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Cofinal $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$ subsets of ω^ω

by

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Abstract. We study properties of $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$ subsets of ω^ω that are cofinal relative to the orders $\le$ ($\le^\star$) of full (eventual) domination. We apply these results to prove that the topological statement "Any compact covering mapping from a Borel space onto a Polish space is inductively perfect" is equivalent to the statement "$\forall \alpha \in \omega^\omega,\ \omega^\omega \cap L(\alpha)$ is bounded for $\le^\star$".

This work is a continuation of [3], in which we studied the validity of the following statement for two separable metric spaces X and Y:

$\mathbb{A}(X,Y)$: "Any compact covering mapping $f : X \to Y$ is inductively perfect".

We recall that if $f : X \to Y$ is a continuous mapping then f is said to be:

• *compact covering* if any compact subset of Y is the direct image of some compact subset of X;

• *perfect* if the inverse image of any compact subset of Y is a compact subset of X;

• *inductively perfect* if there exists a subset X' of X such that the restriction of f to X' is a perfect mapping onto Y.

Notice that, as we showed in [3], the study of these notions can easily be reduced to the case where all the spaces are zero-dimensional, hence subsets of ω^ω or $\mathbf{2}^\omega$.

Obviously, any inductively perfect mapping is compact covering: If K is any compact subset of Y then the set $H = X' \cap f^{-1}(K)$ is compact and clearly $f(H) = K$. The converse statement, that is, $\mathbb{A}(X,Y)$, is false in general but holds under some regularity assumptions on X or Y. The main known results in this direction are the following:

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0.1. *In ZFC*, $\mathbb{A}(X,Y)$ *holds*:

(a) *if* X *is Polish*, *and then* Y *is also Polish*;
(b) *if* Y *is* σ*-compact.*

0.2. *Assume that "all* $\mathbf{\Sigma}^1_1$ *games are determined". Then* $\mathbb{A}(X,Y)$ *holds*:

(a) *if* X *is* $\mathbf{\Pi}^1_1$ (*i.e. coanalytic*) *and* Y *is* $\mathbf{\Sigma}^1_1$ (*i.e. analytic*), *and then* Y *is Borel*;
(b) *if* X *and* Y *are* $\mathbf{\Pi}^1_1$;
(c) *if* X *is Borel*, *and then* Y *is Borel of the same Borel class as* X.

0.3. *Assume that "the constructible reals are uncountable". Then* $\mathbb{A}(X,Y)$ *might not hold*:

(a) *even if* Y *is Polish*;
(b) *even if* X *and* Y *are* $\mathbf{\Pi}^1_1$.

The statement 0.1(a) was proved several years ago (with different formulations) by Christensen [2] and Saint Raymond [8] independently, and was motivated by a problem on the complexity of the hyperspace $\mathcal{K}(X)$ of all compact subsets of a space X (see 1.4 below). On the other hand, 0.1(b) is a quite recent result due to Ostrovskiĭ [7], also to Just and Wicke [4] in the (nontrivial) case where Y is countable, and was motivated by a problem of Michael on countable topological spaces. All the other results, in 0.2 and 0.3, are from [3], to which we refer the reader for a more detailed discussion.

One of the main problems which were not settled in our previous work, and which was first considered by Ostrovskiĭ in [7], is to decide whether the following statement $\mathbb{A}(\mathbf{\Delta}^1_1)$:

$$\mathbb{A}(X,Y) \quad \forall X \text{ Borel}, \ \forall Y$$

(in which one can equivalently replace "$\forall Y$" by "$\forall Y\ \mathbf{\Sigma}^1_1$"), or the weaker statement $\mathbb{A}(\mathbf{\Delta}^1_1,\mathbf{\Delta}^1_1)$:

$$\mathbb{A}(X,Y) \quad \forall X \text{ Borel}, \ \forall Y \text{ Borel},$$

holds in ZFC. Indeed, none of the counter-examples constructed in [3] excludes the validity of such absolute statements. Moreover, such possibilities were strengthened by some other results of [3].

In fact, we prove that this is not the case. For example, in the universe L there exists a compact covering mapping from an $\mathbf{F}_\sigma$ subset X of ω^ω onto $Y=\omega^\omega$ which is not inductively perfect. This counter-example ameliorates far away all the previous ones. Notice that by 0.1(a) and 0.1(b) above, the complexity of the Borel sets X and Y is the best possible. However, this construction will not be obtained in a direct and explicit way, but as a consequence of the general study we make of $\mathbb{A}(X,Y)$. More precisely, we prove that the following statements are equivalent:

(1) $\mathbb{A}(X,Y)$ $\forall X$ $\mathbf{\Pi}^1_1$, $\forall Y$ $\mathbf{P}_\sigma$,
(2) $\mathbb{A}(X,Y)$ $\forall X$ Borel, $\forall Y$ Polish,
(3) $\mathbb{A}(X,Y)$ $\forall X$ $\mathbf{P}_\sigma$, $\forall Y$ Polish,

where by "$\mathbf{P}_\sigma$" we mean "an $\mathbf{F}_\sigma$ subset of some Polish space".

We also prove an equivalent reformulation of $\mathbb{A}(X,Y)$ in terms of properties of cofinal subsets of the following ordered spaces:

$$(\omega^\omega,\le),\qquad (\omega^\omega,\le^\star),\qquad (\mathcal{K}(X),\subset)$$

where the orders $\le$ (of full domination) and $\le^\star$ (of eventual domination) on ω^ω are defined by:

$$x\le y \Leftrightarrow \forall n,\ x(n)\le y(n),$$
$$x\le^\star y \Leftrightarrow \exists m,\ \forall n\ge m,\ x(n)\le y(n),$$

and the hyperspace $\mathcal{K}(X)$ of all compact subsets of the space X is endowed with the Hausdorff topology (see 1.4 below) and ordered by the inclusion $\subset$. In the sequel the notions of "bounded" and "cofinal" in ω^ω are to be understood relative to the order $\le$. When these notions are considered relative to the order $\le^\star$ we use the terms: "$\star$-bounded" and "$\star$-cofinal".

Thus we prove that (1)–(3) above are also equivalent to each of the following:

(4) Any $\mathbf{\Pi}^1_1$ cofinal subset of ω^ω contains a $\mathbf{\Sigma}^1_1$ cofinal subset.
(5) Any $\mathbf{\Pi}^1_1$ $\star$-cofinal subset of ω^ω contains a $\mathbf{\Sigma}^1_1$ $\star$-cofinal subset.
(6) Any $\mathbf{\Pi}^1_1$ cofinal subset of $\mathcal{K}(\omega^\omega)$ contains a $\mathbf{\Sigma}^1_1$ cofinal subset.

But the full set-theoretical strength of all these statements is given by their equivalence to

(0) $\forall\alpha\in\omega^\omega$, the set $\omega^\omega\cap L(\alpha)$ is $\star$-bounded in ω^ω.

Other equivalences, that we do not detail here, can also be found in the paper. For example, in (4)–(6) above one can replace "$\mathbf{\Pi}^1_1$" by "$\mathbf{\Sigma}^1_2$", and (or) "$\mathbf{\Sigma}^1_1$" by "closed".

Surprisingly, for many couples of these statements we do not have a direct proof of their equivalence, though they are syntactically very close. The simplest case is (4)$\Leftrightarrow$(5), or (4)$\Leftrightarrow$(6), for which the only proofs we have pass through (0).

However, we were not able to decide whether all these statements are also equivalent to the initially considered $\mathbb{A}(\mathbf{\Delta}^1_1)$ or $\mathbb{A}(\mathbf{\Delta}^1_1,\mathbf{\Delta}^1_1)$, although some of the results we prove suggest that these two assertions might rather be related to the following one, stronger than (0):

$\forall\alpha\in\omega^\omega$, the set $\omega^\omega\cap L(\alpha)$ is countable.

The paper is organized as follows: In Section 1 we fix basic notations and terminology. In Section 2 we give positive results (in ZFC) about $\mathbf{\Sigma}^1_1$

cofinal subsets in "nice" ordered spaces $(E, \prec)$ (such as $(\omega^\omega, \le)$, $(\omega^\omega, \le^\star)$, $(\mathcal{K}(\omega^\omega), \subset)$). The main result ensures that:

If A is a $\mathbf{\Sigma}^1_1$ cofinal subset of E then:

(a) *A contains a closed cofinal subset.*

(b) *There exists a continuous mapping $f : E \to A$ satisfying $x \prec f(x)$ for all $x \in E$.*

In Sections 3 and 4 we give necessary and sufficient conditions for the validity of (0) which will relate this statement to properties of cofinal $\mathbf{\Pi}^1_1$ sets. In these sections we assume that the reader is familiar with the effective descriptive set theory and the basic classical properties of L (for example, as presented in [6]): representation of Σ^1_2 sets, description of the largest Π^1_1 thin set, absoluteness arguments, ... We apply these results in Section 5 to prove the equivalence between (0) and (3)–(5). In Section 6 we go back to compact covering mappings and prove the equivalence between (0) and (1)–(3). This section is totally independent of the previous ones, and all the arguments used there are purely topological.

1. Descriptive properties

1.1. *Classical descriptive classes.* By a *descriptive class* we mean a class of subsets of Polish spaces which is closed under taking inverse images under continuous mapppings between Polish spaces. The classes that we consider in this work are the following classical ones:

- $\mathbf{\Sigma}^0_2$: the class of all $\mathbf{F}_\sigma$ subsets,
- $\mathbf{\Pi}^0_2$: the class of all $\mathbf{G}_\delta$ subsets,
- $\mathbf{\Delta}^1_1$: the class of all Borel subsets,
- $\mathbf{\Sigma}^1_1$: the class of all analytic subsets,
- $\mathbf{\Pi}^1_1$: the class of all coanalytic subsets,
- $\mathbf{\Sigma}^1_2$: the class of all PCA (projection of coanalytic) subsets.

Let Γ be any of the previous classes, except $\mathbf{\Sigma}^0_2$. When, for a space X, we say that "X is in Γ", we mean that X can be (homeomorphically) embedded in some Polish space P as a Γ subset of P. It is a well known and fundamental fact that this notion is absolute, in the sense that it does not depend on the particular Polish space P nor on the embeddding. For example, it is a classical fact that "X is $\mathbf{\Pi}^0_2$" is equivalent to "X is Polish". However, this does not apply to the class $\mathbf{\Sigma}^0_2$, and we shall use the notation $\mathbf{\Sigma}^0_2$ only when working in some explicitly fixed Polish space.

When the Polish space is ω^ω we shall also consider the *effective classes*:

$$\Delta^1_1, \quad \Sigma^1_1, \quad \Pi^1_1, \quad \Sigma^1_2, \quad \Pi^0_2, \quad \Sigma^0_2.$$

1.2. *The class* $\mathbf{P}_\sigma$ *of* σ*-Polish spaces*

THEOREM 1.2.1. *For a separable metrizable space* X, *the following are equivalent*:

(i) X *is* $\mathbf{F}_\sigma$ *in some Polish space.*

(ii) X *is the union of a countable family of closed Polish subspaces.*

(iii) X *is the difference of two* $\mathbf{F}_\sigma$ *sets in some Polish space.*

(iv) *Whenever* X *is embedded in some metrizable space* E, *it is the difference of two* $\mathbf{F}_\sigma$ *sets in* E.

(iv)$\Rightarrow$(iii) and (i)$\Rightarrow$(ii) are obvious, and (iii)$\Rightarrow$(i) follows from the classical fact that a $\mathbf{G}_\delta$ subset of a Polish space is also Polish. Let us prove that (ii)$\Rightarrow$(iv).

Let E be some metrizable space, $X \subset E$, and (F_n) be a countable covering of X by Polish closed subsets of X. If we denote by $\overline{F}_n$ the closure of F_n in E, we have $F_n = X \cap \overline{F}_n$. Moreover, $A = \bigcup_n \overline{F}_n$ is $\mathbf{\Sigma}^0_2$ in E, and since F_n is Polish, $E_n = \overline{F}_n \setminus X = \overline{F}_n \setminus F_n$ is $\mathbf{\Sigma}^0_2$ in $\overline{F}_n$, hence in E. If we put $B = \bigcup_n E_n$, then B is $\mathbf{\Sigma}^0_2$ in E, and we have $X = A \setminus B$. Hence X is the difference of two $\mathbf{F}_\sigma$ sets in E. ∎

In this paper we make use of the following conventions:

• If the space X is $\mathbf{\Sigma}^0_2$ for *any* embedding in *any* Polish space, then obviously X is σ-compact and we write "X is $\mathbf{K}_\sigma$".

• If the space X is $\mathbf{\Sigma}^0_2$ for *some* embedding in *some* Polish space, then we say that X is σ*-Polish* and write "X is $\mathbf{P}_\sigma$".

These conventions, which will lighten the statement of some results, are not universal. In particular, they are not consistent with [3] where we used the notation $D_2(\mathbf{K}_\sigma)$ for the class $\mathbf{P}_\sigma$.

1.3. *Perfect mappings.* We recall that a continuous mapping f between metrizable spaces X and Y is said to be *perfect* if the inverse image of every compact subset of Y is compact in X. It is well known that any perfect mapping is closed. Let us also recall the following well known result:

THEOREM 1.3.1. *If* P *is a Polish space*, *then there exist a closed subset* E *of* ω^ω *and a perfect mapping* f *from* E *onto* P.

If d is any complete metric on P, it is easy to construct inductively a family $(P_s)_{s\in\omega^{<\omega}}$ of closed subsets of P such that:

(i) $P_\emptyset = P$,

(ii) $P_{s^\frown n} \subset P_s$ and $\operatorname{diam}(P_s) \le 1/|s|$,

(iii) $(P_{s^\frown n})_n$ is a locally finite covering of P_s.

