FUNDAMENTA
MATHEMATICAE
159 (1999)

On products of Radon measures

by

C. Gryllakis and **S. Grekas** (Athens)

Abstract. Let $X = [0,1]^{\Gamma}$ with card $\Gamma \geq c$ (c denotes the continuum). We construct two Radon measures μ, ν on X such that there exist open subsets of $X \times X$ which are not measurable for the simple outer product measure. Moreover, these measures are strikingly similar to the Lebesgue product measure: for every finite $F \subseteq \Gamma$, the projections of μ and ν onto $[0,1]^F$ are equivalent to the F-dimensional Lebesgue measure. We generalize this construction to any compact group of weight $\geq c$, by replacing the Lebesgue product measure with the Haar measure.

1. Introduction. Suppose that (X, Σ, μ) and (Y, T, ν) are topological probability spaces, that is, probability spaces with topologies such that every open set is measurable. We can form product measures on $X \times Y$ in various ways. First, we have the ordinary completed product measure $\mu \times \nu$ derived by Carathéodory's method from the outer measure $(\mu \times \nu)^*$, where

$$(\mu \times \nu)^* C = \inf\Big\{\sum_{n=0}^{\infty} \mu E_n \cdot \nu F_n : E_n \in \Sigma,\ F_n \in T,\ n \in \mathbb{N},\ C \subseteq \bigcup_{n=0}^{\infty} E_n \times F_n\Big\}.$$

It can happen that $\mu \times \nu$ is again a topological measure, that is, every set open in $X \times Y$ for the product topology is $\mu \times \nu$-measurable. This is known as the *product measure problem*. The conditions under which it occurs are not well understood yet. On dyadic spaces the problem has a positive answer for a large class of measures; see [Fr-Gr]. In the negative direction, D. H. Fremlin [Fr$_1$], [Fr$_2$] proved that if (S, ν) is the hyperstonian space of the Lebesgue measure on $[0,1]$, then there are open sets in $S \times S$ which are not measurable for the simple product outer measure. More recent results were obtained by M. Talagrand [T$_2$].

Although the measure on the hyperstonian space has the special advantage of being completion regular, it is somehow irritating that the only

1991 *Mathematics Subject Classification*: Primary 28C15.

known counterexample, up to now, for this deceptively simple problem, is a hyperstonian space. In the present paper, using a lemma of Fremlin [Fr_1] as well as a construction of Erdős and Oxtoby, we construct two (probability) measures μ, ν on $[0,1]^A$ with card $A \geq c$ which furnish a counterexample to the product measure problem. These measures are strikingly similar to the usual Lebesgue product measure on $[0,1]^A$. To be precise, for every finite $F \subseteq A$ the projections $\mathrm{pr}_F(\mu)$ and $\mathrm{pr}_F(\nu)$ are equivalent to the Lebesgue product measure on $[0,1]^F$. Moreover, the relevant density functions are step functions. We notice that this similarity to the Lebesgue product measure is, in a certain sense, best possible; for details see Remark 1. We hope that, in view of this striking similarity, our counterexample will help to bring to light the elusive circumstances under which the answer to the product measure problem is positive. Furthermore, our construction can be extended to the case where X is any compact group of weight $\geq c$. In this extension, we replace the Lebesgue product measure by the Haar measure.

The original impulse to study such questions arose from a simple desire to understand the nature of Radon product measures. However, there are important problems in functional analysis and probability theory which relate to our counterexample; see for instance the distinction between "stable" and "R-stable" sets which is necessary in [T_1] (cf. [T_2]).

2. A counterexample

2.1. *Construction of* μ, ν. We need the following fact, proved by Erdős–Oxtoby [Er-Ox] (see also [Fr_1], p. 286). Given $\varepsilon > 0$, there exists an open subset U^* of the unit square $[0,1]^2$ such that

(i) $\lambda^2 U^* < \varepsilon$ (λ denotes the Lebesgue measure),

(ii) if $E, F \subseteq [0,1]$ are such that $\lambda E \cdot \lambda F > 0$, then there exist $E' \subseteq E$ and $F' \subseteq F$ with $\lambda E' \cdot \lambda F' > 0$ such that $E' \times F' \subseteq U^*$.

Now we come to the definition of the measures μ and ν. We enumerate the family $\mathcal{B} = \{E_i \times F_i : i < c\}$, where E_i and F_i are Borel subsets of $[0,1]$ with $E_i \times F_i \subset U^*$ and $\lambda E_i > 0$, $\lambda F_i > 0$. We consider the spaces $\prod_{i<c} X_i$ and $\prod_{i<c} Y_i$, where $X_i = Y_i = [0,1] = I$ for every $i < c$. For each $x \in I$, $y \in I$ and $i < c$, let μ_i^x and ν_i^y be the Radon probability measures on X_i and Y_i respectively given by

$$\mu_i^x = \begin{cases} \lambda & (= \text{the Lebesgue measure on } X_i = [0,1]) \text{ if } x \in E_i, \\ \lambda' & (= \text{the restriction } \lambda|[1/2,1] \text{ normalized to be a probability} \\ & \quad \text{measure) if } x \notin E_i, \end{cases}$$

$$\nu_i^y = \begin{cases} \lambda & \text{if } y \in F_i, \\ \lambda' & \text{if } y \notin F_i. \end{cases}$$

Now, for any $x \in I$ and $y \in I$, let μ_x and ν_y be the Radon product probability measures $\bigotimes_{i<c}\mu_i^x$ and $\bigotimes_{i<c}\nu_i^y$ on $X=\prod_{i<c}X_i$ and $Y=\prod_{i<c}Y_i$, respectively.

