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A note on Tsirelson type ideals

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Abstract. Using Tsirelson's well-known example of a Banach space which does not contain a copy of c_0 or l_p, for $p \geq 1$, we construct a simple Borel ideal $\mathcal{I}_{\mathrm{T}}$ such that the Borel cardinalities of the quotient spaces $\mathcal{P}(\mathbb{N})/\mathcal{I}_{\mathrm{T}}$ and $\mathcal{P}(\mathbb{N})/\mathcal{I}_0$ are incomparable, where $\mathcal{I}_0$ is the summable ideal of all sets $A \subseteq \mathbb{N}$ such that $\sum_{n\in A} 1/(n+1) < \infty$. This disproves a "trichotomy" conjecture for Borel ideals proposed by Kechris and Mazur.

Introduction. Given Borel equivalence relations E and F on Polish spaces X and Y respectively, we say that E is *Borel reducible* to F and write $E \leq_{\mathrm{Bor}} F$ if there is a Borel function $f : X \to Y$ such that for every x and y in X

$$x\,E\,y \quad \text{iff} \quad f(x)\,F\,f(y).$$

For such f let $f^* : X/E \to Y/F$ be defined by $f^*([x]_E) = [f(x)]_F$. Then f^* is an injection of X/E to Y/F which has a Borel lifting f. We write

$$E \sim_{\mathrm{Bor}} F \quad \text{iff} \quad E \leq_{\mathrm{Bor}} F \,\&\, F \leq_{\mathrm{Bor}} E.$$

By an *ideal* $\mathcal{I}$ on $\mathbb{N}$ we mean an ideal of subsets of $\mathbb{N}$ which is *nontrivial*, i.e. $\mathbb{N} \notin \mathcal{I}$, and *free*, i.e. $\{n\} \in \mathcal{I}$, for all $n \in \mathbb{N}$. We say that $\mathcal{I}$ is *Borel* if it is a Borel subset of $\mathcal{P}(\mathbb{N})$ in the usual product topology. Given a Borel ideal $\mathcal{I}$ on $\mathbb{N}$ we define an equivalence relation $E_{\mathcal{I}}$ on $\mathcal{P}(\mathbb{N})$ by letting

$$X\,E_{\mathcal{I}}\,Y \quad \text{if and only if} \quad X \bigtriangleup Y \in \mathcal{I}.$$

Finally, we write $\mathcal{I} \leq_{\mathrm{Bor}} \mathcal{J}$ iff $E_{\mathcal{I}} \leq_{\mathrm{Bor}} E_{\mathcal{J}}$.

The class $(\mathcal{E}, \leq_{\mathrm{Bor}})$ of all Borel ideals with this notion of reducibility was studied by several authors. Here we identify two ideals which are $\sim_{\mathrm{Bor}}$-equivalent. In [LV] Louveau and the author showed that this structure is very rich by embedding into it the partial ordering $(\mathcal{P}(\mathbb{N}), \subseteq^*)$ (where $X \subseteq^* Y$ iff $X \setminus Y$ is finite). The ideals constructed in this proof are all $F_{\sigma\delta}$ and P-ideals (recall that an ideal $\mathcal{I}$ is a *P-ideal* iff for every sequence $\{A_n : n \in \mathbb{N}\}$ of

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members of $\mathcal{I}$ there is $A \in \mathcal{I}$ such that $A_n \subseteq^* A$ for all n). The interest of looking for P-ideals is that in this case, by a result of Solecki [So], $(\mathcal{I}, \triangle)$ is a Polish group under a suitable topology. The construction from [LV] was later modified by Mazur [Ma1] to obtain F_σ ideals. However, Mazur's ideals are not P-ideals.

