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A forcing construction of thin-tall Boolean algebras

by

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Abstract. It was proved by Juhász and Weiss that for every ordinal α with $0 < \alpha < \omega_2$ there is a superatomic Boolean algebra of height α and width ω. We prove that if κ is an infinite cardinal such that $\kappa^{<\kappa} = \kappa$ and α is an ordinal such that $0 < \alpha < \kappa^{++}$, then there is a cardinal-preserving partial order that forces the existence of a superatomic Boolean algebra of height α and width κ. Furthermore, iterating this forcing through all $\alpha < \kappa^{++}$, we obtain a notion of forcing that preserves cardinals and such that in the corresponding generic extension there is a superatomic Boolean algebra of height α and width κ for every $\alpha < \kappa^{++}$. Consistency for specific κ, like ω_1, then follows as a corollary.

0. Introduction. A *superatomic Boolean algebra* is a Boolean algebra in which every subalgebra is atomic. It is a well-known fact that a Boolean algebra B is superatomic iff its Stone space $S(B)$ is scattered. For every ordinal α, the α-*derivative* of $S(B)$ is defined by induction on α as follows: $S(B)^0 = S(B)$; if $\alpha = \beta + 1$, then $S(B)^\alpha$ is the set of accumulation points of $S(B)^\beta$; and if α is a limit, then $S(B)^\alpha = \bigcap\{S(B)^\beta : \beta < \alpha\}$. Then $S(B)$ is scattered iff $S(B)^\alpha = \emptyset$ for some α. This process can be transferred to the Boolean algebra B, yielding an increasing sequence of ideals I_α, which are defined by transfinite induction as follows: we put $I_0 = \{0\}$; if $\alpha = \beta + 1$, then $I_\alpha =$ the ideal generated by $I_\beta \cup \{b \in B : b/I_\beta \text{ is an atom in } B/I_\beta\}$; and if α is a limit, then $I_\alpha = \bigcup\{I_\beta : \beta < \alpha\}$. Then B is superatomic iff there is an ordinal α such that $B = I_\alpha$.

We define the *height* of a superatomic Boolean algebra B by $\mathrm{ht}(B) =$ the least ordinal α such that B/I_α is finite (which means $B = I_{\alpha+1}$). For every $\alpha < \mathrm{ht}(B)$, we denote by $\mathrm{wd}_\alpha(B)$ the cardinality of the set of atoms of B/I_α, and we define the *width* of B by $\mathrm{wd}(B) = \sup\{\mathrm{wd}_\alpha(B) : \alpha < \mathrm{ht}(B)\}$. If κ is an infinite cardinal and $\eta \neq 0$ is an ordinal, we say that a Boolean algebra B is a (κ, η)-*Boolean algebra* if B is superatomic, $\mathrm{wd}(B) = \kappa$ and

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$\mathrm{ht}(B) = \eta$. If B is a (κ, η)-Boolean algebra for some ordinal $\eta \geq \kappa^+$, we say that B is *κ-thin-tall*.

A construction in ZFC of an ω-thin-tall Boolean algebra was carried out by Rajagopalan and, in a simplified way, by Juhász and Weiss (see [5]). However, it is not known whether there exists an (ω_1, ω_2)-Boolean algebra. Nevertheless, it was shown in [7] that under $V = L$, there is a (κ, κ^+)-Boolean algebra for every regular cardinal κ. On the other hand, Juhász and Weiss proved in [5] that for every ordinal α with $0 < \alpha < \omega_2$, there exists an (ω, α)-Boolean algebra. This result is in a sense best possible, since it is known that CH implies the nonexistence of an (ω, ω_2)-Boolean algebra, and also that in the Cohen model there is no such algebra (see [6] and [11]). Also, Baumgartner and Shelah proved in [3] that the existence of an (ω, ω_2)-Boolean algebra is consistent with ZFC. However, if κ is an uncountable cardinal, it is not known whether the existence of a (κ, κ^{++})-Boolean algebra is consistent with ZFC.

The reader may consult [8] and the survey paper [12] for more information on superatomic Boolean algebras.

Our aim in this paper is to prove that for any specific regular cardinal κ and any specific ordinal $\eta < \kappa^{++}$, the existence of a (κ, η)-Boolean algebra is consistent with ZFC. In the proof of this result, we will extend the technique given in [10] for forcing the existence of a (κ, κ^+)-Boolean algebra by means of a κ-closed and κ^+-c.c. partial order. If $\kappa^+ < \eta < \kappa^{++}$, we will need a more intricate argument to verify the κ^+-chain condition.

In Section 2 of the paper, we present our forcing construction of thin-tall Boolean algebras. And in Section 1, we introduce the combinatorial notions that make the proof of our consistency result work.

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Our set-theoretic terminology is standard. Terms not defined here can be found in [4] or [9].

2. Trees of intervals. We provide here the background necessary for the proof of the main result.

Definition 1.1. 1. By an *ordinal interval* we mean an interval of the form $[\alpha, \beta)$, where α, β are ordinals with $\alpha < \beta$.

2. Given an ordinal interval $I = [\alpha, \beta)$, we write $I^- = \alpha$ and $I^+ = \beta$.

All the intervals considered in this paper will be ordinal intervals.

Definition 1.2. 1. Let η be a nonzero ordinal. A *tree of intervals* on η is a collection $\mathcal{I} = \bigcup_{n<\omega} \mathcal{I}_n$ where:

(1) $\mathcal{I}_0 = \{[0, \eta)\}$.

(2) For every $I, J \in \mathcal{I}$, $I \subseteq J$ or $J \subseteq I$ or $I \cap J = \emptyset$.

(3) If I, J are different elements of $\mathcal{I}$, $I \subseteq J$ and J^+ is a limit, then $I^+ < J^+$.

(4) $\mathcal{I}_n$ partitions $[0, \eta)$ for each $n < \omega$.

(5) $\mathcal{I}_{n+1}$ refines $\mathcal{I}_n$ for each $n < \omega$.

(6) For every $\alpha < \eta$ there is an $I \in \mathcal{I}$ such that $I^- = \alpha$.

2. Now given a tree of intervals $\mathcal{I} = \bigcup_{n<\omega} \mathcal{I}_n$, we make the following definitions:

(1) If $\alpha < \eta$ and $n < \omega$, we write $I(\alpha, n) =$ the unique interval $I \in \mathcal{I}_n$ with $\alpha \in I$.

(2) If $\alpha < \eta$, we write $n(\alpha) =$ the least natural number n such that there is an interval $I \in \mathcal{I}_n$ with $I^- = \alpha$.