Then it is easily checked that $E = \{\alpha \in \omega^\omega : \bigcap_{s \prec \alpha} P_s \neq \emptyset\} = \{\alpha \in \omega^\omega : \forall s \prec \alpha,\ P_s \neq \emptyset\}$ is a closed subset of ω^ω, and that the function f defined by

$$\{f(\alpha)\} = \bigcap_{s \prec \alpha} P_s$$

is continuous from E onto P and perfect.

Corollary 1.3.2. *Let E be a Borel subset of some Polish space P. Then there exist a Borel subset E_0 of ω^ω and a perfect mapping f from E_0 onto E.*

Since there exist a closed subset F of ω^ω and a perfect mapping g from F onto P, we can put $E_0 = g^{-1}(E)$ and $f = g_{|E_0}$. Then f is continuous from the Borel subset E_0 of ω^ω onto E, and for every compact subset K of E, $f^{-1}(K) = g^{-1}(K)$ is compact since g is perfect. Thus f is also perfect.

1.4. *Hyperspaces.* Given a space X we very often consider the hyperspace $\mathcal{K}(X)$ of all compact subsets of the space X endowed with the Hausdorff topology, that is, the coarsest topology on $\mathcal{K}(X)$ for which the subset $\mathcal{K}(A)$ is open (closed) when A is open (closed). We recall some of the basic properties that we shall use:

- *If X is $\mathbf{\Pi}^1_1$ then $\mathcal{K}(X)$ is $\mathbf{\Pi}^1_1$.*
- *If X is $\mathbf{\Pi}^0_2$ then $\mathcal{K}(X)$ is $\mathbf{\Pi}^0_2$.*
- *If $\mathcal{K}(X)$ is $\mathbf{\Sigma}^1_1$ then X (hence $\mathcal{K}(X)$) is $\mathbf{\Pi}^0_2$.*

The first two results follow from elementary complexity computations. The last one is the result of Christensen and Saint Raymond we mentioned in the introduction, and which follows from 0.1(a).

2. $\mathbf{\Sigma}^1_1$ cofinal sets

2.0. *Ordered spaces.* In this paper "order" is used in the sense of "partial pre-order".

(a) If $(E, \prec)$ is an ordered space we identify the relation $\prec$ with its graph $G \subset E \times E$. In particular, if Γ is some descriptive class and $G \in \Gamma$ we simply say that $\prec$ *is in Γ*. Notice that since E is the domain of $\prec$, if (the graph of) $\prec$ is $\mathbf{\Delta}^1_1$ then E is automatically $\mathbf{\Sigma}^1_1$; if moreover the space E is also $\mathbf{\Delta}^1_1$, we say that $(E, \prec)$ is a $\mathbf{\Delta}^1_1$ *ordered space.*

(b) Let $(E, \prec)$ be an ordered space.

(1) An element a is said to be *dominated* by an element b if $a \prec b$.

(2) A subset A is said to be *bounded* if all its elements are dominated by some element: $\exists x \in E,\ \forall y \in A,\ y \prec x$.

(3) A subset A is said to be *cofinal* (in E) if any element in E is dominated by some element in A: $\forall x \in E,\ \exists y \in A,\ x \prec y$.

(4) A *domination function* is a mapping $f : E \to E$ such that $x \prec f(x)$ for all $x \in E$. The range of a domination function is obviously cofinal, and

by the axiom of choice any cofinal set contains the range of some domination function.

(5) The following notions play a crucial role in this paper. Fix some family $\mathcal{F}$ of mappings from E into itself. We say that the subset A of E admits an $\mathcal{F}$ *domination function* if A contains the range of some domination function $f \in \mathcal{F}$. When E is a topological space and $\mathcal{F}$ is the set all continuous transformations of E, we also say that A is *continuously cofinal*.

(c) Although many of the results we prove are stated for abstract ordered spaces, we mainly apply these results to the spaces

$$(\omega^\omega, \leq), \quad (\omega^\omega, \leq^\star), \quad (\mathcal{K}(X), \subset)$$

considered in the introduction, to which we refer as the *canonical examples*. The main property of these examples is that the bounded subsets have a simple topological characterization, that we now recall.

It is clear that a subset of ω^ω is bounded (for $\leq$) iff it is contained in some compact subset of ω^ω. This is also true in $(\mathcal{K}(X), \subset)$: Notice that if A is a compact subset of $\mathcal{K}(K)$ then $\bigcup A$ is a compact subset of X, hence A is bounded; and conversely, if A is bounded and dominated by some element $K \in E$ then A is a subset of the compact set $\mathcal{K}(K)$.

It is also a classical fact that a subset of ω^ω is $\star$-bounded (i.e. bounded for $\leq^\star$) iff it is contained in some σ-compact subset of ω^ω.

THEOREM 2.1. *Let* $(E, \prec)$ *be a* $\mathbf{\Delta}^1_1$ *ordered space in which any compact subset is bounded.*

(a) *Any* $\mathbf{\Sigma}^1_1$ *cofinal subset of* E *admits a Borel domination function of first Baire class.*

(b) *If moreover* E *is zero-dimensional*, *then any* $\mathbf{\Sigma}^1_1$ *cofinal subset of* E *admits a continuous domination function.*

Proof. We first show how to derive (a) from (b). Let A be a cofinal subset of E. Fix a perfect mapping φ from a $\mathbf{\Delta}^1_1$ subset E_0 of ω^ω onto E (see Corollary 1.3.2), and consider on E_0 the order $\prec_0$ defined by

$$x \prec_0 y \Leftrightarrow \varphi(x) \prec \varphi(y).$$

Then $(E_0, \prec_0)$ is a zero-dimensional $\mathbf{\Delta}^1_1$ ordered space in which also any compact subset is bounded. Let $A_0 = \varphi^{-1}(A)$, which is clearly a $\mathbf{\Sigma}^1_1$ cofinal subset of E_0; then by (b), there exists a continuous domination function $f_0 : E_0 \to A_0$. Since $\varphi : E_0 \to E$ is perfect it admits a Borel section $\psi : E \to E_0$ of first Baire class, and one checks that $f = \varphi \circ f_0 \circ \psi$ is a first Baire class domination function on E with range in A.

Now we come to the proof of (b). We can and do assume that E is a subspace of ω^ω. We fix a topological embedding i of $\omega^\omega \times \omega^\omega$ onto a $\mathbf{\Pi}^0_2$

subset H of $\mathbf{2}^\omega$, and a closed subset F in $\omega^\omega \times \omega^\omega$ such that

$$x \in A \Leftrightarrow \exists\alpha,\ (x,\alpha) \in F.$$

Consider the game G where Players I and II choose alternately: an integer (chosen by Player I), then 0 or 1 (chosen by Player II). In a run of this game the players construct thus an element (x,ε) in $\omega^\omega \times \mathbf{2}^\omega$; and Player II wins the run iff

$$x \notin E \quad \text{or} \quad (\varepsilon \in H,\ i^{-1}(\varepsilon) = (y,\alpha),\ (y,\alpha) \in F,\ x \prec y)$$

This game is clearly Borel, hence determined.

A winning strategy for Player II in this game defines a continuous mapping $x \mapsto \varepsilon$ from ω^ω into H which by composition with i^{-1} gives a continuous mapping $x \mapsto (y,\alpha)$ from ω^ω into $\omega^\omega \times \omega^\omega$ such that $y \in A$ whenever $x \in E$; and if f denotes the first component of this mapping $x \mapsto y$, then the restriction of f to E is clearly a continuous domination function with range in A.

Hence by determinacy all we have to show is that Player I has no winning strategy in this game. So suppose for contradiction that he has a winning strategy σ. Since Player II is constructing an element in $\mathbf{2}^\omega$, the set K of all x answered by Player I in all runs compatible with σ is a compact subset of ω^ω, and since σ is winning, K necessarily is a subset of E, hence bounded for $\prec$; and since A is cofinal, we can find $b \in A$ such that $x \prec b$ for all $x \in K$. Fix then β in ω^ω such that $(b,\beta) \in F$ and consider the run of G where Player II plays $i(b,\beta)$ and Player I follows σ. In this run Player I constructs an element $a \in K$, and since $a \prec b$, Player II wins; and this gives the contradiction. ∎

COROLLARY 2.2. *Let $(E,\prec)$ be a $\mathbf{\Delta}^1_1$ ordered space in which the bounded subsets are exactly the relatively compact subsets. Then any $\mathbf{\Sigma}^1_1$ cofinal subset of E contains a closed* (*in E*) *cofinal subset.*

P r o o f. Let A be a fixed $\mathbf{\Sigma}^1_1$ cofinal subset of E.

Assume first that E is zero-dimensional. Then by Theorem 2.1(b) there exists a continuous domination function $f : E \to E$ with range in A. If K is any compact subset of E then by the assumption on $\prec$, K is bounded by some element $a \in E$, and it follows that $f^{-1}(K)$ is also bounded by a; then by the assumption on $\prec$, $f^{-1}(K)$ is relatively compact, hence compact since f is continuous. This shows that f is a perfect mapping, and in particular its range $f(E)$ is a closed subset of E.

For the general case we proceed as in the previous proof, by fixing a perfect mapping φ from a $\mathbf{\Delta}^1_1$ subset E_0 of ω^ω onto E and defining A_0 and $\prec_0$ as above. Applying the result of the zero-dimensional case we can find a closed cofinal subset F_0 of E_0 and contained in A_0, and it is immediate to check that $F = \varphi(F_0)$ is a closed cofinal subset of E contained in A. ∎

REMARKS 2.3. (a) It is clear that in Theorem 2.1(a) one cannot drop the zero-dimensional hypothesis. For example, $\mathbb{Z}$ is cofinal in $E=\mathbb{R}$ for the natural ordering, but any continuous mapping from $E=\mathbb{R}$ to $\mathbb{Z}$ is constant.

(b) It follows from 2.0(c) that the three canonical orders $(\omega^\omega,\le)$, $(\omega^\omega,\le^\star)$, $(\mathcal{K}(X),\subset)$ satisfy the hypothesis

(H): "Any compact subset is bounded"

considered in Theorem 2.1. But only $(\omega^\omega,\le)$ and $(\mathcal{K}(X),\subset)$ satisfy the stronger hypothesis

$(\widetilde{\mathrm{H}})$: "The bounded subsets are exactly the relatively compact subsets"

considered in Corollary 2.2.

THEOREM 2.4. *Let* $(E,\prec)$ *denote one of the ordered spaces* $(\omega^\omega,\le)$, $(\omega^\omega,\le^\star)$, $(\mathcal{K}(X),\subset)$ *with* X *Polish. For a subset* A *of* E *the following are equivalent*:

(i) A *contains a* $\mathbf{\Sigma}^1_1$ *cofinal subset.*
(ii) A *is continuously cofinal.*
(iii) A *contains a closed cofinal subset.*

Proof. As we mentioned in Remark 2.3(b) the orders $\le$ and $\subset$ satisfy $(\widetilde{\mathrm{H}})$ and the equivalences follow from Theorem 2.1 and Corollary 2.2.

Also, the order $<^\star$ satisfies (H) so (i)$\Rightarrow$(ii) follows from Theorem 2.1. Since (iii)$\Rightarrow$(i) is obvious we only need to prove (ii)$\Rightarrow$(iii). Let f be a continuous $\star$-domination function on ω^ω. For any finite sequence $s\in\omega^{<\omega}$ consider the closed set

$$F_s=\{x\in\omega^\omega : s\prec f(x) \text{ and } \forall n\ge|s|,\ x(n)\le f(x)(n)\}.$$

Then the restriction of f to F_s is again a perfect mapping and so $A_s = f(F_s)$ is a closed subset of ω^ω. Since $\omega^\omega=\bigcup_s F_s$, the set $A=\bigcup_s A_s$ (the range of f) is $\star$-cofinal. It follows that one of the closed sets A_s is $\star$-cofinal: otherwise, for any s we can find $x_s\in\omega^\omega$ which is not $\star$-dominated by any element of A_s, and if x is any element of ω^ω $\star$-dominating all the x_s then x would not be $\star$-dominated by any element of A. ∎

REMARK 2.5. We do not know whether in Theorem 2.1 the $\mathbf{\Delta}^1_1$ hypothesis on the order can be relaxed and replaced by $\mathbf{\Sigma}^1_1$. We were able to do this for orders $\prec$ on ω^ω which satisfy the following hypothesis:

(H_0): "The order $\prec$ is coarser than the canonical order $\le$",

that is:

$$x\le y \ \Rightarrow\ x\prec y.$$

Notice that this condition is stronger than (H) and is meaningless when the base space E is not ω^ω. Fix any filter $\mathcal{F}$ on ω and define an order $\le^{\mathcal{F}}$

on ω^ω by

$$x \leq^{\mathcal{F}} y \Leftrightarrow \{n \in \omega : x(n) \leq y(n)\} \in \mathcal{F}.$$

Clearly, the canonical orders $\leq$ and $\leq^\star$ can be defined according to this scheme by taking for $\mathcal{F}$ the trivial filter $\{\omega\}$ in the case of $\leq$, and the Fréchet filter in the case of $\leq^\star$. It is clear that such an order always satisfies (H_0). Moreover, the relation $\leq^{\mathcal{F}}$ has the same descriptive complexity as $\mathcal{F}$. In particular, if the filter $\mathcal{F}$ is $\mathbf{\Sigma}^1_1$ then the following result applies to $\leq^{\mathcal{F}}$.