By $\le_{\mathrm{RK}}$ we denote the *Rudin–Keisler ordering* on ideals, i.e.

$$\mathcal{I} \le_{\mathrm{RK}} \mathcal{J} \quad \text{iff} \quad \exists f : \mathbb{N} \to \mathbb{N}(X \in \mathcal{I} \leftrightarrow f^{-1}(X) \in \mathcal{J}).$$

The *Rudin–Blass ordering* $\le_{\mathrm{RB}}$ is obtained by requiring in the above definition that f be finite-to-one. It is clear that $\mathcal{I} \le_{\mathrm{RB}} \mathcal{J}$ implies $\mathcal{I} \le_{\mathrm{RK}} \mathcal{J}$ and this in turn implies $\mathcal{I} \le_{\mathrm{Bor}} \mathcal{J}$. It is an open question whether $\mathcal{I} \le_{\mathrm{Bor}} \mathcal{J}$ iff there is a set $A \in \mathcal{J}^+$ such that $\mathcal{I} \le_{\mathrm{RB}} \mathcal{J}\restriction A$, the restriction of $\mathcal{J}$ to $\mathcal{P}(A)$. In all known cases, this seems to be true. Mathias [Mat], Jalali-Naini [JN], and Talagrand [Ta] showed that $\mathrm{FIN} \le_{\mathrm{RB}} \mathcal{I}$ for any Borel (in fact, Baire measurable) ideal $\mathcal{I}$, where FIN is the ideal of finite subsets of $\mathbb{N}$. Thus, in a way, the "Borel cardinality" of $\mathcal{P}(\mathbb{N})/\mathrm{FIN}$ is the smallest among all $\mathcal{P}(\mathcal{I})/\mathcal{I}$ for $\mathcal{I}$ a Borel ideal.

Recently, Kechris [Ke2] addressed the issue of finding minimal ideals above FIN under $\le_{\mathrm{Bor}}$. He was motivated by the well-known dichotomy results on Borel equivalence relations. He identified two ideals related to FIN denoted by $\emptyset \times \mathrm{FIN}$ and $\mathrm{FIN} \times \emptyset$ (in fact, these ideals are defined on $\mathbb{N}^2$ but they can be moved to $\mathbb{N}$ by some fixed bijection). Define

$$X \in \emptyset \times \mathrm{FIN} \quad \text{iff} \quad \forall m(\{n : (m,n) \in X\} \text{ is finite}),$$
$$X \in \mathrm{FIN} \times \emptyset \quad \text{iff} \quad \exists m(X \subseteq m \times \mathbb{N}).$$

Thus, it is known and fairly easy to see that $\emptyset \times \mathrm{FIN}$ and $\mathrm{FIN} \times \emptyset$ are incomparable under $\le_{\mathrm{Bor}}$ and strictly above FIN (see [Ke2] for complete references). Say that $\mathcal{I}$ and $\mathcal{J}$ are *isomorphic* iff there is a permutation π of $\mathbb{N}$ such that $X \in \mathcal{I}$ iff $\pi(X) \in \mathcal{J}$. Finally, say that $\mathcal{I}$ is a *trivial variation* of FIN iff there is an infinite set A such that $\mathcal{I} = \{X \subseteq \mathbb{N} : X \cap A \text{ is finite}\}$. Kechris then showed that both $\emptyset \times \mathrm{FIN}$ and $\mathrm{FIN} \times \emptyset$ are minimal above FIN, in the following strong sense.

THEOREM 1 ([Ke2]). *If $\mathcal{I}$ is a Borel ideal and $\mathcal{I} \le_{\mathrm{Bor}} \emptyset \times \mathrm{FIN}$ ($\mathrm{FIN} \times \emptyset$, respectively) then either it is isomorphic to $\emptyset \times \mathrm{FIN}$ ($\mathrm{FIN} \times \emptyset$, respectively) or it is a trivial variation of* FIN.

