A new proof of a Theorem of Zarankiewicz.

By

W. L. Ayres (Ann Arbor, Mich.).

1. A **Peano space** is a metric space which is the continuous image of a closed interval. In his doctor's dissertation [1]), C. Zarankiewicz proved the very interesting property that the set of all cut points of a Peano space is an F_σ, i. e. the sum of a countable number of closed sets. It is the purpose of this note to give a new demonstration of this result. The result follows as a corollary of a lemma which is of some interest in itself and perhaps has other applications. The present proof is also of interest since the closed sets forming the F_σ are characterized by a certain topological property whereas in the proof of Zarankiewicz the closed sets have no particular significance.

2. **Lemma.** *If $\varepsilon > 0$ and H is the set of all points p of the Peano space M such that $M-p$ has at least two components of diameter $\geqq \varepsilon$, then every point of $\overline{H}$ is a cut point of M* [2]).

Let x be a limit point of H. Suppose $M-x$ is connected. There exists a domain D such that $x \epsilon D$, $d(D) < \frac{1}{2}\varepsilon$, $M-D$ is a continuum [3]). As $x \epsilon \overline{H}$, D contains a point y of H. By definition of H, $M-y = A+B+C$, where A and B are components of $M-y$ and $d(A) \geqq \varepsilon$, $d(B) \geqq \varepsilon$. Then A and B contain points a and b such that $(a+b)\cdot D = 0$. As A and B are components of $M-y$, there is no connected subset of $M-y$ containing both a and b. But $M-D \subset M-y$, $a+b \subset M-D$, and $M-D$ is a continuum.

Theorem. (*Zarankiewicz*). *The set of cut points of the Peano space M is an F_σ.*

Let K be the set of cut points of M. Let H_n be the set of all points p of M such that $M-p$ contains at least two components of diameter $\geqq 1/n$. Let $x \epsilon K$ and A and B be two components of $M-x$. If η is the smaller of the numbers $d(A)$ and $d(B)$, then for $1/n < \eta$, $x \epsilon H_n$. By the Lemma, $\overline{H}_n \subset K$ for each n. Then $K = \sum_{n=1}^{\infty} \overline{H}_n$.

3. **Remarks.** In the proof of Zarankiewicz an arbitrary countable set S dense in M was selected. It was shown that every cut point of M separated some two points of S and the set of cut points separating any two fixed points of S is an F_σ, which gave the theorem as S contains only a countable number of pairs of points. An interesting modification of this may be obtained as follows: Every cut point of M separates some two nodes [4]) of M and the set of all cut points of M separating any one pair of nodes is an F_σ. The nodes are of two classes: end points of M and maximal cyclic sets containing at most one cut point of M, those of the last type being at most countable. Now if we take a countable set of end points dense in the end points of M together with all the nodes of the second type, we have a countable set of nodes such that every cut point of M separates two of this countable set. From this it follows that the set of cut points is an F_σ.

1) *Sur les points de division dans les ensembles connexes*, Fund. Math. vol. 9 (1927), p. 163.

2) The form of this lemma stated in the abstract of this note (Bull. Amer. Math. Soc., vol. 36 (1930), p. 203) is incorrect.

3) H. M. Gehman, *Concerning certain types of non-cut points etc.*, Proc. Nat. Acad. Sci., vol. 14 (1928), p. 432.

4) For definition and properties, see G. T. Whyburn, Amer. Jour. of Math. vol. 50 (1928), pp. 167—194.