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Hopfian and strongly hopfian manifolds

by

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Abstract. Let $p : M \to B$ be a proper surjective map defined on an $(n+2)$-manifold such that each point-preimage is a copy of a hopfian n-manifold. Then we show that p is an approximate fibration over some dense open subset O of the mod 2 continuity set C' and $C' \setminus O$ is locally finite. As an application, we show that a hopfian n-manifold N is a codimension-2 fibrator if $\chi(N) \neq 0$ or $H_1(N) \cong \mathbb{Z}_2$.

1. Introduction. Call a closed manifold N *hopfian* if it is orientable and every degree one map $N \to N$ which induces a π_1-isomorphism is a homotopy equivalence. A closed n-manifold N is called a *strongly hopfian manifold* if N_H is hopfian, where H is the intersection of all subgroups with index 2 in $\pi_1(N)$ and N_H is the covering space of N corresponding to H.

In [16] and [17], using the concept of strong hopfianness, Y. Kim showed that all closed strongly hopfian manifolds N with residually finite $\pi_1(N)$ and $\chi(N) \neq 0$ as well as all closed strongly hopfian manifolds with hyperhopfian fundamental group are codimension-2 fibrators. It is well known that every finitely generated residually finite group is hopfian. While every subgroup of a residually finite group is residually finite, a hopfian group can contain non-hopfian subgroups of finite index. For example, consider the group $G = \langle a, b : a^{-1}b^{12}a = b^{18}\rangle$ and a homomorphism $f : G \to \mathbb{Z}_2$ given by $f(a) = 0$, $f(b) = 1$. By the Reidemeister–Schreier method, one can see that the kernel K of f is isomorphic to $\langle c_1, c_2, d : c_i^{-1}d^6c_i = d^9,\ i = 1, 2\rangle$, and $[G : K] = 2$. Baumslag–Solitar [1] tells us that G is hopfian but K

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is non-hopfian. Put $H = \bigcap\{H_i \le G : [G : H_i] = 2\}$. Since $[K : H]$ is finite and K is non-hopfian, H is non-hopfian (see [14]). Of course, this example does not give us the existence of a non-strongly hopfian manifold (see [12]). Consequently, the hopfianness of neither N nor $\pi_1(N)$ guarantees the hopfianness of N_H and $\pi_1(N_H)$.

In this paper, we show that if $p : M \to B$ is a proper surjective map defined on an $(n+2)$-manifold such that each point-preimage is a copy of a hopfian n-manifold, then p is an approximate fibration over some dense open subset O of the mod 2 continuity set C' and $C' \setminus O$ is locally finite. This property enables us to show that, whether or not N_H and $\pi_1(N_H)$ are hopfian, all hopfian manifolds N with hopfian $\pi_1(N)$ and $\chi(N) \neq 0$ are codimension-2 fibrators. Moreover, we can give an affirmative answer to the following question of Chinen [3]: Is every hopfian manifold N with hopfian fundamental group and $H_1(N) \cong \mathbb{Z}_2$ a codimension-2 fibrator?

2. Preliminaries. Throughout this paper, the symbols $\sim$, $\approx$, and $\cong$ denote homotopy equivalence, homeomorphism, and isomorphism, respectively. The symbol χ is used to denote Euler characteristic. All manifolds are understood to be finite-dimensional, connected, metric, and boundaryless. Whenever the presence of boundary is tolerated, the object will be called a manifold with boundary.

A proper map $p : M \to B$ between locally compact ANRs is called an *approximate fibration* if it has the approximate homotopy lifting property (see [4]).

A proper map $p : M \to B$ is N^n-*like* if each fiber $p^{-1}(b)$ is shape equivalent to N. For simplicity, we shall assume that each fiber $p^{-1}(b)$ in an N^n-like map is an ANR having the homotopy type of N^n.