LEMMA 1. (a) *If $U\subseteq\prod_{i<c}X_i$ and $V\subseteq\prod_{i<c}Y_i$ are elementary open sets, then the functions $x\mapsto\mu_xU:[0,1]\to[0,1]$ and $y\mapsto\nu_yV:[0,1]\to[0,1]$ are Borel measurable.*

(b) *There are (unique) Radon measures μ_1 and ν_1 on $I\times X$ and $I\times Y$ respectively such that*

$$\mu_1(D)=\int\mu_xD_x\,\lambda(dx)\quad\textit{and}\quad\nu_1(D)=\int\nu_yD_y\,\lambda(dy)$$

for all Baire sets $D\subset I\times X$, resp. $D\subset I\times Y$, where $D_x=\{u\in X:(x,u)\in D\}$ and $D_y=\{u\in Y:(y,u)\in D\}$.

Proof. Express U as $U=\{u:u(t)\in U_t\ \forall t\in F\}$, where F is a finite set of coordinates and $U_t\subset[0,1]$ is open. For $x\in[0,1]$ we have $\mu_x(U)=\prod_{t\in F}\mu_t^x(U_t)$. Since each map $x\mapsto\mu_t^x(U_t)$, $t\in F$, is clearly Borel measurable, it follows that $x\mapsto\mu_x(U)$ is Borel measurable (the same holds for $y\mapsto\nu_y(V)$).

Let Ω be the class of those sets D for which the map $x\mapsto\mu_xD_x$ is Borel measurable. Then Ω is closed under monotone limits of sequences, and $U\setminus V\in\Omega$ whenever $U,V\in\Omega$ and $V\subset U$. Also Ω contains all elementary open sets in $I\times\prod_{i<c}X_i$, therefore it contains all the Baire sets.

For every Baire set $D\subset I\times\prod_{i<c}X_i$, we define the Baire measures

$$\mu_1D=\int\mu_xD_x\,\lambda(dx),\qquad\nu_1D=\int\nu_yD_y\,\lambda(dy).$$

Clearly μ_1 and ν_1 are Baire measures on $I\times X$, resp. $I\times Y$, which have unique extensions to Radon measures. Let μ and ν be the projections of μ_1 and ν_1 onto the spaces $\prod_{i<c}X_i$ and $\prod_{i<c}Y_i$ respectively (note that we also have

$$(\mu_1\otimes\nu_1)(B)=\int(\mu_x\times\nu_y)(B_{x,y})\,d(x,y),$$

where $B_{x,y}=\{(u,v)\in X\times Y:(x,u,y,v)\in B\}$ and B is any Baire subset of $I\times X\times I\times Y$). The proof of the lemma is complete. ■

2.2. *An open set in $X\times Y$ that is not $\mu\times\nu$-measurable.* Our example of an open set in $\prod_{i<c}X_i\times\prod_{i<c}Y_i$ which is not $\mu\times\nu$-measurable has the very simple form

$$\mathcal{U}=\bigcup_{i<c}U_i\times V_i$$

where $U_i=\mathrm{pr}_i^{-1}[0,1/2)$ and $V_i=\mathrm{pr}_i^{-1}[0,1/2)$ for every $i<c$ and $\mathrm{pr}_i:\prod_{i<c}X_i\to X_i$ is the canonical projection.

2.3. *The $\mu\otimes\nu$-measure of $\mathcal{U}$*

LEMMA 2. *The $\mu\otimes\nu$-measure of $\mathcal{U}$ is less than or equal to ε.*

Proof. We can find $K \subset \mathcal{U}$ compact such that $(\mu \otimes \nu)(\mathcal{U} \setminus K) < \varepsilon$. There exists a finite $M \subset c$ such that $K \subset \bigcup_{i\in M} U_i \times V_i$. We set $E = \bigcup_{i\in M} U_i \times V_i$. Then E is a Baire set. Using the relative formula of Lemma 1 we see immediately that $(\mu \otimes \nu)(E) \le \varepsilon$. ∎

2.4. *The outer $\mu \times \nu$-measure of $\mathcal{U}$.* We set $\mathcal{U}' = \bigcup_{i<c} U_i' \times V_i'$, where $U_i' = \mathrm{pr}_i^{-1}[0,1/4]$ and $V_i' = \mathrm{pr}_i^{-1}[0,1/4]$ for $i < c$. Clearly $U_i' \subset X_i$, $V_i' \subset Y_i$, $\mathcal{U}' \subset \mathcal{U}$ and we shall show that $(\mu \times \nu)^*(\mathcal{U}') = 1$ (hence $(\mu \times \nu)^*(\mathcal{U}) = 1$).

By definition and by the regularity of μ and ν, for some δ small enough we have

$$(\mu \times \nu)^*(\mathcal{U}') + \delta = \sum_{n=0}^{\infty} \mu W_n \cdot \nu T_n,$$

where $W_n \subset X$ and $T_n \subset Y$ are open sets for each $n \in \mathbb{N}$ and $\mathcal{U}' \subset \bigcup_{n=0}^{\infty} W_n \times T_n$. Since each $U_i' \times V_i'$ is compact, for each $i < c$ there is a finite $L_i \subset \mathbb{N}$ such that $U_i' \times V_i' \subset \bigcup_{n\in L_i} W_n \times T_n$.

We consider the family $\{Q \times R \subset \bigcup_{n\in L_i} W_n \times T_n\}$ of measurable rectangles, directed by set-theoretic inclusion; the maximal elements of this family are clearly finitely many, say $Q_1^i \times R_1^i$, $Q_2^i \times R_2^i, \ldots, Q_{n_i}^i \times R_{n_i}^i$. The family $\{Q_j^i \times R_j^i : i < c,\ 1 \le j \le n_i\}$ has countably many pairwise distinct elements; enumerate this family as $\{Q_n \times R_n : n \in \mathbb{N}\}$.