By another result of Solecki [So], if $\mathcal{I}$ is a Borel ideal then $\mathrm{FIN} \times \emptyset \le_{\mathrm{RB}} \mathcal{I}$ iff $\mathcal{I}$ is not a P-ideal. Moreover, if $\mathcal{I}$ is a P-ideal then $\emptyset \times \mathrm{FIN} \le_{\mathrm{RB}} \mathcal{I}$ iff $\mathcal{I}$ is not F_σ. Thus, any ideal which is incomparable with both $\mathrm{FIN} \times \emptyset$ and $\emptyset \times \mathrm{FIN}$ is an F_σ P-ideal. One way of obtaining such ideals is from classical Banach spaces. Fix any $(\alpha_n)_n \in c_0^+ \setminus l_1$, where c_0^+ is the space of all *nonnegative* sequences of reals converging to zero; for concreteness let us say $\alpha_n = 1/(n+1)$ for all n. Define the ideal $\mathcal{I}_0$ by

$$X \in \mathcal{I}_0 \quad \text{iff} \quad \sum_{n\in X} \alpha_n < \infty.$$

Then, clearly, $\mathcal{I}_0$ is an F_σ P-ideal. It is known that $\mathcal{I}_0$ is incomparable in the sense of $\le_{\mathrm{Bor}}$ with both $\mathrm{FIN}\times\emptyset$ and $\emptyset\times\mathrm{FIN}$ (this follows from results of Kechris–Louveau [KL], Hjorth [Hj1], and has also been shown independently by Mazur [Ma2]). Moreover, Hjorth [Hj2] proved that if $\mathcal{I} \le_{\mathrm{Bor}} \mathcal{I}_0$, then either $\mathcal{I} \sim_{\mathrm{Bor}} \mathcal{I}_0$, or else $\mathcal{I}$ is a trivial variation of FIN. In the light of these results Kechris conjectured that the following trichotomy holds.

CONJECTURE 1. *If $\mathcal{I}$ is any Borel ideal on $\mathbb{N}$ and* $\mathrm{FIN} <_{\mathrm{Bor}} \mathcal{I}$ *then either* $\mathrm{FIN}\times\emptyset \le_{\mathrm{Bor}} \mathcal{I}$ *or* $\emptyset\times\mathrm{FIN} \le_{\mathrm{Bor}} \mathcal{I}$ *or* $\mathcal{I}_0 \le_{\mathrm{Bor}} \mathcal{I}$.

As noted in [Ke2], this is equivalent to a conjecture of Mazur [Ma2] which asserts that if $\mathcal{I}$ is an F_σ ideal with $\mathrm{FIN} <_{\mathrm{Bor}} \mathcal{I}$, then $\mathrm{FIN}\times\emptyset \le_{\mathrm{Bor}} \mathcal{I}$ or $\mathcal{I}_0 \le_{\mathrm{Bor}} \mathcal{I}$. In this note we disprove this conjecture by showing that an ideal associated with the Tsirelson space provides a counterexample. This is a Banach space which does not contain an isomorphic copy of the classical Banach spaces c_0 or l_p for $1 \le p < \infty$.

In fact, the picture seems to be much more complicated than suggested by the above conjecture. Thus, apparently, there are no minimal (in the sense of $\le_{\mathrm{Bor}}$) ideals below the ideal $\mathcal{I}_{\mathrm{T}}$ constructed in the next section, but on the other hand, $(\mathcal{P}(\mathbb{N})\ \subseteq^*)$ can be embedded in the class of Tsirelson type ideals ordered by $\le_{\mathrm{Bor}}$, etc. We plan to present these and other related results in a later paper. There is a large literature on Tsirelson's and other related Banach spaces. For a good if somewhat outdated survey we refer the reader to [CS], and for a more recent survey to [OS].

REMARK. A proof of the main result of this paper was found independently by Ilijas Farah in "Ideals induced by Tsirelson submeasures", which appears in this issue of Fundamenta Mathematicae.

1. Tsirelson's space. We now present the Figiel–Johnson version of Tsirelson's space (see [FJ] or [CS]). This is actually the dual of the original space constructed by Tsirelson. We start with some definitions.

(a) If E, F are finite nonempty subsets of $\mathbb{N}$ we let $E \le F$ iff $\max(E) \le \min(F)$. We write $n \le E$ instead of $\{n\} \le E$. Similarly we define $E < F$, etc. We say that a sequence $\{E_i\}_{i=1}^k$ is *admissible* if $k \le E_1 < E_2 < \ldots < E_k$. In general, given an increasing function $h : \mathbb{N} \to \mathbb{N}$ and an integer k we say that a sequence $\{E_i\}_{i=1}^l$ is (h,k)-*admissible* if $k \le E_1 < E_2 < \ldots < E_l$ and $l \le h(k)$.

