

Mapping class group of a handlebody

by

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Abstract. Let B be a 3-dimensional handlebody of genus g . Let \mathcal{M} be the group of the isotopy classes of orientation preserving homeomorphisms of B . We construct a 2-dimensional simplicial complex X , connected and simply-connected, on which \mathcal{M} acts by simplicial transformations and has only a finite number of orbits. From this action we derive an explicit finite presentation of \mathcal{M} .

We consider a 3-dimensional handlebody $B = B_g$ of genus $g > 0$. We may think of B as a solid 3-ball with g solid handles attached to it (see Figure 1). Our goal is to determine an explicit presentation of the mapping class group of B , the group \mathcal{M}_g of the isotopy classes of orientation preserving homeomorphisms of B . Every homeomorphism h of B induces a homeomorphism of the boundary $S = \partial B$ of B and we get an embedding of \mathcal{M}_g into the mapping class group $\text{MCG}(S)$ of the surface S .

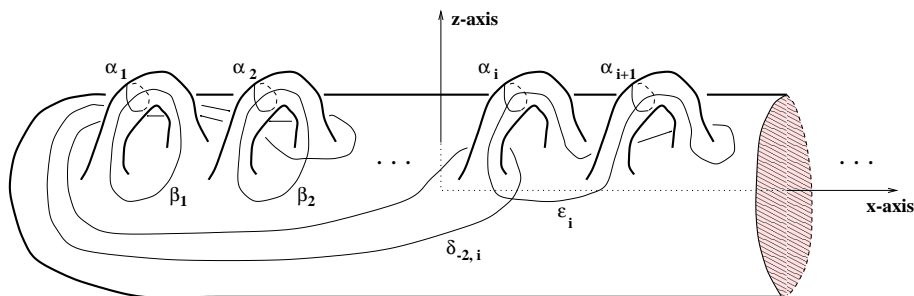


Fig. 1. Handlebody

An explicit and quite simple presentation of $\text{MCG}(S)$ is now known, but it took a lot of time and effort of many people to reach it (see [1], [3], [12],

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[9], [7], [11], [6], [14]). The main step towards an explicit presentation of $\text{MCG}(S)$ was done by Hatcher and Thurston in [7] where they constructed a connected and simply-connected cellular complex—a cut-system complex of S —on which $\text{MCG}(S)$ acts by cellular transformations with a finite number of orbits and well understood stabilizers. A presentation of $\text{MCG}(S)$ was obtained from this action by methods described more explicitly by Laudénbach [11]. The presentation was simplified by Harer [6], and further simplified by Wajnryb [14].

No presentation of \mathcal{M}_g has been known until now. A finite set of generators for \mathcal{M}_g was first obtained by Suzuki [13]. It was quite similar to the set of generators in our Theorem 18, the main result of this paper. The group \mathcal{M}_g has an infinite index in $\text{MCG}(S)$ so the knowledge of $\text{MCG}(S)$ does not help directly in the investigation of \mathcal{M}_g . However, we apply the main idea of Hatcher and Thurston and construct a simplicial, 2-dimensional cut-system complex X of B , similar to the cut-system complex of S , on which \mathcal{M}_g acts by simplicial transformations. We prove by a direct method, quite different from the method of Hatcher and Thurston, that X is connected and simply-connected. We describe the orbits of the action of \mathcal{M}_g on vertices, edges and faces of X . \mathcal{M}_g acts transitively on vertices and has only a finite number of edge-orbits and face-orbits. We describe the stabilizer of a vertex and the stabilizer of each edge and then apply the ideas from [7] and [11] to obtain an explicit presentation of \mathcal{M}_g (Theorem 18). We follow the Master dissertation of Michael Heusner [8] in which the method of [7] and [11] was very clearly and precisely explained. Unfortunately, an explicit presentation of \mathcal{M}_g (Theorem 18) obtained in this way is still rather long and complicated.

1. Cut-system complex. In this section we construct the complex X and prove that it is connected and simply-connected.

Let B be a handlebody of genus g and let S be the boundary of B . By a *curve* on S we always mean a simple closed curve. A curve α is a *meridian curve* if it bounds a disk D in B such that $D \cap S = \partial D = \alpha$. Then D is called a *meridian disk*. If α is a meridian curve, non-separating on S , and if we cut B along a corresponding meridian disk D , we get a handlebody B' of genus $g - 1$. On the boundary of B' we have two copies of D —two distinguished disks. For the purpose of induction on genus we shall consider a handlebody B with a finite number of disjoint distinguished disks on its boundary S and we shall assume that all homeomorphisms and all isotopies of S are fixed on the distinguished disks. If we have disjoint meridian curves on S we can construct pairwise disjoint meridian disks.

We start with disks D_1 and D_2 and put D_2 into general position with respect to D_1 . Then they intersect along disjoint circles. Consider an inner-

most circle C on D_1 . It bounds a disk A_1 in D_1 . It also bounds a disk A_2 in D_2 . Since C is innermost on D_1 the disk A_1 does not meet D_2 outside C . In particular A_1 and A_2 meet only at their common boundary C . Their union forms a sphere. Since B is irreducible the sphere bounds a 3-ball. We can push A_2 across the 3-ball off A_1 . By induction we can get rid of all the intersection of D_1 and D_2 . Now we keep D_1 and D_2 fixed and deform D_3 to make it disjoint from the other two disks, and so on.

A *cut-system* on S is an isotopy class of an unordered collection of g disjoint meridian curves $\alpha_1, \dots, \alpha_g$ which are linearly independent in the homology group $H_1(B)$ and are disjoint from the distinguished disks. When we cut B open along the corresponding disjoint meridian disks we get a 3-ball. We denote a cut-system by $\langle \alpha_1, \dots, \alpha_g \rangle$. We say that two cut-systems are *related by a simple move* if they have $g - 1$ curves in common and the other two curves are disjoint. To simplify notation we drop the symbols for unchanging circles, e.g. $\langle \alpha_i \rangle \rightarrow \langle \alpha'_i \rangle$ means that the curve α_i of a cut-system is replaced by a meridian curve α'_i and the other $g - 1$ curves of the cut-system do not change. It also means that α'_i is disjoint from all the curves of the cut-system and that the new collection of curves forms a new cut-system.

We construct a 2-dimensional complex X . The vertices of X are the cut-systems on S . Two cut-systems are connected by an edge if they are related by a simple move. If three vertices of X have $g - 1$ curves in common and the three remaining curves, one from each cut-system, are pairwise disjoint, then each pair of the vertices is connected by an edge in X and the vertices form a triangle, a closed edge-path with three edges. We glue a face to every triangle and get a 2-dimensional simplicial complex X . The complex X is called the *cut-system complex* of the handlebody B .

The main result of this section is the following

THEOREM 1. *The cut-system complex X is connected and simply-connected.*

A homeomorphism h of B onto itself takes a meridian disk to a meridian disk. It takes a cut-system to a cut-system. If we let $h\langle \alpha_1, \dots, \alpha_g \rangle = \langle h(\alpha_1), \dots, h(\alpha_g) \rangle$ we get an action of the mapping class group \mathcal{M}_g on the vertices of X . The action also takes vertices connected by an edge to vertices connected by an edge, so it is a simplicial action.

Every curve on S is ambient-isotopic to a PL-curve (piecewise-linear). We may assume that all curves in this paper are PL and that in the given, finite collection of curves different curves intersect only in a finite number of points, transversely, and no point belongs to three or more curves. We denote the number of intersection points of α and β by $|\alpha \cap \beta|$. We want to prove that X is simply-connected. We start with a fixed closed edge-path and want to prove that it is null-homotopic in X , i.e. it decomposes into a

sum of triangles. We fix a particular collection of curves representing each vertex and assume that all curves of all these collections intersect transversely at a finite number of points and no point belongs to more than two curves. We may later introduce new curves subject to the same conditions.

