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The Gaussian measure on algebraic varieties

by

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Abstract. We prove that the ring $\mathbb{R}[M]$ of all polynomials defined on a real algebraic variety $M \subset \mathbb{R}^n$ is dense in the Hilbert space $L^2(M, e^{-|x|^2} d\mu)$, where $d\mu$ denotes the volume form of M and $d\nu = e^{-|x|^2} d\mu$ the Gaussian measure on M.

1. Introduction. The aim of the present note is to prove that the ring $\mathbb{R}[M]$ of all polynomials defined on a real algebraic variety $M \subset \mathbb{R}^n$ is dense in the Hilbert space $L^2(M, e^{-|x|^2} d\mu)$, where $d\mu$ denotes the volume form of M and $d\nu = e^{-|x|^2} d\mu$ is the Gaussian measure on M. For $M = \mathbb{R}^n$, the result is well known since the Hermite polynomials constitute a complete orthonormal basis of $L^2(\mathbb{R}^n, e^{-|x|^2} d\mu)$.

2. The volume growth of an algebraic variety and some consequences. We consider a smooth algebraic variety $M \subset \mathbb{R}^n$ of dimension d and denote by $d\mu$ its volume form. Then M has polynomial volume growth: there exists a constant C depending only on the degrees of the polynomials defining M such that for any euclidian ball B_r with center $0 \in \mathbb{R}^n$ and radius $r > 0$ the inequality

$$\mathrm{vol}_d(M \cap B_r) \;\leq\; C r^d$$

holds (see [Brö]). Via the Crofton formulas the above inequality is a consequence of Milnor's results concerning the Betti numbers of an algebraic variety (see [Mi1], [Mi2], in which the stated inequality is already implicitly contained). This estimate yields first of all that the restrictions to M of polynomials on $\mathbb{R}^n$ are square-integrable with respect to the Gaussian measure on M.

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PROPOSITION 1. *Let M be a smooth submanifold of the euclidian space $\mathbb{R}^n$. Suppose that M has polynomial volume growth*, *i.e.*, *there exist constants C and $l \in \mathbb{N}$ such that for any ball B_r*,

$$\mathrm{vol}_d(M \cap B_r) \le C r^l.$$

Then:

1. *The ring $\mathbb{R}[M]$ of all polynomials on M is contained in the Hilbert space $L^2(M, e^{-|x|^2} d\mu)$.*

2. *The functions $e^{\alpha|x|^2}$ for $\alpha < 1/2$ all belong to $L^2(M, e^{-|x|^2} d\mu)$.*

Proof. Throughout this article, denote the distance of the point $x \in \mathbb{R}^n$ to the origin by $r^2 = |x|^2$. We shall prove that the integrals

$$I_m(M) := \int_M r^m e^{-r^2} d\mu < \infty, \qquad m = 1, 2, \ldots,$$

are finite. We have

$$I_m(M) = \sum_{j=0}^{\infty} \int_{M \cap (B_{j+1} - B_j)} r^m e^{-r^2} d\mu$$

and consequently we can estimate $I_m(M)$ as follows:

$$\begin{aligned} I_m(M) &\le \sum_{j=0}^{\infty} (j+1)^m e^{-j^2} [\mathrm{vol}(M \cap B_{j+1}) - \mathrm{vol}(M \cap B_j)] \\ &\le \sum_{r=0}^{\infty} (r+1)^m e^{-r^2} \, \mathrm{vol}(M \cap B_{r+1}). \end{aligned}$$

Using the assumption on the volume growth of M we immediately obtain

$$I_m(M) \le C \sum_{r=0}^{\infty} (r+1)^{m+l} e^{-r^2}.$$

Denoting the summands of the latter series by a_r, we readily see that it converges, since

$$\frac{a_{r+1}}{a_r} = \frac{(r+1)^{m+l} e^{-r^2-2r-1}}{r^{m+l} e^{-r^2}} = \left(\frac{r+1}{r}\right)^{m+l} \frac{1}{e^{2r+1}} \to 0.$$

A similar calculation yields the result for the functions $e^{\alpha r^2}$ with $\alpha < 1/2$. ∎

3. A dense subspace in $\mathcal{C}^0_\infty(S^n)$. The aim of this section is to verify that a certain linear subspace of $\mathcal{C}^0_\infty(S^n)$ is dense therein. Since the family of functions we have in mind cannot be made into an algebra, we have to replace the standard Stone–Weierstraß argument by something different. The idea for overcoming this problem is to use a combination of the well-known theorems of Hahn–Banach, Riesz and Bochner.