For each $n \in \mathbb{N}$ we consider the set $I_n \subset c$ defined by $I_n = \{i < c : U_i' \times V_i' \subset Q_n \times R_n\}$. Then clearly, for every $n \in \mathbb{N}$,

$$\Big(\bigcup_{i\in I_n} U_i'\Big) \times \Big(\bigcup_{i\in I_n} V_i'\Big) \subseteq Q_n \times R_n \subseteq \bigcup_{n=0}^{\infty} W_n \times T_n.$$

Thus, we have proved

LEMMA 3. *Assume the above notation and definitions. Then there exists a countable family I_n, $n \in \mathbb{N}$, of subsets of c such that*

(i) $\bigcup_{n\in\mathbb{N}} I_n = c$,
(ii) *for every* $n \in \mathbb{N}$, $(\bigcup_{i\in I_n} U_i') \times (\bigcup_{i\in I_n} V_i') \subseteq \bigcup_{n=0}^{\infty} W_n \times T_n$.

We set $U_i'' = \mathrm{pr}_i^{-1}[0,1/4)$ and $V_i'' = \mathrm{pr}_i^{-1}[0,1/4)$. Since μ and ν are regular, for every $n \in \mathbb{N}$ there exists a countable $M_n \subset I_n$ such that $\mu(\bigcup_{i\in I_n} U_i'') = \mu(\bigcup_{i\in M_n} U_i'')$ and $\nu(\bigcup_{i\in I_n} V_i'') = \nu(\bigcup_{i\in M_n} V_i'')$. We set $U^n = \bigcup_{i\in M_n} U_i''$ and $V^n = \bigcup_{i\in M_n} V_i''$ for $n \in \mathbb{N}$. We observe that these are Baire sets.

Our next auxiliary result is

LEMMA 4. *Assume the above notations and definitions. For $n \in \mathbb{N}$, set*

$$D_n = \{x \in I : \mu_x(U^n) < 1\}, \quad C_n = \{y \in I : \nu_y(V^n) < 1\}.$$

Then card$\{E_i\times F_i\subseteq U^*: i\in I_n$, *at least one of* $E_i\cap D_n$, $F_i\cap C_n$ *has strictly positive Lebesgue measure*$\}$ *is at most countable.*

Proof. Suppose the contrary. Then the set $\{E_i\times F_i: i\in I_n$, at least one of $E_i\cap D_n$, $F_i\cap C_n$ has strictly positive Lebesgue measure$\}$ is uncountable. Thus at least one of the following sets is uncountable:

$$\{E_i: i\in I_n,\ \lambda(E_i\cap D_n)>0\},\qquad \{F_i: i\in I_n,\ \lambda(F_i\cap C_n)>0\}.$$

Suppose it is the former. Then there exists some $i_0>\sup\{i: i\in M_n\}$ such that $\lambda(E_{i_0}\cap D_n)>0$. It follows easily that $\mu_x(U^n)<\mu_x(U^n\cup U''_{i_0})$ for every $x\in E_{i_0}\cap D_n$. Therefore $\mu(U^n)<\mu(\bigcup_{i\in I_n}U''_i)$, which contradicts $\mu(\bigcup_{i\in I_n}U''_i)=\mu(\bigcup_{i\in M_n}U''_i)$. ∎

We now calculate $(\mu\times\nu)^*(\mathcal{U}')$.

Consider the sets $A_n=[0,1]\setminus D_n=\{x\in I:\mu_x(U^n)=1\}$ and $B_n=[0,1]\setminus C_n=\{y\in I:\nu_y(V^n)=1\}$. We prove that $(\mu\times\nu)(\bigcup_{n\in\mathbb{N}}U^n\times V^n)=1$. Since $(\mu\times\nu)(\bigcup_n U^n\times V^n)\ge(\lambda\times\lambda)(\bigcup_n A_n\times B_n)$ it suffices to verify that $\lambda^2(\bigcup_n A_n\times B_n)=1$. Supposing that $\lambda^2(\bigcup_n A_n\times B_n)<1$, we will arrive at a contradiction of with the help of the key Lemma A of [Fr$_1$], p. 286 (see also proof of 345K Lemma in [Fr$_2$]).

Suppose, if possible, that $(\lambda\times\lambda)(\bigcup_n A_n\times B_n)<1$. In view of Lemma 4, enlarging slightly A_n and B_n, we can assume that $U^*\subseteq\bigcup_n A_n\times B_n$. By Lemma 4, the family $\{E\times F\subseteq U^*$: at least one of $E\cap D_n$, $F\cap C_n$ has positive Lebesgue measure for every $n\}$ is at most countable. We enumerate this family as $\{A'_k\times B'_k: k\in\mathbb{N}\}$.

On $[0,1]$ we consider the second countable topology which has as base the sets of a countable base for the usual topology on $[0,1]$ plus the sets D_n, C_n, $[0,1]\setminus A'_k$, $[0,1]\setminus B'_k$, $n,k\in\mathbb{N}$. Since the Lebesgue measure on $[0,1]$ with this new topology is τ-additive, we can consider its support $K\subseteq[0,1]$. By Lemma A of [Fr$_1$], there exist some $(t,u)\in K\times K$ with $(t,u)\notin\bigcup_n A_n\times B_n\supset\bigcup_k A'_k\times B'_k$ and some $E\times F\subset U^*$ with E and F being open subsets of K in the restriction of the new topology to K, such that $(t,u)\in\widetilde{E}\times\widetilde{F}$, where $\widetilde{C}$ is the closure of C in this topology. Now again by Lemma 4, E and F are almost contained in A_n,A'_k and B_n,B'_k respectively, for some n or $k\in\mathbb{N}$. Since λ is supported on K (with respect to the new topology) and E and F are open in K, it follows that they are contained in the closed sets (for the new topology) $A_n\cap K$, $A'_k\cap K$ and $B_n\cap K$, $B'_k\cap K$ respectively, for some n or $k\in\mathbb{N}$. Thus $\widetilde{E}$ and $\widetilde{F}$ are subsets of A_n,A'_k and B_n,B'_k respectively, for some n or $k\in\mathbb{N}$. This leads to a contradiction, since $(t,u)\in\widetilde{E}\times\widetilde{F}$ and $(t,u)\notin\bigcup_n A_n\times B_n\supset\bigcup_k A'_k\times B'_k$. ∎

