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Ideals induced by Tsirelson submeasures

by

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Abstract. We use Tsirelson's Banach space ([2]) to define an F_σ P-ideal which refutes a conjecture of Mazur and Kechris (see [12, 9, 8]).

1. Introduction. By the dichotomy results of Silver and Harrington–Kechris–Louveau (see [10, 8]), the Borel-cardinality of quotients over Borel equivalence relations on Polish spaces is well-understood below $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$. This cannot be said for the next level of this ordering, even if we restrict our attention to Borel-cardinalities of quotients $\mathcal{P}(\mathbb{N})/\mathcal{I}$ over Borel ideals $\mathcal{I}$. The two natural "successors" of Fin are the Fubini ideals on $\mathbb{N}^2$: $\mathrm{Fin}\times\emptyset$ (also called I_1) consisting of all sets with only finitely many nonempty vertical sections, and $\emptyset\times\mathrm{Fin}$ (also called I_3 and Fin^ω) consisting of all sets all of whose vertical sections are finite. By results of Solecki ([17]), quotients over these two ideals are the critical points for quotients over Borel ideals which are not P-ideals and for Borel P-ideals which are not F_σ, respectively ($\mathcal{I}$ is a *P-ideal* if it is σ-directed under the inclusion modulo finite). In [9], Kechris posed the following trichotomy conjecture for Borel ideals $\mathcal{I}$ such that $\mathcal{P}(\mathbb{N})/\mathcal{I}\not\leq_{\mathrm{B}}\mathcal{P}(\mathbb{N})/\mathrm{Fin}$: at least one of $\mathcal{P}(\mathbb{N}^2)/\mathrm{Fin}\times\emptyset$, $\mathcal{P}(\mathbb{N}^2)/\emptyset\times\mathrm{Fin}$, and $\mathcal{P}(\mathbb{N})/\mathcal{I}_{1/n}$ is $\leq_{\mathrm{B}}\mathcal{P}(\mathbb{N})/\mathcal{I}$ (the summable ideal $\mathcal{I}_{1/n}$ is defined below). By the above results of Solecki, this is equivalent to an earlier dichotomy conjecture of Mazur ([12]): If $\mathcal{I}$ is an F_σ ideal such that $\mathcal{P}(\mathbb{N})/\mathcal{I}\not\leq_{\mathrm{B}}\mathcal{P}(\mathbb{N})/\mathrm{Fin}$, then either $\mathcal{P}(\mathbb{N}^2)/\mathrm{Fin}\times\emptyset$ or $\mathcal{P}(\mathbb{N})/\mathcal{I}_{1/n}$ is $\leq_{\mathrm{B}}\mathcal{P}(\mathbb{N})/\mathcal{I}$.

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Consider an ordering on Borel ideals simpler than $\leq_{\mathrm{B}}$:

$\mathcal{I} \leq^+_{\mathrm{RB}} \mathcal{J}$ if there is $A \subseteq \mathbb{N}$ and $h : A \to \mathbb{N}$
such that $B \in \mathcal{I}$ iff $h^{-1}(B) \in \mathcal{J}$.

If $A = \mathbb{N}$, then we write $\mathcal{I} \leq_{\mathrm{RB}} \mathcal{J}$. Clearly, $\mathcal{I} \leq^+_{\mathrm{RB}} \mathcal{J}$ implies $\mathcal{P}(\mathbb{N})/\mathcal{I} \leq_{\mathrm{B}} \mathcal{P}(\mathbb{N})/\mathcal{J}$, as the mapping $A \mapsto h^{-1}(A)$ verifies. It is rather surprising that the converse is often true; for example, the above Solecki dichotomy results are proved for the $\leq_{\mathrm{RB}}$-ordering (see also Lemma 2.1 below).

Any $\mathcal{I}$ serving as a counterexample to the Kechris–Mazur conjecture, or KMC, would have to be an F_σ P-ideal. Until recently, the only known F_σ P-ideals were the *summable ideals*, that is, ones of the form

$$\mathcal{I}_f = \{A : \nu_f(A) < \infty\} = \{A : \lim_n(\nu_f(A \setminus n)) = 0\}$$

where $\nu_f(A) = \sum_{n\in A} f(n)$ for some $f : \mathbb{N} \to \mathbb{R}^+$. These ideals cannot serve as a counterexample to the Kechris–Mazur conjecture, since we have either $\mathcal{I}_f \leq^+_{\mathrm{RB}}$ Fin or $\mathcal{I}_{1/n} \leq^+_{\mathrm{RB}} \mathcal{I}_f$ (note that this is false for the $\leq_{\mathrm{RB}}$ ordering, and this is why we introduce $\leq^+_{\mathrm{RB}}$). By [17, Theorem 3.3] (see also [11, Lemma 1.2]), all F_σ P-ideals are of the form

$$\mathcal{I} = \{A : \phi(A) < \infty\} = \{A : \lim_i \phi(A \setminus i) = 0\}$$

for some *lower semicontinuous* submeasure ϕ, i.e. a mapping such that $\phi(A) \leq \phi(A \cup B) \leq \phi(A) + \phi(B)$, $\phi(\emptyset) = 0$ and $\lim_i \phi_i(A \cap i) = \phi(A)$ for all A, B. The first non-summable F_σ P-ideals were discovered in [4] (see also [3]). All these ideals were of the form

$$\mathcal{I}_{\{\phi_n\}} = \Big\{A : \sum_i \phi_i(A) < \infty\Big\},$$

where for some sequence $\{n_i\}$ each ϕ_i is a submeasure on the interval $[n_i, n_{i+1})$. But such ideals satisfy the KMC since if $\lim_i \sup_j \phi_i(\{j\}) = 0$ (and this can be assumed without loss of generality by going to a positive set) then there are s_i and m_i such that

$$\phi_k(s_i) \begin{cases} \approx 1/i, & k = m_i, \\ = 0, & \text{otherwise}, \end{cases}$$

so that the map collapsing s_i to i witnesses $\mathcal{I}_{1/n} \leq^+_{\mathrm{RB}} \mathcal{I}_{\{\phi_i\}}$. An F_σ P-ideal which is not of the form $\mathcal{I}_{\{\phi_i\}}$ was later found by Solecki ([16]), who also proved that this ideal is of the form $\mathcal{I}_{\{\phi_i\}}$ when restricted to a positive set, so it is again $\geq^+_{\mathrm{RB}} \mathcal{I}_{1/n}$. Another class of F_σ P-ideals, suggested by Kechris, are ideals of the form

$$\mathcal{I} = \Big\{A : \sqrt[p]{\sum_i \phi_i(A))^p} < \infty\Big\}$$

for a sequence of submeasures ϕ_i as before and $p > 1$, but these again do not serve as a counterexample to KMC, for the same reason as $\mathcal{I}_{\{\phi_i\}}$. (However, using methods and results of [7] it can be proved that the Borel-cardinalities of these quotients are different for different p's.)