(b) Let $\mathbb{R}^{<\omega}$ denote the vector space of all real scalar sequences of finite support and let $\{t_n\}_{n=1}^\infty$ be the canonical unit vector basis of $\mathbb{R}^{<\omega}$. Given a vector $x = \sum_n a_n t_n \in \mathbb{R}^{<\omega}$ we define $Ex = \sum_{n\in E} a_n t_n$, the projection of x onto the coordinates in E.

(c) We define inductively a sequence $(\|\cdot\|_m)_{m=0}^{\infty}$ of norms on $\mathbb{R}^{<\omega}$ as follows. Given $x = \sum_n a_n t_n \in \mathbb{R}^{<\omega}$ let

$$\|x\|_0 = \max_n |a_n|.$$

For $m \ge 0$, we set

$$\|x\|_{m+1} = \max\left\{\|x\|_m, \frac{1}{2}\sup\sum_{j=1}^{k}\|E_j x\|_m : \{E_j\}_{j=1}^{k} \text{ is admissible}\right\}.$$

(d) One verifies that the $\|\cdot\|_m$ are norms on $\mathbb{R}^{<\omega}$, they increase with m, and that for all m,

$$\|x\|_m \le \sum_n |a_n|.$$

Thus, $\lim_m \|x\|_m$ exists and is majorized by the l_1-norm of x. Therefore setting

$$\|x\| = \lim_m \|x\|_m$$

defines a norm on $\mathbb{R}^{<\omega}$.

(e) Finally, Tsirelson's space T is the $\|\cdot\|$ completion of $\mathbb{R}^{<\omega}$.

Recall that $\{t_n\}_{n=1}^{\infty}$ is the canonical unit vector basis of $\mathbb{R}^{<\omega}$. A *block* is a vector y of the form $\sum_{n\in I} a_n t_n$ for some (finite) interval I in $\mathbb{N}$. We now record some basic properties of the space T (cf. [CS, Proposition I.2]).

Proposition 1. (1) *The sequence $\{t_n\}_{n=1}^{\infty}$ is a normalized 1-unconditional Schauder basis for T.*

(2) *For each $x = \sum_n a_n t_n \in T$,*

$$\|x\| = \max\left\{\max_n |a_n|, \frac{1}{2}\sup\sum_{j=1}^{k}\|E_j x\| : \{E_j\}_{j=1}^{k} \textit{ is admissible}\right\}.$$

(3) *For any $k \in \mathbb{N}$, and any k normalized blocks $\{y_i\}_{i=1}^{k}$ such that for some integers $k-1 \le p_1 < p_2 < \ldots < p_{k+1}$, y_i is a linear combination of the base vectors t_n for $p_i < n \le p_{i+1}$, we have*

$$\frac{1}{2}\sum_{i=1}^{k}|b_i| \le \Big\|\sum_{i=1}^{k} b_i y_i\Big\| \le \sum_{i=1}^{k}|b_i|$$

for all scalars $\{b_i\}_{i=1}^{k}$.

We are now ready to define a Tsirelson type ideal $\mathcal{I}_{\mathrm{T}}$. Fix a vector $\alpha = \sum_n \alpha_n t_n \in c_0^+ \setminus T$, for instance, we could again take $\alpha_n = 1/(n+1)$. For a finite subset E of $\mathbb{N}$ define $\tau(E) = \|E\alpha\|$, and for an arbitrary $X \subseteq \mathbb{N}$

let

$$\tau(X) = \sup_n \tau(X \cap n).$$

It is now clear from Proposition 1 that τ is a lower semicontinuous submeasure on $\mathcal{P}(\mathbb{N})$ and that for any X,

$$\tau(X) < \infty \quad \text{iff} \quad \lim_{n\to\infty} \tau(X \setminus n) = 0.$$

Hence the ideal

$$\mathcal{I}_{\mathrm{T}} = \{X : \tau(X) < \infty\}$$

is an F_σ P-ideal.