We first consider the case of genus $g = 1$.

If B is a solid torus then all non-trivial (not contractible on S) meridian curves on the boundary S of B are isotopic. A cut-system is an isotopy class of a single meridian curve and thus X consists of a single point. But if there are distinguished disks on S , which we need for the induction purpose, then different meridian curves may be non-isotopic in the complement of the distinguished disks and they may intersect essentially. In that case the complex X may be quite complicated.

We say that two meridian curves α and β on S form a *2-gon* if there is a disk (a 2-gon) in S bounded by an arc of α and an arc of β . A 2-gon may contain distinguished disks inside. A collection of curves forms a 2-gon if a pair of curves of this collection form a 2-gon. In that case a *minimal 2-gon* is one which does not contain another 2-gon inside. If α and β form a minimal 2-gon then any other curve of the collection crosses the 2-gon along arcs which meet each of the curves α and β once. If two curves on S are homotopic and not disjoint then they form a 2-gon (see [5]).

LEMMA 2. *If $g = 1$ then X is connected.*

PROOF. Let α and β be two meridian curves on S . They are homotopic, so if they are not disjoint they form a 2-gon. Consider a minimal 2-gon, say bounded by arcs a and b of α and β respectively. If we replace the arc a of α by the arc b we get a new curve α' homotopic to α on S (see Figure 2). Then α' is a meridian curve, $|\alpha \cap \alpha'| = 0$ and $|\beta \cap \alpha'| < |\beta \cap \alpha|$. It follows by induction on $|\alpha \cap \beta|$ that α and β can be connected by a sequence of curves such that the consecutive curves are disjoint, and thus connected by an edge in X . ■

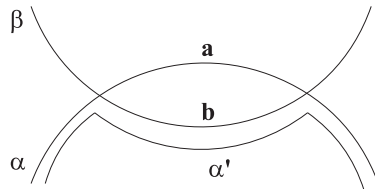


Fig. 2. The curves α and β form a 2-gon

LEMMA 3. *Let $g = 1$ and let $\mathbf{p} = \{\alpha_1, \dots, \alpha_k\}$ be a closed edge-path in X , i.e. $\alpha_1 = \alpha_k$. There exists another closed edge-path $\mathbf{p}' = \{\alpha'_1, \dots, \alpha'_k\}$ homotopic to \mathbf{p} in X and such that the collection of curves $\alpha'_1, \dots, \alpha'_k$ forms no 2-gons.*

Proof. Consider a minimal 2-gon. Suppose it is formed by arcs a and b of curves α_i and α_j respectively. If we replace b by a (as in Figure 2) we get a new curve α'_j , homotopic to α_j on S and disjoint from α_j . So it is a meridian curve and represents a vertex in X . Also $|\alpha'_j \cap \alpha_i| < |\alpha_j \cap \alpha_i|$. By a previous remark, α'_j does not form any new intersections with other curves so it is also disjoint from α_{j-1} and α_{j+1} and is connected by edges to α_{j-1} , α_j , and α_{j+1} . When we replace α_j by α'_j we get a new edge-path and the difference between it and the old path is a sum of two triangles. Thus the new path is homotopic to \mathbf{p} in X and its curves intersect in a smaller number of points. Lemma 3 follows by induction. ■

LEMMA 4. *If $g = 1$ then X is simply-connected.*

Proof. Consider a closed edge-path in X . We want to prove that it can be decomposed into a sum of triangles. By the previous lemma we may assume that it has no 2-gons. But meridian curves on a solid torus are homotopic. If there are no 2-gons then all curves are disjoint, and every pair is connected by an edge. In particular the edge-path is a sum of triangles as required. ■

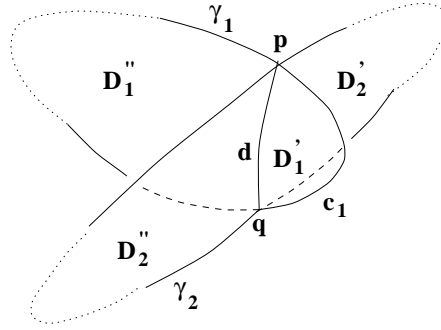
From now on we assume that B has genus g greater than 1.

INDUCTION HYPOTHESIS 1. The cut-system complex of a handlebody of genus less than g is connected and simply-connected.

Our next task is to prove that X is connected. We start with a lemma which will also be used later, in the proof that X is simply-connected, so it may seem unnecessarily complicated at this stage.

LEMMA 5. *Let $\alpha, \beta, \gamma_1, \gamma_2$ be non-separating meridian curves on S such that $|\gamma_1 \cap \gamma_2| = k > 0$, $|\gamma_1 \cap \alpha| < n$, and $|\gamma_2 \cap \alpha| \leq n$. Then there exists a non-separating meridian curve δ such that $|\delta \cap \gamma_1| < k$, $|\delta \cap \gamma_2| < k$, and $|\delta \cap \alpha| < n$. If also $|\gamma_1 \cap \beta| = |\gamma_2 \cap \beta| = 0$, then $|\delta \cap \beta| = 0$.*

Proof. We may put the meridian disks D_1 and D_2 , bounded by γ_1 and γ_2 respectively, into general position. Then they intersect along disjoint arcs and circles. We can get rid of the circles of intersection by moving disks across 3-balls corresponding to innermost circles. Consider now an innermost arc d of the intersection (closest to the boundary) on D_1 . The arc splits D_1 into two disks. One of them, say D'_1 , does not meet D_2 (see Figure 3). The disk D'_1 is bounded by d and by an arc c_1 of γ_1 . The arc c_1 contains m points of the curve α . Consider also a corresponding construction for an innermost arc on D_2 . Among all innermost arcs choose a “minimal” arc d , one for which the number m is minimal. If there are several such “minimal” arcs and one of them is innermost on D_2 we choose that one.

Fig. 3. Meridian disks of γ_1 and γ_2

Suppose first that the minimal arc d is innermost on D_1 (see Figure 3). Let p and q be the end points of d . The arc d also splits D_2 into two disks D'_2 and D''_2 bounded by d and by arcs c'_2 and c''_2 of γ_2 respectively. These disks may intersect D_1 but not inside D'_1 . When we replace the arc c'_2 (or c''_2) of γ_2 by c_1 we get a meridian curve δ' (respectively δ'') on S which bounds a meridian disk $D'_1 \cup D'_2$ (respectively $D'_1 \cup D''_2$). We can push δ' (or δ'') off itself on S , away from the intersections p and q , making it disjoint from γ_2 . We get rid of the intersection points p and q at least, so $|\delta' \cap \gamma_1| < k$ and $|\delta'' \cap \gamma_1| < k$. In the first homology group of S we have $\delta'' + \delta' = \gamma_2$. Since γ_2 is non-separating, i.e. non-trivial in the homology group, one of the curves must be non-separating and we choose this curve for δ . The disks D'_2 and D''_2 must contain innermost arcs (possibly d is innermost on one or both disks), therefore both c'_2 and c''_2 meet α more than m times, by the minimality condition on d . We traded such an arc of γ_2 for the arc c_1 , therefore $|\delta \cap \alpha| < n$.