To begin with, we uniformly approximate the function $e^{-r^2}e^{i\langle k,x\rangle}$ for a fixed vector $k \in \mathbb{R}^n$.

LEMMA 1. *Denote by $p_m(x)$ the polynomial*

$$p_m(x) = \sum_{\alpha=0}^{m-1} i^\alpha \langle k, x\rangle^\alpha/\alpha!.$$

Then the sequence $e^{-r^2}p_m(x)$ converges uniformly to $e^{-r^2}e^{i\langle k,x\rangle}$ on $\mathbb{R}^n$.

Proof. The inequality

$$|p_m(x) - e^{i\langle k,x\rangle}| \le \frac{\|k\|^m\|x\|^m}{m!}\, e^{\|k\|\cdot\|x\|}$$

implies (set $y = \|k\| \cdot \|x\|$)

$$\sup_{x\in\mathbb{R}^n} |e^{-r^2}p_m(x) - e^{-r^2}e^{i\langle k,x\rangle}| \le \sup_{0\le y} \frac{y^m}{m!} e^{y-y^2/\|k\|^2} =: C_m.$$

Therefore, we have to check that for any fixed vector $k \in \mathbb{R}^n$ the sequence C_m tends to zero as $m \to \infty$. For simplicity, denote by k the length of the vector $k \in \mathbb{R}^n$. A direct calculation yields

$$C_m = \frac{1}{m!}\left(\frac{k^2}{4} + \frac{k}{4}\sqrt{k^2+8m}\right)^m \times \exp\left(\frac{k^2}{4} + \frac{k}{4}\sqrt{k^2+8m} - \frac{1}{k^2}\left(\frac{k^2}{4} + \frac{k}{4}\sqrt{k^2+8m}\right)^2\right).$$

We are only interested in the asymptotics of C_m. We will thus ignore all constant factors not depending on m. In this sense, we obtain

$$C_m \approx \frac{1}{m!}\left(\frac{k^2}{4} + \frac{k}{4}\sqrt{k^2+8m}\right)^m \exp\left(\frac{k}{8}\sqrt{k^2+8m} - \frac{k^2+8m}{16}\right).$$

The Stirling formula $m! \approx \sqrt{m}\, m^m e^{-m}$ allows us to rewrite the asymptotics of C_m:

$$C_m \approx \frac{1}{\sqrt{m}\, m^m}\left(\frac{k^2}{4} + \frac{k}{4}\sqrt{k^2+8m}\right)^m \exp\left(\frac{k}{8}\sqrt{k^2+8m} + \frac{m}{2}\right).$$

Since

$$\lim_{m\to\infty}(\sqrt{k^2+8m} - \sqrt{8m}) = 0,$$

we can furthermore replace $\sqrt{k^2+8m}$ by $2\sqrt{2m}$:

$$C_m \approx \frac{1}{\sqrt{m}\, m^m}\left(\frac{k^2}{4} + \frac{k}{4}\sqrt{k^2+8m}\right)^m \exp\left(\frac{k}{4}\sqrt{2m} + \frac{m}{2}\right) =: e^{C_m^*}$$

with

$$C_m^* = m\ln\left(\frac{k^2}{4} + \frac{k}{4}\sqrt{k^2+8m}\right) + \frac{k}{2\sqrt{2}}\sqrt{m} + \frac{m}{2} - m\ln(m) - \frac{1}{2}\ln(m).$$