The main result of this note is the following.

THEOREM 2. *$\mathcal{I}_{\mathrm{T}}$ and $\mathcal{I}_0$ are incomparable under $\le_{\mathrm{Bor}}$.*

Proof. It suffices to show that $\mathcal{I}_0 \not\le_{\mathrm{Bor}} \mathcal{I}_{\mathrm{T}}$. Assume otherwise and fix a Borel function $f : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ witnessing that $\mathcal{I}_0 \le_{\mathrm{Bor}} \mathcal{I}_{\mathrm{T}}$. We first prove the following.

LEMMA 1. *There is an infinite increasing sequence $F_0 < F_1 < \ldots$ of finite sets and a sequence $(\beta_n)_n \in c_0^+ \setminus l_1$ such that for every $X \subseteq \mathbb{N}$,*

$$\tau\Big(\bigcup_{n\in X} F_n\Big) < \infty \quad \textit{iff} \quad \sum_{n\in X} \beta_n < \infty.$$

Proof. First we show that we may assume that f is continuous. To this end, fix a dense G_δ set G such that $f\restriction G$ is continuous. Then, by a standard fact (see [Ke1, §8.9]), there is a partition $\mathbb{N} = X_0 \cup X_1$ and sets $Z_0 \subseteq X_0$, $Z_1 \subseteq X_1$ such that, for any $i \in \{0,1\}$, if $X \cap X_i = Z_i$ then $X \in G$. Fix now i such that $X_i \in \mathcal{I}_0^+$. It follows that the function $g : \mathcal{P}(X_i) \to \mathcal{P}(\mathbb{N})$ defined by

$$g(X) = f(X \cup Z_{1-i})$$

is continuous and witnesses $\mathcal{I}_0\restriction X_i \le_{\mathrm{Bor}} \mathcal{I}_{\mathrm{T}}$. Moreover, it is easily seen that for any $X \in \mathcal{I}_0^+$ we have $\mathcal{I}_0 \le_{\mathrm{RB}} \mathcal{I}_0\restriction X$. Therefore, by composing we can obtain a continuous function witnessing $\mathcal{I}_0 \le_{\mathrm{Bor}} \mathcal{I}_{\mathrm{T}}$.

To simplify notation assume now that f is already continuous. Following [Ve, Lemma 2], we can find a strictly increasing sequence $0 = n_0 < n_1 < \ldots$ of integers, sets $Z_i \subseteq [n_i, n_{i+1})$, and functions $f_i : \mathcal{P}(n_i) \to \mathcal{P}(n_i)$ such that:

(a) for every $X \subseteq \mathbb{N}$, if $X \cap [n_i, n_{i+1}) = Z_i$ then $f(X) \cap n_i = f_i(X \cap n_i)$,

(b) for every $X, Y \subseteq \mathbb{N}$, if $X \cap [n_i, n_{i+1}) = Y \cap [n_i, n_{i+1}) = Z_i$ and $X \triangle Y \subseteq n_i$ then

$$\tau((f(X) \triangle f(Y)) \setminus n_{i+1}) \le 1/2^{i+1}.$$

To see why we can arrange (b) suppose that at some stage i no n_{i+1} and Z_i can be found satisfying (b). Then, as in [Ve, Lemma 2], by using the continuity of f, we can find $X, Y \subseteq \mathbb{N}$ and an infinite increasing sequence $n_i = m_0 < m_1 < \ldots$ such that $X \triangle Y \subseteq n_i$ and for every j,

$$\tau(f(X) \triangle f(Y) \cap [m_j, m_{j+1})) \geq 1/2^{i+1}.$$

But then we would have $X \triangle Y \in \mathcal{I}_0$ while $\tau(f(X) \triangle f(Y)) \notin \mathcal{I}_{\mathrm{T}}$, contradicting the assumption that f is a reduction witnessing $\mathcal{I}_0 \leq_{\mathrm{Bor}} \mathcal{I}_{\mathrm{T}}$.

