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Almost disjoint families and property (a)

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Abstract. We consider the question: when does a Ψ-space satisfy property (a)? We show that if $|\mathcal{A}| < \mathfrak{p}$ then the Ψ-space $\Psi(\mathcal{A})$ satisfies property (a), but in some Cohen models the negation of CH holds and every uncountable Ψ-space fails to satisfy property (a). We also show that in a model of Fleissner and Miller there exists a Ψ-space of cardinality $\mathfrak{p}$ which has property (a). We extend a theorem of Matveev relating the existence of certain closed discrete subsets with the failure of property (a).

1. Introduction. A space X has *property* (a) [13] provided for every open cover $\mathcal{U}$ and dense set D of X there exists a closed discrete (in X) $F \subset D$ such that $\mathrm{st}(F,\mathcal{U}) = X$. Property (a) was introduced by M. Matveev in order to explore the absoluteness condition in the definition of absolutely countable compactness [11]. Some results on property (a) can be found in [9], [13], and [14].

In the first part of this paper we consider the question: Under what conditions does the space $\Psi(\mathcal{A})$ satisfy property (a)? Recall that for an almost disjoint family $\mathcal{A}$ of infinite subsets of ω, $\Psi(\mathcal{A})$ denotes the associated topological space whose underlying set consists of the set of natural numbers ω, and one point x_A for every $A \in \mathcal{A}$. The points in ω are declared to be isolated, and basic neighborhoods of a point x_A are of the form $\{x_A\}\cup(A\setminus n)$ for all $n \in \omega$ (see [6, 5I], or [3, 3.6.I]). Evidently, if $\mathcal{A}$ is countable, then $\Psi(\mathcal{A})$ has property (a); so we are only interested in uncountable almost disjoint families. The statement "$\Psi(\mathcal{A})$ has property (a)" translates into the following set-theoretic statement about $\mathcal{A}$:

$$(*)\qquad (\forall f : \mathcal{A} \to \omega)(\exists P \subset \omega)(\forall A \in \mathcal{A})(0 < |P \cap (A \setminus f(A))| < \omega).$$

We generalize in §5 the following theorem which can be applied to $\Psi(\mathcal{A})$.

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Theorem 1 (M. Matveev [13]). *If X is separable, and contains a closed discrete subset of cardinality $\mathfrak{c}$, then X does not have property* (a).

Corollary 1. *If $|\mathcal{A}| = \mathfrak{c}$, then $\Psi(\mathcal{A})$ does not satisfy property* (a).

It is known, and easy to see, that if $\mathcal{A}$ is *maximal* (i.e., $\Psi(\mathcal{A})$ is pseudocompact), then $\Psi(\mathcal{A})$ does not satisfy property (a) [14]. We recall that there are models of Hechler [7] in which the cardinality of the continuum $\mathfrak{c}$ is arbitrarily large and there exist maximal almost disjoint families of every uncountable cardinality less than or equal to $\mathfrak{c}$. Part of our motivation for this paper was to find what other conditions on $\mathcal{A}$ imply that $\Psi(\mathcal{A})$ has (or does not have) property (a). While this problem is primarily motivated by topological concerns its analysis led us to some new questions about almost disjoint families which may be of independent interest.

Our results yield the following one concerning models of set theory:

Theorem 2. (1) $[\mathfrak{p} = \mathfrak{c}] \Rightarrow \Psi(\mathcal{A})$ *satisfies property* (a) *if and only if* $|\mathcal{A}| < \mathfrak{c}$.

(2) *Adding any number of Cohen reals to a model of CH results in a model where $\Psi(\mathcal{A})$ satisfies property* (a) *if and only if $\mathcal{A}$ is countable.*

(3) *There is a model with an almost disjoint family $\mathcal{A}$ of size $\mathfrak{p}$ where $\Psi(\mathcal{A})$ satisfies property* (a).

It is well known that $\mathfrak{p} = \omega_1$ in the Cohen model used in part (2) of Theorem 2; so in the models used in both parts (1) and (2) of Theorem 2, every Ψ-space of size $\mathfrak{p}$ fails to satisfy property (a). This motivates part (3) of Theorem 2. The proof of Theorem 2 is given in Sections 2, 3, and 4.

To generalize Theorem 1, we introduce in §5 two new cardinal invariants related to density and extent, and we study these further in §7. In §6 we consider the effect of property (a) on the inequality $e(x) < 2^{d(X)}$. Some open questions are given in §8. We use certain small cardinals throughout the paper; see [2] and [5].

2. Martin's Axiom and $\Psi(\mathcal{A})$. Recall that a family $\mathcal{P}$ of subsets of the natural numbers has the *strong finite intersection property* provided every finite intersection of members of $\mathcal{P}$ is infinite. An infinite $S \subset \omega$ is a *pseudointersection* for $\mathcal{P}$ provided $S \setminus P$ is finite for all $P \in \mathcal{P}$ (see [2]).

Definition 1. $\mathfrak{p} = \min\{|\mathcal{P}| : \mathcal{P} \subset [\omega]^\omega$ has the strong finite intersection property, but has no pseudointersection$\}$.