If m is large compared with k, we can estimate $\ln(k^2/4 + (k/4)\sqrt{k^2+8m})$ by $\frac{1}{2}\ln(m) + \alpha$ for some constant α:

$$\begin{aligned} C_m^* &\lessapprox \frac{m}{2}\ln(m) + \alpha m + \frac{k}{2\sqrt{2}}\sqrt{m} + \frac{m}{2} - m\ln(m) - \frac{1}{2}\ln(m) \\ &\le -\frac{m}{2}\ln(m) + (\alpha + 1/2)m + \frac{k}{2\sqrt{2}}\sqrt{m} \\ &\le -\frac{m}{2}\ln(m) + \left(\alpha + 1/2 + \frac{k}{2\sqrt{2}}\right)m \\ &= m\left(\alpha + 1/2 + \frac{k}{2\sqrt{2}} - \frac{1}{2}\ln(m)\right). \end{aligned}$$

Finally, $C_m = \exp(C_m^*)$ converges to zero. ∎

We denote the full ring of polynomials on $\mathbb{R}^n$ by $\mathcal{P}$.

PROPOSITION 2. *The linear space* $\Sigma_\infty := \mathcal{P}\cdot e^{-r^2}$ *is dense in the space* $\mathcal{C}^0_\infty(S^n)$ *of all continuous functions on* $S^n = \mathbb{R}^n \cup \{\infty\}$ *vanishing at infinity.*

Proof. Suppose the closure $\overline{\Sigma}_\infty$ of the linear space Σ_∞ does not coincide with $\mathcal{C}^0_\infty(S^n)$. Then the Hahn–Banach Theorem implies the existence of a continuous linear functional $L : \mathcal{C}^0(S^n) \to \mathbb{R}$ such that

1. $L|_{\Sigma_\infty} = 0$;
2. $L(g_0) \neq 0$ for at least one $g_0 \in \mathcal{C}^0_\infty(S^n)$.

According to Riesz' Theorem (see [Rud, Ch. 6, pp. 129 ff.]), L may be represented by two regular Borel measures μ_+, μ_- on S^n:

$$L(f) = \int_{S^n} f(x)\, d\mu_+(x) - \int_{S^n} f(x)\, d\mu_-(x).$$

In particular, μ_+ and μ_- are finite. The first property $L|_{\Sigma_\infty} = 0$ of L implies

$$\int_{S^n} e^{-r^2} p(x)\, d\mu_+(x) = \int_{S^n} e^{-r^2} p(x)\, d\mu_-(x)$$

for any polynomial $p(x)$. Let us introduce the measures $\nu_\pm = e^{-r^2}\mu_\pm$ on the subset $\mathbb{R}^n \subset S^n$. Then

$$\int_{\mathbb{R}^n} p(x)\, d\nu_+(x) = \int_{\mathbb{R}^n} p(x)\, d\nu_-(x)$$

holds and remains true for any complex-valued polynomial. We may thus choose $p(x) = p_m(x)$ as in the previous lemma:

$$p_m(x) = \sum_{\alpha=0}^{m-1} i^\alpha \langle k, x\rangle^\alpha/\alpha!.$$

But then

$$\int_{S^n} p_m(x)e^{-r^2}\,d\mu_+(x) = \int_{\mathbb{R}^n} p_m(x)\,d\nu_+(x) = \int_{\mathbb{R}^n} p_m(x)\,d\nu_-(x)$$
$$= \int_{S^n} p_m(x)e^{-r^2}\,d\mu_-(x)$$

together with the uniform convergence of $p_m(x)e^{-r^2}$ to $e^{i\langle k,x\rangle}e^{-r^2}$ implies

$$\int_{S^n} e^{i\langle k,x\rangle}e^{-r^2}\,d\mu_+(x) = \int_{S^n} e^{i\langle k,x\rangle}e^{-r^2}\,d\mu_-(x),$$

i.e.,

$$\int_{\mathbb{R}^n} e^{i\langle k,x\rangle}\,d\nu_+(x) = \int_{\mathbb{R}^n} e^{i\langle k,x\rangle}\,d\nu_-(x).$$

