ and $M(y+z)$. If $z' \epsilon M(y+z)$, we have a fixed point by Case I. If z' non-ϵ $M(y+z)$, let p be a limit point of $M(z+z')-M(y+z)$ in $M(y+z)$. Suppose p and z do not belong to the same cyclic element of $M(y+z)$. Then $M(y+z)$ contains a point q which separates p and z. The point q is thus a cut point of $M(y+z)$ and also a cut point of $M(z+z')$. Since q is a cut point of $M(y+z)$, $q \epsilon T(M(x+y))$ and there is a cut point r of $M(x+y)$ such that $r'=q$. Similarly $q \epsilon T(M(y+z))$ and there is a cut point s of $M(y+z)$ such that $s'=q$. Since $M(x+y)$ and $M(y+z)$ have no point in common which is a cut point of both, we have $r \neq s$. But $r'=s'=q$ contrary to the hypothesis that T is a homeomorphism. Then we see that p and z belong to the same cyclic element of M and in much the same way that p is not a cut point of $M(y+z)$. For this reason $M(y+z)+M(z+z')= =M(y+z')$. We may now repeat this with $T(z')$.

Thus at some stage Case II reduces to Case I or there exists an infinite set of distinct points $[x_i]$ such that (a) $x_{i+1}=T(x_i)$, (b) x_i and x_j $(i \neq j)$ do not belong to the same maximal cyclic set of M, (c) $\sum\limits_{i=n}^{i=p}$ arc-set $M(x_i+x_{i+1})=$ arc-set $M(x_n+x_{p+1})$. The set $\sum\limits_i x_i$ has at least one limit point p and from property (c), p is the only limit point and p non-$\epsilon \sum x_i$. Now suppose $p' \neq p$. Let $2\eta=\varrho(p,p')$. There exists an n such that for $i>n$, $\varrho(p,x_i)<\eta$. Since $\lim x_i=p$ and T is a homeomorphism, there exists an m such that for $i>m$, $\varrho(p',x_i')<\eta$. Then as $x_i'=x_{i+1}$ we have by the triangle axiom for $i>n+m+1$,

$$2\eta>\varrho(p,x_i)+\varrho(p',x_i) \geqq \varrho(p,p')=2\eta,$$

a contradiction.

Case III. The point z non-ϵ arc-set $M(x+y)$, and arc-set $M(x+y)+$ $+$arc-set $M(y+z) \neq$ arc-set $M(x+z)$. In this case $M(y+z)-$ $-M(x+y)$ has a limit point p in $M(x+y)$ which is either a cut point of $M(x+y)$ or there is a cut point of $M(x+y)$ separating y and p in $M(x+y)$. As p is a cut point of $M(y+z)$, $p \epsilon T(M)$. Then let q be the point such that $q'=p$, and we see that q is a cut point of $M(x+y)$. There are three possibilities:

(a) q precedes p in $M(x+y)$. Then $T(M(x+q)) \subset M(y+p)$ and $T(M(q+y)) \subset M(p+z)$. As $p \epsilon M(q+y)$ we have $p' \epsilon M(p+z)$ Then we reduce to Case II for we have $T(q)=p$, $T(p)=p'$ and. $M(q+p)+M(p+p')=M(q+p')$ since $M(q+p) \cdot M(p+p')=p$.

(b) p precedes q in $M(x+y)$. In this case p' follows p in $M(x+y)$. The proof may be completed here as in Case I by considering the set of cut points $[s]$ of $M(x+y)$ such that s' does not precede s. The fixed element under T occurs between p and q.

(c) neither p nor q precedes the other in $M(x+y)$. As p is a cut point of $M(y+z)$, it is a cut point of $M'(y+z)$. Then because T is a homeomorphism, q is a cut point of $M(x+y)$. In the definition of order in $M(x+y)$, order was defined for every pair of distinct points in which one was a cut point of $M(x+y)$. Then as order is not defined for p and q, we have $p=q$. As $p=q$ is a cut point of M, it is the desired fixed cyclic element of M under T.

