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Area and Hausdorff dimension of the set of accessible points of the Julia sets of λe^z and $\lambda \sin z$

by

Bogusława Karpińska (Warszawa)

Abstract. The Julia set J_λ of the exponential function $E_\lambda : z \to \lambda e^z$ for $\lambda \in (0, 1/e)$ is known to be a union of curves ("hairs") whose endpoints $\mathcal{C}_\lambda$ are the only accessible points from the basin of attraction. We show that for λ as above the Hausdorff dimension of $\mathcal{C}_\lambda$ is equal to 2 and we give estimates for the Hausdorff dimension of the subset of $\mathcal{C}_\lambda$ related to a finite number of symbols. We also consider the set of endpoints for the sine family $F_\lambda : z \to (1/(2i))\lambda(e^{iz} - e^{-iz})$ for $\lambda \in (0, 1)$ and prove that it has positive Lebesgue measure.

1. Introduction. We consider the complex exponential maps $E_\lambda(z) = \lambda e^z$ where $z \in \mathbb{C}$ and $\lambda \in (0, 1/e)$. The function E_λ has two real fixed points; the attracting fixed point is denoted by p_λ and the repelling one by q_λ. Note that $0 < p_\lambda < 1 < q_\lambda$. The basin of attraction of p_λ is an open, dense and simply connected subset Ω_λ of $\mathbb{C}$.

We choose ν_λ such that $\nu_\lambda < q_\lambda$ and $|E'_\lambda(z)| > 1$ if $\operatorname{Re} z \geq \nu_\lambda$ and denote by H the half-plane $\{z : \operatorname{Re} z \geq \nu_\lambda\}$. The function E_λ maps $\mathbb{C} \setminus H$ into itself. Consequently, this half-plane lies in the basin of attraction Ω_λ and the Julia set of E_λ is contained in H. We divide H (as in [4]) into infinitely many strips: for $k \in \mathbb{Z}$,

$$P(k) = \{z \in \mathbb{C} : \operatorname{Re} z \geq \nu_\lambda,\ (2k-1)\pi \leq \operatorname{Im} z < (2k+1)\pi\}.$$

If the forward orbit of z is completely contained in H then the *itinerary* of z is defined to be the sequence $s = (s_0, s_1, \ldots)$ such that $s_j = k$ if $E^j_\lambda(z) \in P(k)$. But not every sequence corresponds to an actual orbit of E_λ. A sequence $s = (s_0, s_1, \ldots)$ is called *allowable* if there exists $x \in \mathbb{R}$ such that $E^j_\lambda(x) \geq (2|s_j| + 1)\pi$ for each $j = 0, 1, \ldots$

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In [4] Devaney and Krych (see also [3]) proved that there exists $z \in H$ with itinerary s if and only if s is an allowable sequence and the set of points which have s as itinerary forms a curve X_s lying in the Julia set. X_s is the image of $[0,1)$ under a continuous embedding $\phi_s : [0,1) \to \mathbb{C}$, $\phi_s(t) \to \infty$ as $t \to 1$. The set $J_\lambda \setminus \{\infty\}$ consists of a disjoint union of sets X_s. Devaney and Goldberg ([3]) proved that the point $z_s = \phi_s(0)$ is accessible from the basin of attraction (i.e. there exists a path $\gamma : [0,1) \to \Omega_\lambda$ such that $\lim_{t\to 1} \gamma(t) = z_s$) and z_s is the unique accessible point in X_s. We then say that z_s is an *endpoint* of X_s and we denote the set of endpoints by $\mathcal{C}_\lambda$.

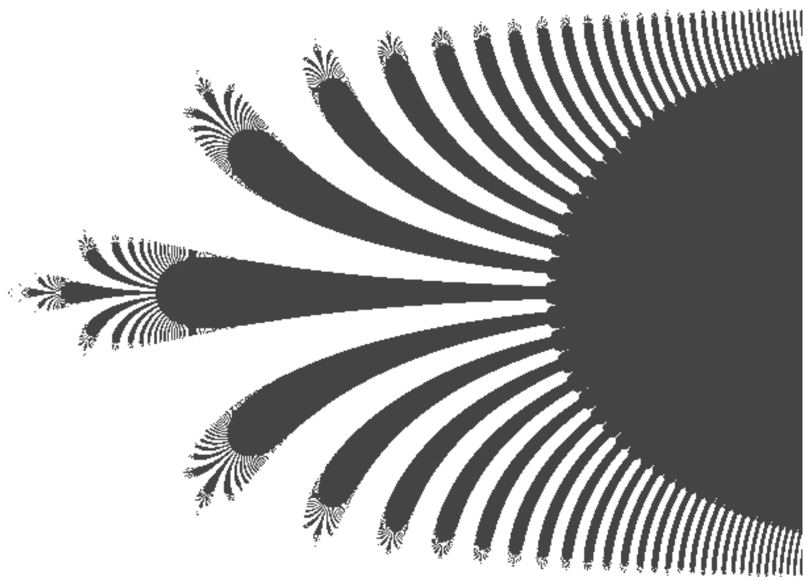

Fig. 1. The Julia set for $0.2e^z$

In [10] McMullen proved that for E_λ as above the Hausdorff dimension of the Julia set J_λ is $\mathrm{HD}(J_\lambda) = 2$. It is known that the Hausdorff dimension of $\mathcal{C}_\lambda$ is greater than or equal to 1; this follows from [9], where it is proved that the topological dimension of the set of endpoints is equal to 1. Another argument is that the harmonic measure ω has its support in the set of accessible points and its Hausdorff dimension is $\mathrm{HD}(\omega) = \inf\{\mathrm{HD}(X) : X \subset J_\lambda,\ \omega(X) = 1\} = 1$ (see [8]).

Note that $\overline{\mathcal{C}}_\lambda = J_\lambda$ because $\mathcal{C}_\lambda$ contains all the repelling periodic points of E_λ, which are dense in J_λ (see [1] and [3]). So one can think that $\mathrm{HD}(\mathcal{C}_\lambda)$ is much larger than 1 if $\mathcal{C}_\lambda$ is "very dense" in J_λ. This question, according to F. Przytycki (see also [9]), has been known since the eighties. Here we give the answer:

Theorem 1. *The Hausdorff dimension of* $\mathcal{C}_\lambda$ *is equal to* 2.

We also prove that the Hausdorff dimension of the subset of $\mathcal{C}_\lambda$ consisting of those endpoints whose itinerary contains a finite number of symbols is greater than 1, but for small λ this dimension is close to 1 (independently of the number of symbols). For $N \in \mathbb{N}$, we define $\Sigma_N = \{s = (s_0, s_1, \ldots) : s_j \in \mathbb{N}$ and $1 \le s_j \le N$ for $j = 0, 1, \ldots\}$. Note that all sequences from Σ_N are allowable. Let $\mathcal{C}_{\lambda,N}$ denote the set of endpoints corresponding to itineraries which belong to Σ_N.

THEOREM 2. *For every $\lambda \in (0, 1/e)$ there exists $N_0 \in \mathbb{N}$ such that for $N > N_0$,*

$$\mathrm{HD}(\mathcal{C}_{\lambda,N}) > 1.$$

Moreover, if λ is sufficiently small then

$$1 + \frac{1}{\log(\log 1/\lambda)} < \mathrm{HD}(\mathcal{C}_{\lambda,N}) < 1 + \frac{1}{\log(\log(\log 1/\lambda))}.$$

REMARK. The above results hold in the case of complex parameters λ such that E_λ has an attracting fixed point, i.e. λ is of the form $\lambda = \xi e^{-\xi}$ for some $\xi \in \mathbb{C}$, $|\xi| < 1$ (with λ replaced by $|\lambda|$ in Theorem 2).

