Hopfian and strongly hopfian manifolds

by

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Abstract. Let $p : M \to B$ be a proper surjective map defined on an $(n + 2)$-manifold such that each point-preimage is a copy of a hopfian $n$-manifold. Then we show that $p$ is an approximate fibration over some dense open subset $O$ of the mod 2 continuity set $C'$ and $C' \setminus O$ is locally finite. As an application, we show that a hopfian $n$-manifold $N$ is a codimension-2 fibrator if $\chi(N) \neq 0$ or $H_1(N) \cong \mathbb{Z}_2$.

1. Introduction. Call a closed manifold $N$ hopfian if it is orientable and every degree one map $N \to N$ which induces a $\pi_1$-isomorphism is a homotopy equivalence. A closed $n$-manifold $N$ is called a strongly hopfian manifold if $N_H$ is hopfian, where $H$ is the intersection of all subgroups with index 2 in $\pi_1(N)$ and $N_H$ is the covering space of $N$ corresponding to $H$.

In [16] and [17], using the concept of strong hopfianness, Y. Kim showed that all closed strongly hopfian manifolds $N$ with residually finite $\pi_1(N)$ and $\chi(N) \neq 0$ as well as all closed strongly hopfian manifolds with hyperhopfian fundamental group are codimension-2 fibrators. It is well known that every finitely generated residually finite group is hopfian. While every subgroup of a residually finite group is residually finite, a hopfian group can contain non-hopfian subgroups of finite index. For example, consider the group $G = \langle a, b : a^{-1}b^{12}a = b^{18} \rangle$ and a homomorphism $f : G \to \mathbb{Z}_2$ given by $f(a) = 0$, $f(b) = 1$. By the Reidemeister–Schreier method, one can see that the kernel $K$ of $f$ is isomorphic to $\langle c_1, c_2, d : c_i^{-1}d^6c_i = d^3, i = 1, 2 \rangle$, and $[G : K] = 2$. Baumslag–Solitar [1] tells us that $G$ is hopfian but $K$...
is non-hopfian. Put $H = \bigcap\{H_i \leq G : [G : H_i] = 2\}$. Since $[K : H]$ is finite and $K$ is non-hopfian, $H$ is non-hopfian (see [14]). Of course, this example does not give us the existence of a non-strongly hopfian manifold (see [12]). Consequently, the hopfianness of neither $N$ nor $\pi_1(N)$ guarantees the hopfianness of $N_H$ and $\pi_1(N_H)$.

In this paper, we show that if $p : M \to B$ is a proper surjective map defined on an $(n+2)$-manifold such that each point-preimage is a copy of a hopfian $n$-manifold, then $p$ is an approximate fibration over some dense open subset $O$ of the mod $2$ continuity set $C'$ and $C' \setminus O$ is locally finite. This property enables us to show that, whether or not $N_H$ and $\pi_1(N_H)$ are hopfian, all hopfian manifolds $N$ with hopfian $\pi_1(N)$ and $\chi(N) \neq 0$ are codimension-2 fibrators. Moreover, we can give an affirmative answer to the following question of Chinen [3]: Is every hopfian manifold $N$ with hopfian fundamental group and $H_1(N) \cong \mathbb{Z}_2$ a codimension-2 fibrator?

2. Preliminaries. Throughout this paper, the symbols $\sim$, $\approx$, and $\cong$ denote homotopy equivalence, homeomorphism, and isomorphism, respectively. The symbol $\chi$ is used to denote Euler characteristic. All manifolds are understood to be finite-dimensional, connected, metric, and boundaryless. Whenever the presence of boundary is tolerated, the object will be called a manifold with boundary.

A proper map $p : M \to B$ between locally compact ANRs is called an approximate fibration if it has the approximate homotopy lifting property (see [4]).

A proper map $p : M \to B$ is $N^n$-like if each fiber $p^{-1}(b)$ is shape equivalent to $N$. For simplicity, we shall assume that each fiber $p^{-1}(b)$ in an $N^n$-like map is an ANR having the homotopy type of $N^n$.