The new F_σ P-ideal which we define here is extracted from *Tsirelson space*, an infinite-dimensional Banach space which does not contain a copy of c_0 or any ℓ_p (see [2]). The study of this space has played a prominent role in the recent striking developments in the theory of infinite-dimensional Banach spaces (see [6], [13, p. 956]). It is likely that other Banach spaces will give rise to interesting examples of analytic P-ideals (see [5]).

After the completion of this paper, we have learned that our main result, Theorem 3.1, was independently proved by B. Veličković ([19]).

The paper is organized as follows. In §2 we prove that $\mathcal{P}(\mathbb{N})/\mathcal{I}_{1/n} \le_{\mathrm{B}} \mathcal{P}(\mathbb{N})/\mathcal{I}$ is equivalent to $\mathcal{I}_{1/n} \le^+_{\mathrm{RB}} \mathcal{I}$. In §3 we introduce the ideals $\mathcal{T}_{fh}$. In §§4–6 various properties of these ideals are proved, and in §7 we conclude the proof that $\mathcal{P}(\mathbb{N})/\mathcal{T}_{fh}$ serves as a counterexample to the Kechris–Mazur conjecture.

A word on notation: If s, t are finite sets of integers and n is an integer, by $s < t$ we denote the fact that $\max s < \min t$, and by $n < s$ ($n > s$) the fact that $n < \min s$ ($n > \max s$, respectively).

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2. The first reduction. A quotient $\mathcal{P}(\mathbb{N})/\mathcal{I}$ has smaller *Borel-cardinality* than the quotient $\mathcal{P}(\mathbb{N})/\mathcal{J}$ (in symbols $\mathcal{P}(\mathbb{N})/\mathcal{I} \le_{\mathrm{B}} \mathcal{P}(\mathbb{N})/\mathcal{J}$) if there is a Borel mapping $F : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ such that $X \triangle Y \in \mathcal{I}$ iff $F(X) \triangle F(Y) \in \mathcal{J}$.

In the following lemma it is proved that the KMC is equivalent to its apparently stronger version, appearing in [4], which states: For every analytic ideal $\mathcal{I}$ such that $\mathcal{I} \not\le_{\mathrm{RB}} \mathrm{Fin}$ one of $\mathrm{Fin} \times \emptyset$, $\emptyset \times \mathrm{Fin}$ or $\mathcal{I}_{1/n}$ is $\le_{\mathrm{RB}} \mathcal{I} \restriction A$ for some $\mathcal{I}$-positive A. (It is well-known that $\mathcal{P}(\mathbb{N})/\mathcal{I} \le_{\mathrm{B}} \mathcal{P}(\mathbb{N})/\mathrm{Fin}$ is equivalent to $\mathcal{I} \le_{\mathrm{RB}} \mathrm{Fin}$.)

Lemma 2.1. *If $\mathcal{J}$ is an analytic P-ideal such that $\mathcal{P}(\mathbb{N})/\mathcal{I}_{1/n} \le_{\mathrm{B}} \mathcal{P}(\mathbb{N})/\mathcal{J}$, then $\mathcal{I}_{1/n} \le^+_{\mathrm{RB}} \mathcal{J}$. Moreover, there are $w_1 < w_2 < \ldots$ in* Fin *such that the map collapsing w_i to i witnesses this.*

Proof. By [17], we can fix a lower semicontinuous submeasure ϕ such that $\mathcal{J} = \{A : \lim_i \phi(A \setminus i) = 0\}$. Let $F : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ be a Borel reduction. By a standard use of stabilizers, similar to the one below, we can assume that F is continuous (see also [18]). Find integers $1 = a_1 < b_1 < a_2 < b_2 < \ldots$, $s_i \subseteq [b_i, a_{i+1})$ and $k_i \in (b_i, a_{i+1})$ so that for all i, all $u, v \subseteq b_i$, and all $X, Y \subseteq [a_{i+1}, \infty)$:

(1) $a_i > 2^i$, $b_i > 2a_i$,
(2) $(F(u \cup s_i \cup X) \triangle F(u \cup s_i \cup Y)) \cap k_i = \emptyset$,
(3) $\phi((F(u \cup s_i \cup X) \triangle F(v \cup s_i \cup X)) \setminus k_i) \le 2^{-i}$.

The method for construction of these sequences is standard, dating back to [15] and [18]: Assume that a_i, b_i ($i \le n$) and s_j, k_j ($j \le n-1$) as above have been chosen, but there are no s_n, k_n and a_{n+1} satisfying the requirements. Condition (1) is easy to satisfy and since F is continuous, (2) will be satisfied for every choice of s_n, a large enough k_n and a large enough a_{n+1}. Therefore we can construct a sequence $b_n < t_1 < l_1 < t_2 < l_2 < \ldots$ so that $l_i \in \mathbb{N}$, $t_i \in \mathrm{Fin}$, and for all i there are $u_i, v_i \subseteq a_n$ such that

$$\phi\Big(\Big(F\Big(u_i \cup \bigcup_{j=1}^{i} t_j \cup t_{i+1}\Big) \triangle F\Big(u_i \cup \bigcup_{j=1}^{i} t_j \cup t_{i+1}\Big)\Big) \setminus l_i\Big) > 2^{-n}.$$

Pick u, v such that $\langle u, v\rangle = \langle u_i, v_i\rangle$ infinitely often. Then $F(u \cup \bigcup_i t_i) \triangle F(v \cup \bigcup_i t_i)$ is not in $\mathcal{J}$—a contradiction.

Assume a_n, b_n, s_n and k_n are chosen to satisfy the above conditions. By (1), $\nu_{1/n}([a_i, b_i]) = \sum_{j=a_i}^{b_i} 1/j \ge 1/n$ and there is $u_i \subseteq (a_i, b_i)$ such that $|\nu_{1/n}(u_i) - 1/i| \le 2^{-i}$ for every i. Let $C = \bigcup_i s_i$ and define $F_1 : P(\bigcup_i [a_i, b_i)) \to \mathcal{P}(\mathbb{N})$ by

$$F_1(B) = F(B \cup C) \triangle F(C).$$

Then $F_1(\emptyset) = \emptyset$ and for $X, Y \subseteq \bigcup_i [a_i, b_i)$ we have $(X \cup C) \triangle (Y \cup C) = X \triangle Y \in \mathcal{I}_{1/n}$ iff $F_1(X) \triangle F_1(Y) = F(X \cup C) \triangle F(Y \cup C) \in \mathcal{J}$. By (2)–(3), for all i, all $u, v \subseteq \bigcup_{i \le m} [a_i, b_i)$, and all $X, Y \subseteq \bigcup_{i \ge m+1} [a_i, b_i)$ we have:

(4) $(F_1(u \cup X) \triangle F_1(u \cup Y)) \cap k_m = \emptyset$,
(5) $\phi((F_1(u \cup X) \triangle F_1(v \cup X)) \setminus k_m) \le 2^{-m}$.