Suppose now that the minimal arc d is innermost on D_2 . We switch the roles of D_1 and D_2 (also in Figure 3) and repeat the construction of δ . There may be some innermost arcs on D_1 for which the corresponding arc of γ_1 also meets α only m times, but not less. In order to construct δ we replace an arc of γ_1 which meets α at least m times by an arc of γ_2 which meets α exactly m times. So δ meets α at most as many times as γ_1 and again $|\delta \cap \alpha| < n$ and the other properties of δ are as in the first case. The assertion about the curve β follows from the construction. ■

LEMMA 6. *If v_1 and v_2 are vertices in X which have one or more curves in common then they can be connected by an edge-path of vertices in X all of which contain the common curves of v_1 and v_2 .*

PROOF. When we cut B open along the disjoint meridian disks bounded by the common curves of v_1 and v_2 we get a handlebody B' of a smaller genus and the remaining curves of v_1 and v_2 form two vertices of the cut-system

complex of B' . By Induction Hypothesis 1 there exists an edge-path \mathbf{p}' in this complex which connects the two vertices. When we add the common curves of v_1 and v_2 to curves of each vertex of the edge-path \mathbf{p}' we get cut-systems on B which form an edge-path in X connecting v_1 and v_2 . ■

LEMMA 7. *If α and β are non-separating meridian curves then there exist vertices v_1 and v_2 of X which are connected by an edge-path in X (possibly $v_1 = v_2$) and contain the curves α and β respectively.*

PROOF. We prove the lemma by induction on $n = |\alpha \cap \beta|$. Assume first that the curves are disjoint. If $\alpha \cup \beta$ does not separate the surface S then the pair α, β can be completed to a cut-system on B representing a vertex $v_1 = v_2$. Suppose now that $\alpha \cup \beta$ separates S . When we cut B along disjoint meridian disks bounded by α and β respectively we get two handlebodies B_1 and B_2 . We choose a cut-system on each of the handlebodies. We get together $g - 1$ meridian curves. When we add α or β to these $g - 1$ curves we get two cut-systems on B which are related by a simple move.

Assume now that $n > 0$. By Lemma 5 we can find a non-separating meridian curve γ with $|\gamma \cap \alpha| < n$ and $|\gamma \cap \beta| < n$. By the induction hypothesis there exists a vertex v_1 of X containing α and connected by an edge-path to a vertex v_3 containing γ . There also exists a vertex v_4 containing γ and connected by an edge-path to a vertex v_2 containing β . Finally, v_3 and v_4 are connected by an edge-path by Lemma 6. ■

As an easy corollary we get

PROPOSITION 8. *X is connected.*

PROOF. If v_1 is a vertex of X containing some curve α and v_2 is a vertex of X containing some curve β then, by Lemma 7, there exist other vertices v_3 and v_4 connected by an edge-path and containing α and β respectively. By Lemma 6 we can connect v_1 to v_3 and v_4 to v_2 by edge-paths in X . ■

We now proceed to prove that X is simply-connected.

We define the distance between two curves α and β as their intersection number $|\alpha \cap \beta|$. The distance between a curve γ and a vertex $\langle \alpha_1, \dots, \alpha_g \rangle$ is equal to the minimum of the distances $|\gamma \cap \alpha_i|$. The *radius* of an edge-path around a curve α is equal to the maximum of the distances from α to the vertices of the edge-path.

A *segment* is a sequence of consecutive vertices of an edge-path which have a curve in common. We say it is an $(\alpha, \beta, \dots, \gamma)$ -*segment* if all vertices of the segment contain the curves $(\alpha, \beta, \dots, \gamma)$.

The general idea of the proof is to start with a closed edge-path in X and reduce it to a sum of simpler edge-paths, and eventually to a sum of triangles, by induction on the radius of the edge-path around a fixed curve α and induction on the number of segments of maximal distance from α .

LEMMA 9. *A closed segment is null-homotopic.*

PROOF. We cut B open along the meridian disk bounded by the common curve α of the segment, getting a handlebody B' of a smaller genus. The remaining curves of each vertex form a vertex of a closed path in the cut-system complex of B' . This path decomposes into a sum of triangles by Induction Hypothesis 1. When we add back the curve α to every vertex we get a decomposition of the initial closed segment into a sum of triangles. ■

LEMMA 10. *A closed path $\mathbf{p} = \langle \alpha, \gamma \rangle \rightarrow \langle \beta, \gamma \rangle \rightarrow \langle \beta, \delta \rangle \rightarrow \langle \alpha, \delta \rangle \rightarrow \langle \alpha, \gamma \rangle$ is null-homotopic in X .*

PROOF. If $g > 2$ then the path is a closed segment, all vertices of \mathbf{p} have at least one additional common curve, so we may assume $g = 2$. All the curves are disjoint. If $\alpha \cup \beta$ (or $\gamma \cup \delta$) does not separate S there is a vertex $\langle \alpha, \beta \rangle$ (respectively $\langle \gamma, \delta \rangle$) which is connected by an edge to every vertex of the path and splits the path into a sum of four triangles. In the other case let us split B open along the meridian disks of $\alpha, \beta, \gamma,$ and δ . Since $\alpha \cup \gamma$ does not separate S we get three connected components and eight boundary components.

There is a connected component with at least three boundary components. By the separation properties of α, β, γ and δ this connected component has three boundary components corresponding to three different curves out of α, β, γ and δ . Two of them belong to one vertex, say α and γ . We can form a connected sum $\alpha + \gamma$ of the two curves inside the connected component. We get a new meridian curve ε disjoint from all four curves. The curve ε forms a vertex with each of the four curves. The path \mathbf{p} decomposes into a sum of ten triangles (see Figure 4). ■

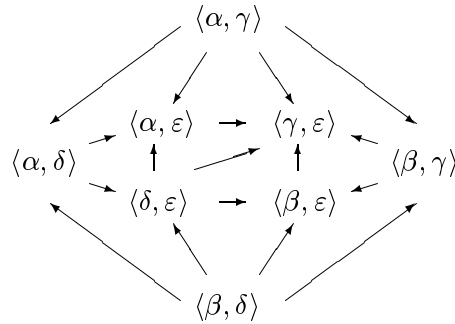


Fig. 4. Reduction of a square path

LEMMA 11. *Let α and β be disjoint, non-separating meridian curves. Let v_1 and v_2 be vertices of X connected by an edge and containing α and β respectively (or $v_1 = v_2$ contains both curves α and β). Let v_3 and v_4 be another pair of vertices connected by an edge and containing α and β*

respectively. Then there exist an α -segment connecting v_1 and v_3 and a β -segment connecting v_2 and v_4 such that the union of the two segments, together with edges connecting v_1 to v_2 and v_3 to v_4 , forms a closed edge-path null-homotopic in X .

Proof. Suppose first that $\alpha \cup \beta$ does not separate S . When we cut B along meridian disks bounded by α and β we get a handlebody B_1 of genus $g - 2$. We can choose $g - 2$ curves out of the common curves of v_1 and v_2 which form a cut-system u_1 on B_1 . We can also choose $g - 2$ curves out of the common curves of v_3 and v_4 which form a cut-system w_1 on B_1 .

We connect u_1 to w_1 by an edge-path in the cut-system complex of B_1 . We add α and β to all vertices of this edge-path. We get an (α, β) -segment. The first vertex of this segment is connected by edges to v_1 and v_2 and forms a triangle (or is equal to v_1 if $v_1 = v_2$). The last vertex is connected by edges to v_3 and v_4 and forms another triangle (see Figure 5, right side). We can extend the (α, β) -segment to an α -segment from v_1 to v_3 and to a β -segment from v_2 to v_4 . The union of the segments and the edges is equal to a sum of two triangles and a trivial path.