THEOREM 2.6. *Let $\prec$ be a $\mathbf{\Sigma}^1_1$ order on ω^ω which is coarser than the canonical order $\leq$. Then any $\mathbf{\Sigma}^1_1$ cofinal subset for $\prec$ is continuously cofinal.*

Proof. The scheme of the proof is essentially the same as the previous one. We fix a topological embedding j of $\omega^\omega \times \omega^\omega \times \omega^\omega \times \omega^\omega$ onto a $\mathbf{\Pi}^0_2$ subset H of $\mathbf{2}^\omega$, a closed subset F of $\omega^\omega \times \omega^\omega$, and a closed subset R of $\omega^\omega \times \omega^\omega \times \omega^\omega$ such that

$$(x \in A \Leftrightarrow \exists \alpha,\ (x,\alpha) \in F) \quad \text{and} \quad (x \prec y \Leftrightarrow \exists \alpha',\ (x,y,\alpha') \in R).$$

Consider the game G' where, as in G, Player I chooses an integer, and Player II chooses 0 or 1. In a run of G' the players construct again an element (x,ε) in $\omega^\omega \times \mathbf{2}^\omega$; and Player II wins the run iff

$$\varepsilon \in H, \quad j^{-1}(\varepsilon) = (x',y,\alpha,\alpha'), \quad x \leq x', \quad (y,\alpha) \in F, \quad (x',y,\alpha') \in R.$$

This game is $\mathbf{\Sigma}^0_2$, hence determined.

As for the game G, a winning strategy for Player II in G' defines a continuous mapping $x \mapsto (x',y,\alpha,\alpha')$ from ω^ω into $\omega^\omega \times \omega^\omega \times \omega^\omega \times \omega^\omega$ with $x \leq x'$, $x' \prec y$ and $y \in A$; and by the hypothesis that $\prec$ is coarser than $\leq$ we also have $x \prec y$. Then the second component of this mapping $x \mapsto y$ is a continuous domination function with range in A.

Also, if σ is a strategy for Player I in G' then the set of all responses of σ is a compact subset K in ω^ω. Then K is bounded for $\leq$ and hence for $\prec$ by some element $b' \in \omega^\omega$, and since A is cofinal for $\prec$ we can find $b \in A$ such that $b' \prec b$; so the hypothesis that $\prec$ is coarser than $\leq$ implies that $x \prec b$ for all $x \in K$. One can finish the argument as in the previous proof by fixing β and β' in ω^ω such that $(b,\beta) \in F$ and $(b',b,\beta') \in R$, and then by considering the run where Player II plays $j(a,b,\alpha,\beta)$ and Player I follows σ, to show that σ is not winning. ∎

The results of this section extend clearly to larger classes of sets than $\mathbf{\Sigma}^1_1$, if one assumes enough determinacy. For example, assuming $\mathrm{Det}(\mathbf{\Sigma}^1_1)$, one can prove that if E is a $\mathbf{\Pi}^1_1$ zero-dimensional space and $\prec$ is an order on E for which any compact subset is bounded, then any $\mathbf{\Sigma}^1_2$ cofinal subset of E is continuously cofinal. In fact, one can easily check that the game defined in the proof of Theorem 2.1 is then $\mathbf{\Sigma}^1_1$, hence determined. In particular, assuming $\mathrm{Det}(\boldsymbol{\Sigma}^1_1)$, if X is a $\boldsymbol{\Pi}^1_1$ subset of ω^ω then any $\boldsymbol{\Sigma}^1_2$ cofinal subset of $\mathcal{K}(X)$ is continuously cofinal.

3. A Π^1_1 cofinal subset in ω^ω with no $\boldsymbol{\Sigma}^1_1$ cofinal subset

THEOREM 3.1. *Suppose that the set $\omega^\omega \cap L$ is not $\star$-bounded in ω^ω. Then there exists a Π^1_1 cofinal subset of ω^ω containing no $\boldsymbol{\Sigma}^1_1$ $\star$-cofinal subset.*

Proof. We fix some recursive embedding $j : \omega^\omega \to \mathbf{2}^\omega$ and define a mapping $\varphi : \omega^\omega \times \omega^\omega \to \omega^\omega$ by

$$\varphi(x, y)(n) = 2x(n) + j(y)(n)$$

for all $n \in \omega$. One checks easily that φ is also a recursive embedding. However, with respect to cofinality and $\star$-cofinality of sets, φ behaves very much like the canonical projection $\pi : \omega^\omega \times \omega^\omega \to \omega^\omega$ onto the first factor ω^ω:

FACT 3.1.1. *For any subset A of $\omega^\omega \times \omega^\omega$ we have the following equivalences*:

(a) ($\varphi(A)$ *is cofinal in* ω^ω) $\Leftrightarrow$ ($\pi(A)$ *is cofinal in* ω^ω),
(b) ($\varphi(A)$ *is $\star$-cofinal in* ω^ω) $\Leftrightarrow$ ($\pi(A)$ *is $\star$-cofinal in* ω^ω).

Proof. Suppose that $\varphi(A)$ is cofinal. Then for all $x \in \omega^\omega$ there exists $(y, z) \in A$ such that $2x \le 2y + j(z)$, hence $x \le y$ and $y \in \pi(A)$.

Conversely, suppose that $\pi(A)$ is cofinal. Then for all $x \in \omega^\omega$ there exists $(y, z) \in A$ such that $x \le y$, hence $x \le y \le \varphi(y, z) = y'$, so $x \le y'$ and $y' \in \varphi(A)$.

The proof of (b) is similar. ■

Denote by C the largest Π^1_1 thin subset of $\omega^\omega \times \omega^\omega$. We recall that

$$C = \{(x, y) \in \omega^\omega \times \omega^\omega : (x, y) \in L_{\omega_1^{(x,y)}}\}.$$

FACT 3.1.2. $\pi(C) = \omega^\omega \cap L$.

Proof. The inclusion $\subset$ follows from the classical fact that $C \subset L$. Conversely, let $x \in \omega^\omega \cap L$, and fix some $\xi < \omega_1$ such that $x \in L_\xi$. Consider now the largest Π^1_1 thin subset of ω^ω, that is, the set

$$C_1 = \{y \in \omega^\omega : y \in L_{\omega_1^y}\},$$

and fix a point y in $C_1 \setminus L_\xi$; such a point exists since $C_1 \in L$, L_ξ is countable in L and C_1 is uncountable in L. Thus $y \in L_{\omega_1^y}$ and since $y \notin L_\xi$ we have $\xi < \omega_1^y$. Hence $x \in L_{\omega_1^y}$ and $(x, y) \in L_{\omega_1^y} \subset L_{\omega_1^{(x,y)}}$; so $x \in \pi(C)$. ■

Let

$$B = \{(x, y) \in \omega^\omega \times \varphi(C) : \neg(y \le^\star x)\}.$$

FACT 3.1.3. $\pi(B) = \omega^\omega$.

Proof. By the previous Fact we have $L \vdash$ "$\pi(C) = \omega^\omega$", so by Fact 3.1.1 we also have $L \vdash$ "$\varphi(C)$ is cofinal in ω^ω". That is,

$$\forall x \in \omega^\omega \cap L,\ \exists y \in \varphi(C), \qquad x \le y$$

and from the assumption that $\omega^\omega \cap L$ is not $\star$-bounded in ω^ω, it follows that $\varphi(C)$ is not $\star$-bounded in ω^ω, which means that $\pi(B) = \omega^\omega$. ■

End of the proof of Theorem 3.1. Let $A = \varphi(B)$; then the set A is Π^1_1, and cofinal by Fact 3.1.1. Again by this Fact, to show that A contains no $\mathbf{\Sigma}^1_1$ $\star$-cofinal subset, it is enough to show that for any $\mathbf{\Sigma}^1_1$ subset B' of B, the set $\pi(B')$ is not $\star$-cofinal in ω^ω.

Fix such a B'. Since $\varphi(C)$ is thin, B' is of the form $B' = \bigcup_n B'_n \times \{y_n\}$. Consider a point y in ω^ω which $\star$-dominates all points y_n. We now prove that y cannot be $\star$-dominated by any point from $\pi(B')$, which will show that $\pi(B')$ is not $\star$-cofinal in ω^ω. So suppose that $y \le^\star x$ for some $x \in B'_n$. Then we would have $y_n \le^\star y \le^\star x$ and this contradicts the fact that $(x, y_n) \in B$. ■

It is clear that the previous construction is uniform and one can prove the following parametrized version:

THEOREM 3.2. *Suppose that for some $\alpha \in \omega^\omega$ the set $\omega^\omega \cap L(\alpha)$ is not $\star$-bounded in ω^ω. Then there exists a $\Pi^1_1(\alpha)$ cofinal subset of ω^ω containing no $\mathbf{\Sigma}^1_1$ $\star$-cofinal subset.*

4. When any $\mathbf{\Sigma}^1_2$ cofinal set contains $\mathbf{\Sigma}^1_1$ cofinal sets

THEOREM 4.1. *Assume that for all $\alpha \in \omega^\omega$ the set $\omega^\omega \cap L(\alpha)$ is $\star$-bounded in ω^ω. Let E be a Polish space endowed with some Borel order $\prec$ for which any compact subset is bounded. Then any $\mathbf{\Sigma}^1_2$ cofinal subset of E contains a $\mathbf{\Sigma}^1_1$ cofinal subset.*

As in the proof of Theorem 2.1 we can reduce the general case to the particular one where $E = \omega^\omega$, and for the latter we shall prove the following effective version:

THEOREM 4.2. *Assume that for some α the set $\omega^\omega \cap L(\alpha)$ is $\star$-bounded in ω^ω. Let E denote the set ω^ω endowed with some $\Delta^1_1(\alpha)$ order $\prec$ for which any compact subset is bounded. Then any $\Sigma^1_2(\alpha)$ cofinal subset of E contains a $\mathbf{\Sigma}^1_1$ cofinal subset.*

Proof. Since all the arguments are uniform we suppose that $\alpha = 0$.

Let A be a Σ^1_2 cofinal subset of E. We prove that A is continuously cofinal (this formally stronger statement is in fact equivalent to the conclusion by Theorem 2.1). We fix in L a tree T on $\omega \times \omega_1$ satisfying

$$x \in A \Leftrightarrow \exists \alpha \in \omega_1^\omega,\ (x, \alpha) \in [T].$$

Consider the following game G in which the moves of each player are indexed by ω separately:

• Player I starts the run. At move n he chooses an integer k_n. Thus in a run Player I defines an infinite sequence $x = (k_n) \in \omega^\omega$, to which we refer as *the sequence constructed by Player I.*

• At each of his moves Player II has two possibilities: He can either pass or choose an element $(m, \xi) \in \omega \times \omega_1$. Thus in a run Player II defines a (finite or infinite) sequence (j_n) in ω giving an increasing enumeration of the moves where he did not pass. If (m_n, ξ_n) denotes the choice he made at move j_n, then we refer to the (finite or infinite) sequence $(y, \alpha) = (m_n, \xi_n)$ as *the sequence constructed by Player II.* Notice that the sequence (y, α) is infinite iff infinitely many times Player II did not pass.

Player II wins the run iff

(y, α) is an infinite branch of T, and $x \prec y$.

These conditions define clearly a Borel, hence determined, game on $\omega \times \omega_1$.

FACT 4.2.1. *If Player II has a winning strategy in G then the set A is continuously cofinal in ω^ω.*

Proof. As in the proof of Theorem 2.1 a winning strategy for Player II in G defines a continuous mapping $x \mapsto (y, \alpha) \in [T]$, with $x \prec y$; and its first component $x \mapsto y$ is a continuous domination function with range in A. ∎

FACT 4.2.2. *In the game G Player I has no winning strategy belonging to L.*

Proof. Let $\sigma \in L$ be an arbitrary strategy for Player I. If $x \in \omega^\omega$ is a sequence constructed by Player I in a run compatible with σ, we say that x is the *response* by σ in this run.

Fix $a^\star \in \omega^\omega$ such that $x \le^\star a^\star$ for all $x \in \omega^\omega \cap L$; such an $a^\star$ exists by hypothesis. Let $a^\emptyset$ be the response by σ in the run where Player II has passed at each move. Finally, let a' be the supremum for the canonical order $\le$ of the pair $\{a^\emptyset, a^\star\}$. By asssumption on $\prec$, we can find in E an element which dominates the compact set $\{x \in \omega^\omega : x \le a'\}$; and since A is cofinal in E we can find such an element in A. So we fix $b \in A$ such that

$$\forall x \in \omega^\omega, \qquad (x \le a' \Rightarrow x \prec b)$$

and then we fix $\beta \in \omega_1^\omega$ such that $(b, \beta) \in [T]$.

Our plan is to define a run of the game G (not necessarily in L) compatible with σ, in which Player II will construct the infinite sequence (b, β) and Player I will construct a sequence $a \le a'$. By the choice of b we will have $a \prec b$ and such a run would be won by Player II, proving that the strategy σ is not winning.

Let us say that a run of the game G is *compatible* with (b,β) if the sequence constructed by Player II in this run is an initial segment of (b,β), possibly infinite and hence equal to (b,β). Denote by U the set of all finite runs u of G of even length and satisfying the following conditions:

(1) $u=\emptyset$, or at the last move in u Player II did not pass.
(2) u is compatible with σ.
(3) u is compatible with (b,β).

Let $u\in U$ have length $2p$. For all $q>p$ we define the finite run $u^{(q)}\in U$ of length $2q$ and extending u, as follows: At all the moves strictly between p and q Player II passes; at the (last) move q of $u^{(q)}$, Player II chooses $(b(0),\beta(0))$ if $u=\emptyset$, otherwise if at the move p he chose $(b(n-1),\beta(n-1))$ then at the move q he chooses $(b(n),\beta(n))$. We also define the infinite run $\widetilde{u}$ extending u, compatible with σ, in which Player II has passed at all moves after p, and denote by $\sigma(\widetilde{u})$ the response by σ in this infinite run; notice that since $\sigma\in L$, both $\widetilde{u}$ and $\sigma(\widetilde{u})$ are in L. Finally, if $v=u^{(q)}$ then for simplicity we denote $\widetilde{v}$ by $\widetilde{u}^{(q)}$.