If $\mathcal{I} = \bigcup_{n<\omega} \mathcal{I}_n$ is a tree of intervals on an ordinal η, then for every $\alpha < \eta$ there is an $n < \omega$ such that $\{\alpha\} \in \mathcal{I}_n$. To check this, note that if $n(\alpha) < n(\alpha + 1)$ then $\{\alpha\} \in \mathcal{I}_{n(\alpha+1)}$, and analogously $n(\alpha) \geq n(\alpha + 1)$ implies $\{\alpha\} \in \mathcal{I}_{n(\alpha)}$.

Definition 1.3. Let $\mathcal{I} = \bigcup_{n<\omega} \mathcal{I}_n$ be a tree of intervals on an ordinal η. Then:

1. For each $n \in \omega$, we define $E_n = \{I^- : I \in \mathcal{I}_n\}$.
2. For each $\alpha < \eta$, we define the *orbit* of α (with respect to $\mathcal{I}$) by

$$o(\alpha) = \bigcup \{E_{m+1} \cap \alpha \cap I(\alpha, m) : m < n(\alpha)\}.$$

Here is an example. Consider $\eta = \omega_2 \cdot \omega_2$. Then we define $\mathcal{I}$ by $\mathcal{I}_0 = \{[0, \omega_2 \cdot \omega_2)\}$, $\mathcal{I}_1 = \{[\omega_2 \cdot \xi, \omega_2 \cdot (\xi + 1)) : \xi < \omega_2\}$ and $\mathcal{I}_n = \{\{\xi\} : \xi < \omega_2 \cdot \omega_2\}$ for $n \geq 2$. In this case, we have $E_0 = \{0\}$, $E_1 = \{\omega_2 \cdot \xi : \xi < \omega_2\}$ and $E_2 = \omega_2 \cdot \omega_2$. Now consider $\alpha = \omega_2 \cdot \omega_1 + \omega^\omega$. It is easy to verify that

$$o(\alpha) = \{\omega_2 \cdot \xi : \xi \leq \omega_1\} \cup \{\omega_2 \cdot \omega_1 + \xi : 0 < \xi < \omega^\omega\}.$$

Proposition 1.1. *Let $\mathcal{I}$ be a tree of intervals on an ordinal η. Suppose that for some $\alpha < \eta$, $\beta \in o(\alpha)$ and $\gamma \in o(\beta)$. Then $\gamma \in o(\alpha)$.*

Proof. Since $\beta \in o(\alpha)$, we have $\beta \in E_{m+1} \cap \alpha \cap I(\alpha, m)$ for some $m < n(\alpha)$. As $\beta \in E_{m+1}$ and $m < n(\alpha)$, we infer $n(\beta) \leq m + 1 \leq n(\alpha)$.

On the other hand, as $\beta \in I(\alpha, m)$ and $\mathcal{I}_m$ is a partition of $[0, \eta)$, we have $I(\alpha, m) = I(\beta, m)$. Thus, since $\mathcal{I}_{k+1}$ refines $\mathcal{I}_k$ for each k, we infer that $I(\alpha, k) = I(\beta, k)$ for all $k \leq m$.

Now, since $\gamma \in o(\beta)$, we have $\gamma \in E_{k+1} \cap \beta \cap I(\beta, k)$ for some $k < n(\beta)$. Thus $k \leq m < n(\alpha)$, and therefore $\gamma \in o(\alpha)$. ∎

Definition 1.4. Let $\mathcal{I}$ be a tree of intervals on an ordinal η.

1. Let $\alpha < \beta < \eta$ and $I \in \mathcal{I}$. Then we say that α, β *separate at* I if for some $n < \omega$, $I = I(\alpha, n) = I(\beta, n)$ and $I(\alpha, n+1) \neq I(\beta, n+1)$.

2. If α, β separate at $I \in \mathcal{I}_n$, we say that n is the *level where* α, β *separate*, and we write $j(\alpha, \beta) = n$.

Let $\mathcal{I}$ be a tree of intervals on an ordinal η. Let $\alpha < \beta < \eta$. Note that α, β separate at some $I \in \mathcal{I}$. Now if we put $k = j(\alpha, \beta)$ and $n = n(\alpha)$, the following two conditions hold:

(1) If $k < n$, then $o(\alpha) \cap \bigcup_{m \leq k} E_m = o(\beta) \cap \bigcup_{m \leq k} E_m$ and $o(\alpha) \cap E_{k+1} \subseteq o(\beta) \cap E_{k+1}$.

(2) If $n \leq k$, then $o(\alpha) \cap \bigcup_{m<n} E_m = o(\beta) \cap \bigcup_{m<n} E_m$ and $o(\beta) \cap E_n = (o(\alpha) \cap E_n) \cup \{\alpha\}$.

Also, it can be proved that if $\mathcal{I}$ is a tree of intervals on η, then $o(\alpha)$ is closed for every $\alpha < \eta$. Since this fact will not be used in the proof of the consistency result of Section 2, we leave its proof to the reader.

Next, we define the kind of tree of intervals we will use later.

Definition 1.5. A tree of intervals $\mathcal{I}$ is *cofinal* if the following two conditions hold:

(1) For every $I \in \mathcal{I}_n$ with I^+ a limit ordinal, $E_{n+1} \cap I$ is a sequence of order type $\mathrm{cf}(I^+)$.

(2) For every $I \in \mathcal{I}_n$ with I^+ a successor ordinal, $E_{n+1} \cap I$ is finite.

Proposition 1.2. *For every ordinal $\eta \neq 0$ there is a cofinal tree of intervals on η.*

Proof. We define a cofinal tree of intervals $\mathcal{I} = \bigcup_{n<\omega} \mathcal{I}_n$ on η by induction on n. We put $\mathcal{I}_0 = \{[0, \eta)\}$. In order to construct $\mathcal{I}_{n+1}$ from $\mathcal{I}_n$, first we define for every $I \in \mathcal{I}_n$ a partition $W(I)$ of I as follows. Suppose that $I \in \mathcal{I}_n$ is not a singleton. Assume that I^+ is a limit ordinal. Put $\lambda = \mathrm{cf}(I^+)$. Now, choose a sequence of ordinals $\langle \alpha_\xi : \xi < \lambda \rangle$ with $\alpha_0 = I^-$ converging to I^+ in a strictly increasing way and such that $\alpha_\xi = \sup\{\alpha_\mu : \mu < \xi\}$ for every limit ordinal $\xi < \lambda$. Then we define $W(I) = \{[\alpha_\xi, \alpha_{\xi+1}) : \xi < \lambda\}$.