Let N and N' be closed n-manifolds and $f : N \to N'$ be a map. If both N and N' are orientable, then the *degree* of f is the nonnegative integer d such that the induced endomorphism $f_* : H_n(N;\mathbb{Z}) \cong \mathbb{Z} \to H_n(N';\mathbb{Z}) \cong \mathbb{Z}$ amounts to multiplication by d, up to sign. In general, the *degree mod* 2 of f is the nonnegative integer d such that the induced endomorphism $f_* : H_n(N;\mathbb{Z}_2) \cong \mathbb{Z}_2 \to H_n(N';\mathbb{Z}_2) \cong \mathbb{Z}_2$ amounts to multiplication by d.

Suppose that N is a closed n-manifold and a proper map $p : M \to B$ is N-like. Let G be the set of all fibers, i.e., $G = \{p^{-1}(b) : b \in B\}$. Put $C = \{p(g) \in B : g \in G$ and there exist a neighborhood U_g of g in M and a retraction $R_g : U_g \to g$ such that $R_g|g' : g' \to g$ is a degree one map for all $g' \in G$ in $U_g\}$, and $C' = \{p(g) \in B : g \in G$ and there exist a neighborhood U_g of g in M and a retraction $R_g : U_g \to g$ such that $R_g|g' : g' \to g$ is a degree one mod 2 map for all $g' \in G$ in $U_g\}$. Call C the *continuity set* of p

and C' the *mod* 2 *continuity set* of p. D. Coram and P. Duvall [6] showed that C and C' are dense, open subsets of B.

A group Γ is said to be *hopfian* if every epimorphism $f : \Gamma \to \Gamma$ is necessarily an isomorphism. A finitely presented group Γ is said to be *hyperhopfian* if every homomorphism $f : \Gamma \to \Gamma$ with $f(\Gamma)$ normal and $\Gamma/f(\Gamma)$ cyclic is an isomorphism (onto). A group Γ is said to be *residually finite* if for any non-trivial element x of Γ there is a homomorphism f from Γ onto a finite group K such that $f(x) \neq 1_K$.

A closed n-manifold N^n is a *codimension*-2 *fibrator* (respectively, a *codimension*-2 *orientable fibrator*) if, whenever $p : M \to B$ is a proper map from an arbitrary (respectively, orientable) $(n+2)$-manifold M to a 2-manifold B such that each $p^{-1}(b)$ is shape equivalent to N, then $p : M \to B$ is an approximate fibration.

All simply connected manifolds, closed surfaces with non-zero Euler characteristic, and closed manifolds N with $\pi_1(N) \cong \mathbb{Z}_2$ (for example, real projective n-spaces, $n > 1$), are known to be codimension-2 fibrators (see [7]).

The following is basic for investigating codimension-2 fibrators.

PROPOSITION 2.1 [7, Proposition 2.8]. *If $p : M \to B$ is a proper surjective map defined on an orientable $(n+2)$-manifold M with closed orientable n-manifolds as point inverses, then B is a 2-manifold and $D = B \setminus C$ is locally finite in B, where C represents the continuity set of p. Moreover, if either M or some point inverses are non-orientable, then B is a 2-manifold with boundary (possibly empty) and $D' = (\operatorname{int} B) \setminus C'$ is locally finite in B, where C' represents the mod 2 continuity set of p.*

The next result summarizes useful information connecting hopfian manifolds and hopfian fundamental groups.

PROPOSITION 2.2 ([8, Theorem 2.2] or [11]). *A closed orientable n-manifold N is a hopfian manifold if any one of the following conditions holds*:

(1) $n \leq 4$;
(2) $\pi_1(N)$ *is virtually nilpotent*;
(3) $\pi_i(N)$ *is trivial for* $1 < i < n-1$.

The following two recent facts play important roles in this paper.