It was shown by McMullen in [10] that the Julia set for maps of the form $f(z) = \gamma e^z + \delta e^{-z}$ where $\gamma, \delta \in \mathbb{C}$ has positive Lebesgue measure. So the natural question is: does the set of endpoints (for parameters such that Cantor bouquets occur) for the sine family have positive Lebesgue measure?

Let $F_\lambda(z) = \lambda(e^z - e^{-z})/2$ where $\lambda \in (0,1)$. Then F_λ has an attracting fixed point at 0 and two repelling fixed points $q_\lambda^+ > 0$ and $q_\lambda^- < 0$ such that $q_\lambda^+ = -q_\lambda^-$. The Julia set for F_λ contains a pair of Cantor bouquets; one in $H^+ = \{z : \operatorname{Re} z > \nu^+\}$ where $0 < \nu^+ < q_\lambda^+$ and one in $H^- = \{z : \operatorname{Re} z < \nu^-\}$ where $q_\lambda^- < \nu^- < 0$ (see [5]). Note that $F_\lambda(z)$ becomes $\lambda \sin z$ in the coordinates $z \to iz$.

We prove the following:

THEOREM 3. *The set of accessible points in the Julia set for maps of the form $F_\lambda(z) = \lambda(e^z - e^{-z})/2$ where $\lambda \in (0,1)$ has positive Lebesgue measure.*

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2. Proof of Theorem 1. For every $n \in \mathbb{N}$ we construct a family $\mathcal{K}_n$ of sets such that the intersection $\mathcal{C}'_\lambda = \bigcap_{n\in\mathbb{N}} \bigcup_{K\in\mathcal{K}_n} K$ is contained in the set of accessible points. Consider the strips

$$S_k = \{z \in H : \operatorname{Im} z \in [-\pi/4 + 2k\pi, \pi/4 + 2k\pi]\}$$

and let $S = \bigcup_{k\in\mathbb{Z}} S_k$. Let ε be fixed, say $\varepsilon = 1/100$. For every integer k and every positive integer j define

$$B_k^j = \{z \in S_k : \nu_\lambda + \pi(j-1)/2 + \varepsilon \le \operatorname{Re} z \le \nu_\lambda + \pi j/2 - \varepsilon\}.$$

Let C be a constant whose choice depends only on λ. Now we only assume that $C > q_\lambda + 2\pi$; we shall indicate further conditions on C as the proof proceeds. The family $\mathcal{K}_n$ consists of the nth preimages of some boxes B_k^j which we have just defined. We take a specific box

$$B_{s_0} = B_{s_0}^j \subset \{z : C < \operatorname{Re} z < 2C\}$$

and we define the collection $\mathcal{K}_n$ inductively:

- $\mathcal{K}_0 = \{B_{s_0}\}$,
- $\mathcal{K}_n$ consists of the sets K_n satisfying the following conditions:
  - (i) there exist $s_n \in \mathbb{Z}$, $j \in \mathbb{N}$ and a box $B_{s_n}^j \subset \{z : \operatorname{Re} z > E_\lambda^n(C)\}$ such that $E_\lambda^n(K_n) = B_{s_n}^j$,
  - (ii) $K_n \subset K_{n-1}$ for some $K_{n-1} \in \mathcal{K}_{n-1}$,
  - (iii) $|s_n| \ge \frac{1}{2}\max\{k : S_k \cap E_\lambda^n(K_{n-1}) \ne \emptyset\}$.

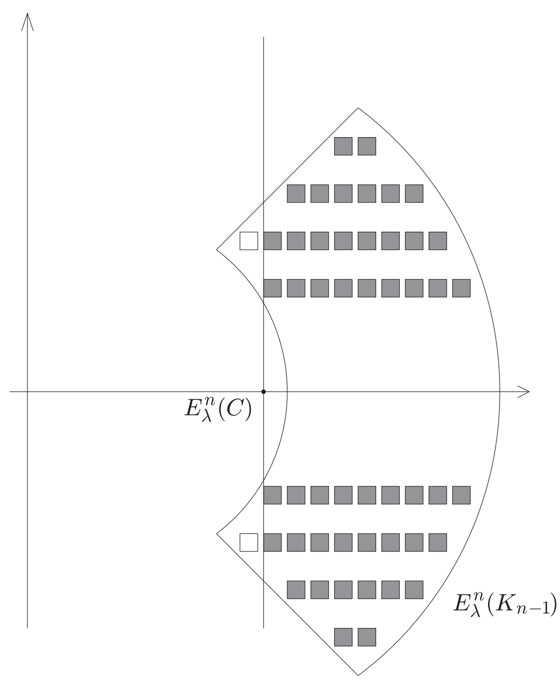

Fig. 2. Boxes from the family $E_\lambda^n(\mathcal{K}_n)$ in $E_\lambda^n(K_{n-1})$

Figure 2 shows $E_\lambda^n(K_{n-1})$ for some $K_{n-1} \in \mathcal{K}_{n-1}$, i.e. the image of some box contained in $\{z : \operatorname{Re} z > E_\lambda^{n-1}(C)\}$. Since $E_\lambda^n(K_{n-1})$ is a part of an annulus whose inner radius is greater than $E_\lambda^n(C)$ it follows that the family $\mathcal{K}_n$ is non-empty.

Let $R = \sup_{z\in E_\lambda^{n+1}(K_n)} \operatorname{Re} z$. It is easy to see that $E_\lambda^{n+1}(K_n)$ is contained in the disk of radius $r = \frac{15}{16}R$ centered at R. The branches of the inverse function of E_λ^{n+1} are univalent in the disk of a greater radius s, one can take $s = \frac{31}{32}R$. Therefore the distortion of E_λ^{n+1}, i.e.

$$\sup_{z_1,z_2\in K_n} \frac{|(E_\lambda^{n+1})'(z_1)|}{|(E_\lambda^{n+1})'(z_2)|},$$

is universally bounded on each $K_n \in \mathcal{K}_n$; this is a consequence of the Koebe distortion theorem (see [6]) which says that if f is a univalent function in $B(w,r) = \{z : |z-w| < r\}$ then for $s \in (0,r)$ the distortion of f on the disk $B(w,s)$ is bounded by $((r+s)/(r-s))^4$.

Our aim is to prove that $\mathcal{C}'_\lambda = \bigcap_{n\in\mathbb{N}} \bigcup_{K\in\mathcal{K}_n} K$ is contained in the set $\mathcal{C}_\lambda$ of endpoints and the Hausdorff dimension of $\mathcal{C}'_\lambda$ is equal to 2. Note that in this way we shall estimate the Hausdorff dimension of a compact subset of $\mathcal{C}_\lambda$ consisting of some endpoints whose itineraries grow superexponentially fast (condition (iii) in the definition of $\mathcal{K}_n$).

Let $L_k : H \to P(k)$ denote the appropriate branch of the inverse function to E_λ.

PROPOSITION 2.1. *For every* $n \in \mathbb{N}$, *every* $K_n \in \mathcal{K}_n$ *and every box*

$$B_{s_{n+1}} \subset E_\lambda^{n+1}(K_n) \cap S_{s_{n+1}} \cap \{z : \operatorname{Re} z \ge E_\lambda^{n+1}(C)\}$$

where $|s_{n+1}| \ge \frac{1}{2}\max\{k : S_k \cap E_\lambda^{n+1}(K_n) \ne \emptyset\}$ *the following holds*:

$$\operatorname{dist}(L_{s_0} \circ \ldots \circ L_{s_n}(B_{s_{n+1}}), L_{s_0} \circ \ldots \circ L_{s_n}(q_\lambda + 2\pi i s_{n+1})) \le 2C2^{-n}$$

(*where* dist *means the Euclidean distance*).