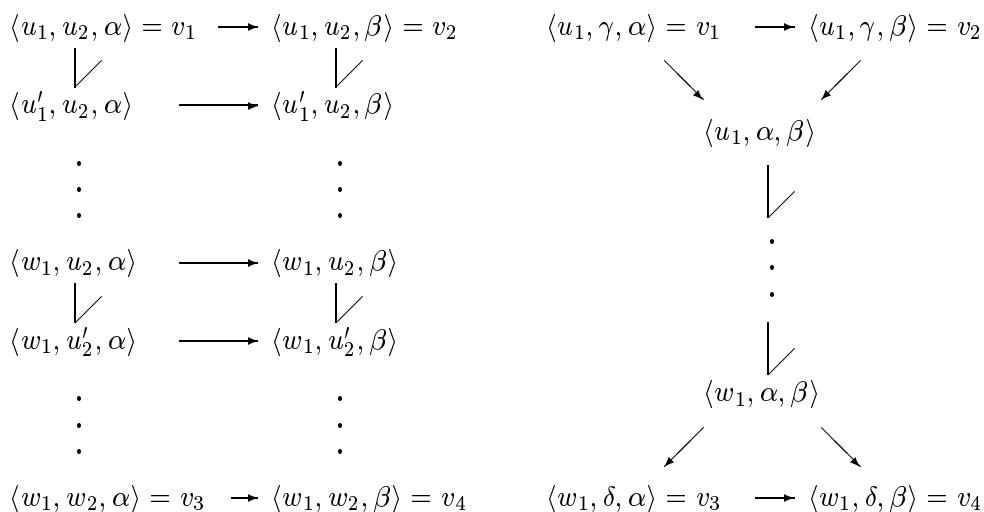


Fig. 5. Parallel segments

Suppose now that $\alpha \cup \beta$ separates S . When we cut B along meridian disks bounded by α and β we get two handlebodies B_1 and B_2 and the sum of the genera of B_1 and B_2 equals $g - 1$. The remaining curves, common to v_1 and v_2 , split into two families; one forms a cut-system u_1 on B_1 and the other forms a cut-system u_2 on B_2 . Also the remaining curves in v_3 and v_4

form a cut-system w_1 on B_1 and a cut-system w_2 on B_2 . We connect u_1 to w_1 by an edge-path in the cut-system complex of B_1 . We add α and the curves of w_2 to all vertices of the edge-path getting an edge-path in X (see Figure 5, left side).

Now we connect u_2 to w_2 by an edge-path in the cut-system complex of B_2 and add α and all curves of w_1 to all vertices of this edge-path. We get a new edge-path in X which extends the first edge-path. Together we get an α -segment connecting v_1 to v_3 in X . Now we replace α by β in all vertices of this segment and get a β -segment connecting v_2 to v_4 in X . Corresponding vertices in the two segments are joined by an edge. The union of the segments together with the edges connecting v_1 to v_2 and v_3 to v_4 splits into squares which are null-homotopic by Lemma 10. ■

LEMMA 12. *A closed edge-path \mathbf{p} of radius zero around a curve α of some vertex of \mathbf{p} is null-homotopic in X .*

Proof. We prove the lemma by induction on the number of segments having a common curve disjoint from α . If all vertices contain α then we have a closed segment. In the other case consider a maximal segment containing α . Let u_1 be its first vertex and let v_1 be its last vertex. Let u_2 be the next vertex of \mathbf{p} . The vertex u_2 contains a curve β disjoint from α , the common curve of the second segment. Let v_2 be the last vertex of the second segment. Suppose that there are only two segments. We are in the situation of Lemma 11. We can connect the two pairs of vertices v_1, u_2 and v_2, u_1 by parallel segments and \mathbf{p} splits into a sum of two closed segments and a path null-homotopic by Lemma 11. So we may assume that there exists a third segment with a common curve γ disjoint from α and β . Let u_3 be the vertex of \mathbf{p} following v_2 . It contains γ . We cut B along meridian disks bounded by α, β, γ .

If we get three components then the genus goes down by one. We choose a cut-system on each component, add them up and get $g-1$ curves. When we also add the curve α (respectively β, γ) we get a cut-system w_1 (respectively w_2, w_3) on B and the cut-systems form a triangle. By Lemma 11 we can connect w_1 to v_1 by an α -segment, w_2 to u_2 and v_2 by β -segments and w_3 to u_3 by a γ -segment. Figure 6 shows that \mathbf{p} can be replaced by an edge-path which does not have a β -segment and the difference of the edge-paths decomposes into a sum of null-homotopic paths by Lemmas 9 and 11. If $\alpha \cup \beta$ does not separate S we can find a vertex $w_1 = w_2$ which contains both curves α and β and is connected by an edge to a vertex w_3 containing γ . The case of $\alpha \cup \gamma$ or $\beta \cup \gamma$ non-separating is similar. If $\alpha \cup \beta \cup \gamma$ does not separate we can find a vertex $w_1 = w_2 = w_3$ containing all three curves. The argument using Figure 6 works in all cases. Lemma 12 follows by induction on the number of segments. ■

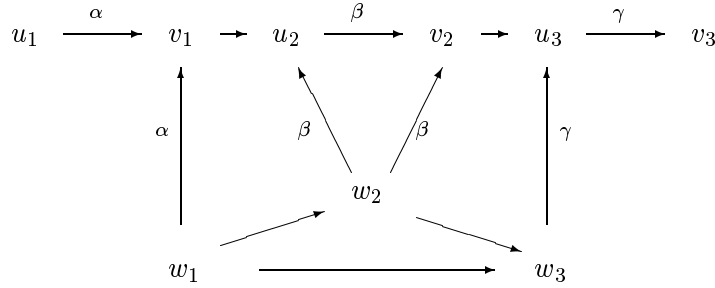


Fig. 6. Reduction of a path of radius 0

The next lemma will be used to make a short-cut from a vertex u_1 to u_2 in the situation of Figure 7.

LEMMA 13. *Let $\alpha, \beta, \gamma_1, \gamma_2$ be non-separating curves on S such that $|\gamma_1 \cap \gamma_2| = k, |\gamma_1 \cap \alpha| < m, |\gamma_2 \cap \alpha| \leq m$ and $|\gamma_1 \cap \beta| = |\gamma_2 \cap \beta| = 0$. Let u_1 and u_2 be vertices of X containing γ_1 and γ_2 respectively. Then there exists a path \mathbf{q} connecting u_1 and u_2 such that all vertices of \mathbf{q} have distance zero from β and have distance less than m from α , except possibly for a final γ_2 -segment of \mathbf{q} which ends at u_2 .*

PROOF. If $|\gamma_1 \cap \gamma_2| = 0$ then we can find vertices w_1 and w_2 connected by an edge in X and containing γ_1 and γ_2 respectively (possibly one vertex contains both). We can connect u_1 to w_1 by a γ_1 -segment and w_2 to u_2 by a γ_2 -segment, by Lemma 6, getting the required edge-path \mathbf{q} .

If $|\gamma_1 \cap \gamma_2| = k > 0$ then, by Lemma 5, we can find a non-separating meridian curve δ such that $|\delta \cap \beta| = 0, |\delta \cap \alpha| < m, |\delta \cap \gamma_1| < k,$ and $|\delta \cap \gamma_2| < k$. Let w be a vertex containing δ . By induction on k we can find an edge-path \mathbf{q}_1 between u_1 and w and an edge-path \mathbf{q}_2 between w and u_2 such that both have radius zero around β , all vertices of \mathbf{q}_1 have distance less than m from α and all vertices of \mathbf{q}_2 have distance less than m from α , except possibly for the final γ_2 -segment which ends at u_2 . Together they form the required edge-path \mathbf{q} . ■

Proof of Theorem 1. X is connected by Proposition 8. We want to prove that every closed edge-path in X is null-homotopic. Suppose that a closed path \mathbf{p} has radius m around some curve α . If $m = 0$ then \mathbf{p} is null-homotpic by Lemma 12. We continue by induction on m . For fixed m we prove the result by induction on the number of segments of \mathbf{p} which have a common curve at distance m from α .