Now assume that sequences $(n_i)_i$ and $(Z_i)_i$ have been found satisfying the above conditions. For $\varepsilon = 0, 1, 2$, let

$$X_\varepsilon = \bigcup \{[n_i, n_{i+1}) : i \equiv \varepsilon \bmod 3\}, \qquad W_\varepsilon = \bigcup \{Z_i : i \equiv \varepsilon \bmod 3\}.$$

Assume for concreteness that $X_0 \notin \mathcal{I}_0$ and define a function $g : \mathcal{P}(X_0) \to \mathcal{P}(\mathbb{N})$ by

$$g(X) = f(X \cup W_1 \cup W_2) \triangle f(W_1 \cup W_2).$$

Then g is continuous and witnesses $\mathcal{I} \restriction X_0 \leq_{\mathrm{Bor}} \mathcal{I}_{\mathrm{T}}$. Now, for each i, define a function $g_i : \mathcal{P}([n_{3i}, n_{3i+1})) \to \mathcal{P}([n_{3i-1}, n_{3i+2}))$ by

$$g_i(X) = g(X) \cap [n_{3i-1}, n_{3i+2})$$

and let

$$g^*(X) = \bigcup_i g_i(X \cap [n_{3i}, n_{3i+1})).$$

Note that (a) and (b) imply that for every $X \subseteq X_0$,

$$\tau(g(X) \triangle g^*(X)) \leq \sum_{i=1}^{\infty} \frac{1}{2^{3i-1}} \leq 1.$$

Now since g witnesses $\mathcal{I} \restriction X_0 \leq_{\mathrm{Bor}} \mathcal{I}_{\mathrm{T}}$ and $g(\emptyset) = \emptyset$ it follows that for any $X \subseteq X_0$,

$$X \in \mathcal{I}_0 \quad \text{iff} \quad g^*(X) \in \mathcal{I}_{\mathrm{T}}.$$

Since $X_0 \notin \mathcal{I}_0$ we can find subsets B_i of $[n_{3i}, n_{3i+1})$ such that if we let

$$\beta_i = \sum_{k \in B_i} \frac{1}{k+1}$$

then $\lim_{i\to\infty} \beta_i = 0$ and $\sum_{i=0}^{\infty} \beta_i = \infty$. Finally, let $F_i = g_i(B_i)$ for each i. Then the sequences $(\beta_i)_i$ and $(F_i)_i$ are as required. ∎

For the remainder of the proof fix sequences $(F_n)_n$ and $(\beta_n)_n$ as in Lemma 1. For a subset X of $\mathbb{N}$ define

$$\varphi(X) = \sum_{n \in X} \beta_n.$$

Then for every such X we have

(1) $$\varphi(X) < \infty \quad \text{iff} \quad \tau\Big(\bigcup_{n \in X} F_n\Big) < \infty.$$

Given a finite subset a of $\mathbb{N}$ let $E_a = \bigcup_{n\in a} F_n$. For a sequence $S = \{a_n\}_{n=1}^\infty$ of finite subsets of $\mathbb{N}$ let $\mathrm{FU}(S)$ denote the family of finite unions of members of S. Call such an S *acceptable* iff $a_1 < a_2 < \ldots$ and

$$\lim_{n\to\infty} \tau(E_{a_n}) = 0 \quad \text{and} \quad \tau\Big(\bigcup_{n=1}^\infty E_{a_n}\Big) = \infty.$$

Given an acceptable sequence $S = \{a_n\}_{n=1}^\infty$ define

$$K(S) = \sup_n \frac{\tau(E_{a_n})}{\varphi(a_n)}.$$

Note that if $S^* \subseteq \mathrm{FU}(S)$ is also acceptable then $K(S^*) \le K(S)$. Finally, let

$$K = \inf\{K(S) : S \text{ acceptable}\}.$$

We first prove the following.