Now assume I^+ is a successor ordinal. Put $\alpha = I^-$. If $I^+ = \alpha + k$ for some $k < \omega$, we put $W(I) = \{[\alpha, \alpha+1), \ldots, [\alpha+k-1, \alpha+k)\}$, and otherwise we consider the limit ordinal γ and the natural number k such that $I^+ = \gamma + k$, and we put $W(I) = \{[\alpha, \gamma), [\gamma, \gamma+1), \ldots, [\gamma+k-1, \gamma+k)\}$. Also, if $I \in \mathcal{I}_n$ is a singleton, we put $W(I) = \{I\}$. Then we define $\mathcal{I}_{n+1} = \bigcup\{W(I) : I \in \mathcal{I}_n\}$. ∎

Proposition 1.3. *Let κ be an infinite cardinal and η an ordinal such that $0 < \eta < \kappa^{++}$. Let $\mathcal{I}$ be a cofinal tree of intervals on η. Then for every $\alpha < \eta$, $|o(\alpha)| \leq \kappa$.*

Proof. We have

$$o(\alpha) = \bigcup\{E_{m+1} \cap \alpha \cap I(\alpha, m) : m < n(\alpha)\}.$$

Fix $m < n(\alpha)$. Put $I = I(\alpha, m)$. Assume that I^+ is a limit. Then, since $E_{m+1} \cap I$ is a sequence of order type $\mathrm{cf}(I^+) \leq \kappa^+$, we infer that the size of $E_{m+1} \cap \alpha \cap I$ is at most κ. ∎

Propositions 1.1–1.3 will be used without explicit mention.

2. The forcing construction. Our purpose here is to prove the following consistency result:

THEOREM 1. *Assume that in M, κ is an infinite cardinal such that $\kappa^{<\kappa} = \kappa$ and η is an ordinal with $0 < \eta < \kappa^{++}$. Then there is a partial order $\mathbb{P}_\eta$ in M that preserves cardinals such that if G is $\mathbb{P}_\eta$-generic over M, then in $M[G]$ there is a (κ, η)-Boolean algebra.*

To prove Theorem 1, we consider an infinite cardinal κ such that $\kappa^{<\kappa} = \kappa$ and an ordinal η with $\kappa^+ \leq \eta < \kappa^{++}$. Note that from König's Lemma we infer that κ is regular. Also, we fix a cofinal tree of intervals $\mathcal{I} = \bigcup_{n<\omega} \mathcal{I}_n$ on the ordinal η.

Recall that if $(X, \leq)$ is a partial order and $s, t \in X$, we say that s, t are *comparable* in $(X, \leq)$ if $s \leq t$ or $t \leq s$, and s, t are *compatible* in $(X, \leq)$ if there is a $u \in X$ such that $u \leq s$ and $u \leq t$.

Now we introduce our notion of forcing for adding a (κ, η)-Boolean algebra.

DEFINITION 2.1. 1. We set $T = \eta \times \kappa$, and for each $\alpha < \eta$ we write $T_\alpha = \{\alpha\} \times \kappa$.

2. We define P_η as the set of all $p = (x_p, \leq_p)$ such that the following conditions are satisfied:

(∗) (1) x_p is a subset of T of size $< \kappa$.

(2) $\leq_p$ is a partial order on x_p such that:

(a) If $s \in T_\alpha$, $t \in T_\beta$ and $s <_p t$, then $\alpha < \beta$.

(b) Every pair s, t of compatible elements in p has an infimum, that is, there is a $v \in x_p$ with $v \leq_p s, t$ and such that for any $u \in x_p$, $u \leq_p s, t$ implies $u \leq_p v$.

(3) If $s, t \in x_p$ are compatible but not comparable in p, v is the infimum of s, t in p and $s \in T_{\alpha_1}$, $t \in T_{\alpha_2}$, $v \in T_\beta$, then $\beta \in o(\alpha_1) \cap o(\alpha_2)$.

(4) Suppose that $s <_p t$, $s \in T_\alpha$, $t \in T_\beta$. Let $J = I(\alpha, n+1)$ where $n = j(\alpha, \beta)$. If $\alpha \neq J^-$ then there is a $v \in x_p \cap T_{J^+}$ such that $s <_p v \leq_p t$.

3. If $p = (x_p, \le_p) \in P_\eta$ and $s, t \in x_p$ are compatible elements in p, we denote the infimum of s, t in p by $i_p\{s, t\}$.

4. If $p, q \in P_\eta$, we put $p \le_\eta q$ iff (a) $x_p \supseteq x_q$; (b) $\le_p \restriction x_q = \le_q$; and (c) if $s, t \in x_q$ and s, t are compatible in p, then s, t are compatible in q and $i_p\{s, t\} = i_q\{s, t\}$.

5. We define $\mathbb{P}_\eta = (P_\eta, \le_\eta)$.

In what follows, if $s \in T_\alpha$ we write $\pi(s) = \alpha$ and $o(s) = o(\pi(s))$.

Lemma 2.1. *If $\mathbb{P}_\eta$ preserves cardinals, then forcing with $\mathbb{P}_\eta$ adjoins a (κ, η)-Boolean algebra.*

Proof. Let G be a $\mathbb{P}_\eta$-generic filter. We put $\le = \bigcup\{\le_p : p \in G\}$. It is easy to check that $T = \bigcup\{x_p : p \in G\}$ and that $\le$ is a partial order on T.

Now suppose that $\alpha < \beta < \eta$ and $t \in T_\beta$. We show that the set $\{s \in T_\alpha : s < t\}$ is infinite. To check this, assume that $p = (x_p, \le_p) \in P_\eta$ and $\xi_0 < \kappa$. We prove that there is a $q \in P_\eta$ such that $q \le_\eta p$ and $(\alpha, \xi) <_q t$ for some ξ with $\xi_0 < \xi < \kappa$. Without loss of generality, we may assume that $t \in x_p$. Since $|x_p| < \kappa$, there is a ξ with $\xi_0 < \xi < \kappa$ and $(\alpha, \xi) \notin x_p$. Put $v = (\alpha, \xi)$. Let $n = j(\alpha, \beta)$. Let $I \in \mathcal{I}_n$ be the interval where α, β separate. Let $J = I(\alpha, n + 1)$.

First, assume $\alpha = J^-$. Then we define $q = (x_q, \le_q)$ by $x_q = x_p \cup \{v\}$, $<_q = <_p \cup \{(v, t') : t \le_p t'\}$. In order to check that $q \in P_\eta$, it is easy to verify $(*)(1)$–(3). We then prove $(*)(4)$. Consider $t' \in x_p$ such that $t <_p t'$. Let $\gamma = \pi(t')$ and $m = j(\alpha, \gamma)$. Let $I' \in \mathcal{I}_m$ be the interval where α, γ separate. Clearly, $m \le n$. Note that if $m = n$, we have $I = I'$ and so we are done. Suppose $m < n$. Let $J' = I(\alpha, m + 1)$. Since $m < n$, we infer that I' does not separate α and β, and thus $\beta \in J'$. Hence, I' is also the interval where β, γ separate. Therefore, by using $(*)(4)$, there is a $u \in x_p \cap T_{(J')^+}$ such that $t <_p u \le_p t'$, and thus $v <_q u \le_q t'$.