Therefore, the Fourier transforms of the measures ν_+ and ν_- coincide. Consequently, by Bochner's Theorem (see [Mau, Ch. XIX, pp. 774 ff.]) we conclude that $\nu_+ = \nu_-$ on $\mathbb{R}^n$. The linear functional $L : \mathcal{C}^0(S^n) \to \mathbb{R}$ must thus be the evaluation of a function at infinity:

$$L(f) = cf(\infty),$$

contrary to the existence of a function $g_0 \in \mathcal{C}^0_\infty(S^n)$ satisfying $L(g_0) \neq 0$. ∎

4. The main result

THEOREM 1. *Let the closed subset $M \subset \mathbb{R}^n$ be a smooth submanifold satisfying the polynomial volume growth condition. Then the ring $\mathbb{R}[M]$ of all polynomials on M is a dense subspace of the Hilbert space $L^2(M, e^{-r^2}d\mu)$.*

Proof. Consider the one-point compactification $\widehat{M} \subset S^n$ of $M \subset \mathbb{R}^n$. Then Tietze's Extension Lemma and Proposition 2 imply that

$$\Sigma_\infty(\widehat{M}) := \mathbb{R}[M] \cdot e^{-r^2/4}$$

is dense in $\mathcal{C}^0_\infty(\widehat{M})$. We introduce the measure $d\nu = e^{-r^2/2}d\mu$, where $d\mu$ is the volume form of M. Since

$$\int_M d\nu = \int_M e^{-r^2/2}\,d\mu = \int_M (e^{r^2/4})^2 e^{-r^2}\,d\mu =: V < \infty,$$

$d\nu$ defines a regular Borel measure $d\widehat{\nu}$ on $\widehat{M}$ (by setting $d\widehat{\nu}(\infty) = 0$). Therefore, the algebra $\mathcal{C}^0_\infty(\widehat{M})$ of all continuous functions on $\widehat{M}$ vanishing at infinity is dense in $L^2(\widehat{M}, d\widehat{\nu})$:

$$\overline{\mathcal{C}^0_\infty(\widehat{M})} = L^2(\widehat{M}, d\widehat{\nu}).$$

For any function f in $L^2(M, e^{-r^2} d\mu)$ we have

$$\int_M |fe^{-r^2/4}|^2 e^{-r^2/2}\, d\mu = \int_M |f|^2 e^{-r^2}\, d\mu < \infty$$

and, therefore, $fe^{-r^2/4}$ lies in $L^2(\widehat{M}, d\widehat{\nu})$. Thus, for a fixed $\varepsilon > 0$, there exists $g \in \mathcal{C}^0_\infty(\widehat{M})$ such that

$$\int_M |fe^{-r^2/4} - g(x)|^2 e^{-r^2/2}\, d\mu < \varepsilon/2.$$

According to Proposition 2 we can find a polynomial $p(x) \in \mathbb{R}[M]$ approximating g:

$$\sup_{x\in\widehat{M}} |g(x) - p(x)e^{-r^2/4}|^2 < \varepsilon/(2V).$$

Using the inequality $\|x+y\|^2 \le 2\|x\|^2 + 2\|y\|^2$ we conclude that

$$\int_M |f(x)e^{-r^2/4} - p(x)e^{-r^2/4}|^2 e^{-r^2/2}\, d\mu < \varepsilon;$$

but this is equivalent to

$$\int_M |f(x) - p(x)|^2 e^{-r^2}\, d\mu < \varepsilon. \ \blacksquare$$

Remark 1. In fact, the smoothness of M is not essential in the proof of Theorem 1. By the same arguments, the main result holds for any manifold M closed in $\mathbb{R}^n$, provided that M admits a volume form such that the condition of polynomial volume growth as formulated in Proposition 1 is satisfied.

5. Examples and final remarks. We now give a few simple examples. Notice that we recover, of course, that the polynomials are dense in $L^2(\mathbb{R}^n, e^{-r^2} d\mu)$ (Hermite polynomials) or in $L^2(M, d\mu)$ for any compact submanifold (Legendre polynomials in the case $M = [-1, 1]$).