Let $w_i = F_1(u_i) \cap [k_{i-1}, k_i)$. Then a map collapsing w_i to i witnesses $\mathcal{I}_{1/n} \le^+_{\mathrm{RB}} \mathcal{J}$. This is implied by the following computations (assume $m \in A$ and let $u_X = \bigcup_{i \in X} u_i$):

$$\begin{aligned}
t_m &= (F_1(u_A) \triangle w_A) \cap [k_{m-1}, k_m)\\
&\subseteq (F_1(u_A) \triangle F_1(u_{A\setminus m})) \cup (F_1(u_{A\setminus m}) \triangle F_1(u_m))) \cap [k_{m-1}, k_m)\\
&\qquad\qquad\qquad\qquad (\text{since } w_A \cap [k_{m-1}, k_m) = w_m)\\
&\subseteq (((F_1(u_A) \triangle F_1(u_{A\setminus m})) \setminus a_m) \cup (F_1(u_{A\setminus m}) \triangle F_1(u_m))) \cap k_m\\
&\qquad\qquad (\text{by (4) and (5), since } u_A \triangle u_{A\setminus m} \subseteq \textstyle\bigcup_{i=1}^{m-1} [a_i, b_i)\\
&\qquad\qquad\qquad \text{and } u_{A\setminus m} \triangle u_m \subseteq \textstyle\bigcup_{i=m+1}^{\infty} [a_i, b_i))\\
&= (F_1(u_A) \triangle F_1(u_{A\setminus m})) \setminus a_m,
\end{aligned}$$

and therefore $\phi(t_m) \le 2^{-m+1}$ if $m \in A$. An analogous computation shows that $\phi(t_m) \le 2^{-m+1}$ also in the case when $m \notin A$, and therefore we have

$$F_1(u_A) \triangle w_A \subseteq \bigcup_m t_m \cup [1, k_1) \in \mathcal{J}.$$

To prove the moreover part, that we can assume that $w_1 < w_2 < \ldots$, find $1 < n_1 < m_1 < n_2 < m_2 < \ldots$ so that $n_i > 2^i$, $\nu_{1/n}([n_i, m_i]) - 1/i| \le 2^{-i}$, and

$$\max \bigcup_{j \in [n_i, m_i]} w_j < \min \bigcup_{j \in [n_{i+1}, m_{i+1}]} w_j.$$

Then the sets $w_i' = \bigcup_{j \in [n_i, m_i]} w_j$ are as required. ∎

Let us digress a little and note that $\emptyset \times \mathrm{Fin}$ also shares the nice property of $\mathcal{I}_{1/n}$ from Lemma 2.1, as its proof readily shows.

LEMMA 2.2. *If $\mathcal{J}$ is an analytic P-ideal such that $\mathcal{P}(\mathbb{N}^2)/\emptyset \times \mathrm{Fin} \le_{\mathrm{B}} \mathcal{P}(\mathbb{N})/\mathcal{J}$, then $\emptyset \times \mathrm{Fin} \le_{\mathrm{RB}}^{+} \mathcal{J} \restriction A$ for some $A \in \mathcal{J}^+$.* ∎

It would be interesting to find more ideals with this property shared by $\mathcal{I}_{1/n}$ and $\emptyset \times \mathrm{Fin}$, since it considerably simplifies some questions about the Borel-cardinality of their quotients. Let us note that a pathological $F_{\sigma\delta}$ P-ideal $\mathcal{J}$ constructed in [3, §6] does not have this property. Namely, by a result of M. R. Oliver ([14]), $E_{\mathcal{J}}$ is Borel-reducible to $E_{\mathcal{Z}_0}$, the equivalence relation induced by the density zero ideal. But the ideal $\mathcal{Z}_0$ is nonpathological (see [3]), and therefore by [3, Proposition 6.5], $\mathcal{J} \le_{\mathrm{RB}}^{+} \mathcal{Z}_0$ would imply that $\mathcal{J}$ is nonpathological as well.

3. Tsirelson submeasures and ideals. Assume that $\{x_n\}$ is an unconditional basic sequence in a Banach space X such that $\lim_n \|\sum_{i=1}^n x_i\| = \infty$. Then

$$\mathcal{J} = \Big\{A : \Big\|\sum_{n \in A} x_n\Big\| < \infty\Big\}$$

is an analytic P-ideal, which we call a *generalized summable ideal*. Many analytic P-ideals are of this form, and ideals $\mathcal{T}_{fh}$ defined below are obtained in this way from the *Tsirelson space*, a Banach space which does not include a copy of c_0 or any ℓ_p (see [2]).

For sets $A, B \subseteq \mathbb{N}$, we often denote by AB their intersection, $A \cap B$. Fix functions $f : \mathbb{N} \to \mathbb{R}$ and an increasing $h : \mathbb{N} \to \mathbb{N}$. A tuple

$$\langle k, E_1, \ldots, E_m \rangle$$

is *h-admissible* if $k \in \mathbb{N}$, $E_i \in \mathrm{Fin}$ for all i, $k < E_1 < E_2 < \ldots < E_m$, and $m \le h(k)$. We abbreviate tuples $\langle k, E_1, \ldots, E_m \rangle$ as $\langle k, \vec{E} \rangle$ and write $m = |\vec{E}|$, so that the necessary condition for the admissibility is $|\vec{E}| \le h(k)$. Let $\mathcal{A}_h$ be the set of all h-admissible tuples. Define a sequence of *Tsirelson* *submeasures* $\tau_n = \tau_{f,h,n}$ $(n \in \mathbb{N} \cup \{\infty\})$ as follows:

$$\begin{aligned}
\tau_{f,h,0}(A) &= \sup_{n\in A} f(n),\\
\tau^k_{f,h,n}(A) &= \sup_{\langle k,\vec{E}\rangle\in\mathcal{A}_h} \sum_{i=1}^{|\vec{E}|} \tau_{f,h,n}(E_iA),\\
\tau_{f,h,n+1}(A) &= \max\left\{\tau_{f,h,n}(A), \tfrac{1}{2}\sup_k \tau^k_{f,h,n}(A)\right\},\\
\tau_{f,h,\infty}(A) &= \sup_n \tau_{f,h,n}(A) = \lim_n \tau_{f,h,n}(A).
\end{aligned}$$

We always omit ∞ in subscripts, so that $\tau_{f,h}$ stands for $\tau_{f,h,\infty}$ and $\tau^k_{f,h}$ stands for $\tau^k_{f,h,\infty}$. Similarly, when f, h are clear from the context we write τ_n instead of $\tau_{f,h,n}$, τ^k instead of $\tau^k_{f,h}$, and so on. The submeasure $\tau = \tau_{f,h}$ defines a *Tsirelson ideal*, $\mathcal{T}_{fh}$, on $\mathbb{N}$ by

$$\mathcal{T}_{fh} = \mathrm{Exh}(\tau) = \{A : \lim_i \tau(A \setminus i) = 0\}.$$

We always assume that $f : \mathbb{N} \to \mathbb{R}^+$, $\lim_i f(i) = 0$, and h is strictly increasing, unless otherwise specified.