We use two basic facts about the cardinal $\mathfrak{p}$:

(1) (Bell's theorem [1], [5, 14C]) if $\kappa < \mathfrak{p}$, then "MA for σ-centered posets of cardinality κ" holds, and

(2) Martin's Axiom implies that $\mathfrak{p} = \mathfrak{c}$ [5, 11D].

We say that an almost disjoint family $\mathcal{A}$ is *soft* if there exists $P \subseteq \omega$ such that for all $A \in \mathcal{A}$, $0 < |P \cap A| < \omega$. It is obvious that any family $\mathcal{A}$ that satisfies $(*)$ is soft, and that there exist soft families of every cardinality $\leq \mathfrak{c}$ (thus a soft family $\mathcal{A}$ of cardinality $\mathfrak{c}$ does not satisfy $(*)$). We use the following observation: If κ is a cardinal such that every almost disjoint family $\mathcal{A}$ with $|\mathcal{A}| < \kappa$ is soft, then every almost disjoint family $\mathcal{A}$ with $|\mathcal{A}| < \kappa$ satisfies $(*)$; hence the corresponding $\Psi(\mathcal{A})$ satisfies property (a).

THEOREM 3. *If* $|\mathcal{A}| < \mathfrak{p}$ *then* $\Psi(\mathcal{A})$ *satisfies property* (a).

Proof. It suffices to prove that any almost disjoint family $\mathcal{A}$ with $|\mathcal{A}| < \mathfrak{p}$ is soft. This can can be deduced directly from Fremlin's "portmanteau theorem" [5, 21A]. We sketch a proof that is a slight variation on the proof that "MA $\Rightarrow \mathfrak{a} = \mathfrak{c}$" [10, 2.16]. Let $\mathcal{A}$ with $|\mathcal{A}| < \mathfrak{p}$ be given. According to [10, 2.7] "the almost disjoint sets partial order" $\mathbb{P}_{\mathcal{A}}$ is defined to be

$$\{(s, F) : s \in [\omega]^{<\omega},\ F \in [\mathcal{A}]^{<\omega}\},$$

where $(s', F') \leq (s, F)$ if and only if

$$s \subset s', \quad F \subset F' \quad \text{and} \quad (\forall A \in F)(A \cap s' \subset s).$$

We use the following subset:

$$\mathbb{Q}_{\mathcal{A}} = \{(s, F) \in \mathbb{P}_{\mathcal{A}} : \text{for every } A \in F,\ s \cap A \neq \emptyset\}.$$

Since $(\mathbb{P}_{\mathcal{A}}, \leq)$ is a σ-centered poset, so is $(\mathbb{Q}_{\mathcal{A}}, \leq)$. Moreover, for each $A \in \mathcal{A}$,

$$D_A = \{(s, F) \in \mathbb{Q}_{\mathcal{A}} : A \in F\}$$

is dense in $\mathbb{Q}_{\mathcal{A}}$. MA says that there is a filter $G \subset \mathbb{Q}_{\mathcal{A}}$ such that $G \cap D_A \neq \emptyset$ for all $A \in \mathcal{A}$. Then $P = \bigcup\{s : (\exists F)((s, F) \in G)\}$ is the desired set.

3. Cohen forcing and $\Psi(\mathcal{A})$

THEOREM 4. *In the Cohen model*, $\Psi(\mathcal{A})$ *does not have property* (a) *whenever* $\mathcal{A}$ *is an uncountable almost disjoint family.*

Proof. We force with $\mathrm{Fn}(\kappa, 2)$ over a model $\mathcal{M}$ of CH. If $\kappa \leq \omega_1$ the result follows from Corollary 1; so we assume that $\kappa > \omega_1$. Fix an $\mathcal{M}$-generic set $G \subseteq \mathrm{Fn}(\kappa, 2)$. We claim that in $\mathcal{M}[G]$ the conclusion of the theorem holds. To show this, fix an uncountable almost disjoint family $\mathcal{A} \in \mathcal{M}[G]$. It is easy to check that if $\Psi(\mathcal{A}_0)$ does not have property (a) for some subset $\mathcal{A}_0 \subseteq \mathcal{A}$ then $\Psi(\mathcal{A})$ will also fail to have property (a). Therefore we may assume that $\mathcal{A}$ is of size ω_1. Therefore there is an $I \subseteq \kappa$ of size ω_1 such that $\mathcal{A} \in \mathcal{M}[G \cap \mathrm{Fn}(I, 2)]$. Note that CH holds in this intermediate model, and that $\mathcal{M}[G]$ is obtained by forcing with $\mathrm{Fn}(\kappa \setminus I, 2)$ over $\mathcal{M}[G \cap \mathrm{Fn}(I, 2)]$. Therefore we may also assume that $\mathcal{A}$ is in the ground model $\mathcal{M}$. Working with the negation of $(*)$, we now construct an open cover of $X = \Psi(\mathcal{A})$, also in the ground model, witnessing that X does not have property (a) in the extension. This open cover will be of the form

$$\mathcal{U}_f = \{\{a\} \cup (a \setminus f(a)) : a \in \mathcal{A}\} \cup \{\{n\} : n \in \omega\}$$

where $f : \mathcal{A} \to \omega$. We need the following lemma.