Proof. The proof is by induction. Since $B_{s_0} \subset \{z : \operatorname{Re} z < 2C\}$ it follows that for $n = 0$ and every box $B_{s_1} \subset E_\lambda(B_{s_0})$,

$$\operatorname{dist}(L_{s_0}(b), L_{s_0}(q_\lambda + 2\pi i s_1)) < 2C$$

for $b \in B_{s_1}$. We prove that for every s_{n+1}, every $B_{s_{n+1}}$ satisfying the assumption of the proposition and every $b \in B_{s_{n+1}}$,

$$\operatorname{dist}(L_{s_0} \circ \ldots \circ L_{s_n}(b), L_{s_0} \circ \ldots \circ L_{s_n}(q_\lambda + 2\pi i s_{n+1})) \le 2C2^{-n}.$$

Note that $B_{s_{n+1}} = E_\lambda^{n+1}(K_{n+1})$ for some $K_{n+1} \in \mathcal{K}_{n+1}$. Let $b \in B_{s_{n+1}}$. Since $B_{s_{n+1}} \subset E_\lambda^{n+1}(K_n) = \{z : \arg z \in [-\pi/4, \pi/4],\ e^{-\pi/2}R \le |z| \le R\}$ for some R, we have $|b| \le R$ and $2\pi|s_{n+1}| \ge R/(2\sqrt{2}) - \pi$. Since $C > q_\lambda$, it follows that $R > C$. We can assume that $C/(2\sqrt{2}) - \pi > C/3$, therefore

$2\pi|s_{n+1}| \geq R/3$. For simplicity we use the following notation:

$$a_{n-k} = L_{s_k} \circ \ldots \circ L_{s_n}(q_\lambda),$$
$$b_{n-k} = L_{s_k} \circ \ldots \circ L_{s_n}(b),$$
$$c_{n-k} = L_{s_k} \circ \ldots \circ L_{s_n}(q_\lambda + 2\pi i s_{n+1}).$$

Our aim is to prove the following inequality:

$$\operatorname{dist}(b_n, c_n) \leq \tfrac{1}{2} \operatorname{dist}(a_n, b_n).$$

We begin with considering a_0, b_0, c_0 (see Fig. 3). Notice that applying L_{s_n}

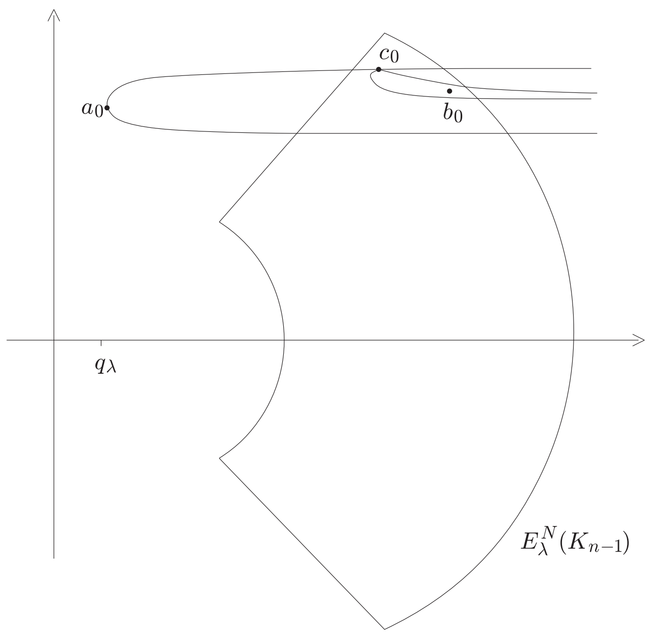

Fig. 3. The first preimages of the triple $q_\lambda, q_\lambda + 2\pi i s_{n+1}, b$

to b and $q_\lambda + 2\pi i s_{n+1}$ we obtain

$$\operatorname{Re} b_0 - \operatorname{Re} c_0 \leq \log \frac{R}{\lambda} - \log \frac{|q_\lambda + iR/3|}{\lambda} \leq \log 3,$$

so

$$\operatorname{dist}(b_0, c_0) \leq |\log 3 + 2\pi i|$$

but

$$\operatorname{dist}(a_0, b_0) \geq \log \frac{|b|}{\lambda} - q_\lambda \geq C - q_\lambda.$$

This means that for a given d_1 (arbitrarily large) if $C \geq q_\lambda + d_1|\log 3 + 2\pi i|$ then

$$d_1 \operatorname{dist}(b_0, c_0) \leq \operatorname{dist}(a_0, b_0).$$

$E_\lambda^n(K_{n-1})$ is the intersection of a sector and an annulus; denote by R' the outer radius of this annulus. It follows from our assumption that the points a_0, b_0, c_0 are contained in the strip $P(s_n)$ such that $2\pi|s_n| \ge R'/(2\sqrt{2}) - \pi$. Thus

$$\operatorname{dist}(a_0, b_0) \le 2\pi + \operatorname{Re} b_0 \le 2\pi + \tfrac{8}{9}|b_0|.$$

But $|b_0| \ge C$ and C is large so we can write

$$\operatorname{dist}(a_0, b_0) \le \tfrac{9}{10}|b_0|.$$

Therefore the disk centered at b_0 of radius $\frac{9}{10}|b_0|$ contains the points a_0 and c_0 (see Fig. 3). Now we can use the Koebe distortion theorem; the inverse branches of E_λ^n are univalent functions in $B\big(b_0, \frac{19}{20}|b_0|\big)$, thus the distortion of E_λ^n on $B\big(b_0, \frac{9}{10}|b_0|\big)$ is bounded by a constant d_2 which does not depend on C.

Hence

$$\frac{\operatorname{dist}(b_n, c_n)}{\operatorname{dist}(a_n, b_n)} \le d_2 \frac{\operatorname{dist}(b_0, c_0)}{\operatorname{dist}(a_0, b_0)} \le \frac{d_2}{d_1}.$$

Choosing $d_1 = 2d_2$ we obtain

$$\operatorname{dist}(b_n, c_n) \le \tfrac{1}{2} \operatorname{dist}(b_n, a_n).$$

But $b_n \in K_{n+1} \subset K_n$ and now a_n plays the role of c_{n-1}, so applying the inductive assumption we see that $\operatorname{dist}(a_n, b_n) \le C 2^{2-n}$. Hence $\operatorname{dist}(b_n, c_n) \le C 2^{1-n}$. ∎

PROPOSITION 2.2. $\bigcap_{n\in\mathbb{N}} \bigcup_{K\in\mathcal{K}_n} K$ *is contained in* $\mathcal{C}_\lambda$, *the set of end-points.*

Proof. Let $z \in \bigcap K_n$ where $K_n \in \mathcal{K}_n$. Then $z \in J_\lambda$ and in fact $\{z\} = \bigcap K_n$ (because E_λ is expanding in H). Let the sequence $s = \{s_i\}_{i=0}^\infty$ be the itinerary of z. It follows from Proposition 2.1 that $z = \lim_{n\to\infty} L_{s_0} \circ \ldots \circ L_{s_{n-1}} \circ L_{s_n}(q_\lambda)$.

Let γ_{s_k} denote the straight line segment joining the point ν_λ to its preimage $\nu_\lambda^{s_k} = \log(\nu_\lambda/\lambda) + 2s_k\pi i$. We may parameterize γ_{s_k} on the interval $[k, k+1]$ in such a way that $\gamma_{s_k}(k) = \nu_\lambda$ and $\gamma_{s_k}(k+1) = \nu_\lambda^{s_k}$. We define the curve ζ_s on the interval $[k, k+1]$ as follows:

- for $k = 0$, $\zeta_s(t) = \gamma_{s_0}(t)$,
- for $k > 0$, $\zeta_s(t) = L_{s_0} \circ \ldots \circ L_{s_{k-1}}(\gamma_{s_k}(t))$.