Now assume $\alpha \ne J^-$. Without loss of generality, we may assume $\beta \ne J^+$. Then we choose a $v^* \in T_{J^+}$ such that $v^* \notin x_p$, and we define $q = (x_q, \le_q)$ by

$$x_q = x_p \cup \{v, v^*\},$$
$$<_q = <_p \cup \{(v, v^*)\} \cup \{(v, t') : t \le_p t'\} \cup \{(v^*, t') : t \le_p t'\}.$$

As above, we can verify that $q \in P_\eta$, and so $q \le_\eta p$.

Now, in order to construct a (κ, η)-Boolean algebra from $(T, \le)$, we define for each $t \in T$ the *cone* of t by $C(t) = \{s \in T : s \le t\}$. Then we can verify that $\{C(t) \setminus (C(t_1) \cup \ldots \cup C(t_n)) : n < \omega,\ t, t_1, \ldots, t_n \in T,\ t_1, \ldots, t_n < t\}$ is a clopen base for a topology $\sigma_\le$ on T such that $(T, \sigma_\le)$ is a locally compact, Hausdorff, scattered space. Let A be the one-point compactification of $(T, \sigma_\le)$. For every $\alpha < \eta$, we have $A^\alpha \setminus A^{\alpha+1} = T_\alpha$. Thus, the algebra of clopen subsets of A is a (κ, η)-Boolean algebra. ∎

So, our aim now is to prove the following result:

LEMMA 2.2. $\mathbb{P}_\eta$ *preserves cardinals.*

It is easy to verify that $\mathbb{P}_\eta$ is κ-closed. To show that $\mathbb{P}_\eta$ has the κ^+-chain condition, we need some preparations.

DEFINITION 2.2. Suppose that $g : y \to z$ is a bijection, where $y, z \in [T]^{<\kappa}$. We say that g is *adequate* if the following two conditions hold:

(1) For every $s, t \in y$, $\pi(s) < \pi(t)$ iff $\pi(g(s)) < \pi(g(t))$.
(2) For every $s = (\alpha, \mu) \in y$, $g(\alpha, \mu) = (\beta, \xi)$ implies $\mu = \xi$.

DEFINITION 2.3. A set $Z \subseteq P_\eta$ is *separated* if the following conditions are satisfied:

(1) $\{x_p : p \in Z\}$ forms a Δ-system with root x.
(2) For each $\alpha < \eta$, either $x_p \cap T_\alpha = x \cap T_\alpha$ for every $p \in Z$, or there is at most one $p \in Z$ such that $x_p \cap T_\alpha \neq \emptyset$.
(3) For every $p, q \in Z$ there is an adequate bijection $h_{pq} : x_p \to x_q$ which satisfies the following:

(a) For any $s \in x$, $h_{pq}(s) = s$.
(b) If $s, t \in x_p$, then $s <_p t$ iff $h_{pq}(s) <_q h_{pq}(t)$.
(c) If s, t are compatible in p, then $h_{pq}(i_p\{s, t\}) = i_q(h_{pq}(s), h_{pq}(t))$.

In what follows, if $Z \subseteq P_\eta$ is a separated set, we denote the root of the set $\{x_p : p \in Z\}$ by $x(Z)$.

Since we are assuming $\kappa^{<\kappa} = \kappa$, we can easily prove by means of a combinatorial argument that every set in $[P_\eta]^{\kappa^+}$ has a separated subset of size κ^+.

The following result will be essential in order to amalgamate two forcing conditions of a separated set correctly.

LEMMA 2.3. *Let $Z \subseteq P_\eta$ be a separated set of size κ^+ such that $x = x(Z)$ is nonempty. If $s, t \in x$ are compatible in a condition of Z, then $i_p\{s, t\} = i_q\{s, t\}$ for every $p, q \in Z$.*

Proof. Since Z is separated, it is clear that if $s, t \in x$ are compatible in some condition of Z, then s, t are compatible in any condition of Z. Now since Z is a separated set of size κ^+ and $|o(\alpha)| \le \kappa$ for every $\alpha < \eta$, we infer from $(*)(3)$ the desired conclusion. ∎

Suppose that $Z \subseteq P_\eta$ is a separated set of size κ^+. By the argument of Lemma 2.1, we may extend the conditions in Z so that each extension p has the property that if $\alpha < \beta$, and $x_p \cap T_\alpha$ and $x_p \cap T_\beta$ are nonempty, then there are elements $s \in x_p \cap T_\alpha$ and $t \in x_p \cap T_\beta$ with $s <_p t$. Then we can find a separated set Z' of size κ^+ with $x = x(Z')$ nonempty such that for every $p \in Z'$ there is a unique $q(p) \in Z$ with $p \le q(p)$ and such that if $p, q \in Z'$

with $p \neq q$, then we may find elements $s \in x_p \setminus x$ and $t \in x_q \setminus x$ such that some of the following conditions holds:

(a) For some $v \in x$, $v <_p s$ and $v <_q t$.
(b) There are $u \in x_p \setminus x$ and $v \in x$ such that $u <_p s$ and $u <_p v <_q t$.
(c) There are $u \in x_q \setminus x$ and $v \in x$ such that $u <_q t$ and $u <_q v <_p s$.

Then the following definition permits us to amalgamate p and q in such a way that any pair of such elements has an infimum in the amalgamation.

Definition 2.4. Suppose that $Z \subseteq P_\eta$ is a separated set and $p, q \in Z$ with $p \neq q$.

1. Let $y \in [T]^{<\kappa}$ be such that $y \cap (x_p \cup x_q) = \emptyset$. Put $x = x(Z)$, $y_1 = x_p \setminus x$ and $y_2 = x_q \setminus x$. Suppose that there are adequate bijections $g_1 : y \to y_1$ and $g_2 : y \to y_2$ such that for every $s \in y$, $\pi(s) < \pi(g_1(s)), \pi(g_2(s))$ and there is no $v \in x$ with $\pi(s) \leq \pi(v) < \max\{\pi(g_1(s)), \pi(g_2(s))\}$. Then we define the *amalgamation of* p, q *via* y as the pair $r = (x_r, \leq_r)$ defined as follows. We set $x_r = x_p \cup x_q \cup y$. If $s, t \in x_r$, we put $s \leq_r t$ iff $s \leq_p t$ or $s \leq_q t$ or one of the following conditions holds:

(a) $s, t \in y$ and $g_1(s) \leq_p g_1(t)$,
(b) $s \in y$, $t \in x_p$ and $g_1(s) \leq_p t$,
(c) $s \in y$, $t \in x_q$ and $g_2(s) \leq_q t$,
(d) $s \in x_p$, $t \in y$ and there is a $v \in x$ such that $s \leq_p v \leq_p g_1(t)$,
(e) $s \in y_1$, $t \in y_2$ and there is a $v \in x$ such that $s \leq_p v \leq_q t$,
(f) $s \in x_q$, $t \in y$ and there is a $v \in x$ such that $s \leq_q v \leq_q g_2(t)$,
(g) $s \in y_2$, $t \in y_1$ and there is a $v \in x$ such that $s \leq_q v \leq_p t$.