THEOREM 3.1. *Assume f, h are as above. Then $\mathcal{P}(\mathbb{N})/\mathcal{I}_{1/n} \not\le_{\mathrm{B}} \mathcal{P}(\mathbb{N})/\mathcal{T}_{fh}$ and $\mathcal{T}_{fh}$ is an F_σ P-ideal.*

The proof of Theorem 3.1 occupies the rest of this note.

LEMMA 3.2. *Assume f, h are as above. Then*

(1) *Either $\tau(A) = \sup_{i\in A} f(i)$ or $\tau(A) = \frac{1}{2}\sup_k \tau^k(A)$.*
(2) *All τ_n and τ are lower semicontinuous submeasures.*
(3) *$\tau(A) < \infty$ if and only if $\lim_n \tau(A \setminus n) = 0$.*
(4) *$\tau_{m+1}(A) < \infty$ if and only if $\lim_n \tau_m(A \setminus n) = 0$.*
(5) *$\mathcal{T}_{fh} = \{A \mid \tau(A) < \infty\}$ and therefore it is an F_σ P-ideal.*

Proof. To prove (1), note that if $\tau(A) > \sup_{i\in A} f(i)$ then we have

$$\tau(A) = \sup_n \left(\frac{1}{2}\sup_k \tau^k_n(A)\right) = \frac{1}{2}\sup_k \tau^k(A).$$

Statement (2) is obvious from the definition. In (3) only the direct implication requires a proof. Assume that $\lim_n \tau(A\setminus n) \neq 0$; then we can find $\varepsilon > 0$ and finite sets $w_1 < w_2 < \ldots$ included in A such that $\tau(w_n) \ge \varepsilon$ for all n. Fix $p \in \mathbb{N}$ and find $\bar{n}$ such that

$$\min(w_{\bar{n}}) > 2p.$$

Then $A_0 = \bigcup_{i=\bar{n}}^{\bar{n}+2p} w_i \subseteq A$ and by (1) we get

$$\tau(A_0) \ge \frac{1}{2}\sum_{i=\bar{n}}^{\bar{n}+2p} \tau(w_i) \ge p\varepsilon.$$

Since p was arbitrary, we have $\tau(A) = \infty$ as required.

The proof of (4) is analogous to that of (3), and (5) follows immediately from (3). ∎

We concentrate on the ideal $\mathcal{T}_{fh}$, but we note that the ideals

$$\mathcal{T}_{f,h,n} = \mathrm{Exh}(\tau_n) = \{A : \lim_i \tau_n(A \setminus i) = 0\}, \qquad n \in \mathbb{N},$$

can turn out to be interesting in their own right. All these are P-ideals which are not F_σ (assuming they are proper ideals, of course), since a mapping witnessing $\emptyset \times \mathrm{Fin} \le_{\mathrm{RB}} \mathcal{T}_{f,h,n}$ can be easily obtained from an $(n-1)$-good sequence (see §6).

4. Properties of Tsirelson submeasures. In this section we show several lemmas which will be used in the proof of Theorem 3.1. First we give a transparent description of how τ_n is computed in Lemma 4.2 below. By $\mathbb{N}^{<\mathbb{N}}$ we denote the set of all finite sequences of integers, and consider it as a tree under the ordering of end-extension. A set $T \subseteq \mathbb{N}^{<\mathbb{N}}$ is a *tree* if it is closed under taking initial segments of its elements. Note that the height of a finite tree T is equal to the maximal length of its elements. By $\langle\rangle$ we denote the empty sequence in $\mathbb{N}^{<\mathbb{N}}$, and $t^\frown i$ is the sequence obtained by concatenating t with $\langle i\rangle$. An *end-node* of T is any $t \in T$ such that $t^\frown i \notin T$ for all i. (Note that end-nodes of T do not necessarily belong to its top level.)

Definition 4.1. A family $\langle E_t : t \in T\rangle$ is an *h-tree* if $T \subseteq \mathbb{N}^{<\mathbb{N}}$ is a finitely branching finite tree, the sets E_t are finite, and for all $t \in T$, if $t^\frown 1, \ldots, t^\frown l$ are the immediate successors of t in T then

(1) $E_{t^\frown 1} < \ldots < E_{t^\frown l}$ and $E_t = \bigcup_{i=1}^{l} E_{t^\frown l}$,
(2) $l \le h(\min E_{t^\frown 1})$, i.e. $\langle \min E_{t^\frown 1}, \ldots, E_{t^\frown l}\rangle$ is h-admissible, and
(3) if t is an end-node of T, then E_t is a singleton.

The *height* of $\langle E_t : t \in T\rangle$ is the height of T. Note that every $i \in E_{\langle\rangle}$ defines a unique branch,

$$B_i = \{t \in T : i \in E_t\},$$

of T. The *length*, $|B_i|$, of this branch is equal to the length of its last node. A function $g : \mathbb{N} \to \{0, 1, \ldots, n, \infty\}$ is an *(h,n)-weight assignment* if there is an h-tree $\langle E_t : t \in T\rangle$ of height at most n such that

(4) $E_{\langle\rangle} = \{i : g(i) \ne \infty\}$,
(5) $g(i) = |B_i|$ for each $i \in E_{\langle\rangle}$.

Lemma 4.2. *Assume that $h : \mathbb{N} \to \mathbb{N}$ is strictly increasing. Then for every s we have*

$$\tau_n(s) = \sup_g \sum_{i \in s} 2^{-g(i)} f(i),$$

where the supremum is taken over all (h,n)-weight assignments g.