The curve ζ_s is contained in the basin of attraction and by [3] has a unique limit point (in J_λ) as $t \to \infty$. But since $L_j : H \to H$ and $|L_j'| \le \delta < 1$ in H we see that $L_{s_0} \circ \ldots \circ L_{s_{k-1}}(\nu_\lambda^k) = L_{s_0} \circ \ldots \circ L_{s_k}(\nu_\lambda)$ and $L_{s_0} \circ \ldots \circ L_{s_k}(q_\lambda)$ have the same limit as $k \to \infty$, namely z. Therefore z is accessible from the basin of attraction. ∎

REMARK. To prove Proposition 2.2 we just need to know that

$$\operatorname{dist}(K_{n+1}, L_{s_0} \circ \ldots \circ L_{s_n}(q_\lambda + 2\pi i s_{n+1})) \le \alpha_n$$

where α_n is a sequence converging to 0 (in our case $\alpha_n = \mathrm{const} \cdot 2^{-n}$). Of course, for α_n going more slowly to 0 we would get larger $\mathcal{K}_n$.

To estimate the Hausdorff dimension we use the following lemma proved by McMullen in [10]:

Lemma 2.3. *For all n let $\mathcal{K}_n$ be a finite collection of disjoint compact subsets of $\mathbb{R}^d$, and define $\widetilde{\mathcal{K}}_n = \bigcup_{K_n \in \mathcal{K}_n} K_n$. Assume that for each $K_n \in \mathcal{K}_n$ there exists $K_{n+1} \in \mathcal{K}_{n+1}$ such that $K_{n+1} \subset K_n$ and a unique $K_{n-1} \in \mathcal{K}_{n-1}$ such that $K_n \subset K_{n-1}$. If for each $K_n \in \mathcal{K}_n$,*

$$\operatorname{diam} K_n \le d_n < 1, \qquad d_n \to 0$$

and

$$\frac{\operatorname{vol}(\widetilde{\mathcal{K}}_{n+1} \cap K_n)}{\operatorname{vol} K_n} \ge \Delta_n$$

then

$$\mathrm{HD}\Big(\bigcap_{n\in\mathbb{N}} \widetilde{\mathcal{K}}_n\Big) \ge d - \limsup_{k\to\infty} \frac{\sum_{i=1}^{k+1} |\log \Delta_i|}{|\log d_k|}.$$

It follows from the definition of $\mathcal{K}_n$ that for $z \in K \in \mathcal{K}_n$ we have $\operatorname{Re} E_\lambda^i(z) > E_\lambda^i(C)$ for $i = 0, \ldots, n$. Therefore for $z \in K$,

$$|(E_\lambda^n)'(z)| \ge E_\lambda(\operatorname{Re} E_\lambda^{n-1}(z)) > E_\lambda^n(C)$$

and for every $K \in \mathcal{K}_n$,

$$\operatorname{diam} K \le d_n = \frac{\pi}{2} \frac{1}{E_\lambda^n(C)}.$$

It is sufficient to prove that there exists a constant $\Delta > 0$ such that for every n,

$$\frac{\operatorname{vol}(\widetilde{\mathcal{K}}_{n+1} \cap K_n)}{\operatorname{vol}(K_n)} \ge \Delta.$$

The distortion of E_λ^{n+1} is bounded on each $K_n \in \mathcal{K}_n$ by a constant L which does not depend on n, hence

$$\frac{\operatorname{vol}(\widetilde{\mathcal{K}}_{n+1} \cap K_n))}{\operatorname{vol}(K_n)} \ge L^2 \frac{\operatorname{vol}(E_\lambda^{n+1}(\widetilde{\mathcal{K}}_{n+1} \cap K_n))}{\operatorname{vol}(E_\lambda^{n+1}(K_n))}$$

and it suffices to show that the last quotient is bounded away from 0. Let $K_n \in \mathcal{K}_n$ and let R denote the radius of $E_\lambda^{n+1}(K_n)$. By the construction of $\mathcal{K}_{n+1}$ we see that

$$\begin{aligned}&\operatorname{vol}(E_\lambda^{n+1}(\widetilde{\mathcal{K}}_{n+1} \cap K_n)) \\ &\quad \ge (1-\varepsilon) \operatorname{vol}\Big(E_\lambda^{n+1}(K_n) \cap \{z : \operatorname{Re} z > E_\lambda^{n+1}(C)\} \cap \bigcup_{|k|\ge p} S_k\Big) - \pi R.\end{aligned}$$

where $p = \frac{1}{2}\max\{k : S_k \cap E_\lambda^{n+1}(K_n) \neq \emptyset\}$. The last term in the above inequality is an estimate of the volume of those boxes which intersect but are not contained in $E_\lambda^{n+1}(K_n) \cap \{z : \operatorname{Re} z > E_\lambda^{n+1}(C)\} \cap \bigcup_{|k|\geq p} S_k$. Since R is much bigger than the width of the strip S_k we have

$$\operatorname{vol}(E_\lambda^{n+1}(\widetilde{\mathcal{K}}_{n+1} \cap K_n)) \geq \frac{1}{10}\operatorname{vol}(E_\lambda^{n+1}(K_n)).$$

This finishes the proof of Theorem 1. ■

Remark. The method of proof carries over to the maps E_λ which have an attracting fixed point, i.e. λ is of the form $\lambda = \xi e^{-\xi}$ for some ξ with $|\xi| < 1$. The maps E_λ which have a single attracting fixed point are quasiconformally conjugate (see [7]), the Julia set is also a union of "hairs" whose endpoints are the only accessible points from the basin of attraction (see [3]). Let D be a small disk with center at the attracting fixed point p_λ such that $E_\lambda(D) \subset \operatorname{int} D$. Consider the sequence of components of $E_\lambda^{-k}(D)$ containing p_λ. Denote by $\widetilde{D}$ the first component containing 0 (we know that 0 belongs to the basin of attraction). There exists a curve γ such that $\gamma = E_\lambda^{-1}(\partial\widetilde{D})$ and $T_{2\pi i}(\gamma) = \gamma$ where $T_{2\pi i}$ is the translation by $2\pi i$. Note that γ is disjoint from $\partial\widetilde{D}$. The left half-plane bounded by γ is mapped by E_λ into itself and the Julia set is contained in the right half-plane H_γ. Now H_γ plays the role of H; we divide it into the strips

$$P(k) = \{z \in H_\gamma : (2k-1)\pi - \arg\lambda \leq \operatorname{Im} z < (2k+1)\pi - \arg\lambda\}$$

where $k \in \mathbb{Z}$ and we define

$$S_k = \{z \in H_\gamma : \operatorname{Im} z \in [-\pi/4 - \arg\lambda + 2k\pi, \pi/4 - \arg\lambda + 2k\pi]\}.$$

Then for $z \in \bigcup_{k\in\mathbb{Z}} S_k$ we have $\arg E_\lambda(z) \in [-\pi/4, \pi/4]$.

3. Proof of Theorem 2. We apply the methods of thermodynamic formalism described e.g. in [11]. Assume that $f : \mathbb{C} \to \mathbb{C}$ is an expanding map (there is a constant a such that $|f'| > a > 1$) and that f is conformal and open. Let X be a compact, f-invariant set. We say that X is a *repeller* if there exists a neighbourhood U of X such that $X = \bigcap_{n\geq 0} f_{|U}^{-n}(U)$.

For $z_0 \in \mathbb{C}$, the *topological pressure* is defined as

$$P(\alpha) = \lim_{n\to\infty} \frac{1}{n} \log S_n(\alpha)$$

where

$$S_n(\alpha) = \sum_{z\in f^{-n}(z_0)} \frac{1}{|(f^n)'(z)|^\alpha}$$

$S_n(\alpha)$ does not depend on the choice of the point z_0 (because of uniformly bounded distortion of the iterates of an expanding map). It is easy to see that the function $\alpha \to P(\alpha)$ is strictly decreasing, convex, and $P(0) > 0$.

Hence, there exists a unique α_0 such that $P(\alpha_0) = 0$. We use the theorem which says that if X is a conformal expanding repeller then $\mathrm{HD}(X) = \alpha_0$ (see [2], [12]).