LEMMA 1. *Assume CH. Let $\mathcal{A}$ be an almost disjoint family of size ω_1. There is a function $f : \mathcal{A} \to \omega$ such that for each* $\mathrm{Fn}(\omega, 2)$*-name τ for a subset of ω, if*

(a) $\mathbf{1} \Vdash (\forall a \in \check{\mathcal{A}})(|\tau \cap a| < \omega)$

then

(b) $\mathbf{1} \Vdash (\exists a \in \check{\mathcal{A}})(\tau \cap (a \setminus \check{f}(a)) = \emptyset)$.

Proof. Enumerate $\mathcal{A}$ as $\{a_{\alpha,n} : \alpha < \omega_1,\ n < \omega\}$, and by CH, enumerate all $\mathrm{Fn}(\omega, 2)$-nice names for subsets of ω as $\{\tau_\alpha : \alpha < \omega_1\}$. Fix $\alpha < \omega_1$. If there is a $p \in \mathrm{Fn}(\omega, 2)$ and an $a \in \mathcal{A}$ such that

$$p \Vdash |\check{a} \cap \tau| = \omega$$

then let $f(a_{\alpha,n}) = 0$ for each $n \in \omega$. So assume that

(c) $\mathbf{1} \Vdash (\forall a \in \check{\mathcal{A}})(|\tau_\alpha \cap a| < \omega)$.

It is well known that $\mathrm{Fn}(\omega, 2)$ does not add any function in ${}^\omega\omega$ dominating all ground model functions in ${}^\omega\omega$. In the language of forcing extensions this means that if G is $\mathrm{Fn}(\omega, 2)$-generic over $\mathcal{M}$ then for each $s \in {}^\omega\omega \cap \mathcal{M}[G]$ there is an $f \in {}^\omega\omega \cap \mathcal{M}$ such that $\{n : f(n) > s(n)\}$ is infinite (see [10], Exercise VII.G7). Therefore

$$E = \{p \in \mathrm{Fn}(\omega, 2) : (\exists f \in {}^\omega\omega)(p \Vdash \text{“}\check{f}(n) > \max(\tau_\alpha \cap \check{a}_{\alpha,n}) \text{ for infinitely many } n\text{”})\}$$

is a dense subset of $\mathrm{Fn}(\omega, 2)$. For each $p \in E$ fix a corresponding $f_p \in {}^\omega\omega$. Since E is countable, we may choose $g_\alpha \in {}^\omega\omega$ such that for each $p \in E$, $g_\alpha(n) > f_p(n)$ for all but finitely many $n \in \omega$. Then

(d) $\mathbf{1} \Vdash \{n \in \omega : \check{g}_\alpha(n) > \max(\tau_\alpha \cap \check{a}_{\alpha,n})\}$ is infinite.

(We will only need the fact that this set is forced to be nonempty). Now define $f : \mathcal{A} \to \omega$ by $f(a_{\alpha,n}) = g_\alpha(n)$ for each $\alpha \in \omega_1$ and each $n \in \omega$. If

(e) $\mathbf{1} \Vdash (\forall a \in \check{\mathcal{A}})(|\tau \cap \check{a}| < \omega)$,

then there is an α such that

(f) $\mathbf{1} \Vdash \tau = \tau_\alpha$.

Therefore, by (d),

$$\mathbf{1} \Vdash (\exists n \in \omega)(\tau_\alpha \cap \check{a}_{\alpha,n} \subseteq \check{g}_\alpha(n)).$$

But this clearly means that

$$\mathbf{1} \Vdash (\exists a \in \check{\mathcal{A}})(\tau_\alpha \cap (a \setminus \check{f}(a)) = \emptyset).$$

This completes the proof of Lemma 1.

To complete the proof of Theorem 4, fix $G \subseteq \mathrm{Fn}(\kappa, 2)$ generic over $\mathcal{M}$, and fix f given by the lemma. We claim that in $\mathcal{M}[G]$, X does not have property (a). By way of contradiction assume otherwise. Then, since ω is dense in X, there is an $F \subseteq \omega$ closed discrete such that $\mathrm{st}(F, \mathcal{U}_f) = X$. By $(*)$, $F \cap a$ is finite for each $a \in \mathcal{A}$ and $F \cap (a \setminus f(a)) \neq \emptyset$ for each $a \in \mathcal{A}$. Since F is countable, there is a countable $I \subseteq \kappa$ such that $F \in \mathcal{M}[G \cap \mathrm{Fn}(I, 2)]$. But as $\mathrm{Fn}(I, 2)$ is isomorphic to $\mathrm{Fn}(\omega, 2)$ and $G \cap \mathrm{Fn}(I, 2)$ is $\mathrm{Fn}(I, 2)$-generic over $\mathcal{M}$, there is an $\mathrm{Fn}(\omega, 2)$-name τ such that

$$\mathbf{1} \Vdash (\forall a \in \check{\mathcal{A}})(|\tau \cap a| < \omega \text{ and } \tau \cap (a \setminus \check{f}(a)) \neq \emptyset).$$

This contradicts Lemma 1.