Proof. Note that every branching of an h-tree corresponds to an application of a step in the recursive definition of τ_{n+1}, and that the nodes of height k come with the weight equal to 2^{-k} because of the k-fold multiplication with 1/2. Condition (3) corresponds to $\tau_0(A) = \sup_{i\in A} f(i)$. Therefore, the lemma is proved by a straightforward induction on n. ∎

Lemma 4.3. *If f, h are as in Lemma 4.2, $n \in \mathbb{N}$ and s is finite, then there is an $s' \subseteq s$ such that $\tau_n(s') = \tau_n(s)$ and $\tau(s') \le 3\tau_n(s')/2$.*

Proof. Since s is finite, the supremum appearing in Lemma 4.2 is attained for some (h,n)-weight assignment g; let $s' = \{i : g(i) \neq \infty\} \cap s$. It suffices to prove that $\tau_m(s') \le 3\tau_n(s')/2$ for every m. Fix m, let g_m be some (h,m)-weight assignment, and let $X = \{i \in s' : g_m(i) > g(i)\}$ and $Y = \{i \in s' : g_m(i) \le g(i)\}$. We claim that

$$(\dagger)\qquad \sum_{i\in Y} 2^{-g_m(i)} f(i) \le \tau_n(s').$$

To verify this, by Lemma 4.2 it suffices to show that the map g' defined by

$$g'(i) = \begin{cases} g_m(i) & \text{if } g_m(i) \le g(i),\\ \infty & \text{if } g_m(i) > g(i) \text{ or } g_m(i) = \infty,\end{cases}$$

is an (h,n)-weight assignment. To see this, let $\langle E_t : t \in T\rangle$ be an h-tree of height m witnessing that g_m is an (h,m)-weight assignment. Then for $T' = \{t \in T : |t| \le n\}$ the family $\langle E_t \cap \{i : g'(i) \neq \infty\} : t \in T'\rangle$ is an h-tree (this follows immediately from the definitions). Since g is an (h,n)-weight assignment, this tree is of height at most n and it witnesses that g'_m is an (h,n)-weight assignment.

Therefore $(\dagger)$ is true and since $X \subseteq s'$ we have

$$\begin{aligned}\tau_m(s') &= \sum_{i\in X} 2^{-g_m(i)} f(i) + \sum_{i\in Y} 2^{-g_m(i)} f(i)\\ &\le \frac{1}{2}\sum_{i\in X} 2^{-g(i)} f(i) + \tau_n(s') \le \frac{1}{2}\tau_n(s') + \tau_n(s').\end{aligned}$$

Since m was arbitrary and $\tau(s') = \sup_m \tau_m(s')$, this concludes the proof. ∎

Recall that $\nu_f(s) = \sum_{i\in S} f(i)$.

Lemma 4.4. *Assume $\lim_n f(n) = 0$ and $h : \mathbb{N} \to \mathbb{N}$ is strictly increasing. Then for all s and n we have $\nu_f(s) \ge 2^{n+1}(\tau_{n+1}(s) - \tau_n(s))$.*

Proof. Fix an $\varepsilon > 0$. Let g be an $(h, n+1)$-weight assignment such that

$$\sum_{i\in s} 2^{-g(i)} f(i) \ge \tau_{n+1}(s) + \varepsilon,$$

as given by Lemma 4.2. Let $s_0 = \{i \in s : g(i) \le n\}$. Then by a weight assignment argument identical to that in the proof of Lemma 4.3 we have

$\tau_n(s) \geq \tau_n(s_0) \geq \sum_{i\in s_0} 2^{-g(i)} f(i)$, and therefore

$$2^{-(n+1)}\nu_f(s) \geq 2^{-(n+1)} \sum_{i\in s\setminus s_0} f(i)$$
$$\geq \sum_{i\in s} 2^{-g(i)} f(i) - \sum_{i\in s_0} 2^{-g(i)} f(i) \geq \tau_{n+1}(s) - \tau_n(s) + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, this concludes the proof. ∎

Recall that if $\{w_i\}$ is a sequence of sets and $A \subseteq \mathbb{N}$, then we write $w_A = \bigcup_{i\in A} w_i$. The following lemma will be very useful (recall the definition of τ_n^k from §3).

LEMMA 4.5. *Assume f, h are as in Lemma 4.2, $w_1 < w_2 < \ldots$ is a sequence of finite sets, and $\delta > 0$. If for all i we have*

(1) $\tau_n(w_i) < \delta/2$, *and*
(2) $\tau_{n-1}^{\max(w_i)}(w_{[i+1,\infty)}) < \delta$ (*taking* $\max(w_0) = 1$),

then for every $A \subseteq \bigcup_i w_i$ we have $\tau_n(A) < \delta$.

Proof. If $\tau_n(A) = \tau_{n-1}(A)$, the conclusion follows from (2) above.

CLAIM 4.6. *Under the above assumptions, if $A \subseteq \bigcup_i w_i$ and $\tau_n(A) > \tau_{n-1}(A)$, then*

$$\tau_n(A) \leq \sup_i \Big(\tau_n(w_i) + \frac{1}{2}\tau_{n-1}^{\max w_i}\Big(\bigcup_{j=i+1}^{\infty} w_j\Big)\Big).$$

Proof. Fix $\varepsilon > 0$. By the assumption, we have

$$\tau_n(A) \leq \frac{1}{2}\sum_{j=1}^{m} \tau_{n-1}(E_jA) + \varepsilon$$

for some $m \leq h(k)$ and $k < E_1 < \ldots < E_m$. Let i be the minimal such that $k \leq \max w_i$. If $\max E_l \leq \max w_i$ for all $l = 1, \ldots, h(k)$, then $\tau_n(A) \leq \tau_n(w_i) + \varepsilon$, and there is nothing to prove. Let l be the minimal such that $\max E_l > \max w_i$. Then

$$\tau_n(A) \leq \frac{1}{2}\sum_{j=1}^{l-1} \tau_{n-1}(E_jA) + \tau_{n-1}(E_lA) + \frac{1}{2}\sum_{j=l}^{m} \tau_{n-1}(E_jA) + \varepsilon$$
$$\leq \frac{1}{2}\sum_{j=1}^{l-1} \tau_{n-1}(E_jA) + \tau_{n-1}(E_l(Aw_i))$$
$$+ \tau_{n-1}(E_l(A \setminus w_i)) + \frac{1}{2}\sum_{j=l}^{m} \tau_{n-1}(E_jA) + \varepsilon$$

$$\leq \tau_n(w_i) + \frac{1}{2}\tau^k_{n-1}\Big(\bigcup_{p=i+1}^{\infty} w_p\Big) + \varepsilon.$$

Since $k \leq \max w_i$ and $\varepsilon > 0$ was arbitrarily small, this completes the proof. ■

Lemma 4.5 follows immediately by the claim. ■

5. The second reduction. The main result of this section is Proposition 5.3, essentially saying that if $\mathcal{I} \leq_{\mathrm{RB}} \mathcal{T}_{fh}$ then $\mathcal{I}$ is of the form $\mathcal{T}_{f'h'}$ for some f', h' (possibly with $\lim_i f'(i) \neq 0$). It is essentially due to Casazza, Johnson and Tzafriri (see [1] and [2, Proposition I.12 and Lemmas II.1 and II.3]), who used the case when h is the identity function, to prove that every infinite-dimensional subspace of Tsirelson space includes a copy of Tsirelson space. We reproduce the proof from [1] for the convenience of the reader.