Remark. If $F\subset c$ is finite, then it is easily seen that, for the projections $\mathrm{pr}_F\mu$ and $\mathrm{pr}_F\nu$ on $\prod_{i\in F}X_i$ and $\prod_{i\in F}Y_i$, we have $\mathrm{pr}_F\mu\ll\lambda_F$

and $\mathrm{pr}_F\nu \ll \lambda_F$ (where $\ll$ denotes absolute continuity and λ_F is the F-dimensional Lebesgue measure on $\prod_{i\in F} X_i$). So, if instead of μ and ν we consider the measures $\mu+m$ and $\nu+m$ (where m is the Lebesgue product measure on $\prod_{i<c} X_i$), then $\mathrm{pr}_F(\mu+m)$ and $\mathrm{pr}_F(\nu+m)$ are both equivalent to the Lebesgue product measure.

This similarity of the measures $\mu+m$ and $\nu+m$ to the Lebesgue product measure is, in a sense, best possible; to be precise, if for two measures η,θ on $\prod_{i<c} X_i$, $\mathrm{pr}_M\eta$ and $\mathrm{pr}_M\theta$ are equivalent to the Lebesgue product measure on $\prod_{i\in M} X_i$ for every countable $M\subset c$, then it is easily seen that these measures satisfy b) of Proposition on p. 564 of [Gry]. Therefore they are completion regular. But for completion regular measures on products of compact metric spaces the answer to the product measure problem is positive; see e.g. [Gry] (cf. [Fr-Gr]).

Note. The above construction can be carried out on every uncountable product of c-many compact metric spaces (and in particular on $\{0,1\}^c$) as follows:

1. Let X_i, $i<c$, be a family of compact metric spaces, with at least two points each, μ_i a strictly positive Radon probability measure on X_i, and $I=[0,1]$. Let also $A_i\subseteq X_i$ be open with $\mu_iA_i<1$. We set $Y_i=X_i$, $\nu_i=\mu_i$, $i<c$.

2. For $x,y\in I$ and $i<c$, let μ_i^x and ν_i^y be the probability measures on X_i and Y_i respectively given by

$$\mu_i^x=\begin{cases}\mu_i, & x\in E_i,\\ \mu_i'\Big(:=\dfrac{1}{\mu_i(A_i^{\mathrm{c}})}\times\mu_i|A_i^{\mathrm{c}}\Big), & x\notin E_i,\end{cases}\qquad \nu_i^x=\begin{cases}\nu_i, & y\in F_i,\\ \nu_i'\ (:=\mu_i'), & y\notin F_i,\end{cases}$$

where E_i and F_i are as in 2.1.

3. We can now define (in analogy) the measures μ_x and ν_y as well as μ and ν on $X=\prod_{i<c}X_i$ and $Y=\prod_{i<c}Y_i$ respectively (note that now X_i and Y_i correspond to $[0,1]$). Then, by the same procedure, the set $\mathcal{U}=\bigcup_{i<c}U_i\times U_i$ ($U_i=\mathrm{pr}_i^{-1}A_i,\ \mathrm{pr}_i:X\to X_i$ the projection) satisfies

(α) $(\mu\otimes\nu)(\mathcal{U})\le\varepsilon$,

(β) $(\mu\times\nu)^*(\mathcal{U})=1$,

(γ) if $M\subset c$ is any finite set, and μ_M and $\widetilde{\mu}_M$ are the projections of μ and $\widetilde{\mu}=\bigotimes_{i<c}\mu_i$, respectively, onto $\prod_{k\in M}X_k$, then μ_M is absolutely continuous with respect to $\widetilde{\mu}_M$.

3. Generalizations

3.1. *The property* (I). From the results of §2 it easily follows that for every cardinal $a\ge c$, there exist positive Radon measures μ and ν on $\{0,1\}^\alpha$

such that (1) their finite-dimensional projections are equivalent to the corresponding Haar measures and (2) some open set in $\{0,1\}^\alpha \times \{0,1\}^\alpha$ is not $\mu \times \nu$-measurable (of course, if we consider $\xi = \mu + \nu$ instead of μ, ν, the same holds; therefore we can assume that $\mu = \nu$). Thus the topological space $X = \{0,1\}^\alpha$ has the following property:

(I) for some positive Radon measure ξ on X, there exist open subsets of $X \times X$ that are not $\xi \times \xi$-measurable.

In this section we describe a large class of compact spaces with the property (I). This class contains every Stonian (compact, extremally disconnected) space as well as every dyadic space of topological weight $\geq c$.

Let X_1 and X_2 be a couple of compact spaces. We consider a family $(A_i \times B_i)_{i \in \Gamma}$, $\operatorname{card} \Gamma \geq c$, of compact rectangles in $X_1 \times X_2$ and a sequence $\{W_n \times T_n\}$ of open rectangles in $X_1 \times X_2$. The following result is clear from the arguments used in §2.

Lemma 5. *There exists a sequence $\Gamma_n \subset \Gamma$, $n \in \mathbb{N}$, such that*

(i) $\bigcup_{n \in \mathbb{N}} \Gamma_n = \Gamma$,
(ii) *for every* $n \in \mathbb{N}$, $(\bigcup_{i \in \Gamma_n} A_i) \times (\bigcup_{i \in \Gamma_n} B_i) \subseteq \bigcup_{n \in \mathbb{N}} W_n \times T_n$.