Let K be defined by

$$K = \left\{ z \in \mathbb{C} : \pi \le \operatorname{Im} z \le (2N+1)\pi,\ \nu_\lambda \le \operatorname{Re} z \le \log \frac{3\pi N}{\lambda} \right\}.$$

Then E_λ maps K onto an annulus which covers K, and $E_\lambda(\{z : \operatorname{Re} z = \log(3\pi N/\lambda)\})$ and K are disjoint. Indeed, for N sufficiently large,

$$E_\lambda\left(\log \frac{3\pi N}{\lambda} \right) \ge \left| \log \frac{3\pi N}{\lambda} + (2N+1)\pi i \right|.$$

From now on we make the assumption that N is so large (N depends only on λ) that the following condition holds:

$$q_\lambda \le \left| \log \frac{3\pi N}{\lambda} + (2N+1)\pi i \right| \le 3\pi N. \tag{1}$$

Thus if $1 \le s \le N$ then $L_s K \subset K$ (as before, L_s denotes the appropriate branch of the inverse function).

Let $\mathcal{K}_i = \bigcup L_{s_0} \circ \ldots \circ L_{s_i}(K)$, where the union is over all finite sequences $(s_0, \ldots, s_i)$ such that $1 \le s_j \le N$, $j = 0, \ldots, i$.

Proposition 3.1. *Assume that N satisfies* (1). *Then*

$$\mathcal{C}_{\lambda,N} = \bigcap_{i \ge 1} \mathcal{K}_i.$$

Proof. For $1 \le s_j \le N$, L_{s_j} maps K into itself and there exists a constant $a < 1$ such that $|L'_{s_j}(z)| < a$ for any $z \in K$. The diameters of $L_{s_0} \circ \ldots \circ L_{s_n}(K)$ shrink to 0 as n tends to infinity so the intersection $\bigcap_{n \in \mathbb{N}} L_{s_0} \circ \ldots \circ L_{s_n}(K)$ is a point which has itinerary $s = (s_0, s_1, \ldots)$. Denote it by z_s.

Thus for a given sequence s there exists a unique point z_s such that $E_\lambda^n(z_s) \in K$ for every n. We claim that z_s is an accessible point in J_λ. Indeed, the straight line segments joining the point ν_λ to its preimages $L_{s_j}(\nu_\lambda)$ for $1 \le s_j \le N$ have a uniformly bounded length and E_λ is expanding on K. Therefore the curve ζ_s constructed in the same way as in the proof of Proposition 2.2 converges to a point z which remains in K under iteration of E_λ and $s(z) = s$. Hence $z = z_s$. ■

Now we give a lower bound for the Hausdorff dimension of $\mathcal{C}_{\lambda,N}$. The set $\mathcal{C}_{\lambda,N}$ is a conformal expanding repeller, $\mathcal{C}_{\lambda,N} = \bigcap_{i \ge 0} \mathcal{K}_i$, so it is sufficient to estimate the zero of the function

$$P_N(\alpha) = \lim_{n \to \infty} \frac{1}{n} \log \sum_{z \in E_\lambda^{-n}(q_\lambda),\, s(z) \in \Sigma_N} \frac{1}{|(E_\lambda^n)'(z)|^\alpha}.$$

Since

$$S_{n+1}^N(\alpha) = \frac{N}{q_\lambda^\alpha} \sum_{z_1 \in E_\lambda^{-1}(q_\lambda)} \frac{1}{|z_1|^\alpha} \Bigg(\sum_{z_2 \in E_\lambda^{-1}(z_1)} \frac{1}{|z_2|^\alpha} \Bigg(\cdots \Bigg(\sum_{z_n \in E_\lambda^{-1}(z_{n-1})} \frac{1}{|z_n|^\alpha} \Bigg) \Bigg) \cdots \Bigg)$$

and for each $1 \le k \le n$,

$$\sum_{z_k \in E_\lambda^{-1}(z_{k-1})} \frac{1}{|z_k|^\alpha} \ge \sum_{k=1}^{N} \frac{1}{[(2k+1)^2\pi^2 + (\log 3\pi N/\lambda)^2]^{\alpha/2}},$$

we have

$$\begin{aligned} P_N(1) &\ge \log \Bigg(\sum_{k=1}^{N} \frac{1}{[(2k+1)^2\pi^2 + (\log cN)^2]^{1/2}} \Bigg) \\ &\ge \log \Bigg(\sum_{k=[\log cN]}^{N} \frac{1}{\sqrt{2}(2k+1)\pi} \Bigg) \end{aligned}$$

where $c = 3\pi/\lambda$. Since

$$\frac{1}{\sqrt{2}\pi} \sum_{k=[\log cN]}^{N} \frac{1}{(2k+1)} \ge \frac{1}{\sqrt{2}\pi} \int_{k=[\log cN]}^{N+1} \frac{dx}{2x+1} = \frac{1}{2\sqrt{2}\pi} \log \frac{2N+3}{2[\log cN]+1},$$

$P_N(1)$ can be large for N sufficiently large.

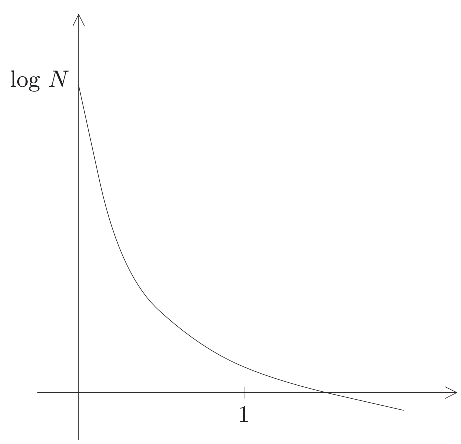

Fig. 4. The graph of the pressure function $P_N(\alpha)$

Hence for every $\lambda \in (0, 1/e)$ there exists N_0 such that for every $N > N_0$ the Hausdorff dimension of $\mathcal{C}_{\lambda,N}$ is greater than 1. Now we prove that there exists λ_0 such that for every $\lambda \in (0, \lambda_0)$,

$$\mathrm{HD}(\mathcal{C}_{\lambda,N}) > 1 + \frac{1}{\log(\log 1/\lambda)}.$$

Assume that λ is so small that condition (1) holds for $N = [1/\lambda]$ and additionally that $3\pi/\log cN \le 1$.

It suffices to estimate

$$\sum_{k=1}^{N} \frac{1}{[(2k+1)^2\pi^2 + (\log cN)^2]^{\alpha/2}} \ge \int_1^{N+1} \frac{dx}{[(2x+1)^2\pi^2 + (\log cN)^2]^{\alpha/2}}.$$

Substituting

$$t = \frac{(2x+1)\pi}{\log cN}, \quad A = \frac{3\pi}{\log cN}, \quad B = \frac{(2N+3)\pi}{\log cN}$$

we see that the latter expression is equal to

$$(2) \qquad \frac{1}{2\pi(\log cN)^{\alpha-1}} \int_A^B \frac{dt}{(t^2+1)^{\alpha/2}}$$

$$\ge \frac{1}{\pi 2^{1+\alpha/2}(\log cN)^{\alpha-1}} \int_1^B \frac{dt}{t^\alpha}$$

$$= \frac{1}{(\alpha-1)\pi 2^{1+\alpha/2}} \left[\frac{1}{(\log cN)^{\alpha-1}} - \left(\frac{1}{(2N+3)\pi} \right)^{\alpha-1} \right].$$

Let $\alpha = 1 + 1/\log(\log 1/\lambda)$ and $N = [1/\lambda]$. If λ is sufficiently small then the last term in (2) is smaller than $1/(2(\log cN)^{\alpha-1})$. Hence for $N = [1/\lambda]$,

$$P_N\left(1 + \frac{1}{\log(\log 1/\lambda)}\right) \ge \log\frac{1}{8\pi} + \log\left(\log\left(\log\frac{1}{\lambda}\right)\right) - \frac{\log(\log 3\pi/\lambda^2)}{\log(\log 1/\lambda)}.$$