2. If r is the amalgamation of p, q via y for some $y \in [T]^{<\kappa}$, we say that r is an *amalgamation* of p, q.

We will need conditions $(*)(3)$ and $(*)(4)$ in order to provide room for the set y of the definition of amalgamation.

Lemma 2.4. *Let $Z \subseteq P_\eta$ be a separated set of size κ^+. Let $p, q \in Z$ with $p \neq q$. Then any amalgamation of p, q is a partially ordered set satisfying $(*)(1)$–(2).*

Proof. Suppose that $r = (x_r, \leq_r)$ is an amalgamation of p, q via y for some $y \in [T]^{<\kappa}$. Clearly, $|x_r| < \kappa$. As above, we put $x = x(Z)$, $y_1 = x_p \setminus x$ and $y_2 = x_q \setminus x$. Consider the corresponding adequate bijections $g_1 : y \to y_1$ and $g_2 : y \to y_2$. Then, since for all $s \in y$, $\pi(s) < \pi(g_1(s)), \pi(g_2(s))$ and there is no $v \in x$ with $\pi(s) \leq \pi(v) < \max\{\pi(g_1(s)), \pi(g_2(s))\}$, it is easy to check that for every $s, t \in x_r$, $s <_r t$ implies $\pi(s) < \pi(t)$.

Now, in order to show that $\leq_r$ is a transitive relation, consider $u, s, t \in x_r$ such that $u \leq_r s \leq_r t$. If $u, s, t \in x_p$ or $u, s, t \in x_q$, we are done. For the rest, we consider the following cases:

Case 1: $u, s \in y_1$, $t \in y$. We have $u \leq_p s$. Since $s \leq_r t$, there is a $v \in x$ such that $s \leq_p v \leq_p g_1(t)$, and therefore $u \leq_p v \leq_p g_1(t)$, and so $u \leq_r t$.

Case 2: $u \in y_1$, $s \in y$, $t \in y_1$. Since $u \leq_r s$, there is a $v \in x$ such that $u \leq_p v \leq_p g_1(s)$. Since $s \leq_r t$, we have $g_1(s) \leq_p t$. Therefore, $u \leq_p t$.

Case 3: $u \in y$, $s \in y_1$, $t \in y$. Since $u \leq_r s$, we have $g_1(u) \leq_p s$. From $s \leq_r t$ we infer that there is a $v \in x$ such that $s \leq_p v \leq_p g_1(t)$. Thus $g_1(u) \leq_p g_1(t)$, whence $u \leq_r t$.

Case 4: $u \in y_1$, $s \in y_2$, $t \in y_1$. As $u \leq_r s$, there is a $v_1 \in x$ such that $u \leq_p v_1 \leq_q s$. Since $s \leq_r t$, there is a $v_2 \in x$ such that $s \leq_q v_2 \leq_p t$. Since $v_1 \leq_q v_2$ and $v_1, v_2 \in x$, we have $v_1 \leq_p v_2$, and so $u \leq_p t$.

Case 5: $u \in y$, $s \in y_1$, $t \in y_2$. As $u \leq_r s$, $g_1(u) \leq_p s$. As $s \leq_r t$, there is a $v \in x$ such that $s \leq_p v \leq_q t$. From $g_1(u) \leq_p v$, we deduce $g_2(u) \leq_q v$, and thus $g_2(u) \leq_q t$, whence $u \leq_r t$.

In all the other cases the considerations are similar.

Next, assume that s, t are compatible but not comparable in r. We show that the pair s, t has an infimum $i_r\{s, t\}$ in r. We distinguish the following two cases:

Case 1: $s, t \in x_p$. Then s, t are compatible in p and $v = i_p\{s, t\}$ is the infimum of s, t in r. To see this, consider $u \in x_r$ such that $u \leq_r s, t$. If $u \in x_p$, we have $u \leq_p s, t$, and thus $u \leq_p v$. If $u \in y$, we infer that $g_1(u) \leq_p s, t$, hence $g_1(u) \leq_p v$, and therefore $u \leq_r v$. Now suppose that $u \in y_2$. It follows that there are $w_1, w_2 \in x$ such that $u \leq_q w_1 \leq_p s$ and $u \leq_q w_2 \leq_p t$. Put $w = i_p\{w_1, w_2\}$. By Lemma 2.3, we have $w = i_q\{w_1, w_2\}$, and thus $w \in x$. Now since $u \leq_q w \leq_p s, t$, we infer $u \leq_q w \leq_p v$, and therefore $u \leq_r v$.

Case 2: $s \in y_1$, $t \in y_2$. Let $t' = h_{qp}(t)$. Then we can show that s, t' are compatible in p, and if we put $v = i_p\{s, t'\}$, then $i_r\{s, t\} = v$ if $v \in x$, and $i_r\{s, t\} = g_1^{-1}(v)$ if $v \in y_1$. To verify this statement, assume that $u \in x_r$ is such that $u \leq_r s, t$. First, assume $u \in x_p$. From $u \leq_r t$ we deduce that there is a $w \in x$ such that $u \leq_p w \leq_q t$. Set $w' = i_p\{s, w\}$. Note that since s, t are not comparable in r, we have $s \not\leq_p w$. Then as $|Z| = \kappa^+$ and $|o(w)| = \kappa$, we infer from $(*)(3)$ that $w' \in x$. Also, as $w \leq_q t$, we have $w \leq_p t'$, and thus $u \leq_p w' \leq_p s, t'$. If $u \in x_q$, the argument is parallel. Finally, note that if $u \in y$, we infer $g_1(u) \leq_p s, t'$.

The considerations are analogous in the remaining cases. For example, suppose that $s \in y_1$, $t \in y$. Put $t' = g_1(t)$. Then, by using an argument similar to the one given in Case 2, we can show that s, t' are compatible in p, and if we set $v = i_p\{s, t'\}$, it follows that $i_r\{s, t\} = v$ if $v \in x$, and $i_r\{s, t\} = g_1^{-1}(v)$ if $v \in y_1$. ∎

Note that we also infer from the proof of Lemma 2.4 that if a forcing condition r is an amalgamation of p, q, then $r \leq_\eta p, q$. However, an amalgamation of two forcing conditions of a large separated set need not satisfy $(*)(3)$–(4). To overcome this problem, we need to refine the notion of a separated set.

For every $n < \omega$, we write $\mathcal{I}_n^+ = \{I \in \mathcal{I}_n : \mathrm{cf}(I^+) = \kappa^+\}$.