Let now X_j, Y_j, $j = 1, 2$, be compact spaces and $p_j : Y_j \to X_j$ continuous surjections. We suppose that μ_j, ν_j are Radon measures on X_j, Y_j resp. with $p_j(\nu_j) = \mu_j$. Then we have

Lemma 6. *If X_1 and X_2 are totally disconnected and $H \subset X_1 \times X_2$ is open with $(\mu_1 \times \mu_2)^*(H) \geq \gamma > 0$, then $(\nu_1 \times \nu_2)^*[(p_1 \times p_2)^{-1} H] \geq \gamma$.*

Proof. Let $\{W_n \times T_n\}$ be any sequence of open rectangles in $Y_1 \times Y_2$ such that $W := (p_1 \times p_2)^{-1} H \subseteq \bigcup_{n \in \mathbb{N}} W_n \times T_n$. We write H as a union of clopen rectangles, i.e. $H = \bigcup_{i \in \Gamma} A_i \times B_i$. Then $W = \bigcup_{i \in \Gamma} (p_1 \times p_2)^{-1}(A_i \times B_i) = \bigcup_{i \in \Gamma} p_1^{-1} A_i \times p_2^{-1} B_i$. By Lemma 5, there is a sequence $\{\Gamma_n\}$ of subsets of Γ such that

(i) $\bigcup_n \Gamma_n = \Gamma$,
(ii) for every n, $S_n := (\bigcup_{i \in \Gamma_n} p_1^{-1} A_i) \times (\bigcup_{i \in \Gamma_n} p_2^{-1} B_i) \subseteq \bigcup_n W_n \times T_n$.

Then

$$\begin{aligned} \gamma &\leq (\mu_1 \times \mu_2)\Big[\bigcup_n \Big(\bigcup_{i \in \Gamma_n} A_i \times \bigcup_{i \in \Gamma_n} B_i\Big)\Big] = (\nu_1 \times \nu_2)\Big[\bigcup_n S_n\Big] \\ &\leq (\nu_1 \times \nu_2)\Big(\bigcup_n W_n \times T_n\Big) \leq \sum_n \nu_1(W_n) \cdot \nu_2(T_n). \ \blacksquare \end{aligned}$$

We now consider the class $\mathcal{Z}$ of compact topological spaces admitting a continuous surjection onto $[0,1]^c$. This class contains every Stonian space [B-F] as well as every dyadic space of weight $\geq c$ (see [E]).

Proposition 1. *Every compact space* $Z \in \mathcal{Z}$ *has the property* (I).

Proof. By Lemma 1.1 of [H], there exists a continuous surjection from some closed subset X of Z onto $\{0,1\}^c$. By Lemma 6, we can find a suitable measure ξ on X, satisfying (I). Then ξ, considered as a measure on the whole space Z, also has the property (I). ∎

The next result implies the result of Fremlin [Fr_1], provided that the hyperstonian space has "large" topological weight.

Corollary 1. *Let* (Y,ν) *be the Stone space of the Lebesgue product measure on* $[0,1]^A$ *with* $\operatorname{card} A \geq c$. *Then there exists an open set* $W \subset Y \times Y$ *such that* $(\nu \otimes \nu)(W) < (\nu \times \nu)^*(W)$.

Proof. Let μ be a Radon measure on $\{0,1\}^c$ with full support and satisfying (I). Then we can easily find a measure μ' on $\{0,1\}^A$ of full support, also satisfying (I) on $\{0,1\}^A$, such that the hyperstonian space of $(\{0,1\}^A, \mu')$ is (Y,ν) (simply, multiply μ by a suitable product measure).

If we now consider the canonical surjection $\pi : Y \to \{0,1\}^A$, then $\pi(\nu) = \mu'$. Lemma 6 yields the result. ∎

Remark 2. Combining Lemma 6 with techniques used by Talagrand [T_2], we obtain the result of Fremlin without any restriction:

If (S,ϱ) is the Stone space of $([0,1],\lambda)$, then by [T_2], there exists some $n \in \mathbb{N}$ such that ϱ^n $(:= \varrho \times \ldots \times \varrho)$ satisfies (I). Since the hyperstonian space of (S^n, ϱ^n) is (S,ϱ), Lemma 6 assures the existence of some open subset of $S \times S$ which is not $\varrho \times \varrho$-measurable.

3.2. *The case of a compact group.* Since every topological group is a dyadic space (see [K], [U]; also Prop. 7.6 in [P]), we already know that there exists some measure ξ on the group satisfying condition (I). However, we have no information on the relation of ξ to the Haar measure. On the other hand, it is well known that every compact group is the projective limit of a (directed) family of compact metric groups ([Mo-Zi], [Pr]) such that the Haar measure is the projective limit of the corresponding Haar measures on the metric "components" (see e.g. [C] and the relevant references). So (in accordance with the measure constructed on $[0,1]^c$ in §2), it is natural to demand that the projection of ξ on every metric group is equivalent to the corresponding Haar measure.

In this subsection we extend the construction of §2 to any compact group G with $w(G) \geq c$, where in place of the Lebesgue product measure we have the (normalized) Haar measure and in place of finite products of unit intervals we have some compact metric groups (precisely, Lie groups having as projective limit the group G).

In the following, for a compact group A, w_A denotes the (normalized) Haar measure on A. The relevant result is

PROPOSITION 2. *Let G be a compact group with $w(G) \geq c$. Then there exist a* (*directed*) *family* $(G_j)_{j\in J}$ *of compact Lie groups and a Radon probability measure μ' on G such that*

(a$_1$) $G \cong \mathrm{proj}_j \lim G_j$ ($\cong$ *denotes topological isomorphism*),

(a$_2$) *there are open sets in $G \times G$ which are not $\mu' \times \mu'$-measurable*,

(a$_3$) *for each $j \in J$, the projection of μ' onto G_j is absolutely continuous with respect to w_{G_j}.*

NOTE. Considering the measure $\frac{1}{2}\mu' + \frac{1}{2}w_G$ instead of μ', we see that every projection of this measure is equivalent to the (corresponding) Haar measure.