Example 1. Consider a revolution surface in $\mathbb{R}^3$ defined by two polynomials f, h,

$$\begin{cases} x = f(u_1)\cos u_2, \\ y = f(u_1)\sin u_2, \qquad f(u_1) > 0,\ (u_1, u_2) \in \mathbb{R}\times[0, 2\pi]. \\ z = h(u_1), \end{cases}$$

Then $d\mu = f\sqrt{f'^2 + h'^2}\, du_1 du_2$ and $r^2 = f^2 + h^2$, and thus

$\mathbb{R}[f\cos u_2, f\sin u_2, h]$ is dense in

$$L^2(\mathbb{R}\times[0, 2\pi], e^{-(f^2+h^2)} f\sqrt{f'^2 + h'^2}\, du_1 du_2).$$

In the special case of a cylinder, i.e. $f = 1$, $h = u_1$, this reduces to the well known fact that the ring

$$\mathbb{R}[u_1, \cos u_2, \sin u_2] = \mathbb{R}[u_1] \otimes \mathbb{R}[\cos u_2, \sin u_2]$$

is indeed dense in the Hilbert space

$$L^2(\mathbb{R} \times [0, 2\pi], e^{-u_1^2} du_1 du_2) = L^2(\mathbb{R}, e^{-u_1^2} du_1) \otimes L^2([0, 2\pi], du_2).$$

Example 2. Let $F : \mathbb{C} \to \mathbb{C}$ be a polynomial and consider the surface defined by

$$f : \mathbb{C} \to \mathbb{R}^3, \qquad f(z) = (x, y, |F(z)|), \qquad z = x + iy.$$

Then one checks that $d\mu = \sqrt{1 + |F'|^2}\, |dz|^2$ and $r^2 = |z|^2 + |F(z)|^2$. Thus

$$\overline{\mathbb{R}[x, y, |F(z)|]} = L^2(\mathbb{R}^2, e^{-(|z|^2 + |F(z)|^2)} \sqrt{1 + |F'|^2}\, |dz|^2).$$

Let us study the polynomial $F = z^{2k}$ in more detail. Here the coordinate ring coincides with the usual polynomial ring $\mathbb{R}[x, y]$ in two variables, and thus we have proved that these are dense in

$$L^2(\mathbb{R}^2, e^{-(|z|^2 + |z|^{4k})} \sqrt{1 + 4k^2 |z|^{2(2k-1)}}\, |dz|^2).$$

Example 3. We finish with a one-dimensional example: the graph $M = \{(x, f(x))\}$ of a polynomial $f : \mathbb{R} \to \mathbb{R}^n$. Then $d\mu = \sqrt{1 + \|f'\|^2}\, dx$, and we obtain

$$\overline{\mathbb{R}[x]} = L^2(\mathbb{R}, e^{-(x^2 + \|f(x)\|^2)} \sqrt{1 + \|f'\|^2}\, dx).$$

Remark 2. The main result raises an interesting analogous problem in complex analysis which, to our knowledge, is still open. It is well known that the polynomials on $\mathbb{C}^n$ are dense in the Fock or Bergman space

$$\mathcal{F}(\mathbb{C}^n) := \{f \in L^2(\mathbb{C}^n, e^{-r^2} d\mu) \mid f \text{ holomorphic}\}.$$

Furthermore, a theorem by Stoll (see [Sto1], [Sto2]) states that among all complex-analytic submanifolds N of $\mathbb{C}^n$, those with polynomial growth are *exactly* the algebraic ones, and thus the only ones for which the elements of the coordinate ring are square-integrable with respect to the Gaussian measure. It is then common to study the space

$$\mathcal{F}(N) := \{f \in L^2(N, e^{-r^2} d\mu) \mid f \text{ holomorphic}\},$$

but we were not able to find any results on whether $\mathbb{C}[N]$ is dense herein.

More elaborate applications of the main result to the situation where M carries a reductive algebraic group action will be discussed in some forthcoming works (see e.g. [Agr]). In this case, one can decompose the ring $\mathbb{R}[M]$ into isotypic components and, via Theorem 1, one obtains a decomposition of $L^2(M, e^{-r^2} d\mu)$ analogous to the classical Frobenius reciprocity.

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