Let v_0 be a vertex of \mathbf{p} containing α . We say that \mathbf{p} begins at v_0 . Let v_1 be the first vertex of \mathbf{p} which has distance m from α . Let \mathbf{q}_0 be the maximal segment of \mathbf{p} which starts at v_1 , contains some fixed curve β satisfying $|\beta \cap \alpha| = m$ and such that no vertex of \mathbf{q}_0 contains a curve β'

satisfying $|\beta' \cap \alpha| < m$. Let v_2 be the last vertex of \mathbf{q}_0 . Let u_1 be the vertex of \mathbf{p} preceding v_1 and let u_2 be the vertex of \mathbf{p} following v_2 (see Figure 7). The vertex u_1 contains a curve γ_1 such that $|\gamma_1 \cap \alpha| < m$. The vertex u_2 contains a curve γ_2 such that $|\gamma_2 \cap \alpha| < m$ or $|\gamma_2 \cap \alpha| = m$ and γ_2 is the common curve of the next segment. Since the curves in the neighbouring vertices are disjoint we have $|\beta \cap \gamma_1| = |\beta \cap \gamma_2| = 0$.

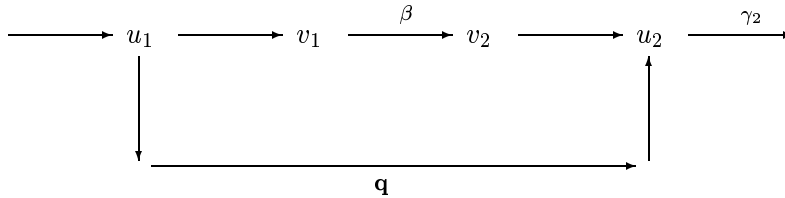


Fig. 7. Reduction of a path of radius m

We can now replace the part of \mathbf{p} from u_1 to u_2 by a path \mathbf{q} from Lemma 13 (see Figure 7). The new path \mathbf{p}_1 differs from \mathbf{p} by a closed path of radius zero around β , so \mathbf{p}_1 is homotopic to \mathbf{p} , and it has no β -segment (although its γ_2 -segment may be longer than in \mathbf{p}). The number of segments in \mathbf{p}_1 at distance m from α is smaller than in \mathbf{p} . Theorem 1 follows by induction. ■

2. A presentation of \mathcal{M}_g . In this section we establish a presentation of the mapping class group of a handlebody. We denote by $B = B_g$ a handlebody of genus g represented in Figure 1. We denote by S the boundary of B , by \mathcal{M} the mapping class group of B and by X the cut-system complex of B .

For $g = 1$ one can compute the mapping class group directly:

THEOREM 14. *The mapping class group \mathcal{M}_1 of a solid torus B is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_2$.*

Proof. If $g = 1$ then the curve α_1 in Figure 1 is the only meridian curve on the boundary S of B , up to isotopy. A homeomorphism h of B either leaves the isotopy class of α_1 invariant or, up to isotopy, h fixes α_1 setwise and reverses its orientation, so that h^2 leaves the isotopy class of α_1 invariant. The z -axis in Figure 1 is shown for g even. For odd genus the z -axis passes through the middle handle. In the case of genus one the rotation z through 180 degrees around the z -axis reverses the orientation of α_1 , and $z^2 = 1$. So for every $h \in \mathcal{M}_1$ either h or zh preserves the isotopy class of α_1 . If a homeomorphism h preserves the isotopy class of α_1 then it is isotopic to a homeomorphism fixed on α_1 . It induces a homeomorphism of the annulus $S - \alpha_1$ and thus it is a power of the Dehn twist a_1 with respect to the curve α_1 (see Definition 1 below). The elements a_1 and z of \mathcal{M}_1 commute. The theorem follows.

From now on we assume that $g > 1$.

To shorten some formulas we adopt the following notation for conjugation: $a * b = aba^{-1}$. As usual, $[a, b] = aba^{-1}b^{-1}$.

REMARK 1. Some proofs of relations between homeomorphisms of a surface are left to the reader. The general idea of the proof is as follows. We split the surface into a union of disks by a finite number of curves (and arcs with the end-points on the boundary if the surface has a boundary). We prove that the given product of homeomorphisms takes each curve (respectively arc) onto an isotopic curve (arc), preserving some fixed orientation of the curve (arc). Then the product is isotopic to a homeomorphism pointwise fixed on each curve and arc. But a homeomorphism of a disk fixed on its boundary is isotopic to the identity homeomorphism, relative to the boundary, by the Lemma of Alexander. Thus the given product of homeomorphisms is isotopic to the identity.

We start with some properties of homeomorphisms of the surface S . Dehn proved in [4] that every homeomorphism of S is isotopic to a product of *twists* of the following type.

DEFINITION 1. A (positive) *Dehn twist* with respect to a simple closed curve α on an oriented surface S is the isotopy class of a homeomorphism h of S , supported in a regular neighbourhood N of α (an annulus), obtained as follows: we cut S open along α , rotate one side of the cut by 360 degrees to the right (this makes sense on an oriented surface) and then glue the surface back together, damping out the rotation to the identity at the boundary of N . The Dehn twist (or simply twist) with respect to α will be denoted by T_α .

LEMMA 15. *Let α be a curve on S , let h be a homeomorphism and let $\alpha' = h(\alpha)$. Then $T_{\alpha'} = hT_\alpha h^{-1}$.*

PROOF. Since h maps α to α' we may assume that (up to isotopy) it also maps a neighbourhood N of α to a neighbourhood N' of α' . The homeomorphism h^{-1} takes N' to N , then T_α maps N to N , twisting along α , and h takes N back to N' . Since T_α is supported in N , the composite map is supported in N' and it is a Dehn twist about α' . ■

LEMMA 16. *Let $\gamma_1, \dots, \gamma_k$ be a chain of curves, i.e. the consecutive curves intersect once and non-consecutive curves are disjoint. Let N denote a regular neighbourhood of the union of these curves. Let c_i denote the twist along γ_i . Then the following relations hold:*

- (i) The “commutativity relation”: $c_1 c_3 = c_3 c_1$.
- (ii) The “braid relation”: $c_1 c_2(\gamma_1) = \gamma_2$, $c_1 c_2 c_1 = c_2 c_1 c_2$.

(iii) The “chain relation”: If k is odd then N has two boundary components, ∂_1 and ∂_2 , and $(c_1 \dots c_k)^{k+1} = T_{\partial_1} T_{\partial_2}$. If k is even then N has one boundary component ∂_1 and $(c_1 \dots c_k)^{2k+2} = T_{\partial_1}$.

$$(iv) (c_2 c_1 c_3 c_2)(c_4 c_3 c_5 c_4)(c_2 c_1 c_3 c_2) = (c_4 c_5 c_3 c_4)(c_2 c_1 c_3 c_2)(c_4 c_3 c_5 c_4).$$

Proof. Relation (i) is obvious. It follows immediately from the definition of Dehn twist that $c_2(\gamma_1) = c_1^{-1}(\gamma_2)$. Both statements of (ii) follow from this and from Lemma 15. Relation (iii) is a little more complicated, but we shall only use it for $k = 2, 3$, for which it is well known (see [14]).