By the continuity of the strategy σ, for any $u\in U$ we have $\sigma(\widetilde{u})=\lim_q\sigma(\widetilde{u}^{(q)})$. In particular, the set $A_u=\{\sigma(\widetilde{u}^{(q)}) : 2q>|u|\}$ is relatively compact in ω^ω, and so admits a supremum a_u relative to the canonical order $\le$. Since $A_u\in L$ we have $a_u\in\omega^\omega\cap L$, hence $a_u\le^\star a^\star\le a'$ and so

$$a_u\le^\star a'.$$

Notice that $a^\emptyset$ introduced at the beginning of the proof is just $\sigma(\widetilde{\emptyset})$, the response by σ to the run in which Player II passed at all moves. On the other hand, $a_\emptyset$ is the supremum of all responses by σ in all runs in which Player II passed at all moves but one, at which he played $(b(0),\beta(0))$. Thus $a^\emptyset\le a_\emptyset\le^\star a'$ but in general these elements are different. However, by the choice of a' we also have

$$a^\emptyset\le a'.$$

We now define by induction on $n\in\omega$ a sequence (u_n) in U and a sequence (q_n) in ω as follows:

(0) $u_0=\emptyset$ and $q_0=0$.
(1) $\forall j>q_{n+1},\ a_{u_n}(j)\le a'(j)$.
(2) $u_{n+1}=u_n^{(q_{n+1})}$.

Condition (0) defines (u_0,q_0). Suppose that (u_n,q_n) is defined. Then by the previous remarks we can find q_{n+1} satisfying (1); moreover, we can assume that $2q_{n+1}>|u_n|$ so that we can define u_{n+1} by (2).

Let $\overline{u}$ be the unique infinite run such that $u_n\prec\overline{u}$ for all n, and let a be the response by σ in $\overline{u}$, which, by definition of U, is compatible with σ.

Then

$$\forall j \le q_1, \quad a(j) = a^{\emptyset}(j) \le a'(j)$$

and, by condition (1) above, for all $n \in \omega$ we have

$$\forall j \in [q_{n+1}, q_{n+2}),\ a(j) = \sigma(\widetilde{u}_{n+1})(j) = \sigma(\widetilde{u}_n^{(q_{n+1})})(j) \le a_{u_n}(j) \le a'(j).$$

Thus $\overline{u}$ is a run compatible with σ and it also follows from the definition of U that in this run Player II has constructed the infinite sequence $(b, \beta) \in [T]$, whereas Player I has constructed a. Since $a \le a'$, we have $a \prec b$ and the run is won by Player II. This finishes the proof of Fact 4.2.2. ■

It is clear that Theorem 4.2 is an immediate consequence of 4.2.1 and 4.2.2 and the following Fact:

FACT 4.2.3. *In the game* G, *the winning Player has a winning strategy in* L.

We recall that the game G is determined in the universe, since it is Borel. What is meant by the statement of the previous Fact is that one of the players has a strategy belonging to L and winning in the universe. Such an absoluteness property, which is standard for closed games (on any set κ), or for Δ^1_1 games on ω, deserves some justification here since G is not of one of these forms.

To prove Fact 4.2.3 we need to reformulate the game G in a more standard way: A run in G will now be viewed as an infinite sequence $(x, y, z, \alpha) \in (\omega \times \omega \times 2 \times \omega_1)^\omega$, where:

- $x(n) \in \omega$ is the integer chosen by Player I at his nth move.
- If Player II did not pass at his nth move, then $z(n) = 1$ and $(y(n), \alpha(n)) \in \omega \times \omega_1$ is the element chosen by Player II at this move.
- If Player II passed at his nth move, then $x(n) = y(n) = z(n) = 0$.

Notice that in this new representation the sequence (y, α) is always infinite, and the sequence constructed by Player II in such a run is just the (finite or infinite) subsequence of (y, α) corresponding to the indices n for which $z(n) = 1$, wheras x is, as before, the sequence constructed by Player I. In this setting Player II wins the run iff

$$(y, \alpha) \in [T'] \quad \text{and} \quad (x, y, z) \in B$$

where $T' \in L$ is a tree on $\omega \times \omega_1$ that one can extract simply from T; and B is the Δ^1_1 set of all $(x, y, z) \in (\omega \times \omega \times 2)^\omega$ such that $x \prec y$, and $z(n) = 1$ for infinitely many indices n. The main point in this condition is that the definition of the Borel set B involves only the coordinates in ω^ω.

Now it is clear that Fact 4.2.3 is a particular case of the following general result, by which the proof of Theorem 4.2 is completed. ■

Proposition 4.3. *Let κ be an ordinal, $T \in L$ a tree on $\kappa \times \omega$, and B a Δ^1_1 subset of ω^ω. Let Γ be the Borel game on $\kappa \times \omega$ in which the win set for Player II is the set $A = [T] \cap (\kappa^\omega \times B)$. Then in Γ one of the players has a winning strategy in L.*

For κ and κ' two ordinals, S a tree on $\kappa \times \kappa'$ and $r \in \kappa^\omega$ we denote by $S(r)$ the tree on κ' defined by

$$\forall n \in \omega,\ \forall t \in \kappa'^n, \qquad (t \in S(r) \Leftrightarrow (r_{|n}, t) \in S).$$

Lemma 4.4. *Let κ and κ' be two ordinals, and $S \in L$ be a tree on $\kappa \times \kappa'$. Let J denote one of the players I or II, let Γ be the game on κ defined by*

$$\forall r \in \kappa^\omega, \qquad (\textit{Player } J \textit{ wins the run } r) \Leftrightarrow (\textit{The tree } S(r) \textit{ is well founded})$$

and let Γ_L be the game in L defined by the same win condition for Player J. Then any strategy for Player J which is winning in the game Γ_L is still winning in the game Γ.

Proof. Denote by J$'$ the opponent Player of J. For any strategy σ for Player J in Γ consider the tree

$$S_\sigma = \{(r, r') \in S : r \text{ is a finite run compatible with } \sigma\}.$$

Then σ is not winning in Γ iff there exists an infinite run r compatible with σ and won by J$'$, and by the win condition this means that the tree $S(r)$ is not well founded. Since the run r is compatible with σ we have $S(r) = S_\sigma(r)$. Thus σ is not winning in Γ iff the tree $S_\sigma(r)$ is not well founded, or equivalently,

$$(\sigma \text{ is winning in } \Gamma) \Leftrightarrow (\text{The tree } S_\sigma \text{ is well founded}).$$

If moreover $\sigma \in L$ then $S_\sigma \in L$ and the previous equivalence stated in L gives

$$(\sigma \text{ is winning in } \Gamma_L) \Leftrightarrow (\text{The tree } S_\sigma \text{ is well founded in L}).$$

The conclusion of the lemma follows from the absoluteness of the formula "The tree S_σ is well founded". ■

Proof of Proposition 4.3. We show that Γ is of the form considered in Lemma 4.4 with $\mathrm{J} = \mathrm{I}$ and $\mathrm{J} = \mathrm{II}$, by defining two suitable trees S^{I} and S^{II} in L satisfying the hypothesis of the lemma.

Fix first in L two trees T_1 and T_2 on $\omega \times \omega$ such that for $x \in \omega^\omega$ we have

$$(x \in B \Leftrightarrow \exists y \in \omega^\omega,\ (x, y) \in [T_1]) \quad \text{and} \quad (x \notin B \Leftrightarrow \exists b \in \omega^\omega,\ (x, y) \in [T_2]).$$

Fix in L a one-to-one mapping $\xi \mapsto s_\xi$ from $\kappa \setminus \{0\}$ onto $(\kappa \times \omega)^{<\omega} \setminus T$. Finally, for any finite sequence t, denote by $t_\star$ and $t^\star$ the left and right shift on t defined by

$$\begin{aligned}
&|t_\star| = |t^\star| = |t| - 1,\\
&t_\star(j) = t(j) \qquad \forall j < |t| - 2,\\
&t^\star(j) = t(j+1) \quad \forall j < |t| - 2.
\end{aligned}$$

Now we define S^{I} and S^{II} by

$$S^{\mathrm{I}} = \{(u, v, w) \in (\kappa \times \omega \times \omega)^{<\omega} : (u, v) \in T \text{ and } (v, w) \in T_1\}$$

and

$$\begin{aligned}
S^{II} = \{(u, v, w) \in (\kappa \times \omega \times \kappa)^{<\omega} : (w(0) = 0 \Rightarrow (v_\star, w^\star) \in T_2)\\
\text{and } (w(0) = \xi > 0 \Rightarrow (s_\xi \text{ is compatible with } (u, v))\}.
\end{aligned}$$

We use the following easily checked Fact:

FACT 4.3.1. *For* $(\overline{u}, \overline{v}) \in (\kappa \times \omega)^\omega$ *we have*:

$$\begin{aligned}
&((\overline{u}, \overline{v}) \in A) \Leftrightarrow (\exists \overline{w} \in \omega^\omega,\ (\overline{u}, \overline{v}, \overline{w}) \in S^{\mathrm{I}}),\\
&((\overline{u}, \overline{v}) \notin A) \Leftrightarrow (\exists \overline{w} \in \kappa^\omega,\ (\overline{u}, \overline{v}, \overline{w}) \in S^{\mathrm{II}}).
\end{aligned}$$

Applying this Fact we have:

$$\begin{aligned}
&(\text{Player I wins the run } r) \Leftrightarrow (\text{The tree } S^{\mathrm{I}}(r) \text{ is well founded}),\\
&(\text{Player II wins the run } r) \Leftrightarrow (\text{The tree } S^{\mathrm{II}}(r) \text{ is well founded}).
\end{aligned}$$

Consider now the game $\varGamma_L$ in L in which the win condition for Player II is defined by the first equivalence. Then by absoluteness the win condition for Player II is also defined by the second equivalence. Thus we can apply Lemma 4.4 for both players.

It also follows from the previous observations that the win set for Player II in $\varGamma_L$ is the set $A \cap L = [T] \cap (\omega_1^\omega \times B)$, and the game $\varGamma_L$ is also Borel in L (by absoluteness of the relation "B is a Borel subset of ω^ω"), hence $\varGamma_L$ is determined in L. So there exists a winning strategy for one of the players in $\varGamma_L$, and applying Lemma 4.4 for this player we see that this strategy is also winning in $\varGamma$. ∎

We have the following extension for $\mathbf{\Sigma}^1_1$ orders on ω^ω satisfying the hypothesis (H_0) introduced in 2.5:

THEOREM 4.5. *Assume that the set* $\omega^\omega \cap L$ *is* $\star$*-bounded in* ω^ω. *Let* $\prec$ *be a* $\varSigma^1_1$ *order on* ω^ω *which is coarser than the canonical order* $\le$. *Then any* $\varPi^1_1$ *cofinal subset for* $\prec$ *contains a* $\mathbf{\Sigma}^1_1$ *cofinal subset.*

Proof. As for Theorem 2.4 the proof is an adaptation of the arguments of the proof of the previous result. Fix in L a tree T on $\omega \times \omega_1$ and a tree S on $\omega \times \omega \times \omega$ such that

$(x \in A \Leftrightarrow \exists \alpha \in \omega_1^\omega,\ (x,\alpha) \in [T])$ and

$$(x \prec y \Leftrightarrow \exists \alpha' \in \omega^\omega,\ (x,y,\alpha') \in [S])$$

and define the game G' in which, similarly to G,

• Player I constructs an infinite sequence $x \in \omega^\omega$.

• Player II has the possibility either to pass, or to choose an element $(m', m, \xi, \xi') \in \omega \times \omega \times \omega_1 \times \omega_1$, constructing thus a (finite or infinite) sequence $(x', y, \alpha, \alpha') \in (\omega \times \omega \times \omega_1 \times \omega_1)^\omega$.

Player II wins the run iff the sequence (x', y, α, α') is infinite and

$$(y,\alpha) \in [T], \quad x \leq x', \quad (x', y, \alpha') \in [S].$$

This game is also Borel, and even $\mathbf{\Sigma}_2^0$, and the three Facts about the game G, established in the proof of Theorem 4.2 above, still hold for the game G'. The proofs are essentially the same for the first and third Fact. Only the proof of the second Fact requires minor modifications. We leave the details to the reader. ∎

5. The size of some sets of constructible reals

THEOREM 5.1. *Let $(E,\prec)$ denote one of the ordered spaces $(\omega^\omega,\leq)$, $(\omega^\omega,\leq^\star)$, $(\mathcal{K}(\omega^\omega),\subset)$. Then, for every $\alpha \in \omega^\omega$, the following are equivalent*:

(i) *The set $\omega^\omega \cap L(\alpha)$ is $\star$-bounded in ω^ω.*

(ii) *Any $\Sigma_2^1(\alpha)$ cofinal subset of E contains a closed cofinal subset.*

(iii) *Any $\Pi_1^1(\alpha)$ cofinal subset of E contains a $\mathbf{\Sigma}_1^1$ cofinal subset.*

Proof. (i)⇒(ii). Apply Theorems 4.2 and 2.4.

(ii)⇒(iii). Obvious.

(iii)⇒(i). We argue by contradiction; so suppose non-(i), that is, the set $\omega^\omega \cap L(\alpha)$ is not $\star$-bounded in ω^ω. By Theorem 3.1 there exists a $\Pi_1^1(\alpha)$ cofinal (for $\leq$) subset A of ω^ω which contains no $\mathbf{\Sigma}_1^1$ $\leq^\star$-cofinal subset. In particular, the same $\Pi_1^1(\alpha)$ set A is cofinal with no $\mathbf{\Sigma}_1^1$ cofinal subset; and is also $\leq^\star$-cofinal with no $\mathbf{\Sigma}_1^1$ $\star$-cofinal subset. This proves non-(iii) when $\leq$ is one of the canonical orders $\leq$ or $\leq^\star$ on ω^ω.