The construction of μ' on G is in analogy with that of the measure μ on $[0,1]^c$ in §2. We only give the basic steps and tools for the construction. For simplicity we assume that $w(G) = c$.

We begin with the following lemmata.

LEMMA 7. *If G is a totally disconnected* (*compact*) *group*, *then the conclusion of Proposition* 2 *holds with G_j being finite groups.*

Proof. Since G is totally disconnected, there exists a directed family $(G_j)_{j\in J}$, $\mathrm{card}\, J = c$, of finite groups such that $G = \mathrm{proj}_j \lim G_j$. Note that $(G_j, w_{G_j})_{j\in J}$ is a projective system of measure spaces which satisfies $w_G = \mathrm{proj}_j \lim w_{G_j}$ ([C], [Mo-Zi]). We now define the measure μ' and we prove that μ' and G_j satisfy our requirements.

For brevity we follow the notations and arguments in [Gr$_1$].

(a) We consider the (directed) set $\Gamma = \{F_j : j \in J\}$ of compact normal subgroups of G, where $G_j = G/F_j$ (see proof of Lemma 2.2 in [Gr$_1$]) and using the F_γ we define the compact normal subgroups H_γ of G, for $\gamma < c$.

(b) Following the proof of Theorem 2.3 of [Gr$_1$], we set $X_0 = G/H_1$, $X_\gamma = H_\gamma/H_{\gamma+1}$ for $\gamma > 0$ (clearly the X_γ are finite groups) and $X = \prod_{\gamma<c} X_\gamma$.

(c) We construct the measures μ_γ, $\gamma < c$, and the map $q_G : X \to G$ exactly as in the proof of Theorem 2.3 of [Gr$_1$]. Then, by the construction of q_G and by Theorem 8 of [Mos], q_G is a homeomorphism between X and G which, in addition, agrees with the projections (i.e. the relations (2.4) of [Gr$_1$] are satisfied). We use the X_i and μ_i, $i < c$, to construct the measure μ as in the note of §2. Then for every finite $M \subset c$, the projection of μ on $\prod_{i\in M} X_i$ is absolutely continuous with respect to $\bigotimes_{i\in M} \mu_i$.

(d) Consider the U_i, $i < c$, as in the note of §2 and set $\mu' = q_G(\mu)$ and $U_i' = q_G(U_i)$. Then it is easily seen that for the set $\mathcal{U}' = \bigcup_{i<c} U_i' \times U_i'$,

we have:

(i) $(\mu' \otimes \mu')(\mathcal{U}') \le \varepsilon$,
(ii) $(\mu' \times \mu')^*(\mathcal{U}') = 1$;

hence (a_1)–(a_3) of Proposition 2 are satisfied. ∎

LEMMA 8. *If G is a connected (compact) group, then the conclusion of Proposition 2 also holds.*

For the proof of this lemma we need some notions and techniques used in [Gr_2] and [Gr-Me].

Let (H, X) be a free compact transformation group [i.e. the compact group H acts freely on the compact space X]; see e.g. [Mo-Zi]. If μ is a (positive) Radon measure on the coset space $Y = X/H$, then the *H-Haar* (or simply, the *Haar*) *lift* $\lambda = \lambda[\mu, H, X]$ of μ is the measure λ defined as follows:

$$\lambda(f) = \int\limits_Y \Big(\int\limits_H f(tx)\, dw_H(t)\Big)\, d\mu(\dot{x}), \qquad f \in C(X),$$

where $H \times X \to X : (t, x) \mapsto tx$ is the (left) action of the group H on X, $\dot{x} = Hx$ the class of $x \in X$ and $C(X)$ the space of (real-valued) continuous function on X (see [Bo]; also [Gr_2] and the references there).

REMARK 3. (1) Let X be a compact group, H a closed normal subgroup of X and μ a Radon measure on X/H with $\mu \ll w_{X/H}$. Suppose that H is a Lie group. Since $\lambda[\mu, H, X]$ is Baire isomorphic to $\mu \times w_H$ (see Lemmas 1.2 and 2.3 of [Gr_2]), we have $\lambda[\mu, H, X] \ll w_X$.

(2) Following the proof of [He-Ro], (25.35), p. 423, we see immediately that

[i_1] every compact abelian group A is topologically isomorphic to some quotient $\prod_{i\in\Gamma} Q_i/R$, where $\operatorname{card}\Gamma = w(A)$, each Q_i is a compact metric group and R is a closed normal subgroup of $\prod_{i\in F} Q_i$.

On the other hand, by the structure theorem of Pontryagin and van Kampen ([Pr], Theorem 6.5.6),

[i_2] every connected compact group is topologically isomorphic to a quotient $(A \times \prod_i L_i)/K$, where each L_i is a Lie group, A is a compact abelian group and K is a closed normal subgroup of $A \times \prod_{i\in F} L_i$.

Combining now [i_1] and [i_2], it follows that (see also Theorem 1.1 of [Gr-Me])

[i_3] every connected compact group G is topologically isomorphic to some quotient G_1/N, where $G_1 = \prod_i G_i'$ with the G_i' compact metric groups (with at least two elements each) and N is a closed normal subgroup of G_1.

(3) Suppose that X_i, Y_i, μ_i, A_i, μ, $\widetilde{\mu}$, X are as in the note of §2. For any ordinal $0 \le \alpha \le c$, we set $Z_0 = X_0$ and $Z_\alpha = \prod_{\gamma<\alpha} X_\gamma$ for $\alpha > 0$. Then clearly $X \cong Z := \operatorname{proj}_{\alpha<c} \lim Z_\alpha$, and we may consider the canonical projection $p_\alpha : X \to Z_\alpha$. Now, with the same reasoning as in the note of §2, we conclude that the set $\mathcal{U} = \bigcup_{\gamma<c} U_\gamma \times U_\gamma$ (where $U_0 = p_0^{-1} A_0$ and $U_\gamma = p_{\gamma+1}^{-1}(Z_\gamma \times A_\gamma)$ for $\gamma > 0$) satisfies $(\mu \otimes \mu)(\mathcal{U}) \le \varepsilon$ and $(\mu \times \mu)^*(\mathcal{U}) = 1$.