LEMMA 2.3 [17, Lemma 3.2]. *Let N be a closed manifold. Suppose that $f : \pi_1(N) \to \pi_1(N)$ is a homomorphism whose induced action on $H_1(N;\mathbb{Z}_2)$ is an automorphism (i.e., $f(H) \subset H$ and the natural map $f' : \pi_1(N)/H \to \pi_1(N)/H$ is an isomorphism). Then*

(1) *f is an epimorphism if and only if $f|H : H \to H$ is an epimorphism.*
(2) *f is an isomorphism if and only if $f|H : H \to H$ is an isomorphism.*

Proposition 2.4 [2, Corollary 3.3]. *Let N be a codimension-2 orientable fibrator. If N has no 2-to-1 covering, then N is a codimension-2 fibrator.*

3. Hopfian manifolds as codimension-2 fibrators

Theorem 3.1. *Let N be a hopfian n-manifold with hopfian fundamental group. Let a proper map $p : M \to B$ defined on an $(n+2)$-manifold M be N-like. Then p is an approximate fibration over some dense open subset O of the mod 2 continuity set C' of p and $C' \setminus O$ is locally finite.*

Proof. Let $G = \{p^{-1}(b) \equiv g_b : b \in B\}$.

Claim. *Any $x \in C'$ has a neighborhood V_x and a dense open subset O_x of V_x such that p is an approximate fibration over O_x and $V_x \setminus O_x$ is locally finite.*

Fix $g_0 \in G$ with $p(g_0) \in C'$. Take a neighborhood U $(\subset C')$ of $p(g_0)$ such that $p^{-1}(U)$ retracts to g_0, and take a smaller connected neighborhood V of $p(g_0)$ such that $p^{-1}(V)$ deformation retracts to g_0 in $p^{-1}(U)$. Call this retraction $R : p^{-1}(V) \to g_0$. If N has no 2-to-1 covering, the claim follows from [2, Proposition 3.2] and [8, Theorem 2.1]. Now we assume that N has a 2-to-1 covering. Take the covering map $q : M^* \to p^{-1}(V)$ corresponding to $R_{\#}^{-1}(H)$, where $H = \bigcap_{i\in I} H_i$ with $I = \{i : [\pi_1(N) : H_i] = 2\}$. Since $[\pi_1(p^{-1}(V)) : R_{\#}^{-1}(H)] = [\pi_1(g_0) : H] < \infty$, q is finite. We see that for all $g \in G$ with $p(g) \in C'$, $q^{-1}(g) \equiv g^*$ is connected and has homotopy type of N_H (see [16, Lemma 3.1] for a detailed proof), where N_H is the covering space of N corresponding to H. Set $G^* = \{g^* : g \in G \text{ with } p(g) \in V\}$. Let $p^* = p \circ q : M^* \to B^* = M^*/G^* = V$ be the composition map. By Proposition 2.1, we see that the continuity set $C(p^*)$ of p^* is dense open in V, and $V \setminus C(p^*)$ is locally finite. So it is enough to show that p^* is an approximate fibration over the continuity set $C(p^*)$ of p^*.

Fix $g_b^* \in G^*$ with $p^*(g_b^*) = p(g_b) = b \in C(p^*)$. Carefully take a small neighborhood W $(\subset C(p^*))$ of b and a retraction $R_b : p^{-1}(W) \to g_b$. Let $R_b^* : W^* \equiv q^{-1}(p^{-1}(W)) \to g_b^*$ be the lifting of R_b.

For any $a \in W$, consider the diagram

$$\begin{array}{ccccc} g_a^* & \longrightarrow & W^* & \xrightarrow{R_b^*} & g_b^* \\ \downarrow{\scriptstyle q|} & & \downarrow{\scriptstyle q|} & & \downarrow{\scriptstyle q|} \\ g_a & \longrightarrow & p^{-1}(W) & \xrightarrow{R_b} & g_b \end{array}$$

Since $(R_b^*|) : g_a^* \to g_b^*$ is a map of degree one, $(R_b|) : g_a \to g_b$ has degree one. The hopfian hypotheses of N and $\pi_1(N)$ yield that $(R_b|)$ is a homotopy equivalence. In particular, $(R_b|)_{\#} : \pi_1(g_a) \to \pi_1(g_b)$ is an isomorphism.