4. A Ψ-space of size $\mathfrak{p}$ with property (a). We prove that it is consistent that there is an almost disjoint family A of size $\mathfrak{p}$ with property (a). Since the existence of a Q-set implies the existence of a normal Ψ-space of size ω_1, the existence of A follows from the next two theorems. Recall that $\mathfrak{d}$ denotes the smallest cardinality of a dominating family of functions from ω to ω with respect to the mod finite order (e.g., see [2]).

Theorem 5. *Suppose that $A \subseteq [\omega]^\omega$ is an almost disjoint family such that $|A| < \mathfrak{d}$ and such that $\Psi(A)$ is normal. Then $\Psi(A)$ has property* (a).

Theorem 6 (Fleissner–Miller). *If ZFC is consistent then so is ZFC + (there is a Q-set) + $(\mathfrak{d} = 2^\omega = \omega_2)$ + $(\mathfrak{p} = \omega_1)$.*

Proof of Theorem 5. We need some notation. For a finite sequence $s \in \omega^n$ and $k \in \omega$, $s \frown k \in \omega^{n+1}$ is the sequence extending s whose $(n+1)$st element is k. For sequences $s, t \in \omega^n$ we write $s \leq t$ if $s(k) \leq t(k)$ for each $k < n$.

Fix A as in the hypothesis of the theorem. It suffices to prove that any such A is soft. For each $s \in \omega^{<\omega}$ define $A_s \subseteq A$ and open sets $U_s, V_s \subseteq \Psi(A)$ as follows. $A_\emptyset = \emptyset$, $U_\emptyset = \emptyset$ and $V_\emptyset = \Psi(A)$. For each $n \in \omega$ let $A_{\langle n\rangle} = \{a \in A : a \cap n \neq \emptyset\}$. Using normality of $\Psi(A)$ fix disjoint open sets $U_{\langle n\rangle} \supseteq A_{\langle n\rangle}$ and $V_{\langle n\rangle} \supseteq A \setminus A_{\langle n\rangle}$. In addition we choose these open sets so that

(a) $n < m$ implies $U_{\langle n\rangle} \subseteq U_{\langle m\rangle}$ and $V_{\langle m\rangle} \subseteq V_{\langle n\rangle}$.

Fix $n > 1$ and suppose that A_s, U_s and V_s have been defined for each $s \in \omega^n$ so that

(b) $A_s \subseteq A$, U_s and V_s are open in $\Psi(A)$ such that $U_s \cap V_s = \emptyset$ for each $s \in \omega^{\leq n}$.

(c) $U_s \supseteq \bigcup_{i \leq n} A_{s|i}$ and $V_s \supseteq A \setminus \bigcup_{i \leq n} A_{s|i}$ are disjoint open sets.

(d) $U_{s|i} \subseteq U_{s|j}$ and $V_{s|i} \supseteq V_{s|j}$ for each $i < j \leq n$ and for each $s \in \omega^n$.

Fix $s \in \omega^{n+1}$. Let
$$A_s = \Big\{a \in A \setminus \bigcup_{i \leq n} A_{s|i} : a \cap (s(n) \setminus s(n-1)) \cap V_{s|n} \neq \emptyset\Big\}.$$

Using normality, fix a clopen $U \subseteq \Psi(A)$ such that $U \cap A = A_s$, $U \cap U_{s|n} = \emptyset$ and $U \subseteq V_{s|n}$. Let $U_s = U_{s|n} \cup U$ and let $V_s \subseteq V_{s|n}$ be an open set such that $A \setminus \bigcup_{i \leq n+1} A_{s|i} \subseteq V_s$ and $U_s \cap V_s = \emptyset$. This completes the construction of the family $\{A_s, U_s, V_s : s \in \omega^{<\omega}\}$. Notice that the sets A_s satisfy the following properties.

(e) For each $s \in \omega^n$ and for each $k > j \geq s(n-1)$, $A_{s\frown j} \subseteq A_{s\frown k}$.

(f) For each $s \in \omega^n$, $A = \bigcup_{i \leq n} A_{s|i} \cup \bigcup_{k > s(n-1)} A_{s\frown k}$.

For each $a \in A$ define $f_a : \omega \to \omega$ as follows. Let $f_a(0)$ be the minimum k such that $a \in A_{\langle k \rangle}$. For $n > 1$, having defined $f_a|n$ let $f_a(n)$ be the minimum k such that for each $s \in \omega^n$, if $s \leq f_a|n$ and $a \notin \bigcup_{i \leq n} A_{s|i}$, then $a \in A_{s\frown k}$. Clause (f) and the fact that the set of such s is finite guarantee that $f_a(n)$ is well defined.

Using $|A| < \mathfrak{d}$, fix an increasing $f \in \omega^\omega$ such that for each $a \in A$ there is an $n \in \omega$ such that $f(n) \geq f_a(n)$.