After all this discussion we may proceed to the

Proof of Lemma 8. (a) Consider G_1, N, G'_i as in [i_3]. We prove that

[j_1] there exists a (directed) family $\Psi = \{G_j\}$ of compact metric groups and a Radon (probability) measure μ' on G such that (a_1)–(a_3) of Proposition 2 are satisfied.

Every G'_j is of the form $G'_j = G_1/F_j$ ($j \in J$), where F_j is some closed normal subgroup of G_1 (i.e. $F_j \cong \prod_{i\ne j} G'_i$). If we set $N_j = G_1/NF_j$, then we can assume that the N_j are pairwise different (see [Gr_2], discussion 1.3 and proof of Theorem 1.1). We note that $\bigcap_{i\in J} F_i = \{\mathrm{id}_{G_1}\}$ and $\bigcap_{j\in J} NF_j = N$, in other words, the families

$$\Phi = \{G_1/F_{i_1} \cap \ldots \cap F_{i_k} : i_1, \ldots, i_k \in J\},$$
$$\Psi = \{G_1/NF_{i_1} \cap \ldots \cap NF_{i_k} : i_1, \ldots, i_k \in J\}$$

have as projective limits G_1 and G_1/N respectively.

(b) We enumerate the family $\{N_j : j \in J\}$ as $\{N_\alpha : \alpha < c\}$. We can assume without loss of generality that

[j_2] for every $\alpha < c$, $$K_\alpha := \Big(\bigcap_{\gamma<\alpha} NF_\gamma\Big) \cap NF_\alpha \ne \bigcap_{\gamma<\alpha} NF_\gamma$$

(if not, then we can make inductively a suitable choice of N_α and work with those N_α).

In agreement with the notation of Remark 3(2), we set $X_i = G'_i$, $\mu_i = w_{G'_i}$, $i < c$ (of course $\widetilde{\mu} = w_{G_1}$). Note that $q_j(w_{G'_j}) = w_{N_j}$, where q_j is the canonical projection from G'_j to N_j. By (j_2), for every $a < c$, we can find an open set $B'_\alpha \subseteq G_1/K_\alpha$ such that $w_{G_1/K_\alpha}(B'_\alpha) < 1$ and $B'_\alpha = r_\alpha^{-1} r_\alpha(B'_\alpha)$, where r_α is the projection of G_1/K_α to $G_1/NF_\alpha = N_\alpha$. Finally, we set $A_\alpha = q_\alpha^{-1}(r_\alpha B'_\alpha)$, $\alpha < c$.

(c) Then the A_α satisfy the conditions of Remark 3, therefore we can define (as there) the measure μ and the sets U_α and $\mathcal{U}$. Then it is routine to see that for the measure $p(\mu) = \mu'$ (where $p : G_1 \to G$ is the projection) and for the set $\mathcal{U}' = \bigcup_{\gamma<c} U'_\gamma \times U'_\gamma$ (where $p(U_\gamma) = U'_\gamma$), we have

(I_1) $(\mu' \otimes \mu)'(\mathcal{U}') \le \varepsilon$,

(I_2) the projection of μ' onto any $G_1/NF_{i_1} \cap \ldots \cap NF_{i_1}$ is absolutely continuous with respect to the corresponding Haar measure. (This holds because for every $a < c$, the projections of μ on the finite subproducts of $G_1 = \prod_i G'_i$ are absolutely continuous with respect to the projections of $\widetilde{\mu}$).

Thus, $[\mathrm{j}_1]$ is true for the family Ψ.

(d) To complete the proof of Lemma 8 it suffices to observe that every $G_j \in \Psi$, as a metric compact group, is the projective limit of some sequence of (compact) Lie groups [Mo-Zi], say $G_j = \mathrm{proj}_{n\in\mathbb{N}} \lim G/K_j^n$, where the K_j^n can be chosen to be closed normal subgroups of the connected group G. Then clearly $\bigcap_{n,j} K_j^n = \{\mathrm{id}_G\}$; therefore, G is the projective limit of Lie groups, of the form G/L each, where L is a finite intersection of the K_j^n. This completes the proof. ■

Remark 4. Assume that, for a compact group G with $w(G) \geq \omega$ and a Radon measure τ on G, there exists a family $\{G/M_i\}$ of Lie groups (with M_i normal closed subgroups of G) such that

(c_1) $\bigcap_i M_i = \{\mathrm{id}_G\}$,

(c_2) for every i, the projection of τ of G/M_i is absolutely continuous with respect to (the Haar measure) w_{G/M_i}.

Then one can check the following:

$[\mathrm{j}_3]$ For every compact Lie group of the form G/M (where M is a closed normal subgroup of G), the projection of τ onto G/M is absolutely continuous with respect to $w_{G/M}$.

For a compact group B, we denote by B_0 the component of the identity in B.

Proof of Proposition 2. Suppose that G is any compact group with $w(G) = c$. It suffices to construct a measure τ on G with the property (a_2) of Proposition 2 and such that $[\mathrm{j}_3]$ holds. By a theorem of Mostert ([Mos], Theorem 8), G is homeomorphic to $G/G_0 \times G_0$.

Case 1: $w(G_0) < c$ (and therefore, $w(G/G_0) = c$). By Lemma 8, there exist a directed system $\{G_i\}$ of finite groups and a measure μ' on G/G_0 such that $G/G_0 = \mathrm{proj}_i \lim G_i$, and for each i, the projection of μ' onto G_i is absolutely continuous with respect to w_{G_i}.