Definition 2.5. Let $Z \subseteq P_\eta$ be a separated set. Set $x = x(Z)$. Then:

1. Assume that $I \in \mathcal{I}_n^+$. Let $\langle \alpha_\mu : \mu < \kappa^+ \rangle$ be the strictly increasing enumeration of $E_{n+1} \cap I$. Put

$$\xi = \text{the least } \mu \text{ such that } \alpha_\mu \supseteq \pi[x] \cap I.$$

Then we define $\gamma(I) = \alpha_{\xi+\kappa}$.

2. For every $\alpha < \eta$, if there is an $n < \omega$ and an $I \in \mathcal{I}_n^+$ with $\alpha \in I$ such that $\gamma(I) \leq \alpha$, we define $k(\alpha)$ as the least natural number n with this property. Otherwise, we put $k(\alpha) = \infty$.

3. For each $\alpha < \eta$, we define the interval $I(\alpha)$ as follows. If $k(\alpha) < \infty$, we put $I(\alpha) = I(\alpha, k(\alpha))$, and if $k(\alpha) = \infty$, we set $I(\alpha) = \{\alpha\}$.

For every $v \in T$, we write $k(v) = k(\pi(v))$ and $I(v) = I(\pi(v))$.

Next, we introduce the central notion of this section.

Definition 2.6. A separated set $Z \subseteq P_\eta$ is *pairwise equivalent* if for all $p, q \in Z$ and all $s \in x_p$, we have $k(s) = k(h_{pq}(s))$ and $I(s) = I(h_{pq}(s))$.

Lemma 2.5. *Every set in* $[P_\eta]^{\kappa^+}$ *has a pairwise equivalent subset of size* κ^+.

Proof. Let Z be a subset of P_η of size κ^+. We may assume that Z is separated. Then, by using the assumption that $\kappa^{<\kappa} = \kappa$, we prove that Z has a pairwise equivalent subset of size κ^+. First, note that since $\{k(\alpha) : \alpha < \eta\}$ is countable and $|Z| = \kappa^+$, by thinning out Z if necessary we may suppose that for every $p, q \in Z$ and every $s \in x_p$, $k(s) = k(h_{pq}(s))$.

Next, for every ordinal $\alpha < \eta$ we define the *type* of α as follows. If $k(\alpha) < \infty$, we put $\mathrm{tp}(\alpha) = \langle I(\alpha, n) : n \leq k(\alpha) \rangle$. If $k(\alpha) = \infty$, we consider $l(\alpha) =$ the least natural number n such that $\{\alpha\} \in \mathcal{I}_n$, and then we define $\mathrm{tp}(\alpha) = \langle I(\alpha, n) : n \leq l(\alpha) \rangle$. Now if $\mathrm{tp}(\alpha) = \langle I_0, \ldots, I_k \rangle$, for each $n < \omega$ we write $\mathrm{tp}(\alpha, n) = I_n$ if $n \leq k$ and $\mathrm{tp}(\alpha, n) = I_k$ if $n > k$. Then we write $\mathrm{TP}(n) = \{\mathrm{tp}(\alpha, n) : \alpha < \eta\}$ for $n < \omega$.

We prove by induction on n that $|\mathrm{TP}(n)| \leq \kappa$ for each $n < \omega$. The case $n = 0$ is immediate. Suppose that $|\mathrm{TP}(n)| \leq \kappa$. Note that for every $I \in \mathrm{TP}(n)$ and every $\alpha < \eta$ such that $\mathrm{tp}(\alpha, n) = I$, if $\mathrm{cf}(I^+) = \kappa^+$ and $\alpha \geq \gamma(I)$ then $\mathrm{tp}(\alpha, n+1) = I$, and otherwise $\mathrm{tp}(\alpha, n+1) = J$ where $J \subseteq I$ is the interval of $\mathcal{I}_{n+1}$ that contains α. Therefore, for every $I \in \mathrm{TP}(n)$ we have

$$|\{\mathrm{tp}(\alpha, n+1) : \alpha < \eta \text{ and } \mathrm{tp}(\alpha, n) = I\}| \le \kappa.$$

Thus, since $|\mathrm{TP}(n)| \le \kappa$ we infer that $|\mathrm{TP}(n+1)| \le \kappa \otimes \kappa = \kappa$.

Now, we write $a_p = \pi[x_p]$ and $\theta_p = \langle \mathrm{tp}(\alpha) : \alpha \in a_p \rangle$ for $p \in Z$. Consider the cardinal $\lambda < \kappa$ such that $|a_p| = \lambda$ for each $p \in Z$. We deduce that

$$|\{\theta_p : p \in Z\}| \le (\kappa^{<\omega})^\lambda = \kappa.$$

Therefore, there is a subset Y of Z of size κ^+ such that $\theta_p = \theta_q$ for all $p, q \in Y$. Clearly, Y is as required. ∎

Now, to show that $\mathbb{P}_\eta$ is κ^+-c.c., we prove the following result:

LEMMA 2.6. *A pairwise equivalent set $Z \subseteq P_\eta$ of size κ^+ is linked.*

Proof. As above, we denote by x the root of $\{x_p : p \in Z\}$. We write $a_p = \pi[x_p]$ for $p \in Z$, and $a = \pi[x]$.

Consider $p, q \in Z$ with $p \ne q$. Our aim is to show that there is an amalgamation r of p, q such that $r \in P_\eta$. First, for $n < \omega$, we choose for every interval $I \in \mathcal{I}_n^+$ a set $D(I) \subseteq E_{n+1} \cap [I^-, \gamma(I))$ of size κ such that if $\alpha \in (a_p \cup a_q) \cap \gamma(I)$ and $\beta \in D(I)$, then $\alpha < \beta$.

Put $\delta = \mathrm{o.t.}(a_p \setminus a) = \mathrm{o.t.}(a_q \setminus a)$. Let $\{\alpha_\xi : \xi < \delta\}$ and $\{\alpha'_\xi : \xi < \delta\}$ be the strictly increasing enumerations of $a_p \setminus a$ and $a_q \setminus a$ respectively. Note that for every $\xi < \delta$, since $I(\alpha_\xi) = I(\alpha'_\xi)$ and $\alpha_\xi \ne \alpha'_\xi$, we have $k(\alpha_\xi) = k(\alpha'_\xi) < \infty$. Now, proceeding by transfinite induction on ξ, we associate with each pair α_ξ, α'_ξ an ordinal $\beta_\xi \in o(\alpha_\xi) \cap o(\alpha'_\xi)$ as follows. For any $\xi < \delta$, we consider $I = I(\alpha_\xi) = I(\alpha'_\xi)$, and then we define β_ξ as the least element of $D(I) \setminus \{\beta_\mu : \mu < \xi\}$.