For any $x \in \omega^\omega$ let $\theta(x) = \{y \in \omega^\omega : y \leq x\}$. Then the mapping $\theta : \omega^\omega \to \mathcal{K}(\omega^\omega)$ is clearly a one-to-one Δ_1^1 embedding of ω^ω onto a cofinal subset of $\mathcal{K}(\omega^\omega)$. Moreover, θ is increasing for these ordered spaces. It then follows that the set $\theta(A) = \{\theta(x) : x \in A\}$ is also a $\Pi_1^1(\alpha)$ cofinal subset of $\mathcal{K}(\omega^\omega)$ with no $\mathbf{\Sigma}_1^1$ cofinal subset. This proves non-(iii) when $(E,\prec) = (\mathcal{K}(\omega^\omega),\subset)$. ∎

THEOREM 5.2. *Let $(E,\prec)$ denote one of the ordered spaces*: $(\omega^\omega,\leq)$, $(\omega^\omega,\leq^\star)$, $(\mathcal{K}(X),\subset)$ *with X a Polish non-$\mathbf{K}_\sigma$ space. Then the following are equivalent*:

(i) *For all* $\alpha \in \omega^\omega$ *the set* $\omega^\omega \cap L(\alpha)$ *is* $\star$*-bounded in* ω^ω*.*
(ii) *Any* $\mathbf{\Sigma}^1_2$ *cofinal subset of* E *contains a closed cofinal subset.*
(iii) *Any* $\mathbf{\Pi}^1_1$ *cofinal subset of* E *contains a* $\mathbf{\Sigma}^1_1$ *closed subset.*

Proof. When $(E, \prec)$ is one of the ordered spaces $(\omega^\omega, \leq)$, $(\omega^\omega, \leq^\star)$, $(\mathcal{K}(\omega^\omega), \subset)$, the result follows from Theorem 5.1 by relativization.

Suppose that $(E, \prec)$ is $(\mathcal{K}(X), \subset)$ with X a general Polish non-$\mathbf{K}_\sigma$ space. Then, as in the previous proof, (i)$\Rightarrow$(ii) follows from Theorems 4.1 and 2.4, and (ii)$\Rightarrow$(iii) is obvious. We now prove (iii)$\Rightarrow$(i).

Notice that since the Polish space X is not $\mathbf{K}_\sigma$, by the Hurewicz Theorem we can find in X a closed subset X' homeomorphic to ω^ω. Consider then the mapping $\Phi : \mathcal{K}(X) \to \mathcal{K}(X')$ defined by $\Phi(K) = K \cap X'$. Since Φ is increasing and onto, it is easy to check that the direct image under Φ of any cofinal subset of $\mathcal{K}(X)$ is a cofinal subset of $\mathcal{K}(X')$. We now show that the inverse image under Φ of any cofinal B subset of $\mathcal{K}(X')$ is also a cofinal subset of $\mathcal{K}(X)$. Let

$$A = \Phi^{-1}(B) = \{K \in \mathcal{K}(X) : K \cap X' \in B\}.$$

Then for any $K \in \mathcal{K}(X)$ we can find $K' \in B$ such that $K \cap X' \subset K' \subset X'$, hence

$$(K \cup K') \cap X' = (K \cap X') \cup (K' \cap X') = K'.$$

Since $K' \in B$ this shows that $K'' = K \cup K' \in A$; but obviously $K \subset K''$.

So suppose non-(i). Then by the case $X = \omega^\omega$ we can find a $\mathbf{\Pi}^1_1$ cofinal subset B of $\mathcal{K}(X')$ with no $\mathbf{\Sigma}^1_1$ cofinal subset. Since Φ is clearly Borel, $A = \Phi^{-1}(B)$ is $\mathbf{\Pi}^1_1$, and by the previous remarks A is also cofinal in $\mathcal{K}(X)$ and contains no $\mathbf{\Sigma}^1_1$ cofinal subset. This proves non-(iii) in the general case. ∎

Remark 5.3. If in the previous statement the Polish space X is $\mathbf{K}_\sigma$, we still have (i)$\Rightarrow$(ii)$\Rightarrow$(iii). But (iii)$\Rightarrow$(i) is false. For example, if X is compact then (iii) always holds since any cofinal subset contains a cofinal subset which is a singleton, namely $\{X\}$!

5.4. *Closed sets with code in L.* Let F be a closed subset of ω^ω. By a *code* for F we mean any tree T on ω such that $F = [T]$; if such a tree T exists in L (i.e. if $T \in L$) then we say that F *has a code in L*. We discuss here briefly the main properties of this notion that we shall use.

Notice that if F has a code in L then the *canonical* (in fact, minimal) code

$$\Sigma(F) = \{s \in \omega^{<\omega} : F \cap N_s \neq \emptyset\}$$

is also in L. This is shown by standard absoluteness arguments from the following equivalence:

$$s \in \Sigma(F) \Leftrightarrow \text{the tree } T \cap \Sigma_s \text{ is not well founded}$$

where Σ_s denotes the tree of all sequences extending s.

Obviously, if $F \in L$ then $\Sigma(F) \in L$, hence F has a code in L. But unless $V = L$ there are closed sets not in L having a code in L. For example, the whole space ω^ω and the compact set $\mathbf{2}^\omega$ have codes in L. A large class of such closed sets is given by the following:

If F is $\Sigma^1_1(\alpha)$ for some $\alpha \in \omega^\omega \cap L$, then F has a code in L.

In fact, in this case $\Sigma(F)$ is clearly a $\Sigma^1_1(\alpha)$ subset of $\omega^{<\omega}$ and by classical results $\Sigma(F) \in L(\alpha) = L$. In particular,

If F is Σ^1_1 then F has a code in L.

For any subset A of ω^ω we denote by $\mathcal{K}_L(A)$ the set of all compact subsets of A with code in L. Thus by the previous remarks

$$\mathcal{K}(A) \cap L \subset \mathcal{K}_L(A)$$

but unless $V = L$ the previous inclusion is always strict.

THEOREM 5.5. *The following are equivalent*:

(i) *The set $\omega^\omega \cap L$ is $\star$-bounded in ω^ω.*

(ii) *For any Π^0_2 subset A of ω^ω, there exists a countable subset $\mathcal{C}$ of $\mathcal{K}(A)$ such that*

$$\forall K \in \mathcal{K}_L(A),\ \exists K' \in \mathcal{C}, \qquad K \subset K'.$$

(iii) *For any Σ^0_2 subset A of ω^ω, there exists a $\mathbf{K}_\sigma$ subset B of A such that*

$$\forall K \in \mathcal{K}_L(A), \qquad K \subset B.$$

(iv) *For any Σ^1_1 subset A of ω^ω, there exists a $\mathbf{K}_\sigma$ subset B of A such that*

$$A \cap L \subset B.$$

Proof. (i)⇒(ii). Suppose first that $A = \omega^\omega$. Fix $a \in \omega^\omega$ such that $\omega^\omega \cap L$ is $\star$-dominated by a. For any $s \in \omega^{<\omega}$ let a_s denote the element of ω^ω defined by

$$a_s(j) = \begin{cases} s(j) & \text{if } j < |s|, \\ a_s(j) = a(j) & \text{if } j \geq |s|. \end{cases}$$

Then the subset

$$\mathcal{C}_0 = \{\{x \in \omega^\omega : x \leq a_s\} : s \in \omega^{<\omega}\}$$

of $\mathcal{K}(\omega^\omega)$ clearly satisfies (ii).

For the general case, fix a Δ^1_1 perfect mappping φ from some Π^0_1 subset F of ω^ω onto A and let

$$\mathcal{C} = \{\varphi(K \cap F) : K \in \mathcal{C}_0\}.$$

Then for any $K \in \mathcal{K}(A)$ with code T, the compact set $\varphi^{-1}(K)$ also has a code $\Delta^1_1(T)$. In particular, if $K \in \mathcal{K}_L(A)$ then $\varphi^{-1}(K) \in \mathcal{K}_L(\omega^\omega)$; and one easily derives property (ii) for $\mathcal{C}$ from the same property for $\mathcal{C}_0$.

(ii)⇒(iii). Suppose $A = \bigcup A_n$ with all the A_n's $\varPi^0_1$. For all n let $\mathcal{C}_n$ be a countable subset of $\mathcal{K}(A_n)$ satisfying (ii) for the $\varPi^0_2$ set A_n, and consider the $\mathbf{K}_\sigma$ subset

$$B = \bigcup_n \Big(\bigcup \mathcal{C}_n\Big)$$

of A. Since each A_n has a code in L, for any $K \in \mathcal{K}_L(A)$ the compact set $K \cap A_n$ also has a code in L, hence $K = \bigcup_n K \cap A_n \subset B$.

(iii)⇒(iv). First notice that applying (iii) to $A = \omega^\omega$ we can find a $\mathbf{K}_\sigma$ set B_0 such that $\omega^\omega \cap L \subset B_0 \subset \omega^\omega$.

If A is any $\varSigma^1_1$ subset of ω^ω, fix a continuous $\varDelta^1_1$ mapping φ in L from ω^ω onto A, and let $B = \varphi(B_0)$, which is also a $\mathbf{K}_\sigma$ set. Then by absoluteness of the statement

$$\forall y \in A,\ \exists x \in \omega^\omega, \qquad y = \varphi(x)$$

we have

$$A \cap L = \varphi(\omega^\omega \cap L) \subset \varphi(B_0) = B \subset A$$

and B is clearly $\mathbf{K}_\sigma$.

(iv)⇒(i). Since $\mathcal{K}_L(\omega^\omega)$ contains all singletons from $\omega^\omega \cap L$, applying (iv) to $A = \omega^\omega$ we see that $\omega^\omega \cap L$ is contained in some $\mathbf{K}_\sigma$ subset of ω^ω and hence is $\star$-bounded. ■

REMARKS 5.6. (a) If we suppose that "The set $\omega^\omega \cap L$ is countable" then (ii)–(iv) hold for any subset A of ω^ω without any descriptive hypothesis on A. In fact, in this case the set $\mathcal{K}_L(\omega^\omega)$ is also countable, hence we can realize (ii) with $\mathcal{C} = \mathcal{K}_L(A)$ and (iii) with $C = \bigcup \mathcal{C}$, which is $\mathbf{K}_\sigma$, whereas (iv) is obvious by taking $B = A \cap L$, which is now countable. But as we shall see through the following remarks, any minor strengthening of (ii), (iii) or (iv) implies that "The set $\omega^\omega \cap L$ is countable" and hence that (ii)–(iv) hold for any A.

(b) Assume that (iv) holds for any $\varPi^1_1$ subset A of $\omega^\omega \cap L$. Let A be the largest thin $\varPi^1_1$ subset of $\mathbf{2}^\omega$ and $B \subset A$ as in (iv). Since $A \subset L$, in this case we find that $A = B$ is $\mathbf{K}_\sigma$ hence countable, as A is thin. But it is a classical fact that such a set A is in one-to-one correspondence (even in L) with $\mathbf{2}^\omega \cap L$, which is then also countable. And since there is a $\varDelta^1_1$ embedding j of ω^ω into $\mathbf{2}^\omega$, $\omega^\omega \cap L = j^{-1}(\mathbf{2}^\omega \cap L)$ is countable as well.

(c) Assume that (ii) holds for $A = \mathbb{Q}$, some $\varDelta^1_1$ (homeomorphic) copy in ω^ω of the set of all rational numbers, and fix a countable family $\mathcal{C}$ satisfying (ii). Since the set $\mathcal{K}_L(\mathbb{Q})$ is $\mathbf{\Pi}^1_1$ complete, for any $\varPi^1_1$ set $A_0 \subset \mathbf{2}^\omega \cap L$ there exists a $\varDelta^1_1$ continuous mapping $\varPhi : \mathbf{2}^\omega \to \mathcal{K}_L(\omega^\omega)$ such that for all α,

$$\alpha \in A_0 \Leftrightarrow \varPhi(\alpha) \in A_0.$$

Moreover, since $A_0 \subset L$, for any $\alpha \in A_0$ the compact set $\varPhi(\alpha)$ is $\varDelta^1_1(\alpha)$ hence has a code in L; so there exists $C \in \mathcal{C}$ such that $\varPhi(\alpha) \subset C$. Conversely, such an inclusion obviously implies by the hypothesis on $\varPhi$ that $\alpha \in \mathbb{Q}$. Thus

$$\alpha \in A_0 \Leftrightarrow \exists C \in \mathcal{C},\ \varPhi(\alpha) \subset C.$$

This shows that any $\varPi^1_1$ subset A_0 of $\mathbf{2}^\omega \cap L$ is $\mathbf{K}_\sigma$, hence by (b) above that $\omega^\omega \cap L$ is countable.

(d) If A is a $\varDelta^1_1$ subset of ω^ω which is a $\mathbf{P}_\sigma$ space, then it follows from general structural results (see [5]) that one can write $A = A' \cap A''$ where A' is $\varPi^0_2(\alpha)$ and A'' is $\varSigma^0_2(\alpha)$ for some $\alpha \in \omega^\omega \cap L$. Then one can easily adapt the arguments of Theorem 5.5 to show that (ii) implies that condition (iii) holds for any set A satisfying the hypothesis above. But we shall see that this is the exact limit for extending (iii). We recall that a typical subset of $\omega^\omega \times \omega^\omega$ which is not a $\mathbf{P}_\sigma$ space is given by the complement of the product $\mathbb{P} \times \mathbb{Q}$, where $\mathbb{P} \subset \omega^\omega$ is a copy of the space ω^ω with empty interior and $\mathbb{Q}$ is as above a copy of the rationals.