Let $N$ and $N'$ be closed $n$-manifolds and $f : N \to N'$ be a map. If both $N$ and $N'$ are orientable, then the degree of $f$ is the nonnegative integer $d$ such that the induced endomorphism $f_* : H_n(N; \mathbb{Z}) \cong \mathbb{Z} \to H_n(N'; \mathbb{Z}) \cong \mathbb{Z}$ amounts to multiplication by $d$, up to sign. In general, the degree mod $2$ of $f$ is the nonnegative integer $d$ such that the induced endomorphism $f_* : H_n(N; \mathbb{Z}_2) \cong \mathbb{Z}_2 \to H_n(N'; \mathbb{Z}_2) \cong \mathbb{Z}_2$ amounts to multiplication by $d$.

Suppose that $N$ is a closed $n$-manifold and a proper map $p : M \to B$ is $N$-like. Let $G$ be the set of all fibers, i.e., $G = \{p^{-1}(b) : b \in B\}$. Put $C = \{p(g) : g \in G\}$ and there exist a neighborhood $U_g$ of $g$ in $M$ and a retraction $R_g : U_g \to g$ such that $R_g|g' : g' \to g$ is a degree one map for all $g' \in G$ in $U_g$, and $C' = \{p(g) : g \in G\}$ and there exist a neighborhood $U_g$ of $g$ in $M$ and a retraction $R_g : U_g \to g$ such that $R_g|g' : g' \to g$ is a degree one mod $2$ map for all $g' \in G$ in $U_g$. Call $C$ the continuity set of $p$.
Hopfian manifolds


A group $\Gamma$ is said to be hopfian if every epimorphism $f : \Gamma \to \Gamma$ is necessarily an isomorphism. A finitely presented group $\Gamma$ is said to be hyperhopfian if every homomorphism $f : \Gamma \to \Gamma$ with $f(\Gamma)$ normal and $\Gamma/f(\Gamma)$ cyclic is an isomorphism (onto). A group $\Gamma$ is said to be residually finite if for any non-trivial element $x$ of $\Gamma$ there is a homomorphism $f$ from $\Gamma$ onto a finite group $K$ such that $f(x) \neq 1_K$.

A closed $n$-manifold $N^n$ is a codimension-2 fibrator (respectively, a codimension-2 orientable fibrator) if, whenever $p : M \to B$ is a proper map from an arbitrary (respectively, orientable) $(n + 2)$-manifold $M$ to a 2-manifold $B$ such that each $p^{-1}(b)$ is shape equivalent to $N$, then $p : M \to B$ is an approximate fibration.

All simply connected manifolds, closed surfaces with non-zero Euler characteristic, and closed manifolds $N$ with $\pi_1(N) \cong \mathbb{Z}_2$ (for example, real projective $n$-spaces, $n > 1$), are known to be codimension-2 fibrators (see [7]).

The following is basic for investigating codimension-2 fibrators.

**Proposition 2.1** [7, Proposition 2.8]. If $p : M \to B$ is a proper surjective map defined on an orientable $(n + 2)$-manifold $M$ with closed orientable $n$-manifolds as point inverses, then $B$ is a 2-manifold and $D = B \setminus C$ is locally finite in $B$, where $C$ represents the continuity set of $p$. Moreover, if either $M$ or some point inverses are non-orientable, then $B$ is a 2-manifold with boundary (possibly empty) and $D' = (\text{int } B) \setminus C'$ is locally finite in $B$, where $C'$ represents the mod 2 continuity set of $p$.

The next result summarizes useful information connecting hopfian manifolds and hopfian fundamental groups.

**Proposition 2.2** ([8, Theorem 2.2] or [11]). A closed orientable $n$-manifold $N$ is a hopfian manifold if any one of the following conditions holds:

1. $n \leq 4$;
2. $\pi_1(N)$ is virtually nilpotent;
3. $\pi_i(N)$ is trivial for $1 < i < n - 1$.

The following two recent facts play important roles in this paper.

**Lemma 2.3** [17, Lemma 3.2]. Let $N$ be a closed manifold. Suppose that $f : \pi_1(N) \to \pi_1(N)$ is a homomorphism whose induced action on $H_1(N; \mathbb{Z}_2)$ is an automorphism (i.e., $f(H) \subset H$ and the natural map $f' : \pi_1(N)/H \to \pi_1(N)/H$ is an isomorphism). Then

1. $f$ is an epimorphism if and only if $f|H : H \to H$ is an epimorphism.
2. $f$ is an isomorphism if and only if $f|H : H \to H$ is an isomorphism.
Proposition 2.4 [2, Corollary 3.3]. Let $N$ be a codimension-2 orientable fibrator. If $N$ has no 2-to-1 covering, then $N$ is a codimension-2 fibrator.