LEMMA 5.1. *Assume $f : \mathbb{N} \to \mathbb{R}$ is nonnegative and $h : \mathbb{N} \to \mathbb{N}$ is strictly increasing, and let $h^+(n) = h(n + h(n))$. Then for every A and n we have*

$$\tau_{f,h,n}(A) \leq \tau_{f,h^+,n}(A) \leq 3\tau_{f,h,n}(A).$$

Proof. The left-hand side inequality is obvious, since $h^+ \geq h$. We prove the following strengthening of the right-hand side inequality by induction:

$(*)$ For all A and n there are sets $F_1 < F_2 < F_3$ such that

$$\tau_{f,h^+,n}(A) \leq \sum_{j=1}^{3} \tau_{f,h,n}(F_j A).$$

The case when $n = 0$ is trivial, so let us assume the lemma is proved for some n and prove it for $n + 1$. If $\tau_{f,h^+,n+1}(A) = \tau_{f,h^+,n}(A)$, then there is nothing to prove, so we can assume

$$\tau_{f,h^+,n+1}(A) = \frac{1}{2}\sum_{l=1}^{h^+(k)} \tau_{f,h^+,n}(E_l A) \qquad \text{for some } h^+\text{-admissible } \langle k, \vec{E}\rangle,$$

$$\leq \frac{1}{2}\sum_{l=1}^{h^+(k)}\sum_{j=1}^{3} \tau_{f,h,n}(F_{lj}E_l A) \quad \text{for some } F_{l1} < F_{l2} < F_{l3}.$$

We can assume $F_{lj}E_l = F_{lj}$ for all l, j. For $G_{3(l-1)+j} = F_{lj}$ we have $G_1 < G_2 < \ldots < G_{3h^+(k)}$. We can assume all G_l's are nonempty, possibly by eliminating the empty ones from the sequence. Let $k^* = k + h(k)$ (so that $h^+(k) = h(k^*)$) and

$$F_1 = \bigcup_{l=1}^{h(k)} G_l, \quad F_2 = \bigcup_{l=h(k)+1}^{h(k)+h(k^*)} G_l, \quad F_3 = \bigcup_{l=h(k)+h(k^*)+1}^{3h^+(k)} G_l.$$

Then $\langle k, G_1, \ldots, G_{h(k)}\rangle$ is h-admissible, so we have

$$\tau_{f,h,n+1}(F_1 A) \geq \frac{1}{2}\sum_{l=1}^{h(k)} \tau_{f,h,n}(G_l A).$$

Note that, since each G_l is nonempty, we have $\min(G_{h(k)+1}) \geq \min G_1 + h(k) \geq k^*$, and therefore the tuple associated with F_2 is h-admissible and we have

$$\tau_{f,h,n+1}(F_2 A) \geq \frac{1}{2}\sum_{l=h(k)+1}^{h(k^*)} \tau_{f,h,n}(G_l A).$$

Like before, $\min G_{h(k)+h(k^*)+1} \geq k^* + h(k^*)$. Since $k \leq h(k)$ and $h(2k) \leq h(k^*)$, we have (note that $h(i)+j \leq h(i+j)$, since h is strictly increasing)

$$3h^+(k) \leq h(k^*) + 2h(k^*) \leq h(k^*) + h(k^* + h(k^*))$$

and $3h^+(k) - h(k^*) \leq h(k^* + h(k^*))$, so the tuple associated with F_3 is h-admissible, and we have

$$\tau_{f,h^+,n+1}(A) \leq \frac{1}{2}\sum_{l=1}^{3h(2k))} \tau_{f,h,n}(G_l A) \leq \sum_{j=1}^{3} \tau_{f,h,n+1}(F_j A),$$

completing the inductive proof. ∎

LEMMA 5.2. *Assume f, h are as in Lemma 5.1 and that $w_1 < w_2 < \ldots$ are finite sets. Let $f'(i) = \tau_{f,h}(w_i)$, $h'(i) = h(\min w_i)$, and $h''(i) = h(\max w_i)$. Then for all $A \subseteq \mathbb{N}$ we have*

$$\tau_{f',h'}(A) \leq \tau_{f,h}(w_A) \leq 6\tau_{f',h''}(A).$$

P r o o f. We first prove the left-hand side inequality, by proving

$$(*) \qquad \tau_{f',h',n}(A) \leq \tau_{f,h}(w_A)$$

using induction on n. Since $\tau_{f',h'}(A) = \lim_n \tau_{f',h',n}(A)$, this will suffice. In the case when $n = 0$ for some $i \in A$ we have $\tau_{f',h',0}(A) = \tau_{f,h}(w_i) \leq \tau_{f,h}(\bigcup_{j\in A} w_j)$.

Now assume $(*)$ is true for n. If $\tau_{f',h',n+1}(A) = \tau_{f',h',n}(A)$, there is nothing to prove. So we can assume

$$\tau_{f',h',n+1}(A) = \frac{1}{2}\sum_{j=1}^{h'(k)} \tau_{f',h',n}(E_j A) \qquad \text{for some } h\text{-admissible } \langle k, \vec{E}\rangle.$$

(Assuming that $|\vec{E}| = h'(k)$ is clearly not a loss of generality.) Since h' is increasing, we can assume $k = \min E_1$, and therefore $h'(\min E_1) = h(\min w_k)$. Let $E'_j = \bigcup_{j\in E_i} w_j$. Then $\langle \min w_k, E'_1, \ldots, E'_{h'(k)}\rangle$ is h-admissible, by the inductive assumption we have

$$\frac{1}{2}\sum_{j=1}^{h'(k)} \tau_{f',h',n}(E_jA) \le \frac{1}{2}\sum_{j=1}^{h(\min w(k))} \tau_{f,h}(E'_j w_A) \le \tau_{f,h}(w_A),$$

and this ends the verification of the left-hand side inequality.