(e) Assume that (iii) holds for some set A which is a $\varDelta^1_1$ (homeomorphic) copy of the space $\omega^\omega \times \omega^\omega \setminus \mathbb{P} \times \mathbb{Q}$ above. Then by the proof of Théorème 6 in [9] for any $\varPi^1_1$ set $A_0 \subset \mathbf{2}^\omega \cap L$ there exists a $\varDelta^1_1$ lower semi-continuous mapping $\varPhi : \mathbf{2}^\omega \to \mathcal{K}_L(\omega^\omega)$ such that for all α,

$$\begin{aligned}&\alpha \in A_0 \Rightarrow \varPhi(\alpha) \subset A,\\ &\alpha \notin A_0 \Rightarrow \varPhi(\alpha) \setminus A \text{ dense in } \varPhi(\alpha).\end{aligned}$$

Fix a $\mathbf{K}_\sigma$ set B such that $A \cap L \subset B \subset A$, and let $B = \bigcup_n K_n$ where all the K_n's are compact. Again for any $\alpha \in A_0$ the compact set $\varPhi(\alpha)$ has a code in L; so $\varPhi(\alpha) \subset B = \bigcup_n K_n \subset A$ and by Baire's Category Theorem there exist n and s such that $\emptyset \neq N_s \cap \varPhi(\alpha) \subset K_n$. On the other hand, if $\alpha \notin A_0$ then for all n the set $\varPhi(\alpha) \setminus K_n \supset \varPhi(\alpha) \setminus A$ is dense in $\varPhi(\alpha)$. Thus

$$\alpha \in A_0 \Leftrightarrow \exists n, s,\ \emptyset \neq N_s \cap \varPhi(\alpha) \subset K_n.$$

This again shows that any $\varPi^1_1$ subset A_0 of $\mathbf{2}^\omega \cap L$ is $\mathbf{K}_\sigma$, hence that $\omega^\omega \cap L$ is countable.

6. Compact covering and inductively perfect mappings. We recall the following notation for two separable metric spaces X and Y:

$\mathbb{A}(X, Y)$: "Any compact covering mapping $f : X \to Y$ is inductively perfect".

One can easily check that a mapping $f : X \to Y$ is compact covering (inductively perfect) iff the projection mapping from the graph of f (viewed as a subspace of $Y \times X$) is compact covering (inductively perfect); this reduces the general study of these notions to the particular case of projection mappings.

6.1. *Projection mappings.* A *projection mapping* is a mapping $\pi_X : X \to Y$ where X is a subset of some product space $Y \times Z$, π_X is the restriction to X of the canonical projection π and $Y = \pi(X) = \pi_X(X)$.

If $\mathcal{X}$ and $\mathcal{Y}$ are two classes of spaces, by a *projection* from $\mathcal{X}$ onto $\mathcal{Y}$ we mean a projection mapping with domain in $\mathcal{X}$ and range in $\mathcal{Y}$.

(a) *Reducing to zero-dimensional projections.* It follows from the previous remark that if the classes $\mathcal{X}$ and $\mathcal{Y}$ satisfy

($\star$) $\forall X \in \mathcal{X},\ \forall Y \in \mathcal{Y},\ \forall Z$ a closed subset of $X \times Y,\ Z \in \mathcal{X}$

then the following are equivalent:

(i) $\mathbb{A}(X, Y)$ $\forall X \in \mathcal{X},\ \forall Y \in \mathcal{Y}$.

(ii) Any compact covering projection from $\mathcal{X}$ onto $\mathcal{Y}$ is inductively perfect.

If moreover $\mathcal{X}$ and $\mathcal{Y}$ are descriptive classes then it follows from the the proof of [3], Theorem 3.1 that (i) and (ii) above are also equivalent to the statements obtained by restricting in (i) and (ii) all spaces to be zero-dimensional.

(b) *Inductively perfect projections.* Notice that in the study of a π_X for $X \subset Y \times Z$, we can always assume that the space Z is compact. In this case the following observation, which follows easily from the definitions, is very convenient and will be constantly used in the rest of this section to verify that a projection mapping is inductively perfect:

If Z is compact then π_X is perfect if and only if X is closed in $Y \times Z$.

(c) *Strongly compact covering projections.* We say that the projection mapping $\pi_X : X \to Y$ is *strongly compact covering* if for any compact subset K of Y there exists $z \in Z$ such that $K \times \{z\} \subset X$.

Observe that this notion is meaningless for mappings which are not projection mappings. As we shall see this strengthening of "compact covering" is linked to the following strengthening of "inductively perfect".

Let $X \subset Y \times Z$ and consider $\widetilde{X} \subset Y \times \mathcal{K}(X)$ defined by

$$\widetilde{X} = \{(y, H) \in Y \times \mathcal{K}(X) : y \in \pi(H)\}.$$

Then one can easily check that

$$\pi_X \text{ is compact covering} \Leftrightarrow \pi_{\widetilde{X}} \text{ is strongly compact covering},$$
$$\pi_X \text{ is inductively perfect} \Leftrightarrow \pi_{\widetilde{X}} \text{ is inductively perfect.}$$

(d) *Continuous sections.* By a *section* of a mapping $f : X \to Y$ we mean a mapping $g : Y \to X$ such that

$$\forall y \in Y, \quad f(g(y)) = y.$$

It is clear that if f has a continuous section then f is inductively perfect; but the converse is false.

The following variation of Ostrovskiĭ's Theorem can be proved in a totally similar way as the original result (see [7] or [4]).

THEOREM 6.2. *Any strongly compact covering projection onto a σ-compact space admits a continuous section.*

THEOREM 6.3. *For a $\mathbf{\Pi}_1^1$ zero-dimensional space Y, the following are equivalent*:

(i) *Any strongly compact covering projection from a $\mathbf{\Pi}_1^1$ space onto Y has a continuous section.*

(ii) *For any cofinal $\mathbf{\Pi}_1^1$ subset A of $\mathcal{K}(Y)$ there exists a continuous mapping $f : Y \to A$ satisfying $y \in f(y)$ for all $y \in Y$.*

Proof. (i)⇒(ii). Take A as in (ii) and consider the $\mathbf{\Pi}_1^1$ set

$$X = \{(y, H) \in Y \times A : y \in H\}.$$

If K is any compact subset of Y, then since A is cofinal there exists $H \in A$ such that $K \subset H$, so $K \times \{H\} \subset X$ and $f(K \times \{H\}) = K$. This proves that the projection mapping π_X is strongly compact covering, hence by (i) it admits a continuous section which clearly provides a continuous mapping $f : Y \to A$ satisfying (ii).

(ii)⇒(i). Let $\pi_X : X \to Y$ be a strongly compact covering projection of X a $\mathbf{\Pi}_1^1$ subset of $Y \times Z$ onto Y. We want to prove that π_X is inductively perfect. By Theorem 6.2 above we may and do assume that Y is not σ-compact. In particular, its Cantor second derivative Y'' is not compact. So fix an infinite partition (U_n) of Y'' into clopen (in Y'') nonempty sets. Then there is a partition (Y_n) of Y into clopen (in Y) subsets of Y such that $U_n = Y'' \cap Y_n$. Pick in each U_n some point a_n, and an infinite, one-to-one sequence $(a_n^{(k)})_{k\in\omega}$ of points of $Y_n \cap Y'$ converging to a_n. Then the set

$$F = \{a_n^{(k)} : k, n \in \omega\} \cup \{a_n : n \in \omega\}$$

is a closed subset of Y with empty interior.

Consider the mappings $\nu : \mathcal{K}(Y) \to \omega$ and $F_n : \mathbf{2}^\omega \to \mathcal{K}(Y)$ for $n \in \omega$ defined by

$$\nu(K) = \min\{n : \forall m \geq n,\ K \cap Y_m = \emptyset\},$$
$$F_n(\varepsilon) = \{a_n^{(k)} : k \in \omega \text{ with } \varepsilon(k) = 1\} \cup \{a_n\}$$

and define the mapping $\Phi : \mathcal{K}(Y) \times \mathbf{2}^\omega \to \mathcal{K}(Y)$ by

$$\Phi(K, \varepsilon) = K \cup F_{\nu(K)}(\varepsilon).$$

Notice that for all (K, ε) we have

$$K \subset \Phi(K, \varepsilon) \subset K \cup F.$$

FACT 6.3.1. *Φ is a homeomorphism from $\mathcal{K}(Y)\times\mathbf{2}^\omega$ onto a cofinal subset of $\mathcal{K}(Y)$*

Proof. Since F_n is clearly a homeomorphism from $\mathbf{2}^\omega$ onto a compact subset of $\mathcal{K}(Y)$ and ν is locally constant, Φ is continuous. Moreover, if $K'=\Phi(K,\varepsilon)$ then $\nu(K')$ is locally constant and the same holds for $N=\nu(K)=\nu(K')-1$; hence

$$\varepsilon=F_N^{-1}(K'\cap Y_N)\quad\text{and}\quad K=K'\cap\bigcup_{n\le N}Y_n.$$

This shows that Φ is one-to-one and since the Y_n's are clopen it follows from the last formula that Φ^{-1} is also continuous. ∎

We can obviously suppose that the projection of $X\subset Y\times Z$ on the second factor is the whole space Z, in particular that Z is a $\mathbf{\Sigma}^1_2$ space. Fix then a $\mathbf{\Pi}^1_1$ subset E of $\mathbf{2}^\omega$ and a continuous mapping $\varphi:E\to Z$ onto Z, and consider the $\mathbf{\Pi}^1_1$ set

$$A=\{K'=\Phi(K,\varepsilon)\in\mathcal{K}(Y):\varepsilon\in E\text{ and }K\times\{\varphi(\varepsilon)\}\subset X\}.$$

Since π_X is strongly compact covering, it follows from the inclusion $K\subset\Phi(K,\varepsilon)$ that A is a cofinal subset of $\mathcal{K}(Y)$.

Let $f:Y\to A$ be a continuous mapping as in (ii); we now define a continuous mapping $g:Y\to Z$ which is a section of π_X. Let $y\in Y$ and $f(y)=K'\in A$; put $K'=\Phi(K,\varepsilon)$ with $\varepsilon\in E$, and define $g(y)=\varphi(\varepsilon)\in Z$ and $H(y)=K$. Since Φ is a homeomorphism it is clear that the mappings $g:Y\to Z$ and $H:Y\to\mathcal{K}(Y)$ are continuous, and all we have to prove is that $(y,g(y))\in X$ for all $y\in Y$.

Notice that for all $y\in Y$ we have, with the previous notations,

$$y\in f(y)=H(y)\cup F_{\nu(K)}(\varepsilon);$$

in particular, $y\in H(y)$ if $y\in Y\setminus F$. Hence by the density of $Y\setminus F$ for all $y\in Y$ we have $y\in H(y)$; but since $H(y)\times\{g(y)\}\subset X$ it follows that $(y,g(y))\in X$. ∎

THEOREM 6.4. *The following are equivalent*:

(i) $\mathbb{A}(X,Y)$ $\forall X$ $\mathbf{\Pi}^1_1$, $\forall Y$ *Polish.*

(ii) *Any strongly compact covering projection from a $\mathbf{\Pi}^1_1$ space onto any Polish space has a continuous section.*

(iii) *For any Polish space Y and any cofinal $\mathbf{\Pi}^1_1$ subset A of $\mathcal{K}(Y)$ there exists a continuous mapping $f:Y\to A$ satisfying $y\in f(y)$ for all $y\in Y$.*

(iv) *For any Polish space Y and any cofinal $\mathbf{\Pi}^1_1$ subset A of $\mathcal{K}(Y)$ there exists a continuous mapping $F:\mathcal{K}(Y)\to A$ satisfying $K\subset F(K)$ for all $K\in\mathcal{K}(Y)$.*

Proof. (iv)⇒(iii) is obvious and (iii)⇒(ii) follows from Theorem 6.3.

(ii)⇒(i). Let Y be a Polish space and π_X be a compact covering projection from a $\mathbf{\Pi}^1_1$ space $X \subset Y \times Z$ onto Y, with Z compact. Consider the set

$$\widetilde{X} = \{(y,H) \in Y \times \mathcal{K}(X) : y \in \pi(H)\}.$$

If K is any compact subset of Y and H a compact subset of X such that $\pi(H) = K$ then obviously $K \times \{H\} \subset \widetilde{X}$. Hence the projection mapping $\pi_{\widetilde{X}}$ is strongly compact covering, so by (ii), $\pi_{\widetilde{X}}$ has a continuous section $g : Y \to \mathcal{K}(X)$. Then it is clear that the subset

$$H = \{(y,z) \in Y \times Z : z \in g(y)\}$$

of X is closed in $Y \times Z$ and that $\pi(H) = Y$.

(i)⇒(iv). Let Y be a Polish space and A be some $\mathbf{\Pi}^1_1$ cofinal subset of $\mathcal{K}(Y)$. Consider the $\mathbf{\Pi}^1_1$ set

$$X = \{(K,K') \in \mathcal{K}(Y) \times A : K \subset K'\}.$$

It is easy to see that the projection mapping π_X is compact covering (even strongly) onto $\mathcal{K}(Y)$, which is also Polish, hence by (i), π_X is inductively perfect. Let $H \subset X$ be a closed subset of $\mathcal{K}(Y) \times \mathcal{K}(Y)$ with total projection on the first factor $\mathcal{K}(Y)$; then the projection of H on the second factor is a $\mathbf{\Sigma}^1_1$ subset of A which is also cofinal. The conclusion then follows from Theorem 2.4. ■

THEOREM 6.5. *The following are equivalent*:

(i) $\mathbb{A}(X,Y)$ $\forall X$ $\mathbf{\Pi}^1_1$, $\forall Y$ $\mathbf{P}_\sigma$.