We prove (iv). The proof involves a long sequence of braid transformations (relations (i) and (ii)):

$$\begin{aligned} (c_2 c_1 c_3 c_2)(c_4 c_3 c_5 c_4)(c_2 c_1 c_3 c_2) &= c_2 c_3 c_1 c_2 c_4 c_5 c_3 c_4 c_2 c_3 c_1 c_2 \\ &= c_2 c_3 c_4 c_1 c_5 c_2 c_3 c_2 c_4 c_3 c_1 c_2 = c_2 c_3 c_4 c_3 c_1 c_5 c_2 c_3 c_4 c_3 c_1 c_2 \\ &= c_4 c_2 c_3 c_4 c_1 c_5 c_2 c_4 c_3 c_1 c_2 c_4 = c_4 c_2 c_3 c_1 c_4 c_5 c_4 c_2 c_3 c_1 c_2 c_4 \\ &= c_4 c_5 c_2 c_3 c_1 c_4 c_2 c_3 c_1 c_2 c_5 c_4 = c_4 c_5 c_2 c_3 c_1 c_2 c_1 c_4 c_3 c_2 c_5 c_4 \\ &= c_4 c_5 c_3 c_2 c_3 c_4 c_1 c_2 c_3 c_2 c_5 c_4 = c_4 c_5 c_3 c_2 c_3 c_4 c_3 c_1 c_2 c_3 c_5 c_4 \\ &= c_4 c_5 c_3 c_4 c_2 c_3 c_4 c_1 c_2 c_3 c_5 c_4 = (c_4 c_5 c_3 c_4)(c_2 c_1 c_3 c_2)(c_4 c_3 c_5 c_4). \blacksquare \end{aligned}$$

LEMMA 17. Let U be a disk with outer boundary ∂ and with n inner holes bounded by curves $\partial_1, \dots, \partial_n$. For $1 \leq i < j \leq n$ let $\alpha_{i,j}$ be the simple closed curve in U shown in Figure 8, separating two holes ∂_i and ∂_j from the other holes. Let d be the twist along ∂ , let d_i be the twist along ∂_i , and let $a_{i,j}$ be the twist along $\alpha_{i,j}$. Then

$$d = (a_{1,2} a_{1,3} \dots a_{1,n} a_{2,3} \dots a_{2,n} a_{3,4} \dots a_{n-2,n-1} a_{n-2,n} a_{n-1,n})(d_1 \dots d_n)^{2-n}.$$

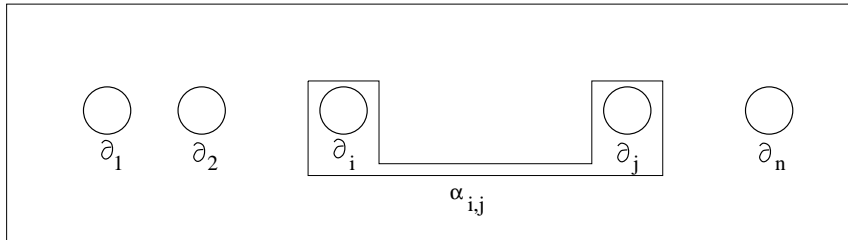


Fig. 8. Disk with holes and the curves $\alpha_{i,j}$

Proof. If we allow isotopies which rotate the curves ∂_i then the group of the isotopy classes of homeomorphisms of U is isomorphic to the group of the isotopy classes of homeomorphisms which preserve punctures of a disk with n punctures. This is the classical pure braid group on n strings. The elements $a_{i,j}$ represent the standard generators of the pure braid group and

d is equal to the generator Δ^2 of the centre. Therefore, *modulo the twists* d_i ,

$$d = (a_{1,2}a_{1,3} \dots a_{1,n}a_{2,3} \dots a_{2,n}a_{3,4} \dots a_{n-2,n-1}a_{n-2,n}a_{n-1,n}).$$

The factor corresponding to the twists d_i can be checked by induction (by drawing many pictures). ■

The special case of the above lemma, for $n = 3$, was observed by Dennis Johnson [10] and called a *lantern relation*.

We now fix some curves on S . The handlebody B consists of g handles attached to a 3-ball. For $i = 1, \dots, g$ and $j = 1, \dots, g - 1$ we fix curves $\alpha_i, \beta_i, \varepsilon_j$ (see Figure 1). The curve α_i is a meridian curve across the i th handle, β_i is a curve along the i th handle and ε_i runs along the i th handle and the $(i + 1)$ st handle. The curves $\alpha_1, \dots, \alpha_g$ form a cut-system.

We denote by I_0 the set of indices $\{-g, 1 - g, \dots, -1, 1, 2, \dots, g\}$.

When we cut S open along the curves $\alpha_1, \dots, \alpha_g$ we get a sphere S_0 with $2g$ holes bounded by curves $\partial_i, i \in I_0$, where ∂_i and ∂_{-i} correspond to the same curve α_i on S (see Figure 10). The glueing back map identifies ∂_i with ∂_{-i} according to the reflection with respect to the x -axis. Curves on S can be represented on S_0 . If a curve on S meets some curves α_i then it is represented on S_0 by a disjoint union of several arcs. In particular ε_i is represented by two arcs joining ∂_{-i} to ∂_{-i-1} and ∂_i to ∂_{i+1} . We denote by $\delta_{i,j}, i < j \in I_0$, the curves represented in Figures 1 and 10. The curve $\delta_{i,j}$ separates the holes ∂_i and ∂_j from the other holes on S_0 .

DEFINITION 2. For each subset $I = \{i_1, \dots, i_n\}$ of I_0 with $i_1 < \dots < i_n$ we define a *standard curve* δ_I in S_0 (see Figure 9), which separates the holes $\partial_{i_1}, \dots, \partial_{i_n}$ from the other holes in S_0 . We see that if $i, j \in I$ then $\delta_{i,j}$ is contained inside δ_I . If I contains just two indices i, j then $\delta_I = \delta_{i,j}$, as in Figure 10. For $i \leq j \in I_0$ and $I = \{k \in I_0 \mid i \leq k \leq j\}$ we define a curve $\gamma_{i,j} = \delta_I$.

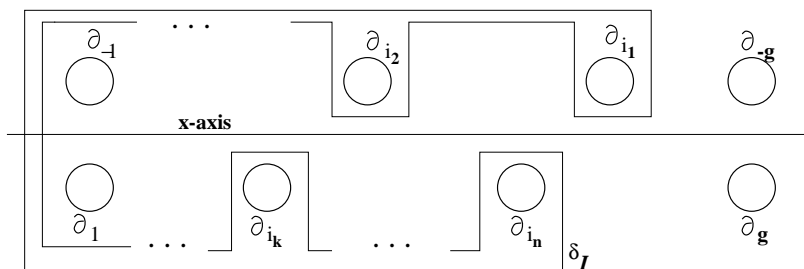
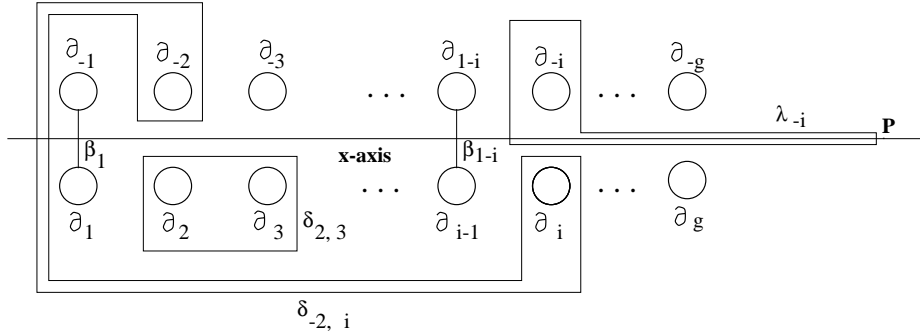


Fig. 9. Standard curve δ_I on S_0 for $I = \{i_1, \dots, i_n\}$

We now fix some elements of \mathcal{M} . A homeomorphism h of S represents an element in \mathcal{M} if it takes curves of some cut-system $\langle \gamma_1, \dots, \gamma_g \rangle$ to curves

Fig. 10. Curves $\delta_{i,j}$ and λ_k on S_0

of a cut-system. Indeed, we can extend h to disjoint meridian disks bounded by $\gamma_1, \dots, \gamma_g$. When we cut B open along the meridian disks we get a ball and h can be extended from the boundary of the ball to its interior. A twist with respect to a curve α belongs to \mathcal{M} if and only if α is a meridian curve. A curve which is disjoint from all curves of some cut-system is a meridian curve.