Now we prove the right-hand side inequality. Let $h^*(i) = 2h''(i)$. Since $2h''(i) \le h''(i + h''(i))$, Lemma 5.1 implies $\tau_{f',h^*,n} \le 3\tau_{f'h'',n}$ for all n, and therefore it suffices to prove

$$\tau_{f,h,n}(w_A) \le 2\tau_{f',h^*,n}(A)$$

using induction on n. When $n = 0$ for some $i \in A$ we have

$$\tau_{f,h,0}(w_A) = \tau_{f,h,0}(w_i) \le \tau_{f,h}(w_i) = \tau_{f',h'',0}(i) \le 2\tau_{f',h^*,0}(A).$$

Now we assume the lemma is true for n and prove it for $n+1$. Again we can assume that $\tau_{f,h,n+1}(w_A) > \tau_{f,h,n}(w_A)$, therefore for some h-admissible $\langle k, \vec{E}\rangle$ (without loss of generality, we can assume that $k = \min E_1$ and $|\vec{E}| = h(k)$) we have

$$\tau_{f,h,n+1}(w_A) = \frac{1}{2}\sum_{l=1}^{h(k)} \tau_{f,h,n}(E_l w_A).$$

We can assume $w_A \subseteq \bigcup_{l=1}^{h(k)} E_l$. Let $E_l^- = \{i : \min(w_i) \in E_l\}$ and $E_l^+ = \{i : \max(w_i) \in E_l\}$. Then by the inductive assumption

$$\begin{aligned}\frac{1}{2}\sum_{l=1}^{h(k)} \tau_{f,h,n}(E_l w_A) &\le \frac{1}{2}\sum_{l=1}^{h(k)} 2\tau_{f,h,n}(E_l^+ A) + \frac{1}{2}\sum_{l=1}^{h(k)} 2\tau_{f,h,n}(E_l^- A)\\ &\le \frac{1}{2}\sum_{l=1}^{h(k)} 2\tau_{f',h^*,n}(E_l^+ A) + \frac{1}{2}\sum_{l=1}^{h(k)} 2\tau_{f',h^*,n}(E_l^- A).\end{aligned}$$

If $j = \min(\bigcup_{l=1}^{h(k)}(E_l^+ \cup E_l^-))$, then $h''(j) = h(\max w_j) \ge h(\min E_1) = h(k)$, also $h^*(j) = 2h''(j) \ge 2h(k)$, so that $\langle j, \vec{E}^*\rangle$ (where $\vec{E}^*$ is the increasing enumeration of $\bigcup_{l=1}^{h(k)}(E_l^- \cup E_l^+)$) is h^*-admissible, so that the right-hand side is equal to at most $2\tau_{f',h^*,n+1}(A)$. As pointed out earlier, this concludes the proof since $\tau_{f',h^*,n+1} \le \tau_{f',h'',n+1}$ for all n. ∎

We are now prepared for the main result of this section.

Proposition 5.3. *Assume $f : \mathbb{N} \to \mathbb{R}$ is nonnegative and $h : \mathbb{N} \to \mathbb{N}$ is strictly increasing. If $w_1 < w_2 < \ldots$ are finite sets, then for $f'(i) = \tau_{f,h}(w_i)$ and $h'(i) = h(\min w_i)$ we have $\mathcal{T}_{f',h'} = \{A : w_A \in \mathcal{T}_{fh}\}$.*

Proof. Let $h''(i) = \max(w_i)$ and $(h')^+(i) = h'(i + h'(i))$. Since $h'' \leq (h')^+$, Lemmas 5.1 and 5.2 imply that for every A we have

$$\tau_{f',h'}(A) \leq \tau_{f,h}(w_A) \leq 6\tau_{f',h''}(A) \leq 6\tau_{f'(h')^+}(A) \leq 18\tau_{f',h'}(A),$$

thus $\tau_{f,h}(w_A) = \infty$ if and only if $\tau_{f',h'}(A) = \infty$, and the two ideals coincide. ∎

6. Good sequences. An important part of the proof of Theorem 3.1 is in proving its weaker version:

PROPOSITION 6.1. *If h is strictly increasing, the ideals $\mathcal{T}_{fh}$ and $\mathcal{I}_{1/n}$ are different.*

Proof. Assume the contrary, that $\mathcal{T}_{fh} = \mathcal{I}_{1/n}$. Note that we can assume $\lim_i f(i) = 0$, for otherwise there would be an infinite set A none of whose infinite subsets is in $\mathcal{T}_{fh}$, but there is no such set for $\mathcal{I}_{1/n}$.

CLAIM 6.2. *There is a sequence $t_1 < t_2 < \ldots$ and $N \in \mathbb{N}$ such that*

(a) $\nu_f \leq N\nu_{1/n}$ *on* $\bigcup_i t_i$, *and*
(b) $\inf_i(\tau_1(t_i)) > 0$.

Proof. Let $I_m = [m, m + h(m))$ and note that $\tau(I_m) = \tau_1(I_m)/2 = \nu_f(I_m)/2$. We shall prove that there is an $N \in \mathbb{N}$ such that if $t_m^0 = \{k \in I_m : kf(k) < N\}$, then all but finitely many t_m^0 satisfy $\nu_{1/n}(t_m^0) \geq (\ln 2)/2$. If we can find such a sequence, then we can take t_i to be its subsequence satisfying $t_1 < t_2 < \ldots$, and (a) will be satisfied. To assure (b), note that $\liminf_i \nu_f(t_i) > 0$, since otherwise there would be an infinite $C \subseteq \mathbb{N}$ such that $Y = \bigcup_{i\in C} t_i$ satisfies $\nu_f(Y) < \infty$, but then $Y \in \mathcal{T}_{fh}\setminus\mathcal{I}_{1/n}$, contradicting our assumptions. Therefore we can find a subsequence of t_i which satisfies $\inf_i \nu_f(t_i) > 0$, and since $t_i \subseteq [m_i, m_i + h(m_i))$ for some m_i, this implies (b).

Assume that N as above does not exist. Then we can find $\{m(N)\}_{N=1}^{\infty}$ satisfying

(1) $\nu_{1/n}\{k \in I_{m(N)} : kf(k) < 2^N\} < (\ln 2)/2$,
(2) $k > 2^N$ for all $k \in I_{m(N)}$ with $f(k) < 2^{-N}$, and
(3) the intervals $I_{m(N)}$ are pairwise disjoint.

Since $\nu_{1/n}(I_m) \geq \ln(m + h(m)) - \ln(m) \geq \ln 2$ for all m, by (1) we have

$$\nu_{1/n}\{k \in I_{m(N)} : kf(k) \geq 2^N\} > (\ln 2)/2.$$

Let $s_N \subseteq I_{m(N)}$ be such that

(4) $kf(k) \geq 2^N$ for all $k \in s_N$, and
(5) $|\nu_{1/n}(s_N) - 2^{-N+1}| < 2^{-N}$.