Thus for all sufficiently small λ,

$$P_N\left(1 + \frac{1}{\log(\log 1/\lambda)}\right) \ge 0$$

and

$$\mathrm{HD}(\mathcal{C}_{\lambda,N}) \ge 1 + \frac{1}{\log(\log 1/\lambda)}.$$

Now we prove the last inequality in Theorem 2. Let

$$\Sigma'_N = \{s = (s_0, s_1, \ldots) : \forall j,\ s_j \in \mathbb{Z},\ |s_j| \le N\},$$

and let $\mathcal{C}'_{\lambda,N}$ denote the set of endpoints whose itineraries belong to Σ'_N. We show that there exists $\lambda_0 \in (0, 1/e)$ such that for $\lambda \in (0, \lambda_0)$,

$$\mathrm{HD}\Big(\bigcup_{N\ge1} \mathcal{C}'_{\lambda,N}\Big) \le 1 + \frac{1}{\log(\log(\log 1/\lambda))}.$$

We have

$$S_n^N(1+\varepsilon) = \sum_{z\in E_\lambda^{-n}(q_\lambda),\, s(z)\in\Sigma_N'} \frac{1}{|(E_\lambda^n)'(z)|^{1+\varepsilon}}.$$

We can write the sum S_n^N in the same form as before. For every $N \in \mathbb{N}$ and $1 \le l \le n$ we have

$$\sum_{z_l\in E_\lambda^{-1}(z_{l-1})} \frac{1}{|z_l|^{1+\varepsilon}} \le 2\sum_{k=1}^{N} \frac{1}{[\nu_\lambda^2 + (2k-1)^2\pi^2]^{(1+\varepsilon)/2}} + \frac{1}{\nu_\lambda^{1+\varepsilon}}.$$

Hence

$$P_N(1+\varepsilon) \le \log\left(2\sum_{k=1}^{N} \frac{1}{[\nu_\lambda^2 + (2k-1)^2\pi^2]^{(1+\varepsilon)/2}} + \frac{1}{\nu_\lambda^{1+\varepsilon}}\right).$$

It is easy to show that

$$\begin{aligned}
&\frac{1}{\nu_\lambda^{1+\varepsilon}} + 2\sum_{k=1}^{\infty} \frac{1}{[\nu_\lambda^2 + (2k-1)^2\pi^2]^{(1+\varepsilon)/2}} \\
&\qquad\le \frac{1}{\nu_\lambda^{1+\varepsilon}} + 2\int_0^\infty \frac{dx}{[\nu_\lambda^2 + (\pi x)^2]^{(1+\varepsilon)/2}} \\
&\qquad\le \frac{1}{\nu_\lambda^{1+\varepsilon}} + \frac{2}{\pi\nu_\lambda^\varepsilon}\left(\int_0^1 \frac{dt}{[1+t^2]^{(1+\varepsilon)/2}} + \int_1^\infty \frac{dt}{[1+t^2]^{(1+\varepsilon)/2}}\right) \\
&\qquad\le \frac{1}{\nu_\lambda^{1+\varepsilon}} + \frac{2}{\pi\nu_\lambda^\varepsilon}\left(c_0 + \frac{1}{\varepsilon}\right).
\end{aligned}$$

where

$$c_0 = \int_0^1 \frac{dt}{[1+t^2]^{(1+\varepsilon)/2}}.$$

Since $\nu_\lambda > \log 1/\lambda$ we see that for small λ and $\varepsilon = 1/\log(\log\nu_\lambda)$ we have

$$\frac{1}{\nu_\lambda^{1+\varepsilon}} + \frac{2}{\pi\nu_\lambda^\varepsilon}\left(c_0 + \frac{1}{\varepsilon}\right) \le 1$$

(because the left hand side tends to 0 as $\lambda \to 0$). Therefore for λ sufficiently small

$$\mathrm{HD}\Big(\bigcup_{N\ge1} \mathcal{C}_{N,\lambda}'\Big) \le 1 + \frac{1}{\log(\log\nu)} \le 1 + \frac{1}{\log(\log(\log 1/\lambda))}.$$

This completes the proof of Theorem 2. ∎

Remark. The theorem remains true for $\lambda = \xi e^{-\xi}$ where $\xi \in \mathbb{C}$, $|\xi| < 1$ (if we replace λ by $|\lambda|$). If $|\lambda| < 1/e$ (in particular, in the second part of the

Theorem 2 we can assume that $|\lambda| < 1/e$) then the only modification in the proof is that we define

$$K = \left\{ z : \pi - \arg\lambda \le \operatorname{Im} z \le (2N+1)\pi - \arg\lambda,\ \nu_\lambda \le \operatorname{Re} z \le \log\frac{3\pi N}{\lambda} \right\}.$$

We take an arbitrary point $z_0 \in K$ and we estimate the zero of the pressure function starting from the point z_0. If $|\lambda| \ge 1/e$ then we need to ensure that $E_\lambda(K) \supset K$. Let n_λ be an integer such that $(2n_\lambda - 1)\pi \ge |E_\lambda(\nu_\lambda)|$ and define

$$K = \left\{ z : (2n_\lambda - 1)\pi - \arg\lambda \le \operatorname{Im} z \le (2N+1)\pi - \arg\lambda, \right.$$
$$\left. \nu_\lambda \le \operatorname{Re} z \le \log\frac{3\pi N}{\lambda} \right\}.$$

Hence if $n_\lambda \le s \le N$ then $L_s(K) \subset K$. We consider the subset of $\mathcal{C}_{\lambda,N}$ consisting of those endpoints which never visit the strips $P(0), \ldots, P(n_\lambda - 1)$ under iteration of E_λ and in the same way as before we prove that for N sufficiently large the Hausdorff dimension of this subset is greater than 1.

4. The set of endpoints for the sine family. In the proof of Theorem 3 we use a method analogous to that for Theorem 1. We follow the notation used in the introduction: q_λ^+ and q_λ^- are the repelling fixed points (real), $q_\lambda^+ = -q_\lambda^-$. For $k \in \mathbb{Z}$ define

$$S_k^+ = \{z \in \mathbb{C} : \operatorname{Re} z \ge q_\lambda^+,\ \operatorname{Im} z \in (-\pi/2 + k\pi, \pi/2 + k\pi)\},$$
$$S_k^- = \{z \in \mathbb{C} : \operatorname{Re} z \le q_\lambda^-,\ \operatorname{Im} z \in (\pi/2 + k\pi, 3/2\pi + k\pi)\}.$$

Let $S_k = S_k^+ \cup S_k^-$ and $S = \bigcup_{k\in\mathbb{Z}} S_k$.

The part of the preimage of the vertical line $V^+ = \{z \in \mathbb{C} : \operatorname{Re} z = q_\lambda^+\}$ contained in S_k has two components: one in S_k^+ and one in S_k^-. Note that k must be even. Hence there are two branches of the inverse function mapping the right half-plane H^+ into S_k for k even; we use the same notation $L_k : H^+ \to S_k$ for both. Similarly, $V^- = \{z \in \mathbb{C} : \operatorname{Re} z = q_\lambda^-\}$ has two preimages in S_k for k odd, $L_k : H^- \to S_k$.

We pack every S_k with boxes that have sides of length π:

$$B_{k,j}^1 = \{z \in S_k^+ : q_\lambda^+ + j\pi < \operatorname{Re} z < q_\lambda^+ + (j+1)\pi\} \quad \text{for } j = 0, 1, \ldots,$$
$$B_{k,j}^{-1} = \{z \in S_k^- : q_\lambda^- + (j-1)\pi < \operatorname{Re} z < q_\lambda^- + j\pi\} \quad \text{for } j = 0, -1, \ldots$$

Let C be a constant such that

$$\lambda(e^C - e^{-C})/2 > C > 10^4/\lambda. \tag{3}$$

Let $g(x) = e^x$. Note that for every $x \ge C$ we have $F_\lambda(2x) > \lambda e^{2x}/4 > 2g(x)$.