We can write every G_i in the form $G_i = G/L_i$, where L_i is a closed normal subgroup of G. We now claim that the Haar lift $\tau = \lambda[\mu, G_0, G]$ is the required measure. Indeed, using Remark 4, we can easily see that the projection of τ onto any Lie group G/M is absolutely continuous with respect to $w_{G/M}$ (see $[\mathrm{j}_3]$). Finally, using Lemma 6, we get the claim.

Case 2: $w(G_0) = c$. By Lemma 8 and Remark 4, there is a measure μ_0 on G_0 such that

(d_1) there exist non-$\mu_0 \times \mu_0$-measurable subsets of $G_0 \times G_0$,

(d_2) for every continuous homomorphism g of G onto any compact Lie group H, we have $g(\mu_0) \ll w_H$.

The measure μ_0 on G_0 may be viewed as a measure on G. We denote this measure by τ; it is clear that there exist open subsets of $G \times G$ which are not $\tau \times \tau$-measurable. We claim that τ is the required measure.

Indeed, it suffices to prove that [j_3] holds. Let G/M be any quotient as in [j_3] and $\pi : G \to G/M$ be the canonical projection. Then $\pi(G_0) = (G/M)_0$ (see e.g. [Mo-Zi] or Theorem 7.12 of [He-Ro]), therefore $\pi(\tau) = \pi|G_0(\mu_0)$. But $\pi|G_0(\mu_0)$ is absolutely continuous with respect to $w_{(G/M)_0}$, because $(G/M)_0$ is a clopen subgroup of G/M and $w_{(G/M)_0}$ is absolutely continuous with respect to $w_{G/M}$. This completes the proof. ■

The previous arguments imply the following general theorem.

THEOREM. *If G is a compact group with $w(G) \geq c$, then there exists a Radon probability measure τ on G such that*

(1) $\sup \tau = G$,

(2) *there exist open sets in $G \times G$ which are not $\tau \times \tau$-measurable*,

(3) *for every (compact) Lie group H of the form $H = G/M$ (with M a closed normal subgroup of G), the projection of τ onto H is equivalent to the Haar measure on H.*

Acknowledgments. The authors are indebted to D. H. Fremlin for reading early versions of this note and for many valuable suggestions. Thanks are also due the referee for his patience.

References

[B-F] B. Balcar and F. Franek, *Independent families in complete Boolean algebras*, Trans. Amer. Math. Soc. 274 (1982), 607–618.

[Bo] N. Bourbaki, *Intégration*, Ch. 8, Hermann, Paris, 1959–1967.

[C] J. R. Choksi, *Recent developments arising out of Kakutani's work on completion regularity of measures*, in: Contemp. Math. 26, Amer. Math. Soc., Providence, R.I., 1984, 81–94.

[E] B. A. Efimov, *Mappings and embeddings of dyadic spaces*, Mat. Sb. 103 (1977), 52–68 (in Russian).

[Er-Ox] P. Erdős and J. C. Oxtoby, *Partitions of the plane into sets having positive measure in every non-null measurable product set*, Trans. Amer. Math. Soc. 79 (1955), 91–102.

[Fr_1] D. H. Fremlin, *Products of Radon measures: a counter-example*, Canad. Math. Bull. 19 (1976), 285–289.

[Fr_2] —, *Measure Theory*, University of Essex, Colchester, 1994.

[Fr-Gr] D. H. Fremlin and S. Grekas, *Products of completion regular measures*, Fund. Math. 147 (1995), 27–37.

[Gr_1] S. Grekas, *Structural properties of compact groups with measure-theoretic applications*, Israel J. Math. 87 (1994), 89–95.

[Gr_2] —, *Measure-theoretic problems in topological dynamics*, J. Anal. Math. 65 (1995), 207–220.

[Gr-Me] S. Grekas and S. Mercourakis, *On the measure theoretic structure of compact groups*, Trans. Amer. Math. Soc. 350 (1998), 2779–2796.

[Gry] C. Gryllakis, *Products of completion regular measures*, Proc. Amer. Math. Soc. 103 (1988), 563–568.

[H] R. Haydon, *On Banach spaces which contain $l^1(\tau)$ and types of measures on compact spaces*, Israel J. Math. 28 (1977), 313–324.

[He-Ro] E. Hewitt and K. Ross, *Abstract Harmonic Analysis I*, Springer, Berlin, 1963.

[K] V. Kuz'minov, *On a hypothesis of P. S. Aleksandrov in the theory of topological groups*, Dokl. Akad. Nauk SSSR 125 (1959), 727–729 (in Russian).

[Mo-Zi] D. Montgomery and L. Zippin, *Topological Transformation Groups*, Interscience, 1955.

[Mos] P. S. Mostert, *Sections in principal fibre spaces*, Duke Math. J. 23 (1956), 57–71.

[P] A. Pełczyński, *Linear extensions, linear averagings, and their applications to linear topological classification of spaces of continuous functions*, Dissertationes Math. 58 (1968).

[Pr] J. F. Price, *Lie Groups and Compact Groups*, Cambridge Univ. Press, 1977.

[T_1] M. Talagrand, *Pettis integral and measure theory*, Mem. Amer. Math. Soc. 307 (1984).

[T_2] —, *On liftings and the regularization of stochastic processes*, Probab. Theory Related Fields 78 (1988), 127–134.

[U] V. V. Uspenskiĭ, *Why compact groups are dyadic*, in: General Topology and its Relations to Modern Analysis and Algebra VI, Proc. Sixth Prague Topological Symposium 1986, Z. Frolík (ed.), Heldermann, Berlin, 1988, 601–610.

Department of Mathematics
University of Athens
Panepistimiopolis 15784 Athens, Greece
E-mail: sgrekas@eudoxos.dm.uoa.gr

Received 12 August 1997;
in revised form 16 February 1998 and 29 June 1998