(ii) $\mathbb{A}(X,Y)$ $\forall X$ $\mathbf{\Pi}^1_1$, $\forall Y$ *Polish.*

(iii) $\mathbb{A}(X,Y)$ $\forall X$ $\mathbf{\Delta}^1_1$, $\forall Y$ *Polish.*

(iv) $\mathbb{A}(X,Y)$ $\forall X$ $\mathbf{P}_\sigma$, $\forall Y$ *Polish.*

Proof. Obviously, (i)⇒(ii)⇒(iii)⇒(iv). We shall not prove (iv)⇒(i) directly, but by showing (iv)⇒(ii)⇒(i), and we start with the second implication, which is simpler than the first.

(ii)⇒(i). Since the classes $\mathcal{X} = \mathbf{\Pi}^1_1$ and $\mathcal{Y} = \mathbf{P}_\sigma$ satisfy condition $(\star)$ of 6.1(a), it is enough to prove (i) for zero-dimensional projections.

So let $\pi_X : X \to Y$ $(X \subset Y \times Z)$ be a compact covering mapping where X is a $\mathbf{\Pi}^1_1$ space, and $Y = \pi(X)$ a $\mathbf{P}_\sigma$ space. We embed the spaces Y and Z in $\mathbf{2}^\omega$. Then there exists a $\mathbf{\Pi}^0_2$ subset $\widetilde{Y}$ of $\mathbf{2}^\omega$ containing Y as a $\mathbf{\Sigma}^0_2$ subset. Consider now the projection mapping $\pi_{\widetilde{X}}$ associated with the set

$$\widetilde{X} = X \cup (\widetilde{Y} \setminus Y) \times \mathbf{2}^\omega.$$

Then clearly $\widetilde{X}$ is $\mathbf{\Pi}^1_1$ and $\pi(\widetilde{X}) = \widetilde{Y}$ is Polish. Moreover, since $X = \widetilde{X} \cap (Y \times Z)$, to prove that π_X is inductively perfect it is enough to prove that

$\pi_{\widetilde{X}}$ is inductively perfect, so by (ii) it is enough to prove that $\pi_{\widetilde{X}}$ is compact covering.

So let K be any compact subset of $\widetilde{Y}$. Then $Y_0 = K \cap Y$ is σ-compact; moreover, the projection π_{X_0} from $X_0 = X \cap (Y_0 \times \mathbf{2}^\omega)$ onto Y_0 is still compact covering. Hence by Ostrovskiĭ's Theorem π_{X_0} is inductively perfect, and we can find a subset H_0 of X_0 of total projection $\pi(H_0) = Y_0$ which is closed in $Y_0 \times \mathbf{2}^\omega$.

Let H_1 denote the closure of H_0 in $K \times \mathbf{2}^\omega$; since H_0 is closed in $Y_0 \times \mathbf{2}^\omega$ it follows that

$$H_1 \subset H_0 \cup [(K \setminus Y_0) \times \mathbf{2}^\omega],$$

hence $H_1 \subset \widetilde{X}$. Let $K_1 = \pi(H_1)$; since $K_1 \supset K \cap Y$ we have $K \setminus K_1 \subset \widetilde{Y} \setminus Y$, hence

$$(K \setminus K_1) \times \mathbf{2}^\omega \subset \widetilde{X}.$$

Consider now

$$H = \{(y,z) \in K \times \mathbf{2}^\omega : \forall z' \in \mathbf{2}^\omega,\ d((y,z),H_1) \le d((y,z'),H_1)\}$$

where $d(a,A)$ denotes the distance from the point a to the set A for some fixed metric on $\mathbf{2}^\omega \times \mathbf{2}^\omega$. It is clear that H is a compact subset of $K \times \mathbf{2}^\omega$ with $\pi(H) = K$; moreover, $H \cap (K_1 \times \mathbf{2}^\omega) = H_1$ so that $H \subset \widetilde{X}$. This shows that π_X is compact covering and finishes the proof of (i).

(iv)$\Rightarrow$(ii). As in the previous implication, by 6.1(a) it is enough to prove (ii) for zero-dimensional projections.

So let $\pi_X : X \to Y$ (with $X \subset Y \times Z$) be a projection mapping where X is a $\mathbf{\Pi}^1_1$ space, $Y = \pi(X)$ a zero-dimensional Polish space that we embed as a $\mathbf{\Pi}^0_2$ subset of ω^ω, and Z a zero-dimensional compact space. We define another zero-dimensional projection $\pi_{\widetilde{X}} : \pi_{\widetilde{X}} \to \widetilde{Y}$ (with $\widetilde{X} \subset \widetilde{Y} \times \widetilde{Z}$) where $\widetilde{X}$ is a $\mathbf{P}_\sigma$ space and $\widetilde{Y}$ is Polish such that:

(a) If π_X is compact covering then $\pi_{\widetilde{X}}$ is compact covering.
(b) If $\pi_{\widetilde{X}}$ is inductively perfect then π_X is inductively perfect.

This will clearly prove the implication. Notice that Y is a subspace of $\widetilde{Y}$ as in the previous implication, but X is not in any sense the restriction of $\widetilde{X}$ to Y; in fact, $\widetilde{Z} \ne Z$.

(1) *Definition of* $\widetilde{Y}$ *and* $\widetilde{Z}$. We let

$$\widetilde{Z} = \mathbf{2}^\omega \times \mathbf{2}^\omega \quad \text{and} \quad \widetilde{Y} = Y \cup Y^\star$$

where $Y^\star$ is a countable set that we define below, together with the topology on $\widetilde{Y}$.

For this we need to fix some notations. As usual, we denote by π the projection from $\widetilde{Z}$ onto the first factor. Let

$$\mathcal{F} = \{E \in \mathcal{K}(\widetilde{Z}) : \pi(E) = \mathbf{2}^\omega\}$$

and for all n let

$$\mathcal{E}_n = \Big\{E \in \mathcal{F} : \exists T \subset 2^n \times 2^n,\ E = \bigcup_{t\in T} N_t\Big\}.$$

Then $\mathcal{E} = \bigcup_n \mathcal{E}_n$ is just the set of all clopen subsets of $\widetilde{Z} = \mathbf{2}^\omega \times \mathbf{2}^\omega$ with total projection on the first factor; notice in particular that $\mathcal{E}$ is dense in $\mathcal{F}$. We now define

$$Y^\star = \{(s, E) \in \Sigma(Y) \times \mathcal{K}(\widetilde{Z}) : E \in \mathcal{E}_{|s|}\}$$

where $\Sigma(Y)$ denotes the canonical tree of Y viewed as a closed subset of ω^ω. We endow $\widetilde{Y}$ with the unique topology τ satisfying:

- The restriction of τ to Y is the discrete topolgy.
- For any $y \in Y \subset \omega^\omega$ a basis of neighbourhoods for y in $\widetilde{Y}$ is given by the family $(V_s)_{s\prec y}$ where

$$V_s = \{y' \in Y : s \prec y'\} \cup \{y^\star = (s', E) \in Y^\star : s \prec s'\}.$$

It is then easy to embed homeomorphically $\widetilde{Y}$ onto a closed subset of $\omega^\omega \times \omega^\omega$ sending Y onto $Y \times \{0\}$. In particular, $\widetilde{Y}$ is a Polish space and Y is a closed subset of $\widetilde{Y}$.

(2) *Definition of* $\widetilde{X}$. We fix an embedding j of the Polish space $\mathcal{K}(Y \times Z)$ onto a $\mathbf{\Pi}^0_2$ subset of $\mathbf{2}^\omega$ which is the first factor of $\widetilde{Z}$. If $j(H) = u \in \mathbf{2}^\omega$ then we simply say that u is the *code* of the compact set $H \subset Y \times Z$. Notice that the set of all (y, u) satisfying

$$y \in Y, \quad u \text{ is the code of } H \in \mathcal{K}(Y \times Z), \quad y \in \pi(H)$$

is a $\mathbf{\Pi}^0_2$ subset G_0 of $Y \times \mathbf{2}^\omega$.

Moreover, since X is a $\mathbf{\Pi}^1_1$ subset of $Y \times \mathbf{2}^\omega$ we see that the set $A = \mathbf{2}^\omega \setminus j(\mathcal{K}(X))$ is $\mathbf{\Sigma}^1_1$, and we can fix a $\mathbf{\Pi}^0_2$ subset G of $\mathbf{2}^\omega \times \mathbf{2}^\omega$ such that

$$u \in A \Leftrightarrow \exists v \in \mathbf{2}^\omega,\ (u, v) \in G,$$

or equivalently,

$$u \text{ is the code of } H \in \mathcal{K}(X) \Leftrightarrow (\{u\} \times \mathbf{2}^\omega) \cap G = \emptyset.$$

We define

$$\widetilde{X} = \Big((G_0 \times \mathbf{2}^\omega) \cup \bigcup_{y^\star=(s,E)\in Y^\star} \{y^\star\} \times E\Big) \setminus (Y \times G).$$

Clearly, $\widetilde{X}$ is a subset of $\widetilde{Y} \times \widetilde{Z}$ which is the difference of two $\mathbf{\Pi}^0_2$ subsets, hence a $\mathbf{P}_\sigma$ space.

(3) *If* π_X *is compact covering then* $\pi_{\widetilde{X}}$ *is compact covering.* Let $\widetilde{K}$ be a compact subset of $\widetilde{Y}$. Since Y is closed in $\widetilde{Y}$ we see that $K = \widetilde{K} \cap Y$ is a compact subset of Y, hence there exists a compact subset H of X such

that $\pi(H) = K$. Let $u \in \mathbf{2}^\omega$ be the code of H; then for all $y \in K$ we have $(y, u) \in G_0$, and since $H \subset X$ we see that $(u, v) \notin G$ for all $v \in \mathbf{2}^\omega$; so

$$K \times \{u\} \times \mathbf{2}^\omega \subset X.$$

Moreover, it follows from the definition of $\widetilde{X}$ that

$$\forall y^\star \in Y^\star,\ \forall u \in \mathbf{2}^\omega,\ \exists v = \varphi(y^\star, u) \in \mathbf{2}^\omega, \qquad (y^\star, u, \varphi(y^\star, u)) \in \widetilde{X}.$$

Then let

$$\widetilde{H} = (K \times \{u\} \times \mathbf{2}^\omega) \cup \bigcup_{y^\star \in K \cap Y^\star} \{(y^\star, u, \varphi(y^\star, u))\}.$$

It is clear that $\widetilde{H}$ is a compact subset of $\widetilde{X}$ and that $\pi(\widetilde{H}) = \widetilde{K}$.

(4) *If* $\pi_{\widetilde{X}}$ *is inductively perfect then so is* π_X. Let $\widetilde{F}$ be a subset of $\widetilde{X}$ with total projection onto $\widetilde{Y}$, and which is closed in $\widetilde{Y} \times \widetilde{Z}$, and define

$$F = \{(y, z) \in Y \times Z : \exists u \text{ a code of some compact set } H \subset Y \times Z \text{ such that } (y, z) \in H \text{ and } \forall v \in \mathbf{2}^\omega,\ (y, u, v) \in \widetilde{F}\}.$$

Notice that if $(y, u, v) \in \widetilde{F} \subset \widetilde{X}$ then necessarily u is the code of a (unique) compact set H and that $H \subset X$. From this remark one can easily derive that $F \subset X$, and that F is closed in $Y \times Z$. Hence to prove that π_X is inductively perfect we only need to show that $\pi(F) = Y$.

So fix y in Y and consider

$$\widetilde{F}(y) = \{(u, v) \in \mathbf{2}^\omega \times \mathbf{2}^\omega : (y, u, v) \in \widetilde{F}\},$$

which is a compact subset of $\widetilde{Z}$.

Fact 6.5.1. $\forall E \in \mathcal{F}$, $E \cap \widetilde{F}(y) \neq \emptyset$.

Proof. We can choose a sequence (E_n) such that $E = \lim_n E_n$ and for all n, $E_n \in \mathcal{E}_n$. Let $s_n = y|_n$ so that $y^\star_n = (s_n, E_n) \in Y^\star$, and $y = \lim_n y^\star_n$. Since $\pi(\widetilde{F}) \supset Y^\star$ we can pick for all n some $(u_n, v_n) \in \widetilde{F}(y^\star_n)$; then $(y^\star_n, u_n, v_n) \in \widetilde{X}$, hence by definition of $Y^\star$ we also have $(u_n, v_n) \in E_n$. Then any cluster point of the sequence (u_n, v_n) will be in $E \cap \widetilde{F}(y)$; hence this set is nonempty. ∎

Fix $u \in \mathbf{2}^\omega$ such that

$$\widetilde{F}(y) \supset \{u\} \times \mathbf{2}^\omega.$$

Such a u exists for otherwise the open set $\mathbf{2}^\omega \times \mathbf{2}^\omega \setminus \widetilde{F}(y)$ would have a total projection on the first factor, hence would contain a clopen set E_0 with total projection, that is, $E_0 \in \mathcal{F}$ with $E_0 \cap \widetilde{F}(y) = \emptyset$, thus contradicting the previous fact.

It then follows from the definition of $\widetilde{X}$ that u is the code of a compact set $H \subset Y \times Z$ and that

$$(\{u\} \times \mathbf{2}^\omega) \cap G = \emptyset$$

hence by the definition of G we have in fact $H \subset X$. It also follows from the definition of $\widetilde{X}$ that $(y, u) \in G_0$, so $y \in \pi(H)$ and we can find $z \in Z$ such that $(y, z) \in H$, hence $z \in F(y)$. ∎

7. Conclusion and open questions. For any two descriptive classes $\mathcal{X}$ and $\mathcal{Y}$ consider the following statements:

$\mathbb{A}(\mathcal{X},\mathcal{Y})$: "For any $X \in \mathcal{X}$ and any $Y \in \mathcal{Y}$, any compact covering mapping $f : X \to Y$ is inductively perfect".