Lemma 2. *$K = 0$ or $K = \infty$.*

Proof. We show that if there is an acceptable S such that $K(S)$ is finite then there is another acceptable $S^* \subseteq \mathrm{FU}(S)$ such that

$$K(S^*) \le \frac{119}{120} K(S).$$

The proof of this follows closely that of Lemma 2.1 of [FJ] or Proposition 1.3 of [CS]. To begin, fix an acceptable $S = \{a_n\}_{n=1}^\infty$ such that $K(S)$ is finite. Note that since $\tau(\bigcup_{n=1}^\infty E_{a_n}) = \infty$ and $\lim_{n\to\infty} \tau(E_{a_n}) = 0$ we know that for $n > 0$ and every integer k we can find some $b \in \mathrm{FU}(S)$ such that $k \le E_b$ and $15/(16n) \le \tau(E_b) \le 17/(16n)$.

Claim 1. *For every $n \ge N$ and k there is $b \in \mathrm{FU}(S)$ such that $k \le E_b$,*

$$\tau(E_b) \le \frac{119}{64n} \quad \textit{and} \quad \varphi(b) \ge \frac{30}{16nK(S)}.$$

Note that using this claim we can easily produce an increasing sequence $b_1 < b_2 < \ldots$ of members of $\mathrm{FU}(S)$ such that

$$\frac{\tau(E_{b_n})}{\varphi(b_n)} \le \frac{119}{120} K(S), \qquad \sum_{n=1}^\infty \varphi(b_n) = \infty, \qquad \lim_{n\to\infty} \tau(E_{b_n}) = 0.$$

Then $S^* = \{b_n\}_{n=1}^\infty$ is acceptable and $K(S^*) \le \frac{119}{120} K(S)$, as desired.

Proof of Claim 1. Fix $n \ge N$ and k. First find some $b_0 \in \mathrm{FU}(S)$ such that $k \le E_{b_0}$ and

$$\frac{15}{16n} \le \tau(E_{b_0}) \le \frac{17}{16n}.$$

Set $n_0 = \max E_{b_0}$. Now let $r = 2n_0$ and find sets $b_i \in \mathrm{FU}(S)$, for $1 \le i \le r$, such that $b_0 < b_1 < \ldots < b_r$ and, for every $1 \le i \le r$,

$$\frac{15}{16nr} \le \tau(E_{b_i}) \le \frac{17}{16nr}.$$

Finally, let $b' = \bigcup_{i=1}^r b_i$ and $b = b_0 \cup b'$. We claim that b is as required.

Consider an admissible sequence $l \le H_1 < \ldots < H_l$ for some l. If $l > n_0$ then

$$\sum_{j=1}^{l} \tau(H_j \cap E_b) = \sum_{j=1}^{l} \tau(H_j \cap E_{b'}) \le 2\tau(E_{b'}) \le 2\sum_{j=1}^{l} \tau(E_{b_j}) \le \frac{34}{16n}.$$

If $l \le n_0$ we define

$$A = \{i > 0 : H_j \cap E_{b_i} \ne \emptyset \text{ for at least two values of } j\},$$
$$B = \{i > 0 : H_j \cap E_{b_i} \ne \emptyset \text{ for at most one value of } j\}.$$

Then, since A has at most l elements, we have

$$\begin{aligned}\sum_{j=1}^{l} \tau(H_j \cap E_b) &\le \sum_{j=1}^{l} \tau(H_j \cap E_{b_0}) + \Big(\sum_{i\in A}\sum_{j=1}^{l} + \sum_{i\in B}\sum_{j=1}^{l}\Big)\tau(H_j \cap E_{b_i})\\ &\le 2\tau(E_{b_0}) + 2\sum_{i\in A} \tau(E_{b_i}) + \sum_{i\in B}\tau(E_{b_i})\\ &\le \frac{34}{16n} + (2l + r - l)\frac{17}{16nr} \le \frac{17}{16n}\left(2 + \frac{r+l}{r}\right)\\ &\le \frac{17}{16n}\left(3 + \frac{n_0}{r}\right) = \frac{119}{32n}.\end{aligned}$$

From these two inequalities it now follows that

$$\tau(E_b) = \sup\left\{\frac{1}{2}\sum_{j=1}^{l} \tau(H_j \cap E_b) : \{H_j\}_{j=1}^{l} \text{ is admissible }\right\} \le \frac{119}{64n}.$$

On the other hand, notice that

$$\varphi(b) = \sum_{i=0}^{r} \varphi(b_i) \ge \frac{1}{K(S)} \sum_{i=0}^{r} \tau(E_{b_i}) \ge \frac{30}{16nK(S)}.$$

This completes the proof of Claim 1 and Lemma 2. ∎

We now show that (1) fails in both cases of Lemma 2, thus arriving at a contradiction.