3. Hopfian manifolds as codimension-2 fibrators

Theorem 3.1. Let $N$ be a hopfian $n$-manifold with hopfian fundamental group. Let a proper map $p : M \to B$ defined on an $(n + 2)$-manifold $M$ be $N$-like. Then $p$ is an approximate fibration over some dense open subset $O$ of the mod 2 continuity set $C'$ of $p$ and $C' \setminus O$ is locally finite.

Proof. Let $G = \{ p^{-1}(b) \equiv g_b : b \in B \}$.

Claim. Any $x \in C'$ has a neighborhood $V_x$ and a dense open subset $O_x$ of $V_x$ such that $p$ is an approximate fibration over $O_x$ and $V_x \setminus O_x$ is locally finite.

Fix $g_0 \in G$ with $p(g_0) \in C'$. Take a neighborhood $U (\subset C')$ of $p(g_0)$ such that $p^{-1}(U)$ retracts to $g_0$, and take a smaller connected neighborhood $V$ of $p(g_0)$ such that $p^{-1}(V)$ deformation retracts to $g_0$ in $p^{-1}(U)$. Call this retraction $R : p^{-1}(V) \to g_0$. If $N$ has no 2-to-1 covering, the claim follows from [2, Proposition 3.2] and [8, Theorem 2.1]. Now we assume that $N$ has a 2-to-1 covering. Take the covering map $q : M^* \to p^{-1}(V)$ corresponding to $R_\#^{-1}(H)$, where $H = \bigcap_{i \in I} H_i$ with $I = \{ i : [\pi_1(N) : H_i] = 2 \}$. Since $[\pi_1(p^{-1}(V)) : R^{-1}_\#(H)] = [\pi_1(g_0) : H] < \infty$, $q$ is finite. We see that for all $g \in G$ with $p(g) \in C'$, $q^{-1}(g) \equiv g^*$ is connected and has homotopy type of $N_H$ (see [16, Lemma 3.1] for a detailed proof), where $N_H$ is the covering space of $N$ corresponding to $H$. Set $G^* = \{ g^* : g \in G \text{ with } p(g) \in V \}$. Let $p^* = p \circ q : M^* \to B^* = M^*/G^* = V$ be the composition map. By Proposition 2.1, we see that the continuity set $C(p^*)$ is dense open in $V$, and $V \setminus C(p^*)$ is locally finite. So it is enough to show that $p^*$ is an approximate fibration over the continuity set $C(p^*)$ of $p^*$.

Fix $g_b^* \in G^*$ with $p^*(g_b^*) = p(g_b) = b \in C(p^*)$. Carefully take a small neighborhood $W (\subset C(p^*)$) of $b$ and a retraction $R_b : p^{-1}(W) \to g_b$. Let $R_b^*: W^* \equiv q^{-1}(p^{-1}(W)) \to g_b^*$ be the lifting of $R_b$.

For any $a \in W$, consider the diagram

$$
\begin{array}{cccc}
g_a^* & \to & W^* & \xrightarrow{R_b^*} & g_b^* \\
p \downarrow & & \downarrow q & & \downarrow q \\
g_a & \to & p^{-1}(W) & \xrightarrow{R_b} & g_b
\end{array}
$$

Since $(R_b^*) : g_a^* \to g_b^*$ is a map of degree one, $(R_b) : g_a \to g_b$ has degree one. The hopfian hypotheses of $N$ and $\pi_1(N)$ yield that $(R_b)$ is a homotopy equivalence. In particular, $(R_b)_{\#} : \pi_1(g_a) \to \pi_1(g_b)$ is an isomorphism.
By Lemma 2.3, we see that \((R_b^*)_\# : \pi_1(g_a^*) \to \pi_1(g_b^*)\) is an isomorphism. Moreover, since for \(i \geq 2\), the homomorphism
\[
\pi_i(g_a^*) \cong \pi_i(g_a) \xrightarrow{(R_b^*)_\#} \pi_i(g_b) \cong \pi_i(g_b^*)
\]
is an isomorphism, by the Whitehead Theorem \((R_b^*)\) is a homotopy equivalence. It follows from [8, Theorem 2.1] and [4] that \(p^* = p \circ q\) is an approximate fibration over the continuity set \(C(p^*)\) of \(p^*\).