$\mathbb{A}^\star(\mathcal{X},\mathcal{Y})$: "For any $X \in \mathcal{X}$ and any $Y \in \mathcal{Y}$, if $X \subset \mathcal{K}(Y)$ is cofinal then there exists a continuous mapping $f : Y \to X$ such that $y \in f(y)$".

$\mathbb{A}^{\star\star}(\mathcal{X},\mathcal{Y})$: "For any $X \in \mathcal{X}$ and any $Y \in \mathcal{Y}$, if $X \subset \mathcal{K}(Y)$ is cofinal then there exists a continuous mapping $F : \mathcal{K}(Y) \to X$ such that $T \subset F(T)$".

We restrict ourselves to the case where the classes $\mathcal{X}$ and $\mathcal{Y}$ are *zero-dimensional* and satisfy the condition considered in 6.1:

($\star$) $\forall X \in \mathcal{X},\ \forall Y \in \mathcal{Y},\ \forall Z$ a closed subset of $X \times Y,\ Z \in \mathcal{X}$.

We recall that under these assumptions we have (see 6.1)

$\mathbb{A}(\mathcal{X},\mathcal{Y}) \Leftrightarrow$ Any compact covering projection from $\mathcal{X}$ onto $\mathcal{Y}$ is inductively perfect.

Also, under the same assumptions the proof of Theorem 6.2 shows that

$\mathbb{A}^\star(\mathcal{X},\mathcal{Y}) \Leftrightarrow$ Any strongly compact covering projection from $\mathcal{X}$ onto $\mathcal{Y}$ has a continuous selection.

The remarks in 6.1(a) show that if $\mathcal{X}$ is $\mathbf{\Pi}^0_2$ or $\mathbf{\Pi}^1_1$ then

$$\mathbb{A}^\star(\mathcal{X},\mathcal{Y}) \Rightarrow \mathbb{A}(\mathcal{X},\mathcal{Y}).$$

More generally, this implication holds for any class $\mathcal{X}$ such that $\kappa(\mathcal{X}) \subset \mathcal{X}$, where

$$\kappa(\mathcal{X}) = \{\mathcal{K}(X) : X \in \mathcal{X}\}.$$

But there is no general condition on the classes $\mathcal{X}$ and $\mathcal{Y}$ which ensures the converse of the last implication.

The comparison of $\mathbb{A}^\star$ and $\mathbb{A}^{\star\star}$ is clearer. In fact,

$$\mathbb{A}^\star(\mathcal{X},\kappa(\mathcal{Y})) \Rightarrow \mathbb{A}^{\star\star}(\mathcal{X},\mathcal{Y}) \Rightarrow \mathbb{A}^\star(\mathcal{X},\mathcal{Y}).$$

The second implication is obvious. For the first one let $X \in \mathcal{X}$, $Y \in \mathcal{Y}$, and suppose that $X \subset \mathcal{K}(Y)$ is cofinal. Consider the set $\widetilde{X} = \{H \in \mathcal{K}(Y) : \bigcup H \in X\}$. Since the mapping $H \mapsto \bigcup H$ is continuous from $\mathcal{K}(Y)$ into Y we have $\widetilde{X} \in \mathcal{X}$; moreover, since X is cofinal in $\mathcal{K}(Y)$, for any $H \in \mathcal{K}(Y)$ we can find $K \in X$ such that $\bigcup H \subset K$. Then one easily checks that $H \subset \mathcal{K}(K) \in \widetilde{X}$, which proves that $\widetilde{X}$ is cofinal in $\mathcal{K}(Y)$. So there exists a continuous mapping $\widetilde{f} : \mathcal{K}(Y) \to \widetilde{X}$ satisfying $K \in \widetilde{f}(K)$ for all K, and we define the mapping $F : \mathcal{K}(Y) \to X$ by $F(K) = \bigcup \widetilde{f}(K)$. This F is continuous and satisfies $K \subset F(K)$ for all K.

In particular, if $\kappa(\mathcal{Y}) \subset \mathcal{Y}$ (so if $\mathcal{Y}$ is $\mathbf{\Pi}^0_2$ or $\mathbf{\Pi}^1_1$) then

$$\mathbb{A}^{\star\star}(\mathcal{X}, \mathcal{Y}) \Leftrightarrow \mathbb{A}^{\star}(\mathcal{X}, \mathcal{Y}).$$

We can now summarize the main results of the paper through the following diagram:

$$\begin{array}{ccccccc}
\mathrm{Det}(\mathbf{\Sigma}^1_1) \Rightarrow & \mathbb{A}^{\star\star}(\mathbf{\Pi}^1_1, \mathbf{\Pi}^1_1) & \Rightarrow & \mathbb{A}^{\star\star}(\mathbf{\Pi}^1_1, \mathbf{\Delta}^1_1) & \Rightarrow & \mathbb{A}^{\star\star}(\mathbf{\Pi}^1_1, \mathbf{\Pi}^0_2) \\
& \Updownarrow & & \Downarrow & & \Updownarrow \\
& \mathbb{A}^{\star}(\mathbf{\Pi}^1_1, \mathbf{\Pi}^1_1) & \Rightarrow & \mathbb{A}^{\star}(\mathbf{\Pi}^1_1, \mathbf{\Delta}^1_1) & \Rightarrow & \mathbb{A}^{\star}(\mathbf{\Pi}^1_1, \mathbf{\Pi}^0_2) \\
& \Downarrow & & \Downarrow & & \Updownarrow \\
& \mathbb{A}(\mathbf{\Pi}^1_1, \mathbf{\Pi}^1_1) & \Rightarrow & \mathbb{A}(\mathbf{\Pi}^1_1, \mathbf{\Delta}^1_1) & \Rightarrow & \mathbb{A}(\mathbf{\Pi}^1_1, \mathbf{\Pi}^0_2) \\
& & & \Updownarrow & & \Updownarrow \\
& \Downarrow & & \mathbb{A}(\mathbf{\Delta}^1_1, \mathbf{\Delta}^1_1) & \Rightarrow & \mathbb{A}(\mathbf{\Delta}^1_1, \mathbf{\Pi}^0_2) \\
& & & & & \Updownarrow \\
\multicolumn{6}{c}{\forall\alpha,\ \omega^\omega \cap L(\alpha) \text{ is countable} \ \Rightarrow\ \forall\alpha,\ \omega^\omega \cap L(\alpha) \text{ is } \star\text{-bounded}}
\end{array}$$

In the following comments we consider the diagram as a matrix where $\mathrm{Det}(\mathbf{\Sigma}^1_1)$ is column 0. The implication $\mathrm{Det}(\mathbf{\Sigma}^1_1) \Rightarrow \mathbb{A}^{\star\star}(\mathbf{\Pi}^1_1, \mathbf{\Pi}^1_1)$ is just Remark 2.7. The other horizontal implications are obvious. In the first column the last implication $\mathbb{A}(\mathbf{\Pi}^1_1, \mathbf{\Pi}^1_1) \Rightarrow$ "$\forall\alpha,\ \omega^\omega \cap L(\alpha)$ is countable" is the result 0.3(b) mentioned in the introduction (see [3], Theorem 7.2); the other follow from the remarks at the beginning of this section. In the second column the only nontrivial implication is $\mathbb{A}(\mathbf{\Delta}^1_1, \mathbf{\Delta}^1_1) \Rightarrow \mathbb{A}(\mathbf{\Pi}^1_1, \mathbf{\Delta}^1_1)$, which can be proved by exactly the same method as (iv)$\Rightarrow$(ii) in Theorem 6.5. Finally, the equivalences of the last column relate Theorems 5.2, 6.4, 6.5, which also give the equivalence with $\mathbb{A}(\mathbf{\Delta}^1_1, \mathbf{P}_\sigma)$ and $\mathbb{A}(\mathbf{P}_\sigma, \mathbf{\Pi}^0_2)$.

We finish by some informal remarks on the plausibility of the missing arrows in this diagram. These remarks are based on some unsuccessful attempts to complete the picture.

(1) First consider the subdiagram constituted only by the statements involving $\mathbb{A}$, $\mathbb{A}^\star$, $\mathbb{A}^{\star\star}$, thus excluding $\mathrm{Det}(\mathbf{\Sigma}^1_1)$ and the statements about the size of the $\omega^\omega \cap L(\alpha)$'s.

It might be possible that the statements in each column are equivalent (this is already the case for the third column). One cannot even exclude that all the statements of the first two columns might be equivalent. But it seems highly improbable that any statement from the second (and a fortiori the first) column could be equivalent to the statements of the third column. Still between all these possible equivalences the less expected one is between $\mathbb{A}^{\star\star}(\mathbf{\Pi}^1_1,\mathbf{\Delta}^1_1)$ and $\mathbb{A}^{\star}(\mathbf{\Pi}^1_1,\mathbf{\Delta}^1_1)$. In any case we are convinced that no proof of any of these possible equivalences can be direct, but has to go through some extra statement such as some assumption on the size of the $\omega^\omega\cap L(\alpha)$'s, or even $\mathrm{Det}(\mathbf{\Sigma}^1_1)$.

(2) But a more interesting result would be to obtain any new comparison between "$\forall\alpha,\ \omega^\omega\cap L(\alpha)$ is countable" and one of the $\mathbb{A}$, $\mathbb{A}^\star$, $\mathbb{A}^{\star\star}$ statements considered above, although one cannot exclude the possibility that some of these statements (at least $\mathbb{A}^{\star\star}(\mathbf{\Pi}^1_1,\mathbf{\Pi}^1_1)$) could be equivalent to $\mathrm{Det}(\mathbf{\Sigma}^1_1)$.

The most interesting implication would be "$\forall\alpha,\ \omega^\omega\cap L(\alpha)$ is countable $\Rightarrow \mathbb{A}(\mathbf{\Delta}^1_1,\mathbf{\Delta}^1_1)$", for which we have some evidence. Notice that the converse of this implication is also open.

Another challenging statement is $\mathbb{A}^{\star\star}(\mathbf{\Pi}^1_1,\mathbb{Q})$ where $\mathbb{Q}$ denotes the space of all rational numbers. Notice that as for $\mathbb{A}(X,\mathbb{Q})$ one can prove in ZFC that $\mathbb{A}^\star(X,\mathbb{Q})$ holds for all X (see for example the proof of $\mathbb{A}(X,\mathbb{Q})$ in [3]); but it is quite clear that this should not be the case for $\mathbb{A}^{\star\star}(X,\mathbb{Q})$ and not even for $\mathbb{A}^{\star\star}(\mathbf{\Pi}^1_1,\mathbb{Q})$.

Post Scriptum. 1. Recently we were able to prove the following implications:

$$\mathbb{A}(\mathbf{\Delta}^1_1,\mathbf{\Delta}^1_1)\Rightarrow \text{“}\forall\alpha,\ \omega^\omega\cap L(\alpha)\text{ is countable”}\Leftrightarrow \mathbb{A}^\star(\mathbf{\Pi}^1_1,\mathbf{\Pi}^0_3),$$
$$\mathbb{A}^{\star\star}(\mathbf{\Pi}^1_1,\mathbf{\Delta}^1_1)\Rightarrow \mathrm{Det}(\mathbf{\Sigma}^1_1),$$

and even

$$\mathbb{A}^{\star\star}(\mathbf{\Pi}^1_1,\mathbb{Q})\Rightarrow \mathrm{Det}(\mathbf{\Sigma}^1_1),$$

which answers some of the questions raised in Section 7.

2. We are grateful to S. Todorčević who informed us about the papers [10] by Spinas and [1] by Brendle, Hjorth and Spinas concerning some properties of $\star$-cofinal sets (dominating sets in their terminology).

In fact, their results are formally incomparable with the results of the present paper. For example, Theorem 1.1 of [1] implies Corollary 2.2 above in the particular case of the order $\le^\star$, but on the other hand, in this restricted context the conclusion of Theorem 1.1 is more precise but meaningless in the general context of a Borel ordering. However, even in the case of the order $\le^\star$, one cannot derive the main result (Theorem 4.5 above) from Theorem 2.2 of [1].

References

[1] J. Brendle, G. Hjorth and O. Spinas, *Regularity properties for dominating projective sets*, Ann. Pure Appl. Logic 72 (1995), 291–307.

[2] J. P. R. Christensen, *Necessary and sufficient conditions for the measurability of certain sets of closed sets*, Math. Ann. 200 (1973), 189–193.

[3] G. Debs and J. Saint Raymond, *Compact covering and game determinacy*, Topology Appl. 68 (1996), 153–185.

[4] W. Just and H. Wicke, *Some conditions under which tri-quotient or compact-covering maps are inductively perfect*, ibid. 55 (1994), 289–305.

[5] A. Louveau, *A separation theorem for* $\mathbf{\Sigma}^1_1$ *sets*, Trans. Amer. Math. Soc. 260 (1980), 363–378.

[6] Y. N. Moschovakis, *Descriptive Set Theory*, North-Holland, Amsterdam, 1980.

[7] A. V. Ostrovskiĭ, *On new classes of mappings associated with k-covering mappings*, Vestnik Moskov. Univ. Ser. I Mat. Mekh. 1994, no. 4, 24–28 (in Russian); English transl.: Moscow Univ. Math. Bull. 49 (1994), no. 4, 29–23.

[8] J. Saint Raymond, *Caractérisation d'espaces polonais d'après des travaux récents de J. P. R. Christensen et D. Preiss*, Sém. Choquet, 11^e–12^e années, Initiation à l'Analyse, exp. no. 5, Secrétariat Mathématique, Paris, 1973, 10 pp.

[9] —, *La structure borélienne d'Effros est-elle standard?*, Fund. Math. 100 (1979), 201–210.

[10] O. Spinas, *Dominating projective sets in the Baire space*, Ann. Pure Appl. Logic 68 (1995), 327–342.

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