3. **Theorem.** *If every cyclic element C of the Peano continuum M has the property that every homeomorphism* [1]) *carrying C into a subset of itself has an invariant point, then the entire continuum M has the same property.*

[1]) *Note from the Editors.*

As shown in the Thesis of M. Borsuk, the term „homeomorphism" may be replaced in theorems 3—5 by „continuous transformation", which in case of theorems 4 and 5 gives a stronger result. The paper of M. Borsuk will appear in this Journal.

See also a note by S. Lefschetz in C. R. Paris vol. 190 of 2. I. 1930.

Let T be a homeomorphism such that $T(M)\subset M$. By the preceding theorem there is a cyclic element C of M such that $T(C)\subset C$ Then by our assumed property there is a point p of C such that $T(p)=p$.

As far as homeomorphisms are concerned, this last result answers a question of C. Kuratowski [1]). The converse of this theorem is not true, i. e. there exist Peano continua M having the property that every homeomorphism of M into a subset of itself has an invariant point but containing cyclic elements not having this property. For an example take a circle plus an interval having one end point on the circle.

4. ***Theorem.*** *If every cyclic element of the Peano continuum M is an n-dimensional simplex (n may vary for different elements), then every homeomorphism of M into a subset of itself has a fixed point.*

This result follows immediately from the result of § 3 and the Brouwer fixed-point theorem.

5. ***Theorem.*** *If the Peano continuum M lies in a plane and does not separate the plane, then every homeomorphism of M into a subset of itself has an invariant point.*

Since M does not separate the plane, it follows by a theorem of Whyburn (II, p. 188) that every maximal cyclic set of M is a simple closed curve plus its interior, a two dimensional simplex. All other cyclic elements of M are points. Hence our theorem follows from the result of § 4.

This result is a partial solution of the well-known problem as to whether a general bounded continuum not separating its plane has this property. We exhibit here the result for any continuum which is locally connected.

[1]) *Quelques applications d'eléments cycliques de M. Whyburn*, Fund. Math. vol. 14 (1929), p. 139, footnote [2]).

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Sur les courbes d'ordre c.

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Le but de cette Note est de démontrer l'*existence d'une courbe péanienne* [1]) *plane K possédant les propriétés suivantes*:

I. $K^c = K$, *c. à d. tout point de K est d'ordre c*;

II. *K contient un sous-ensemble dénombrable A tel que $K-A$ est punctiforme.*

Il résulte de II, que:

II[a]. *K ne contient aucune famille non dénombrable de continus disjoints deux à deux* [2]).

Le plan euclidien sera désigné par R_2, la frontière d'un ensemble arbitraire U par $F(U)$, son diamètre par $\delta(U)$. J'appelle *domaine jordanien* tout domaine plan, borné dont la frontière est une ligne simple fermée.

A tout domaine jordanien G et à tout nombre $\eta > 0$ nous ferons correspondre une suite de domaines jordaniens $\{G_k(G,\eta)\}$ et un ensemble dénombrable $B(G,\eta)$ de manière à satisfaire aux conditions suivantes:

(C$_1$) On a: $\overline{G_k(G,\eta)}\subset G$; $G_k(G,\eta)\times G_l(G,\eta)=0$ pour $k\neq l$; $\lim \overline{G_k(G,\eta)} = F(G)$, et l'ensemble: $\Phi(G,\eta)=F(G)+\sum_{k=1}^{\infty}\overline{G_k(G,\eta)}$ est un continu.

[1]) c. à d. d'un continu localement connexe, de dimension 1.

[2]) J'ai démontré dans une Note antérieure (Fund. Math. XV p. 222 ss.) l'existence d'un continu péanien plan K possedant la propriété II[a]) et tel que $K^c \neq 0$. Mais dans ce cas on avait $K^{cc}=0$.