(Note that (5) can be assured by using (2).) Then we have

$$\tau(s_N) = \nu_f(s_N)/2 \geq 2^N\nu_{1/n}(s_N) \geq 2^N \cdot 2^{-N} = 1,$$

therefore $\bigcup_N s_N \in \mathcal{I}_{1/n} \setminus \mathcal{T}_{fh}$, contradicting our assumption and completing the proof of Claim 6.2. ■

We will use $\{t_i\}$ given by Claim 6.2 to find $\{w_j\}$ such that for all j,

(6) $w_1 < w_2 < \ldots$ are included in $\bigcup_i t_i$,
(7) $\tau_{j+1}(w_j) \geq 2^{-j+1}$, and
(8) $\tau_j(w_j) \leq 2^{-j}$.

Assume $\{w_j\}$ satisfy (6)–(8) above, and find $v_j \subseteq w_j$ such that $\tau(v_j) \leq 3\tau_{j+1}(w_j)/2$ and $\tau_{j+1}(v_j) = \tau_{j+1}(w_j)$. Therefore $\tau(v_j) < 2^{-j+2}$ and by Lemma 4.4 we have $\nu_f(v_j) \geq 2^{j+1}(2^{-j+1} - 2^{-j}) = 2$. Note that $\tau \leq \nu_f$ and $\nu_f \leq N\nu_{1/n}$ on the set $X = \bigcup_j v_j \subseteq \bigcup t_i$, and therefore $\lim_n \tau(X \setminus n) < \infty$ and $\lim_n \nu_f(X \backslash n) = \infty$. This implies $X \in \mathcal{T}_{fh} \backslash \mathcal{I}_f \subseteq \mathcal{T}_{fh} \backslash \mathcal{I}_{1/n}$, contradicting our assumptions.

Therefore it suffices to find $\{w_i\}$ satisfying (6)–(8) above. We do this by using the following notion (recall that $u_A = \bigcup_{i\in A} u_i$).

Definition 6.3. A sequence $u_1 < u_2 < \ldots$ of finite sets is *m-good for f, h* (or simply *m-good* if f, h are clear from the context) if there exists a $\delta > 0$ such that:

(i) the set $\{\tau_m(u_i) : i \in \mathbb{N}\}$ is dense in $[0, \delta]$, and
(ii) $\lim_i \tau_{m-1}(u_{[i,\infty)}) = 0$.

Lemma 6.4. *Assume* $\lim_n f(n) = 0$ *and h is strictly increasing. If a sequence $\{u_i\}$ is m-good and $a, \varepsilon > 0$, then there is a finite $v \subseteq \bigcup_i u_i$ satisfying $|\tau_{m+1}(v) - a| < \varepsilon$ and $\tau_m(v) < \varepsilon$.*

Proof. We can assume that $\varepsilon < \delta$, where δ is as in the definition of good sequence. By going to a subsequence we can also assume that for all i we have

$$\tau_{m-1}^{\max u_i}(u_{[i+1,\infty)}) < \varepsilon \qquad (\text{taking } \max(u_0) = 1).$$

The set $A = \{i : \tau_m(u_i) < \varepsilon/2\}$ is infinite and $\limsup_{k\in A} \tau_m(u_k) = \varepsilon/2$ (because the sequence $\{u_i\}$ is m-good and the corresponding δ is bigger than $\varepsilon/2$). By Lemma 3.2(4) we have $\tau_{m+1}(u_A) = \infty$. Therefore we can find $v \subseteq u_A$ such that $\tau_{m+1}(v) \geq a$, $\tau_{m+1}(v') < a$ for every $v' \subsetneq v$, and $f(j) < \varepsilon$ for every $j \in v$. Then by the subadditivity of τ_{m+1} we have $|\tau_{m+1}(v) - a| = \tau_{m+1}(v) - a < \varepsilon$. By Lemma 4.5 (applied with $\delta = \varepsilon$) we have $\tau_m(v) < \varepsilon$, therefore v is as required. ■

Lemma 6.5. *Assume* $\lim_n f(n) = 0$ *and h is strictly increasing. Then for every m-good sequence $\{u_i\}$ there is an $(m+1)$-good sequence $\{v_i\}$ such that $\bigcup_i v_i \subseteq \bigcup_i u_i$.*

Proof. Let q_i ($i \in \mathbb{N}$) be an enumeration of all rationals in the interval $(0, 1)$. By using Lemma 6.4, we can recursively find $v_1 < v_2 < \ldots$ included in

$\bigcup_i u_i$ and such that $|\tau_{m+1}(v_i) - q_i| < 2^{-i}$ and $\tau_m(v_i) < 2^{-i}$. Then for every i we have $\tau_m(v_{[i,\infty)}) < 2^{-i+1}$, therefore the sequence v_i is $(m+1)$-good. ∎

Let t_i be as in Claim 6.2, and let $\varepsilon = \inf_i \tau_1(t_i) > 0$. Since $\lim_n f(n) = 0$, we can find $u_i \subseteq t_i$ $(i \in \mathbb{N})$ so that $\{\tau_1(u_i) : i \in \mathbb{N}\}$ is dense in $[0, \varepsilon]$. Since the condition $\lim_i \tau_0(u_i) = 0$ reduces to our assumption that $\lim_k f(k) = 0$, this sequence is 1-good. Therefore the sequence w_i as in (6)–(8) can now be constructed recursively by using Lemmas 6.5 and 6.4. As explained before, this implies that the two ideals differ and concludes the proof of Proposition 6.1. ∎

The above proof gives the following proposition of independent interest (see the proof of [5, Proposition 3.6]).

PROPOSITION 6.6. *If* $\lim_n f(n) = 0$, $h : \mathbb{N} \to \mathbb{N}$ *is strictly increasing and*

$$\liminf_n \nu_f([n, n + h(n)) > 0,$$

then for every $m \geq 1$ *there is an* m*-good sequence.* ∎

7. Proof of Theorem 3.1. The ideal $\mathcal{T}_{fh}$ is, by Lemma 3.2(5), an F_σ P-ideal. Therefore it remains to prove that $\mathcal{P}(\mathbb{N})/\mathcal{I}_{1/n} \not\leq_{\mathrm{B}} \mathcal{P}(\mathbb{N})/\mathcal{T}_{fh}$. By Lemma 2.1, it suffices to prove that there is no sequence of finite sets $w_1 < w_2 < \ldots$ such that for all $A \subseteq \mathbb{N}$ we have

$$A \in \mathcal{I}_{1/n} \quad \text{if and only if} \quad \bigcup_{i \in A} w_i \in \mathcal{T}_{fh}.$$

Assume that such a sequence exists. By Proposition 5.3, for some strictly increasing h' and $f'(n) = \tau_{f,h}(w_n)$ the ideals $\mathcal{I}_{1/n}$ and $\mathcal{T}_{f'h'}$ coincide, but this contradicts Proposition 6.1 and completes the proof. ∎

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