DEFINITION 3. We denote by a_i, b_i, e_j the Dehn twists along the curves $\alpha_i, \beta_i, \varepsilon_j$ respectively and we denote by $d_{1,2}$ the Dehn twist with respect to $\delta_{1,2}$. For $i \in I_0$ we denote by d_i the twist of S_0 along the curve ∂_i . It can be identified with the twist $a_{|i|}$ of S . We fix the following elements of \mathcal{M} :

$$s = b_1 a_1 a_1 b_1,$$

$$t_i = e_i a_i a_{i+1} e_i \quad \text{for } i = 1, \dots, g-1.$$

For $i < j \in I_0$ we let

$$d_{i,j} = (t_{i-1} t_{i-2} \dots t_1 t_{j-1} t_{j-2} \dots t_2) * d_{1,2} \quad \text{if } i > 0,$$

$$d_{i,j} = (t_{-i-1}^{-1} t_{-i-2}^{-1} \dots t_1^{-1} s^{-1} t_{j-1} t_{j-2} \dots t_2) * d_{1,2} \quad \text{if } i < 0 \text{ and } i+j > 0,$$

$$d_{i,j} = (t_{-i-1}^{-1} t_{-i-2}^{-1} \dots t_1^{-1} s^{-1} t_j t_{j-1} \dots t_2) * d_{1,2}$$

$$\quad \text{if } i < 0, j > 0 \text{ and } i+j < 0,$$

$$d_{i,j} = (t_{-j-1}^{-1} t_{-j-2}^{-1} \dots t_1^{-1} t_{-i-1}^{-1} t_{-i-2}^{-1} \dots t_2^{-1} s^{-1} t_1^{-1} s^{-1}) * d_{1,2} \quad \text{if } j < 0,$$

$$d_{i,j} = (t_{j-1}^{-1} d_{j-1,j} t_{j-2}^{-1} d_{j-2,j-1} \dots t_1^{-1} d_{1,2}) * (s^2 a^4) \quad \text{if } i+j = 0,$$

$$d_I = (d_{i_1, i_2} d_{i_1, i_3} \dots d_{i_1, i_n} d_{i_2, i_3} \dots d_{i_2, i_n} d_{i_3, i_4} \dots d_{i_{n-1}, i_n}) (a_{i_1} \dots a_{i_n})^{2-n}$$

where $I = \{i_1, \dots, i_n\} \subset I_0$, $i_1 < \dots < i_n$,

$$c_{i,j} = d_I, \quad \text{where } i \leq j \in I_0 \text{ and } I = \{k \in I_0 \mid i \leq k \leq j\},$$

$$r_{i,j} = b_j a_j c_{i,j} b_j, \quad \text{where } i \leq j \in I_0 \text{ and } I = \{k \in I_0 \mid i \leq k \leq j\},$$

$$k_j = a_j a_{j+1} t_j d_{j,j+1}^{-1} \quad \text{for } j = 1, \dots, g-1,$$

$$s_1 = s, \quad \text{and } s_j = (k_{j-1} k_{j-2} \dots k_1) * s_1 \quad \text{for } j = 2, \dots, g,$$

$$z = a_1 a_2 \dots a_g s t_1 t_2 \dots t_{g-1} s t_1 \dots t_{g-2} s \dots s t_1 s d_I \quad \text{where } I = \{1, \dots, g\},$$

$$z_j = (k_{j-1} k_{j-2} \dots k_{g+1-j}) * z \quad \text{if } j > g/2.$$

Most of the products in the above definition represent very simple elements of \mathcal{M} and we now explain their meaning. We first define special homeomorphisms of S_0 .

DEFINITION 4. A *half-twist* $h_{i,j}$ along a curve $\delta_{i,j}$ is the isotopy class (on S_0 relative to its boundary) of a homeomorphism of S_0 which is fixed outside $\delta_{i,j}$ and is equal to a counterclockwise “rotation” through 180 degrees inside $\delta_{i,j}$. In particular $h_{i,j}$ switches the two holes ∂_i and ∂_j inside $\delta_{i,j}$ so it is not fixed on the boundary of S_0 , but $h_{i,j}^2$ is fixed on the boundary of S_0 and is isotopic to the full Dehn twist along $\delta_{i,j}$.

One can check that $t_i = h_{i,i+1} h_{-i-1,-i} a_i^{-1} a_{i+1}^{-1}$ and $s = h_{-1,1} a_1^{-2}$. So t_i switches the holes ∂_i and ∂_{i+1} rotating the interior of $\delta_{i,i+1}$ counterclockwise and it switches the holes ∂_{-i-1} and ∂_{-i} rotating the interior of $\delta_{-i-1,-i}$ counterclockwise, and s switches the holes ∂_1 and ∂_{-1} rotating the interior of $\delta_{-1,1}$ counterclockwise. This interpretation of t_i and s is very useful in the investigation of their action on curves in S_0 .

It follows from Lemma 16 that $s(\alpha_i) = \alpha_i$ for $i = 1, \dots, g$, $t_i(\alpha_i) = \alpha_{i+1}$, $t_i(\alpha_{i+1}) = \alpha_i$, and $t_i(\alpha_j) = \alpha_j$ for $j \neq i, i + 1$. Thus the elements a_i , s , and t_i belong to \mathcal{M} .

We now consider the products $d_{i,j}$. For $i \neq -j$ we start with $\delta_{1,2}$ and apply consecutive factors t_i and s , one at a time. We check that the result is $\delta_{i,j}$ so $d_{i,j}$ is the twist along $\delta_{i,j}$, by Lemma 15. For $d_{-1,1}$ we use Lemma 16(iii). The curve $\delta_{-1,1}$ is the boundary of a regular neighbourhood of $\alpha_1 \cup \beta_1$ and $(a_1 b_1)^6 = s^2 a_1^4$ by Lemma 16(i), (ii), therefore $d_{-1,1} = (a_1 b_1)^6$ is the twist along $\delta_{-1,1}$ by Lemma 16(iii). Now we apply a suitable product of t_i 's and $d_{i,i+1}$'s to $\delta_{-1,1}$ and get $\delta_{-i,i}$. Thus $d_{i,j}$ is the twist along $\delta_{i,j}$ for all i, j .

It follows from Lemma 17 that d_I is equal to the twist along δ_I . In particular $c_{i,j}$ is the twist along the curve $\gamma_{i,j}$.

Now it follows from Lemma 16 that $r_{i,j}(\alpha_j) = \gamma_{i,j}$. Also the curves $\alpha_1, \dots, \alpha_{j-1}, \gamma_{i,j}, \alpha_{j+1}, \dots, \alpha_g$ form a cut-system. Thus $r_{i,j}$ belongs to \mathcal{M} .