Case 1. $K = \infty$. We consider two subcases.

Subcase 1a. Suppose there exist $N \in \mathbb{N}$ and $\varepsilon > 0$ such that for every $k \ge N$ there is N_k such that for every a if $N_k \le E_a$ and $2/k \le \tau(E_a) < 4/k$ then $\varphi(a) \ge \varepsilon/k$. In this case we can produce an infinite increasing sequence

$S = \{a_k\}_{k=N}^{\infty}$ of finite subsets of $\mathbb{N}$ such that $2/k \le \tau(E_{a_k}) \le 4/k$ and $\varphi(a_k) \ge \varepsilon/k$ for every $k \ge N$. It follows that S is acceptable and that $K(S) \le 4/\varepsilon$, contradicting the fact that $K = \infty$.

SUBCASE 1b. Suppose Subcase 1a does not hold. We first show the following.

CLAIM 2. *For every $N \in \mathbb{N}$ and $\varepsilon > 0$ there is a finite set a of integers such that $N \le E_a$, $\tau(E_a) \ge 1$, and $\varphi(a) < \varepsilon$.*

Proof. Fix $N \in \mathbb{N}$ and $\varepsilon > 0$. By our assumption that Subcase 1a does not hold we can find $k \ge N$ and sets $\{a_i\}_{i=1}^k$ such that $\max\{k, N\} \le E_{a_1} < \ldots < E_{a_k}$, $2/k \le \tau(E_{a_i}) < 4/k$ and $\varphi(a_i) < \varepsilon/k$ for $i = 1, \ldots, k$. But then, since the sequence $\{E_{a_i}\}_{i=1}^k$ is admissible, by setting $a = \bigcup_{i=1}^k a_i$ and using Proposition 1 we have

$$\tau(E_a) \ge \frac{1}{2} \sum_{i=1}^{k} \tau(E_{a_i}) \ge \frac{1}{2}\, k\, \frac{2}{k} = 1.$$

On the other hand,

$$\varphi(a) = \sum_{i=1}^{k} \varphi(a_i) < k \frac{\varepsilon}{k} = \varepsilon.$$

Thus we have $N \le E_a$, $\tau(E_a) \ge 1$, and $\varphi(a) < \varepsilon$. ∎

Now by using Claim 2 and Proposition 1 again, we can easily produce an infinite set X such that $\varphi(X) < \infty$ and $\tau(\bigcup_{n\in X} F_n) = \infty$. A contradiction.

CASE 2. $K = 0$. We first show that for every integer N and $\varepsilon > 0$ there is a finite subset a of $\mathbb{N}$ such that $N \le E_a$, $\tau(E_a) < \varepsilon$, and $\varphi(a) \ge 1$. To see this, fix an acceptable $S = \{a_n\}_{n=1}^{\infty}$ such that $K(S) < \varepsilon/2$. Moreover, by thinning out if necessary, we may assume that $N \le E_{a_1}$ and that $\varphi(a_n) \le 1$ for all n. Now there is an integer k such that letting $a = \bigcup_{i=1}^k a_i$ we have $1 \le \varphi(a) \le 2$. On the other hand, using the fact that τ is subadditive and that $\tau(E_{a_i})/\varphi(a_i) < \varepsilon/2$ for every i, we have $\tau(E_a) < \varepsilon$.

Now we easily produce an infinite set X such that $\varphi(X) = \infty$, but $\tau(\bigcup_{n\in X} F_n) < \infty$. A contradiction. ∎

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