Now let \(O = \bigcup_{x \in C'} O_x\) and \(C' = \bigcup_{x \in C'} V_x\). Then we are done.

**Remark 1.** The conclusion of Theorem 3.1 is best possible, in the following sense: there are proper maps from \(S^1 \times \mathbb{R}^2 \to \mathbb{R}^2\) with fiber \(S^1\) which are not approximate fibrations over \(C' = \mathbb{R}^2\) (see [6] or [7, Example 3.6]).

**Remark 2.** Let \(N\) be a hopfian \(n\)-manifold with some properties. The most common procedure of showing that \(N\) is a codimension-2 fibration can be described as follows: Take any \(N\)-like proper map \(p : M \to B\) from an \((n + 2)\)-manifold onto a 2-manifold. First show that \(p\) is an approximate fibration over the mod 2 continuity set \(C'\) of \(p\), and then show that \(p\) is an approximate fibration over \(\text{int } B\) and \(\partial B = \emptyset\). The usefulness of Theorem 3.1 is that, showing that \(p\) is an approximate fibration over the mod 2 continuity set \(C'\) of \(p\), we can localize the situation so that \(C'\) is an open disk containing \(b_0 = p(g_0)\) and \(p\) is an approximate fibration over \(C' \setminus b\). Also, we may assume that \(R : p^{-1}(C') \to g_0\) is a strong deformation retraction.

**Corollary 3.2.** Let \(N\) be a hopfian \(n\)-manifold with hopfian \(\pi_1(N)\). Then \(N\) is a codimension-2 fibration if

1. \(\chi(N) \neq 0\), or
2. \(H_1(N) \cong \mathbb{Z}_2\).

**Proof.** Let a proper map \(p : M \to B\) from an \((n + 2)\)-manifold onto a 2-manifold with boundary be \(N\)-like. Set \(G = \{p^{-1}(b) : b \in B\}\).

**Proof of (1).** Applying the method of the proof of [16, Theorem 3.3] to \(p|C'\), we see that \(p\) is an approximate fibration over the mod 2 continuity set \(C'\) of \(p\). Then copy the proofs of [16, Lemma 3.2] and [16, Theorem 3.3].

**Proof of (2)**

**Claim (i).** \(p\) is an approximate fibration over the mod 2 continuity set \(C'\) of \(p\).

Localize the situation so that \(C'\) is an open disk containing \(b_0 = p(g_0)\) and \(p\) is an approximate fibration over \(C' \setminus b_0\). Also, we may assume that \(R : p^{-1}(C') \to g_0\) is a strong deformation retraction. If for any \(g \in G\) with \(p(g) \in C'\), \((R)|_\# : \pi_1(g) \to \pi_1(g_0)\) is an epimorphism, we are done (see [5]). So now assume that there is a \(g (\neq g_0) \in G\) with \(p(g) \in C'\) such
that \((R\rangle)_\# : \pi_1(g) \to \pi_1(g_0)\) is not an epimorphism. Take the covering \(q : M^* \to p^{-1}(C')\) corresponding to \(R_{\#}^{-1}(H)\), where \(H = \bigcap_{i \in I} H_i\) with \(I = \{i : [\pi_1(N) : H_i] = 2\}\). Here note that \(H\) is the commutator subgroup of \(\pi_1(N)\), for \(H_1(N) = \mathbb{Z}_2\). From the fact that \(\pi_1(g_0)/\langle R\rangle_\#(\pi_1(g))\) is cyclic, we see that \((R\rangle)_\#(\pi_1(g)) = H\), which contradicts the fact that \((R\rangle)_\#^{-1}(H) = H\) (see [17, Lemma 3.1]).