It can be shown that $\mu < \xi < \delta$ implies $\beta_\mu < \beta_\xi$. To see this, observe that either $\alpha_\xi \in I(\alpha_\mu)$ or $j(\alpha_\mu, \alpha_\xi) < k(\alpha_\mu)$. In both cases, it is easy to check that $\beta_\mu < \beta_\xi$.

We set $b = \{\beta_\xi : \xi < \delta\}$. Now, for every $\alpha \in b$ we consider the ordinal ξ with $\alpha = \beta_\xi$ and then we define the set $x^{(\alpha)} = \{(\alpha, \mu) \in T_\alpha : (\alpha_\xi, \mu) \in x_p\} = \{(\alpha, \mu) \in T_\alpha : (\alpha'_\xi, \mu) \in x_q\}$. We put

$$y = \bigcup \{x^{(\alpha)} : \alpha \in b\}, \quad y_1 = x_p \setminus x, \quad y_2 = x_q \setminus x.$$

We consider the corresponding adequate bijections $g_1 : y \to y_1$ and $g_2 : y \to y_2$.

Note that for every $s \in y$, if we set $I = I(g_1(s)) = I(g_2(s))$, then by the definition of $\gamma(I)$, there is no $v \in x$ such that $\pi(s) \le \pi(v) < \max\{\pi(g_1(s)), \pi(g_2(s))\}$. So, let us denote by $r = (x_r, \le_r)$ the amalgamation of p and q via y. Our purpose is to show that $r \in P_\eta$. By Lemma 2.4, we know that r satisfies $(*)(1)$–(2).

Next, we show that if s, t are compatible but not comparable in r and v is the infimum of s, t in r, then $\pi(v) \in o(s) \cap o(t)$. If $s, t \in x_p$, we proved in Lemma 2.4 that if s, t are compatible and not comparable in r, then s, t are compatible and not comparable in p and $i_r\{s, t\} = i_p\{s, t\}$. Thus, the cases

$s, t \in x_p$ and $s, t \in x_q$ are immediate. For the rest, we consider the following cases:

CASE 1: $s \in y_1$, $t \in y_2$ and $s, h_{qp}(t)$ are not comparable in p. Put $t' = h_{qp}(t)$. Since s, t are compatible in r, it follows that s, t' are compatible in p. Assume that $v' = i_p\{s, t'\} \in y_1$. Set $v = g_1^{-1}(v')$. It follows from the proof of Lemma 2.4 that $v = i_r\{s, t\}$. Now since s and t' are not comparable in p, we infer from $(*)(3)$ that $\pi(v') \in o(s)$. Also, as $g_1(v) = v'$ we deduce that $\pi(v) \in o(v')$. Therefore, $\pi(v) \in o(s)$. Now consider $v'' = g_2(v)$. Then $\pi(v'') \in o(t)$ and $\pi(v) \in o(v'')$, and hence $\pi(v) \in o(t)$. Thus, $\pi(v) \in o(s) \cap o(t)$. If $i_p\{s, t'\} \in x$, the considerations are analogous.

CASE 2: $s \in y_1$, $t \in y_2$ and $s, h_{qp}(t)$ are comparable in p. We assume $s \leq_p h_{qp}(t)$. If $h_{qp}(t) \leq_p s$, we would proceed in a similar fashion. Put $t' = h_{qp}(t)$. Let $s^* = g_1^{-1}(s)$. It follows from Lemma 2.4 that $s^* = i_r\{s, t\}$. Without loss of generality, we may assume $s \neq t'$. Note that from $g_1(s^*) = s$ we infer that $\pi(s^*) \in o(s)$. We need to verify that $\pi(s^*) \in o(t)$. It is enough to show that $\pi(t') \in I(s)$. Note that in this situation $\pi(s^*) \in o(t')$, and now since $I(t) = I(t')$, we infer that $\pi(s^*) \in o(t)$.

Suppose on the contrary that $\pi(t') \notin I(s)$. Let n be the level where $\pi(s), \pi(t')$ separate and assume $\pi(s) \in J \in \mathcal{I}_{n+1}$. Clearly, $n < k(s)$. Note that if $\pi(s) = J^-$, then since $n < k(s)$ and $J^- \in E_{n+1}$, we infer $k(s) = \infty$. But on the other hand, from $s \in y_1$ we deduce $k(s) < \infty$. Thus, $\pi(s) \neq J^-$. It follows from $(*)(4)$ that there is a $v \in x_p \cap T_{J^+}$ such that $s <_p v \leq_p t'$. Since $n < k(s)$ and $J^+ \in E_{n+1}$, we infer $k(v) = \infty$, and thus $v \in x$. But then we would have $s <_r t$, which contradicts the assumption that s, t are not comparable in r.

CASE 3: $s \in y_1$, $t \in y$ and $s, g_1(t)$ are not comparable in p. Put $t' = g_1(t)$ and $v' = i_p\{s, t'\}$. Let $I = I(t')$. Since $t' = g_1(t)$, we have $\pi(t) \in D(I)$. First, assume that $v' \in y_1$. Consider $v = i_r\{s, t\} = g_1^{-1}(v')$. Note that $\pi(v) \in o(s)$, as $\pi(v') \in o(s)$ and $\pi(v) \in o(v')$. Since $v' \in y_1$, we have $k(v') < \infty$. Then, as $\pi(v') \in o(t')$ it follows that $\pi(v') \in I$ and $\pi(v') \geq \gamma(I)$. Now since $v' = g_1(v)$, we deduce that $\pi(v) \in D(I)$. But from $\pi(t), \pi(v) \in D(I)$ and $\pi(v) < \pi(t)$, it follows that $\pi(v) \in o(t)$.

Now assume that $v' \in x$. Then $v' = i_r\{s, t\}$. We have to prove that $\pi(v') \in o(t)$. Suppose that $\pi(v') \in I$. Let n be such that $I \in \mathcal{I}_n$. Since $\pi(v') \in o(t')$, we have $\pi(v') \in E_{n+1} \cap I$. As $\pi(t) \in D(I)$ we deduce that $\pi(t) > \pi(u)$ for all $u \in x \cap I$, and therefore $\pi(v') \in o(t)$. If $\pi(v') \notin I$, the considerations are analogous.

All the other cases are similar.

Now, let $s, t \in x_r$ be such that $s <_r t$. Suppose that n is the level where $\pi(s), \pi(t)$ separate and $\pi(s) \in J \in \mathcal{I}_{n+1}$. Assume $\pi(s) \neq J^-$. We show that there is a $v \in x_r \cap T_{J^+}$ such that $s <_r v \leq_r t$. If $s, t \in x_p$ or $s, t \in x_q$, we are done. For the rest, we consider the following cases:

CASE 1: $s \in y_1,\ t \in y_2$. Consider $w \in x$ such that $s <_p w <_q t$. Suppose that $\pi(w) \notin J$. Clearly, the pairs $\pi(s), \pi(t)$ and $\pi(s), \pi(w)$ separate at the same interval. Then since $s <_p w$, there is a $v \in x_p \cap T_{J^+}$ such that $s <_p v \leq_p w$, whence $s <_p v <_r t$. If $\pi(w) \in J$, we can use a similar argument.