Lemma 2. $A = \bigcup_{n \in \omega} A_{f|n}$.

Proof. Fix $a \in A$. Fix n minimal so that $f(n) \geq f_a(n)$. Suppose that $a \notin \bigcup_{i \leq n} A_{f|i}$. Since $f|n \leq f_a|n$, $a \in A_{(f|n)\frown f_a(n)}$ by the definition of $f_a(n)$. By clause (e), $a \in A_{(f|n)\frown k}$ for each $k \geq f_a(n)$ so $a \in A_{f|n+1}$.

Now we define X that will witness that A is soft. Let $S_0 = f(0)$. For each $n > 0$ let $S_n = (f(n) \setminus f(n-1)) \cap V_{f|n}$ and let $X = \bigcup_{n \in \omega} S_n$. Fix $a \in A_{f|n+1}$. By the definition of $A_{f|n+1}$, $a \cap S_n \neq \emptyset$ and by (b) and (d), $a \in U_{f|n+1} \subseteq U_{f|m}$ for all $m > n$. By our construction we also have $U_{f|n+1} \cap V_{f|m} = \emptyset$ for each $m > n$. Therefore, $a \cap \bigcup_{m > n} S_m \subseteq a \setminus U_{f|n+1}$. But since $U_{f|n+1}$ is an open set containing a, $a \setminus U_{f|n+1}$ is finite and therefore $a \cap \bigcup_{m>n} S_m$ is finite. Therefore $0 < |a \cap X| < \omega$ as required.

Proof of Theorem 6. One of the models of [4] satisfies the conclusion of the theorem. Fleissner and Miller construct a model where there is a Q-set concentrated on a countable set. Therefore $\mathfrak{b} = \omega_1$ in this model (see Theorem 10.2 in [2]) and hence $\mathfrak{p} = \omega_1$. The forcing used to obtain the model is an ω_2 length finite support iteration of CCC partial orders. Therefore any set of reals of size ω_1 appears at some initial stage $\alpha < \omega_2$. And since it is a finite support iteration, $V^{P_{\alpha+\omega}}$ contains a Cohen real over V^{P_α} which is therefore not dominated by any real in V^{P_α}. So no set of reals of size ω_1 is dominating.

5. Density discreteness and cofinality. Recall the definition of density of a space X:

$$d(X) = \min\{|D| : D \text{ is a dense subset of } X\} + \omega.$$

Matveev's proof of Theorem 1 yields the following more general statement.

THEOREM 7 (Matveev). *If X has a closed discrete set F with $|F| \geq 2^{d(X)}$, then X does not satisfy property* (a).

In order to extend this result we introduce two new cardinal invariants.

NOTATION. For a dense set $D \subset X$, let $\mathcal{F}_D$ denote the set of all closed discrete (in X) subsets of D.

DEFINITION 2. The *density discreteness number* of a space X is

$$dd(X) = \min\{|\mathcal{F}_D| : D \text{ is dense in } X\} + \omega.$$

Clearly, $dd(X) \leq 2^{d(X)}$, and if X is a T_1-space, then $d(X) \leq dd(X) \leq 2^{d(X)}$.

Let $\mathcal{F}$ be a family of sets. We recall that $\mathcal{C} \subset \mathcal{F}$ is said to be *cofinal* in $\mathcal{F}$ provided for every $F \in \mathcal{F}$ there exists $C \in \mathcal{C}$ such that $F \subset C$.

DEFINITION 3. The *density discreteness cofinality* of a space X is the following number:

$$ddc(X) = \min\{|\mathcal{C}| : D \text{ is dense in } X \text{ and } \mathcal{C} \text{ is cofinal in } \mathcal{F}_D\} + \omega.$$

Obviously, $ddc(X) \leq dd(X)$, but $ddc(X) < d(X)$ is possible. Indeed, consider the following

EXAMPLE 1. *There exists a metrizable space X such that $ddc(X) < d(X) < dd(X)$.*

Proof. Let $X = (\omega_1 \times \omega) \cup \{\infty\}$ where the points of $\omega_1 \times \omega$ are isolated, and the sets $B(\infty, n) = \{\infty\} \cup \{(\alpha, i) : i > n,\ \alpha < \omega_1\}$ for $n < \omega$ form a base at ∞. Now $D = \omega_1 \times \omega$ is dense in X and $\mathcal{C} = \{\omega_1 \times n : n < \omega\}$ is cofinal in $\mathcal{F}_D$ since every set in X that intersects infinitely many levels $(\omega_1 \times \{n\})$ has ∞ as a limit point; so $ddc(X) = \omega < \omega_1 = d(X) < 2^{\omega_1} = dd(X)$.

THEOREM 8. *If X has a closed discrete subset F with $|F| \geq ddc(X)$, and the interior of F is empty, then X does not satisfy property* (a).