By Lemma 2.3, we see that $(R_b^*|)_\# : \pi_1(g_a^*) \to \pi_1(g_b^*)$ is an isomorphism. Moreover, since for $i \geq 2$, the homomorphism

$$\pi_i(g_a^*) \cong \pi_i(g_a) \xrightarrow[(R_b|)_\#]{\cong} \pi_i(g_b) \cong \pi_i(g_b^*)$$

is an isomorphism, by the Whitehead Theorem $(R_b^*|)$ is a homotopy equivalence. It follows from [8, Theorem 2.1] and [4] that $p^* = p \circ q$ is an approximate fibration over the continuity set $C(p^*)$ of p^*.

Now let $O = \bigcup_{x \in C'} O_x$ and $C' = \bigcup_{x \in C'} V_x$. Then we are done.

Remark 1. The conclusion of Theorem 3.1 is best possible, in the following sense: there are proper maps from $S^1 \times \mathbb{R}^2 \to \mathbb{R}^2$ with fiber S^1 which are not approximate fibrations over $C' = \mathbb{R}^2$ (see [6] or [7, Example 3.6]).

Remark 2. Let N be a hopfian n-manifold with some properties. The most common procedure of showing that N is a codimension-2 fibrator can be described as follows: Take any N-like proper map $p : M \to B$ from an $(n+2)$-manifold onto a 2-manifold. First show that p is an approximate fibration over the mod 2 continuity set C' of p, and then show that p is an approximate fibration over int B and $\partial B = \emptyset$. The usefulness of Theorem 3.1 is that, showing that p is an approximate fibration over the mod 2 continuity set C' of p, we can localize the situation so that C' is an open disk containing $b_0 = p(g_0)$ and p is an approximate fibration over $C' \setminus b$. Also, we may assume that $R : p^{-1}(C') \to g_0$ is a strong deformation retraction.

Corollary 3.2. *Let N be a hopfian n-manifold with hopfian $\pi_1(N)$. Then N is a codimension-2 fibrator if*

(1) $\chi(N) \neq 0$, *or*
(2) $H_1(N) \cong \mathbb{Z}_2$.

Proof. Let a proper map $p : M \to B$ from an $(n+2)$-manifold onto a 2-manifold with boundary be N-like. Set $G = \{p^{-1}(b) : b \in B\}$.

Proof of (1). Applying the method of the proof of [16, Theorem 3.3] to $p|C'$, we see that p is an approximate fibration over the mod 2 continuity set C' of p. Then copy the proofs of [16, Lemma 3.2] and [16, Theorem 3.3].

Proof of (2)

Claim (i). *p is an approximate fibration over the mod 2 continuity set C' of p.*

Localize the situation so that C' is an open disk containing $b_0 = p(g_0)$ and p is an approximate fibration over $C' \setminus b_0$. Also, we may assume that $R : p^{-1}(C') \to g_0$ is a strong deformation retraction. If for any $g \in G$ with $p(g) \in C'$, $(R|)_\# : \pi_1(g) \to \pi_1(g_0)$ is an epimorphism, we are done (see [5]). So now assume that there is a $g\,(\neq g_0) \in G$ with $p(g) \in C'$ such that $(R|)_{\#} : \pi_1(g) \to \pi_1(g_0)$ is not an epimorphism. Take the covering $q : M^* \to p^{-1}(C')$ corresponding to $R_{\#}^{-1}(H)$, where $H = \bigcap_{i\in I} H_i$ with $I = \{i : [\pi_1(N) : H_i] = 2\}$. Here note that H is the commutator subgroup of $\pi_1(N)$, for $H_1(N) = \mathbb{Z}_2$. From the fact that $\pi_1(g_0)/(R|)_{\#}(\pi_1(g))$ is cyclic, we see that $(R|)_{\#}(\pi_1(g)) = H$, which contradicts the fact that $(R|)_{\#}^{-1}(H) = H$ (see [17, Lemma 3.1]).

Claim (ii). *p is an approximate fibration over* $\operatorname{int} B$.