Hence for every $n \in \mathbb{N}$,

$$F_\lambda(2g^n(C)) > 2g^{n+1}(C). \tag{4}$$

We take a box $B_{s_0} \in \{z : 2C < \operatorname{Re} z < 3C\}$ (i.e. $B^1_{s_0,j}$ for some j) and we inductively define the following collection $\mathcal{K}_n$ of sets:

- $\mathcal{K}_0 = \{B_{s_0}\}$,
- $\mathcal{K}_n$ consists of the connected sets K_n satisfying the following conditions:
  (i) there exists a box $B_{s_n} \subset \{z : |\operatorname{Re} z| > 2g^n(C)\}$ such that
  $$F^n_\lambda(K_n) = B_{s_n},$$
  (ii) $K_n \subset K_{n-1}$ for some $K_{n-1} \in \mathcal{K}_{n-1}$ and
  $$\pi|s_n| \ge \big(\sup_{z \in F^n_\lambda(K_{n-1})} |\operatorname{Im} z|\big)^{3/4}.$$

Condition (4) guarantees that $\mathcal{K}_{n+1}$ is nonempty for every $n \in \mathbb{N}$ ($F^{n+1}_\lambda(K_n)$ lies outside the disk of radius $F_\lambda(2g^n(C))$; see Fig. 5).

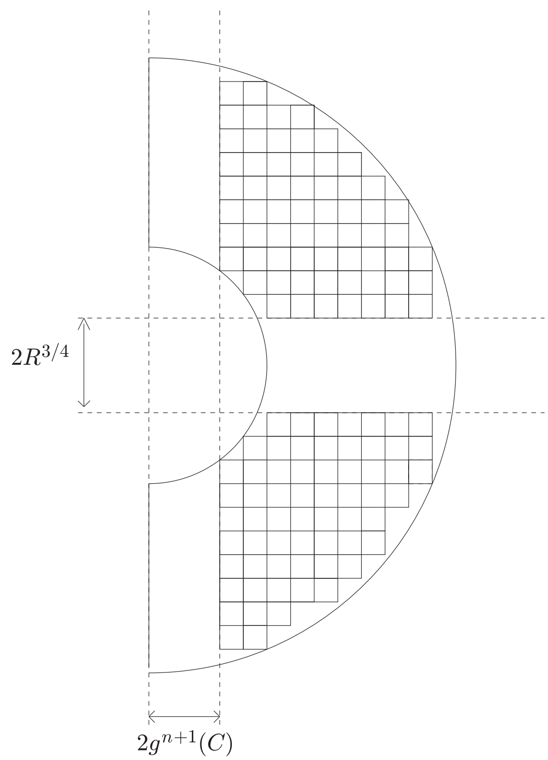

Fig. 5. $F^{n+1}_\lambda(K_n)$ packed with boxes which belong to $F^{n+1}_\lambda(\mathcal{K}_{n+1})$

If $K_{n+1} \in \mathcal{K}_{n+1}$ then $K_{n+1} = L_{s_0} \circ \ldots \circ L_{s_n}(B^\varepsilon_{s_{n+1}})$ for some box $B^\varepsilon_{s_{n+1}}$, where $\varepsilon = \pm 1$. The choice of the branches of L_{s_i} in the above composition (we choose the branch going to $S^+_{s_i}$ or to $S^-_{s_i}$) depends only on the parity of the numbers s_i. Note that $\varepsilon = 1$ (resp. -1) if and only if s_n is even (resp. odd). If s_{i-1} (for $i = 1, \ldots, n$) is even then we choose L_{s_i} going to $S^+_{s_i}$, otherwise we take L_{s_i} going to $S^-_{s_i}$. Since $K_{n+1} \subset H^+$, L_{s_0} goes to $S^+_{s_0}$.

PROPOSITION 4.1. *Let $n \in \mathbb{N}$ and $\varepsilon \in \{-1, 1\}$. Then for every box*

$$B^\varepsilon_{s_{n+1}} \subset F^{n+1}_\lambda(K_n) \cap \{z : |\mathrm{Re}\, z| \geq 2g^{n+1}(C)\} \cap S_{s_{n+1}}$$

where $\pi|s_{n+1}| \geq (\sup_{z \in F^{n+1}_\lambda(K_n)} |\mathrm{Im}\, z|)^{3/4}$ the following holds:

$$\mathrm{dist}(L_{s_0} \circ \ldots \circ L_{s_n}(B^\varepsilon_{s_{n+1}}), L_{s_0} \circ \ldots \circ L_{s_n}(q_\lambda + \pi i s_{n+1})) \leq 3C(3/4)^n$$

where $q_\lambda = q^+_\lambda$ if $\varepsilon = 1$ and $q_\lambda = q^-_\lambda$ if $\varepsilon = -1$.

Proof. For every box $B^\varepsilon_{s_1} \subset F_\lambda(B_{s_0})$ and for every $b \in B^\varepsilon_{s_1}$ we have $\mathrm{dist}(L_{s_0}(b), L_{s_0}(q_\lambda + \pi i s_1)) \leq 3C$.

Let $B^\varepsilon_{s_{n+1}}$ be a box satisfying the assumption and let $b \in B^\varepsilon_{s_{n+1}} = F^{n+1}_\lambda(K_{n+1})$. We prove by induction that

$$\mathrm{dist}(L_{s_0} \circ \ldots \circ L_{s_n}(b), L_{s_0} \circ \ldots \circ L_{s_n}(q_\lambda + \pi i s_{n+1})) \leq 3C(3/4)^n.$$

We use the same notation as in the proof of Proposition 2.1:

$$a^n_{n-k} = L_{s_k} \circ \ldots \circ L_{s_n}(q_\lambda),$$
$$b^n_{n-k} = L_{s_k} \circ \ldots \circ L_{s_n}(b),$$
$$c^n_{n-k} = L_{s_k} \circ \ldots \circ L_{s_n}(q_\lambda + \pi i s_{n+1}),$$

where $q_\lambda = q^+_\lambda$ if $\varepsilon = 1$ and $q_\lambda = q^-_\lambda$ if $\varepsilon = -1$.

If $t = \exp(b^n_0)$ then t satisfies the equation $t^2 - 2bt/\lambda - 1 = 0$.

Since $\mathrm{Re}\, b$ is greater than $2g^{n+1}(C)$ it is easy to see that

$$\log \frac{|b|}{3\lambda} \leq |\mathrm{Re}\, b^n_0| \leq \log \frac{3|b|}{\lambda}.$$

Moreover,

$$|b| \geq R^{3/4} - \pi \quad \text{where} \quad R = \sup_{z \in F^{(n+1)}_\lambda(K_n)} |\mathrm{Im}\, z|.$$

Hence

$$|\mathrm{Re}\, b^n_0 - \mathrm{Re}\, a^n_0| = |\mathrm{Re}\, b^n_0 - q_\lambda| \geq \log \frac{|b|}{3\lambda} - |q_\lambda| \geq \frac{3}{4} \log R - (\log \lambda + |q_\lambda| + 2).$$

Since $|b| \leq R$ and $\pi|s_{n+1}| \geq R^{3/4}$ we see that

$$|\mathrm{Re}\, b^n_0 - \mathrm{Re}\, c^n_0| \leq \log 9 \frac{|b|}{|q_\lambda + \pi i s_{n+1}|} \leq \log(9R^{1/3}).$$

Hence

$$\text{dist}(b_0^n, c_0^n) \le \tfrac{1}{2}|\text{Re}\, b_0^n - \text{Re}\, a_0^n|. \tag{5}$$

Let $k \ge 0$ denote the first time when $\text{dist}(a_k^n, b_k^n) < d$ (here d is a fixed constant with $d > 4 + 2\pi$), i.e.:

$$\forall i = 0, \ldots, k-1, \quad \text{dist}(a_i^n, b_i^n) \ge d \quad \text{and} \quad \text{dist}(a_k^n, b_k^n) < d. \tag{6}$$

If $k = 0$ then it follows from (5) that the points a_0, b_0, c_0 lie in some set of diameter smaller than $2d$. Therefore by the Koebe distortion theorem (F_λ has only two critical values: $\lambda i, -\lambda i$, all of its postcritical values are attracted to 0, and it has no finite asymptotic values) the distortion for the iterates of F_λ is bounded. The distortion is smaller than 10/9 (because C satisfies (3)). Hence $\text{dist}(b_n^n, c_n^n) \le \frac{3}{4}\,\text{dist}(b_n^n, a_n^n)$.