Proof. The proof uses the main idea in Matveev's proof of Theorem 1. Let D be dense and $\mathcal{C}_D$ cofinal in $\mathcal{F}_D$ such that $ddc(X) = |\mathcal{C}_D|$. Since the interior of F is empty, $D \setminus F$ is dense in X, $\mathcal{F}_{D\setminus F} \subset \mathcal{F}_D$,

$$\mathcal{C} = \{H \setminus F : H \in \mathcal{C}_D\}$$

is cofinal in $\mathcal{F}_{D\setminus F}$, $C \cap F = \emptyset$ for all $C \in \mathcal{C}$, and $|\mathcal{C}| \leq |\mathcal{C}_D| = ddc(X)$; so $|\mathcal{C}| = ddc(X)$.

Let $ddc(X)=\kappa$, list $\mathcal{C}$ as $\{H_\alpha:\alpha<\kappa\}$, and list κ points of F as $G=\{x_\alpha:\alpha<\kappa\}$. For each $\alpha<\kappa$, get an open set U_α such that $U_\alpha\cap G=\{x_\alpha\}$, and $U_\alpha\cap H_\alpha=\emptyset$. Then the open cover $\mathcal{U}=\{U_\alpha:\alpha<\kappa\}\cup\{X\setminus G\}$ and the dense set $D\setminus F$ demonstrate that X does not satisfy property (a): If $P\subset D\setminus F$ is closed discrete in X, then for some $\alpha<\kappa$,

$$\mathrm{st}(P,\mathcal{U})\subset\mathrm{st}(C_\alpha,\mathcal{U})\subset X\setminus\{x_\alpha\}.$$

This completes the proof.

The hypothesis "F has nonempty interior" in Theorem 8 cannot be deleted. The space X in Example 1 provides a counterexample since any space with exactly one nonisolated point obviously satisfies property (a).

Corollary 2. *If X has a closed discrete subset F with $|F|\ge dd(X)$, then X does not satisfy property* (a).

Proof. Put $G=\{x\in F:\{x\}\text{ is open in }X\}$. It suffices to show that $|G|<|F|$ because we can then apply Theorem 8 to $F\setminus G$. Suppose $|G|=|F|$. Since G is a set of isolated points, $G\subset D$ for every dense set D. Thus $|\mathcal{F}_D|\ge 2^{|G|}=2^{|F|}$; so we have $2^{|F|}\le dd(X)$. But this leads to the contradiction

$$dd(X)\le|F|<2^{|F|}\le dd(X).$$

Since $dd(X)\le 2^{d(X)}$, Corollary 2 shows immediately that Matveev's Theorem 7 is a corollary to Theorem 8. Also, we note that Corollary 2 can be used to show that $\Psi(\mathcal{A})$ does not have property (a) whenever $|\mathcal{A}|=\mathfrak{c}$ or $\mathcal{A}$ is maximal.

6. Extent. We recall the definitions of extent of a space X:

$$e(X)=\sup\{|F|:F\text{ is a closed discrete subset of }X\}+\omega.$$

In general the two numbers $e(X)$ and $ddc(X)$ are not related. For the space X in Example 1, we have $ddc(X)<e(X)$, and for the space $X=L(\omega_1)$, the one-point Lindelöfization of the discrete space of size ω_1, we have $e(X)<ddc(X)$. Both of these spaces satisfy property (a).

Corollary 3. *If X has property* (a), *then $e(X)\le dd(X)$. In particular*, $e(X)\le 2^{d(X)}$.

Proof. By contradiction, if $e(X)>dd(X)$, then there exists a closed discrete set F with $|F|\ge dd(X)$; so by Corollary 2, X does not satisfy property (a), which is a contradiction.

It is known that for any regular space X, $e(X)\le w(X)\le 2^{d(X)}$ [8, 3.3(b)]. Thus property (a) and regularity both imply $e(X)\le 2^{d(X)}$. On the other hand, property (a) implies $e(X)\le dd(X)$, and regularity does not (if $X=\Psi(\mathcal{A})$ where $\mathcal{A}$ is maximal, then $dd(X)=\omega<e(X)\ge\omega_1$). Of course, regularity does not yield a strict inequality in either case (if $X=\Psi(\mathcal{A})$

where $\mathcal{A}$ is not maximal and $|\mathcal{A}| = \mathfrak{c}$, then $e(X) = dd(X) = 2^{d(X)} = \mathfrak{c}$). Likewise, property (a) does not imply a strict inequality in the first case (if X is a compact separable space, then $e(X) = dd(X) = \omega$).

We are left with the following question: Does property (a) imply $e(X) < 2^{d(X)}$? We show that an affirmative answer to this question is consistent with and (assuming a certain kind of inaccessible cardinal) independent of the usual axioms of ZFC. Let $\mathcal{S}$ stand for the following statement:

$$\mathcal{S}: \text{“If } X \text{ has property (a), then } e(X) < 2^{d(X)}\text{”}.$$

We will use the set-theoretic assumption “2^κ is a successor cardinal for each cardinal κ”. This assumption is implied by GCH, and is consistent with MA + ¬CH. We also use the assumption “$\mathfrak{p} = \mathfrak{c}$ and $\mathfrak{c}$ is weakly inaccessible” (concerning the consistency of this assumption, see [10, VII, Cor. 6.5]).