CASE 2: $s \in y_1,\ t \in y$. Put $t' = g_1(t)$. Let $I' \in \mathcal{I}_m$ be the interval where $\pi(s), \pi(t')$ separate and J' the interval such that $\pi(s) \in J' \in \mathcal{I}_{m+1}$. Since $\pi(t) \in o(t')$, it follows that $\pi(t) \notin J'$. So, I' is also the interval where $\pi(s), \pi(t)$ separate, and thus $J' = J$. As $s <_r t$, there is a $w \in x$ such that $s <_p w <_p t'$, and hence for every $i \leq n$ and every $K \in \mathcal{I}_i^+$ such that $\pi(s), \pi(t') \in K$, we have $\pi(s) < \gamma(K)$. Now, consider $v \in x_p \cap T_{J^+}$ such that $s <_p v \leq_p t'$. Then, since $\pi(v) \in E_{n+1}$, we infer $k(v) = \infty$, so $v \in x$, and therefore $s <_r v <_r t$.

CASE 3: $s \in y,\ t \in x$. Let $s' = g_1(s)$. Consider the interval $I' \in \mathcal{I}_m$ where $\pi(s'), \pi(t)$ separate and the interval $J' \in \mathcal{I}_{m+1}$ with $\pi(s') \in J'$. Note that for $i \leq m$, if $\pi(s') \in K \in \mathcal{I}_i^+$, then since $t \in x$ it follows that $\pi(s') < \pi(t) < \gamma(K)$. Therefore, as $g_1(s) = s'$ we deduce that $\pi(s) \in J'$. But then I' is the interval where $\pi(s), \pi(t)$ separate and $J' = J$. Now as $s' <_p t$ and $\pi(s') \neq J^-$, there is a $v \in x_p \cap T_{J^+}$ such that $s' <_p v \leq_p t$, and hence $s <_r v \leq_r t$.

CASE 4: $s, t \in y$. Put $s' = g_1(s)$ and $t' = g_1(t)$. Note that if $\pi(t') \in I(s')$, then $\pi(s), \pi(t) \in D(I(s'))$, and hence $I(s')$ would be the interval where $\pi(s), \pi(t)$ separate and $\pi(s) = J^-$. Thus, $\pi(t') \notin I(s')$. Then it is not difficult to check that the pairs $\pi(s), \pi(t)$ and $\pi(s'), \pi(t')$ separate at the same interval and that $\pi(s') \in J$. Now consider $v \in x_p \cap T_{J^+}$ such that $s' <_p v \leq_p t'$. Since $\pi(t') \notin I(s')$, we deduce $n < k(s')$, hence we infer $k(v) = \infty$, and therefore $v \in x$. Thus, $s <_r v <_r t$.

All the other cases are similar. ∎

So, we have completed the proofs of Lemma 2.2 and Theorem 1.

In the following result, we show that Theorem 1 can be strengthened by means of an iterated forcing argument.

THEOREM 2. *Assume that in M, κ is an infinite cardinal such that $\kappa^{<\kappa} = \kappa$. Then there is a partial order $\mathbb{P}$ in M that preserves cardinals such that if G is $\mathbb{P}$-generic over M, then in $M[G]$ there is a (κ, η)-Boolean algebra for every $\eta < \kappa^{++}$.*

P r o o f. Following the terminology of [2], we define in M the iterated forcing $\langle \mathbb{R}_\alpha : 1 \leq \alpha \leq \kappa^{++} \rangle$ as follows. We put $\mathbb{R}_1 = \mathbb{P}_{\kappa^+}$. If $\alpha = \beta + 1,\ \beta \geq 1$, we define $\mathbb{R}_\alpha = \mathbb{R}_\beta \otimes \mathbb{Q}_\beta$, where $\mathbb{Q}_\beta = (\mathbb{P}_{\kappa^+ + \beta})^M$. If α is a limit and $\mathrm{cf}(\alpha) < \kappa$, we define $\mathbb{R}_\alpha$ as the inverse limit of the $\mathbb{R}_\beta$ ($\beta < \alpha$). And if α is a limit and $\mathrm{cf}(\alpha) \geq \kappa$, we define $\mathbb{R}_\alpha$ as the direct limit of the $\mathbb{R}_\beta$ ($\beta < \alpha$).

Now we prove by transfinite induction on $\alpha \leq \kappa^{++}$ that $\mathbb{R}_\alpha$ is κ-closed and κ^+-c.c. The case $\alpha = 1$ is clear. Suppose $\alpha = \beta + 1$, $\beta \geq 1$. By the induction hypothesis, $\mathbb{R}_\beta$ is κ-closed and κ^+-c.c. Let G be an $\mathbb{R}_\beta$-generic filter. Then it is easy to check that $(\mathbb{P}_{\kappa^+ + \beta})^{M[G]} = (\mathbb{P}_{\kappa^+ + \beta})^M$, hence $\Vdash_{\mathbb{R}_\beta}$ ($\mathbb{Q}_\beta$ is κ-closed and κ^+-c.c.), and therefore $\mathbb{R}_\alpha$ is κ-closed and κ^+-c.c. Now assume that α is a limit. Clearly, $\mathbb{R}_\alpha$ is κ-closed. Suppose that X is a subset of $\mathbb{R}_\alpha$ of cardinality κ^+. Since $|\mathrm{support}(p)| < \kappa$ for every $p \in \mathbb{R}_\alpha$, we may assume that $\{\mathrm{support}(p) : p \in X\}$ is a Δ-system. Now, by using the argument given in the proof of Theorem 1, we can show that there is a subset Y of X of cardinality κ^+ such that any two elements of Y are compatible. Thus, $\mathbb{R}_{\kappa^{++}}$ is the desired partial order. ■

If κ is a specific uncountable cardinal, we do not know whether the existence of a (κ, κ^{++})-Boolean algebra is consistent with ZFC. However, we want to remark that if κ, λ are specific infinite cardinals with $\kappa < \lambda$, then it is consistent with ZFC that there exists a superatomic Boolean algebra with exactly κ atoms and height λ. To see this, note that an immediate consequence of [1, Theorem 6.1] is that it is consistent with ZFC that there exists a family F such that $F \subseteq \mathcal{P}(\kappa)$, $|F| = \lambda$, $|X| = \kappa$ for each $X \in F$ and $X \cap Y$ is finite for every $X, Y \in F$ with $X \neq Y$. But from the existence of the family F we can easily construct by transfinite induction on $\eta \leq \lambda$ a superatomic Boolean algebra with exactly κ atoms and height η.

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