Now assume that $k > 0$. First we show that $\text{dist}(b_1^n, c_1^n) \le 2 + \pi$. It follows from (5) that

$$\text{dist}(c_0^n, b_0^n) \le |c_0^n| \quad \text{and} \quad \frac{1}{2} \le \frac{|b_0^n|}{|c_0^n|} \le 2.$$

Because $|b_0^n| > 2g^n(C)$ and $|c_0^n| > g^n(C)$, we have

$$|\text{Re}\, b_1^n - \text{Re}\, c_1^n| \le 2.$$

Thus for each $i > 1$, $\text{dist}(b_i^n, c_i^n) \le 2 + \pi$ and the condition (6) means that

$$\forall i = 0, \ldots, k-1, \quad \text{dist}(b_i^n, c_i^n) \le \tfrac{1}{2}\,\text{dist}(a_i^n, b_i^n). \tag{7}$$

The points a_k^n, b_k^n, c_k^n are contained in a set A_k of a fixed diameter so it is enough to show

$$\text{dist}(b_k^n, c_k^n) \le \tfrac{2}{3}\,\text{dist}(a_k^n, b_k^n). \tag{8}$$

Assume that the above inequality is false. Then we can use the bounded distortion argument to obtain a contradiction with (7):

$$\frac{9}{10} \cdot \frac{\text{dist}(a_{k-1}^n, b_{k-1}^n)}{\text{dist}(b_{k-1}^n, c_{k-1}^n)} \le \frac{\text{dist}(a_k^n, b_k^n)}{\text{dist}(b_k^n, c_k^n)} \le \frac{3}{2}.$$

Since the distortion of the iterates of F_λ on A_k is bounded by 10/9 it follows from (8) that

$$\text{dist}(b_n^n, c_n^n) \le \tfrac{3}{4}\,\text{dist}(a_n^n, b_n^n).$$

But $a_n^n = c_{n-1}^{n-1}$, hence by induction

$$\text{dist}(b_n^n, c_n^n) \le 3C(3/4)^n.$$

Another way to prove the above inequality is to apply the Koebe one-quarter theorem to (5) (this remark is due to F. Przytycki). ■

It follows from the above proof that for every n,

$$\operatorname{dist}(c_1^n, c_2^{n+1}) \le \operatorname{dist}(c_1^n, b_1^n) + \operatorname{dist}(b_1^n, b_2^{n+1}) + \operatorname{dist}(b_2^{n+1}, c_2^{n+1}) \le 4 + 3\pi.$$

We apply this simple observation to prove the following:

PROPOSITION 4.2. *The points from* $\bigcap_{n\in\mathbb{N}} \bigcup_{K\in\mathcal{K}_n} K$ *are accessible from the basin of attraction along curves of universally bounded length.*

Proof. Let $z \in \bigcap_{n\in\mathbb{N}} \bigcup_{K\in\mathcal{K}_n} K$ and let $s(z) = (s_0, s_1, \ldots)$ be the itinerary of z. By Proposition 4.1, $z = \lim_{n\to\infty} L_{s_0} \circ \ldots \circ L_{s_n}(q_\lambda + \pi i s_{n+1}) = \lim_{n\to\infty} c_n^n$. For every n and every pair of points c_1^n, c_2^n we can find a pair of points ξ_1^n, ξ_2^n such that $\operatorname{Re}\xi_1^n = \operatorname{Re} c_1^n$, $\operatorname{Re}\xi_2^n = \operatorname{Re} c_2^{n+1}$, $\operatorname{Im}\xi_1^n = \operatorname{Im}\xi_2^n$ and the straight line segment γ_n joining ξ_1^n and ξ_2^n is contained in the basin of attraction. For every n the length of γ_n is bounded by $4 + 3\pi$ and $\gamma_n \subset \{z : |\operatorname{Re} z| \ge 2g^{n-1}(C) - d\}$.

Now we define the curve ζ_s taking the preimages of segments γ_n in the same way as for exponential maps (see the proof of Proposition 2.2). Since F_λ is expanding in the region $\{z : |\operatorname{Re} z| \ge 2g^n(C) - d\}$, the curve $\zeta_s(t)$ has the unique limit point z. ■

Now we show that the set $\bigcap_{n\in\mathbb{N}} \bigcup_{K\in\mathcal{K}_n} K$ has positive Lebesgue measure.

PROPOSITION 4.3. *There exists a constant* $\Delta > 0$ *such that*

$$\operatorname{vol}\Big(\bigcap_{n\in\mathbb{N}} \bigcup_{K\in\mathcal{K}_n} K\Big) \ge \Delta \operatorname{vol} B_{s_0}.$$

Proof. Let $\widetilde{\mathcal{K}}_n = \bigcup_{K\in\mathcal{K}_n} K$. Since the distortion of F_λ^{n+1} on K_n is bounded, we have

$$\frac{\operatorname{vol}(K_n \cap \widetilde{\mathcal{K}}_{n+1})}{\operatorname{vol}(K_n)} \ge 1 - O\bigg(\frac{\operatorname{vol}(F_\lambda^{n+1}(K_n \setminus \widetilde{\mathcal{K}}_{n+1}))}{\operatorname{vol}(F_\lambda^{n+1}(K_n))}\bigg).$$

Let $R = \sup_{z\in F_\lambda^{(n+1)}(K_n)} |\operatorname{Im} z|$. By the definition of the family $\mathcal{K}_n$,

$$R > \frac{\lambda}{2}(e^{2g^n(C)} - e^{-2g^n(C)}) > \frac{\lambda}{4} e^{2g^n(C)}.$$

We have (see Fig. 5)

$$\operatorname{vol}(F_\lambda^{n+1}(K_n \setminus \widetilde{\mathcal{K}}_{n+1})) = O(Rg^{n+1}(C)) + O(R^{7/4})$$

and therefore

$$\begin{aligned}\frac{\operatorname{vol}(F_\lambda^{n+1}(K_n \setminus \widetilde{\mathcal{K}}_{n+1}))}{\operatorname{vol}(F_\lambda^{n+1}(K_n))} &= O\bigg(\frac{g^{n+1}(C)}{R}\bigg) + O(R^{-1/4}) \\ &\le O\bigg(\frac{1}{e^{g^n(C)}} + \frac{1}{(e^{g^n(C)})^{1/4}}\bigg).\end{aligned}$$

We can take C large enough to guarantee that the sums $\sum_{n=1}^{\infty} 1/g^n(C)$ and $\sum_{n=1}^{\infty} 1/(g^n(C))^{1/4}$ are small. Therefore there exists a constant $\Delta > 0$ such that

$$\frac{\operatorname{vol}(\bigcap_{n\in\mathbb{N}} \bigcup_{K\in\mathcal{K}_n} K)}{\operatorname{vol} B_{s_0}} \geq \prod_{n=1}^{\infty} \left(1 - O\left(\frac{1}{e^{g^n(C)}} + \frac{1}{(e^{g^n(C)})^{1/4}}\right)\right) \geq \Delta. \ \blacksquare$$

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Institute of Mathematics
Technical University of Warsaw
Pl. Politechniki 1
00-661 Warszawa, Poland
E-mail: bkarpin@snowman.impan.gov.pl

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