THEOREM 9. (i) *If 2^κ is a successor cardinal for each cardinal κ then $\mathcal{S}$.*
(ii) *If $\mathfrak{p} = \mathfrak{c}$ and $\mathfrak{c}$ is weakly inaccessible then $\neg\mathcal{S}$.*

Proof of (i). We prove $\mathcal{S}$ by contrapositive; so we assume that $e(X) \geq 2^{d(X)}$. By our set-theoretic assumption, $2^{d(X)} = \lambda^+$ for some λ. Thus there exists a closed discrete $F \subset X$ with $|F| = \lambda^+ = 2^{d(X)}$. By Theorem 7, X does not have property (a).

Proof of (ii). We assume “$\mathfrak{p} = \mathfrak{c}$ and $\mathfrak{c}$ is weakly inaccessible”, and construct a counterexample to the statement $\mathcal{S}$. Our counterexample will be of the form $\Psi(\mathcal{A}) \cup \{\infty\}$, where $\mathcal{A}$ is a special kind of almost disjoint family, and “∞” is one additional point.

Let $\mathcal{T}$ be a tower on ω. By “$\mathfrak{p} = \mathfrak{c}$”, we may assume without loss of generality that $\mathcal{T} = \{T_\alpha : \alpha < \mathfrak{c}\}$, and for all $\alpha < \beta < \mathfrak{c}$, $T_\beta \setminus T_\alpha$ is finite, and $A_\alpha = T_\alpha \setminus T_{\alpha+1}$ is infinite. Clearly, $\mathcal{A} = \{A_\alpha : \alpha < \mathfrak{c}\}$ is an almost disjoint family. Let x_α be the point of $\Psi(\mathcal{A})$ associated with A_α. Put $X = \Psi(\mathcal{A}) \cup \{\infty\}$, let points of $\Psi(\mathcal{A})$ have their usual neighborhoods, and define basic neighborhoods of “∞” for each $\alpha < \mathfrak{c}$ and finite $H \subset \omega$ by

$$W(\alpha, H) = \{\infty\} \cup (T_\alpha \setminus H) \cup \{x_\beta : \alpha \leq \beta < c\}.$$

It is straightforward to show that X is a zero-dimensional T_2-space.

Since $W(\alpha, \emptyset)$ misses the closed discrete set $\{x_\beta : \beta < \alpha\}$, and $\mathfrak{c}$ is a limit cardinal, it is clear that $e(X) \geq \mathfrak{c}$, and since $|X| = \mathfrak{c}$, in fact $e(X) = \mathfrak{c}$ (however, X has no closed discrete set of cardinality $\mathfrak{c}$ since “∞” is a limit point of any set of cardinality $\mathfrak{c}$). Since X is separable, $d(X) = \omega$, and thus we have $e(X) = \mathfrak{c} = 2^{d(X)}$. It remains to show that X satisfies property (a). Let $\mathcal{U}$ be an open cover of X, and D a dense set. There exists $\alpha < \mathfrak{c}$, $U_\alpha \in \mathcal{U}$, and finite $H_\alpha \subset \omega$ such that $W(\alpha, H_\alpha) \subset U_\alpha$. Pick $d_0 \in T_\alpha \setminus H_\alpha \subset U_\alpha$. For $\beta < \alpha$, there exist finite sets F_β and $U_\beta \in \mathcal{U}$ such that

$$\{x_\beta\} \cup (A_\beta \setminus F_\beta) \subset U_\beta \quad \text{and} \quad A_\beta \setminus F_\beta \subset \omega \setminus T_\alpha.$$

Now $\mathcal{A}' = \{A_\beta \setminus F_\beta : \beta < \alpha\}$ is an almost disjoint family with $|\mathcal{A}'| < \mathfrak{c} = \mathfrak{p}$; so by Theorem 3, $\Psi(\mathcal{A}')$ satisfies property (a). Thus there exists a closed discrete $P \subset \omega$ such that $\mathrm{st}(P, \mathcal{U}) = \Psi(\mathcal{A}')$. Finally, we note that

$$Z = \omega \setminus (\mathrm{st}(P, \mathcal{U}) \cup (T_\alpha \setminus H_\alpha))$$

is a closed discrete subset of X. Thus $P \cup Z \cup \{d_0\}$ is closed discrete in X, $P \cup Z \cup \{d_0\} \subset \omega \subset D$, and

$$\mathrm{st}(P \cup Z \cup \{d_0\}, \mathcal{U}) = X.$$

Thus X satisfies property (a), and this completes the proof.

Remark 1. In the proof of Theorem 9(ii), we used the statement "$\mathfrak{c}$ is a limit cardinal." In the presence of "$\mathfrak{p} = \mathfrak{c}$", this statement is equivalent to "$\mathfrak{c}$ is weakly inaccessible" because $\mathfrak{p}$ is regular (see [2, 3.1] or [5, 21E]).