**Claim (ii).** \(p\) is an approximate fibration over \(\text{int } B\).

In light of Proposition 2.1, we localize the situation so that \(\text{int } B\) is an open disk containing \(b_0 = p(g_0)\) and \(p\) is an approximate fibration over \(\text{int } B \setminus b_0\). Also, we may assume that \(R : p^{-1}(\text{int } B) \to g_0\) is a strong deformation retraction. It suffices to show that for any \(g \in G\), \((R\rangle)_* : H_1(g) \to H_1(g_0)\) is an isomorphism (see [8, Lemma 5.2] or [15]). So now assume that there is a \(g \neq g_0\) such that \((R\rangle)_* : H_1(g) \to H_1(g_0)\) is not an isomorphism. Then, since \(H_1(N) = \mathbb{Z}_2\), \((R\rangle)_* : H_1(g) \to H_1(g_0)\) is trivial. Take the covering \(q : M^* \to p^{-1}(C')\) corresponding to \(R_{\#}^{-1}(H)\), where \(H = \bigcap_{i \in I} H_i\) with \(I = \{i : [\pi_1(N) : H_i] = 2\}\). Then we see that for all \(g \neq g_0\) in \(G\), \(q^{-1}(g)\) has two components which are homeomorphic to \(N\) and \(q^{-1}(g_0)\) is the covering space of \(N\) corresponding to \(H\).

Since \(\pi_1(g_0)/\langle R\rangle_\#(\pi_1(g))\) is cyclic (and so abelian) and \(H\) is the commutator subgroup of \(\pi_1(N)\), \((R\rangle)_\#(\pi_1(g))\) contains \(H\). So we have \((R\rangle)_\#(\pi_1(g)) = H\), because \(H_1(N) = \mathbb{Z}_2\) and \((R\rangle)_\#\) is not an epimorphism. Hence, by the fact that \(H = (q\rangle)_\#(q^{-1}(g_0))\), we have the lifting \(\overline{R}\) of \(R\) so that \((q\rangle) \circ \overline{R} = R\). Hence we have an epimorphism \(\mathbb{Z}_2 \cong H_1(N) = H_1(g) \to H_1(q^{-1}(g_0)) \cong H_1(N_H)\). So \(H_1(N_H)\) is either trivial or \(\mathbb{Z}_2\).

**Case 1:** \(H_1(N_H)\) is trivial. From the homology exact sequence

\[ \mathbb{Z} \cong H_2(M^*, M^* \setminus q^{-1}(g_0)) \to H_1(M^* \setminus q^{-1}(g_0)) \to H_1(M^*) = 0, \]

we see that \(H_1(M^* \setminus q^{-1}(g_0))\) is cyclic. On the other hand, since \((p \circ q) : M^* \setminus q^{-1}(g_0)\) is an approximate fibration, we see that \(H_1(M^* \setminus q^{-1}(g_0)) \cong i_*\langle H_1(q^{-1}(g_0)) \rangle \otimes \mathbb{Z}\) for the inclusion \(i : q^{-1}(g_0) \to M^* \setminus q^{-1}(g_0)\). By [3, Theorem 2.5], we deduce that \(i_*\) is a monomorphism so that \(H_1(M^* \setminus q^{-1}(g_0)) \cong H_1(q^{-1}(g_0)) \otimes \mathbb{Z} \cong \mathbb{Z}_2 \otimes \mathbb{Z}\), which is not cyclic.

**Case 2:** \(H_1(N_H) = \mathbb{Z}_2\). Let \(K\) be the commutator subgroup of \(\pi_1(N_H) = H\). Then \(K\) is a normal subgroup of \(\pi_1(N)\) with index 4. Since \(\pi_1(N)/K\) is abelian, \(K\) contains the commutator subgroup \(\pi_1(N_H)\) of \(\pi_1(N)\), which is a contradiction.

Therefore, \(p\) is an approximate fibration over \(\text{int } B\).

**Claim (iii).** The boundary of \(B\) is empty.
Suppose not. As in the proof of [16, Theorem 3.3], we then could have a map \( N \to N_H \) with degree one. But since \( H_1(N) = \mathbb{Z}_2 \), we know that \( H_1(N_H) \) is either trivial or \( \mathbb{Z}_2 \). As in the proof of [3, Lemma 6.7], we see that the case of \( H_1(N_H) = 0 \) cannot happen. Also, as before, we can show that the case of \( H_1(N_H) = \mathbb{Z}_2 \) cannot happen.

**Corollary 3.3.** Let \( N^n \) be a closed orientable \( n \)-manifold with hopfian \( \pi_1(N) \) and \( \chi(N) \neq 0 \). Then \( N \) is a codimension-2 fibrator if \( N \) is aspherical or \( n = 4 \).

**References**


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