In light of Proposition 2.1, we localize the situation so that $\operatorname{int} B$ is an open disk containing $b_0 = p(g_0)$ and p is an approximate fibration over $\operatorname{int} B \setminus b_0$. Also, we may assume that $R : p^{-1}(\operatorname{int} B) \to g_0$ is a strong deformation retraction. It suffices to show that for any $g \in G$, $(R|)_* : H_1(g) \to H_1(g_0)$ is an isomorphism (see [8, Lemma 5.2] or [15]). So now assume that there is a $g\,(\neq g_0) \in G$ such that $(R|)_* : H_1(g) \to H_1(g_0)$ is not an isomorphism. Then, since $H_1(N) = \mathbb{Z}_2$, $(R|)_* : H_1(g) \to H_1(g_0)$ is trivial. Take the covering $q : M^* \to p^{-1}(C')$ corresponding to $R_{\#}^{-1}(H)$, where $H = \bigcap_{i\in I} H_i$ with $I = \{i : [\pi_1(N) : H_i] = 2\}$. Then we see that for all $g\,(\neq g_0) \in G$, $q^{-1}(g)$ has two components which are homeomorphic to N and $q^{-1}(g_0) \sim N_H$, where N_H is the covering space of N corresponding to H.

Since $\pi_1(g_0)/(R|)_{\#}(\pi_1(g))$ is cyclic (and so abelian) and H is the commutator subgroup of $\pi_1(N)$, $(R|)_{\#}(\pi_1(g))$ contains H. So we have $(R|)_{\#}(\pi_1(g)) = H$, because $H_1(N) = \mathbb{Z}_2$ and $(R|)_{\#}$ is not an epimorphism. Hence, by the fact that $H = (q|)_{\#}(q^{-1}(g_0))$, we have the lifting $\overline{R|}$ of $R|$ so that $(q|) \circ \overline{R|} = R|$. Hence we have an epimorphism $\mathbb{Z}_2 \cong H_1(N) = H_1(g) \to H_1(q^{-1}(g_0)) \cong H_1(N_H)$. So $H_1(N_H)$ is either trivial or $\mathbb{Z}_2$.

Case 1: $H_1(N_H)$ is trivial. From the homology exact sequence

$$\mathbb{Z} \cong H_2(M^*, M^* \setminus q^{-1}(g_0)) \to H_1(M^* \setminus q^{-1}(g_0)) \to H_1(M^*) = 0,$$

we see that $H_1(M^* \setminus q^{-1}(g_0))$ is cyclic. On the other hand, since $(p \circ q) : M^* \setminus q^{-1}(g_0)$ is an approximate fibration, we see that $H_1(M^* \setminus q^{-1}(g_0)) \cong i_*(H_1(q^{-1}(g)_C)) \oplus \mathbb{Z}$ for the inclusion $i : q^{-1}(g)_C \to M^* \setminus q^{-1}(g_0)$. By [3, Theorem 2.5], we deduce that i_* is a monomorphism so that $H_1(M^* \setminus q^{-1}(g_0)) \cong H_1(q^{-1}(g)_C) \oplus \mathbb{Z} \cong \mathbb{Z}_2 \oplus \mathbb{Z}$, which is not cyclic.

Case 2: $H_1(N_H) = \mathbb{Z}_2$. Let K be the commutator subgroup of $\pi_1(N_H) = H$. Then K is a normal subgroup of $\pi_1(N)$ with index 4. Since $\pi_1(N)/K$ is abelian, K contains the commutator subgroup $\pi_1(N_H)$ of $\pi_1(N)$, which is a contradiction.

Therefore, p is an approximate fibration over $\operatorname{int} B$.

Claim (iii). *The boundary of B is empty.*

Suppose not. As in the proof of [16, Theorem 3.3], we then could have a map $N \to N_H$ with degree one. But since $H_1(N) = \mathbb{Z}_2$, we know that $H_1(N_H)$ is either trivial or $\mathbb{Z}_2$. As in the proof of [3, Lemma 6.7], we see that the case of $H_1(N_H) = 0$ cannot happen. Also, as before, we can show that the case of $H_1(N_H) = \mathbb{Z}_2$ cannot happen. ■

COROLLARY 3.3. *Let N^n be a closed orientable n-manifold with hopfian $\pi_1(N)$ and $\chi(N) \neq 0$. Then N is a codimension-2 fibrator if N is aspherical or $n = 4$.*

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