7. Using only dense sets of smallest cardinality. It seems natural to ask if we can find the numbers $dd(X)$ and $ddc(X)$ by looking only at dense sets of cardinality $d(X)$. To consider this, we define

$$dd_1(X) = \min\{|\mathcal{F}_D| : D \text{ is dense and } |D| = d(X)\},$$

and

$$ddc_1(X) = \min\{|\mathcal{C}| : D \text{ is dense and } |D| = d(X) \text{ and } \mathcal{C} \text{ is cofinal in } \mathcal{F}_D\}.$$

We ask whether $dd(X) = dd_1(X)$, and $ddc(X) = ddc_1(X)$.

Theorem 10. (1) *If GCH holds, then for every T_1-space X, $dd(X) = dd_1(X)$.*

(2) *If $\mathfrak{p} > \omega_1$ then there exists a $T_{3.5}$-space X such that*

(i) *$dd(X) < dd_1(X)$, and*

(ii) *$ddc(X) < ddc_1(X)$.*

Proof of (1). By T_1,

$$d(X) \le dd(X) \le dd_1(X) \le 2^{d(X)} = d(X)^+.$$

Thus we need only consider the case $d(X) = dd(X)$. In this case, there exists a dense set D such that $|\mathcal{F}_D| = d(X)$. By T_1, $|D| \le |\mathcal{F}_D|$; so $|D| = d(X)$.

Proof of (2)(i). The example is a subspace of 2^{ω_1} with the product topology, and we use well known properties of this space. Let $C \subset 2^{\omega_1}$ be a countable dense set. Let $D = \{f \in 2^{\omega_1} : |\{\alpha < \omega_1 : f(\alpha) \ne 0\}| \le \omega\}$. Put $X = C \cup D$ as a subspace of the space 2^{ω_1}. It is well known that D is countably compact. Let E be a countable dense subset of X. Then $E \cap D \subset D_\alpha$ for some $\alpha < \omega_1$ where

$$D_\alpha = \{f \in 2^{\omega_1} : f(\beta) = 0 \text{ for all } \alpha \le \beta < \omega_1\}.$$

Since D_α is a closed nowhere dense set in 2^{ω_1}, $E \cap (C \setminus D_\alpha)$ is dense in C, therefore dense in 2^{ω_1}. Pick any $y \in 2^{\omega_1} \setminus X$. Since $\chi(y, 2^{\omega_1}) = \omega_1$, and $\mathfrak{p} > \omega_1$, it is well known that there exists a countably infinite set $H \subset E$ such that H converges to y. It follows that H has no limit points in X (since $y \notin X$); so H is an infinite closed discrete subset of E. Thus $|\mathcal{F}_E| = 2^\omega$, and since E was an arbitrary countable dense subset of X, we have $dd_1(X) = 2^\omega$. On the other hand, $D_0 = \{f \in 2^{\omega_1} : |\{\alpha < \omega_1 : f(\alpha) = 1\}| < \omega\}$ is dense in 2^{ω_1}, and has no infinite closed discrete set in X since $D_0 \subset D$ (and D is countably compact). Hence $\mathcal{F}_{D_0} = [D_0]^{<\omega}$; so $|\mathcal{F}_{D_0}| = \omega_1$. Thus $dd(X) \le \omega_1$ (in fact, $dd(X) = \omega_1$). Thus we have

$$dd(X) = \omega_1 < \mathfrak{p} \le 2^\omega = dd_1(X).$$

Proof of (2)(ii). We first note that by the countable compactness of D, we have $ddc(X) \le \omega_1$ (in fact, $ddc(X) = \omega_1$). Now we show that $ddc_1(X) > \omega_1$. Let E be a countable dense subset of X, and $\{H_\alpha : \alpha < \omega_1\}$ a family of closed discrete (in X) subsets of E. Since X is dense in 2^{ω_1}, it follows that $\operatorname{cl} H_\alpha$, the closure of H_α in 2^{ω_1}, is nowhere dense in 2^{ω_1}. Since $\mathfrak{p} < \omega_1$, 2^{ω_1} is not the union of ω_1 nowhere dense sets [5, 14.2]; so there exists a point $y \in 2^{\omega_1}$ such that $y \notin \bigcup\{\operatorname{cl} H_\alpha : \alpha < \omega_1\}$. Again using $\mathfrak{p} > \omega_1$, there exists a sequence in E converging to y, which gives us a closed discrete (in X) set that is almost disjoint from (and hence not a subset of) each H_α.

The space $X = C \cup D$ in Theorem 10(2) does not have property (a). This follows from a result of Matveev [12, Proposition 2].

8. Questions

1. Can the consistency of $\neg\mathcal{S}$ be established without the use of large cardinals?

2. By Theorem 3, every almost disjoint family $\mathcal{A}$ with $|\mathcal{A}| < \mathfrak{p}$ is soft. Does $|\mathcal{A}| < \mathfrak{a}$ imply that $\mathcal{A}$ is soft (this question was raised by K. Kunen)? If the answer is "no", then we ask the question for $|\mathcal{A}| < b$ (recall that $\mathfrak{p} \le \mathfrak{b} \le \mathfrak{a} \le \mathfrak{c}$).

3. Can the set-theoretic assumptions in Theorem 8 be weakened, or in the case of Theorem 10(2)(ii), be eliminated entirely?

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