One can check, by drawing pictures, that $k_j(\alpha_j) = \alpha_{j+1}$ and $k_j(\beta_j) = \beta_{j+1}$, hence $s_j = b_j a_j^2 b_j$ is essentially the half-twist along a curve surrounding the holes ∂_j and ∂_{-j} (see Definition 4 and the interpretation of s).

We may assume that the handlebody B has a rotational symmetry with respect to the z -axis (the vertical axis in Figure 1), so the 180 degrees rotation around the z -axis takes the B onto itself and takes the i th handle onto the $(g + 1 - i)$ th handle. This rotation induces the rotation of S_0 through 180 degrees around the origin, which takes ∂_i onto ∂_{i-g-1} and ∂_{-i} onto ∂_{g+1-i} for $i = 1, \dots, g$. We now check that the element z from

Definition 3 represents the rotation. Let z' be the rotation. Consider loops around the holes ∂_i , $i < 0$, which descend from above, from the common base point at ∞ , and are disjoint outside ∞ . Consider also loops around the holes ∂_i , $i > 0$, which ascend from below, from the common base point at ∞ , and are disjoint outside ∞ . It is easy to check, using Definition 4, that z and z' act in the same way on these loops. Since the complement of these loops is a union of a disk and of annuli around ∂_i 's, it follows that z and z' may only differ by a product of a_i 's. It is easy to check that $z(\beta_1) = z'(\beta_1) = \beta_g$. Since $a_g(\beta_g) \neq \beta_g$ the power of a_g in z must be correct. Drawing some more pictures one can check that $z(\beta_j) = z'(\beta_j) = \beta_{g+1-j}$, therefore the power of a_{g+1-j} in z must be correct for $j = 1, \dots, g$. So $z = z'$ is the rotation. From this one can check that z_j leaves the curves α_j and $\gamma_{j-g,j}$ invariant.

We can now state our main result, a finite presentation for \mathcal{M} :

THEOREM 18. *The mapping class group \mathcal{M}_g of a handlebody of genus g admits the following presentation:*

The set of generators consists of $a_1, \dots, a_g, d_{1,2}, s, t_1, \dots, t_{g-1}$ and $r_{i,j}$ for $i < j \in I_0$ and $i = 1$ or $(i < 0, i + j > 0, j - i \leq g)$.

The set of defining relations is:

- (P1) $[a_i, a_j] = 1$ and $[a_i, d_{j,k}] = 1$ for all $i, j, k \in I_0$.
- (P2) *Pure braid relations*
 - (a) $d_{r,s}^{-1} * d_{i,j} = d_{i,j}$ if $r < s < i < j \in I_0$ or $i < r < s < j \in I_0$,
 - (b) $d_{r,i}^{-1} * d_{i,j} = d_{r,j} * d_{i,j}$ if $r < i < j \in I_0$,
 - (c) $d_{i,s}^{-1} * d_{i,j} = (d_{i,j} d_{s,j}) * d_{i,j}$ if $i < s < j \in I_0$,
 - (d) $d_{r,s}^{-1} * d_{i,j} = [d_{r,j}, d_{s,j}] * d_{i,j}$ if $r < i < s < j \in I_0$.
- (P3) $d_{I_0} = 1$.
- (P4) $d_{I_k} = a_{|k|}$ for every $k \in I_0$ where $I_k = I_0 - \{k\}$.
- (P5) $t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}$ for $i = 1, \dots, g-2$ and $[t_i, t_j] = 1$ if $1 \leq i < j - 1 < g - 1$.
- (P6) $s^2 = d_{-1,1} a^{-4}$ and $t_i^2 = d_{i,i+1} d_{-i-1,-i} a_i^{-2} a_{i+1}^{-2}$ for $i = 1, \dots, g-1$.
- (P7) $[s, a_i] = 1$ for $i = 1, \dots, g$, $t_i * a_i = a_{i+1}$ for $i = 1, \dots, g-1$, $[a_i, t_j] = 1$ for $j \neq i, i-1$ and $[t_i, s] = 1$ for $i = 2, \dots, g-1$.
- (P8) $[s, d_{2,3}] = 1$, $[s, d_{-2,2}] = 1$, $st_1 st_1 = t_1 st_1 s$, and $[t_i, d_{1,2}] = 1$ for $i \neq 2$.
- (P9) *If $i < j \in I_0$ and $i = 1$ or $(i < 0, i + j > 0, j - i \leq g)$ then $r_{i,j}^2 = s_j c_{i,j} s_j c_{i,j}^{-1}$.*
- (P10) *If $i < j \in I_0$ and $i = 1$ or $(i < 0, i + j > 0, j - i \leq g)$ then*
 - (a) $r_{i,j} * a_j = c_{i,j}$ and $[r_{i,j}, a_k] = 1$ for $k \neq j$,
 - (b) $[r_{i,j}, t_k] = 1$ if $k < |i|$ or $|i| < k < j - 1$ or $k > j$ or $k = i = 1 < j - 1$,
 - (c) $[r_{i,j}, s_k] = 1$ if $k < |i|$ or $k > j$ or $k = -i$,

- (d) $[r_{i,j}, d_{k,m}] = 1$ if $k, m \in \{i, \dots, j-1\}$ or $k, m \notin \{-j, i, i+1, \dots, j\}$,
(e) $[r_{i,j}, z_j] = 1$ if $j = g$ or $j - i = g$,
(f) $r_{i,j} * d_{i,j} = d_J$ where $J = \{k \in I_0 : i < k \leq j\}$,
(g) $r_{1,j} * d_{-j,1-j} = (t_{j-2}t_{j-3} \dots t_1) * c_{-1,j}$,
(h) if $i < 0$ and $j + i > 1$ then $r_{i,j} * d_{-j,1-j} = (t_{j-2}t_{j-3} \dots t_{1-i}) * c_{i-1,j}$,
(k) if $j < g$ then $r_{i,j}^{-1} * d_{-j-1,-j} = s_{j+1}^{-1} * c_{i,j+1}$.

(P11) If $i < j \in I_0$ and $i = 1$ or $(i < 0, i + j > 0, j - i \leq g)$ then $r_{i,j} * t_{j-1} = t_{j-1}^{-1} * r_{i,j}$.

(P12) For $g \geq j > 2$ and $i = 1$ or $(i < 0, i + j > 0, j - i \leq g)$

$$r_{1,j} = s_j c_{1,j} s_j c_{1,j}^{-1} k_{j-1} a_j c_{1,j-2} t_{j-1} c_{1,j-1}^{-1} t_{j-1}^{-1} r_{1,j-1}^{-1} s_{j-1} h_2 r_{1,2}^{-1} h_2^{-1} k_{j-1}^{-1} \text{ with}$$

$$h_2 = k_{j-1}^{-1} t_{j-2}^{-1} t_{j-3}^{-1} \dots t_1^{-1} k_{j-1} k_{j-2} \dots k_2,$$

$$r_{-1,j} = h_3 r_{1,2}^{-1} h_3^{-1} s_j r_{1,j}^{-1} c_{-1,j-1}^{-1} c_{1,j-1} a_1 s_j c_{-1,j} s_j c_{-1,j}^{-1} \text{ with}$$

$$h_3 = s_1 k_{j-1} k_{j-2} \dots k_2,$$

$$r_{i,j} = h_3 r_{1,2}^{-1} h_3^{-1} s_j r_{i+1,j}^{-1} c_{i,j-1}^{-1} c_{i+1,j-1} a_{-i} s_j c_{i,j} s_j c_{i,j}^{-1} \text{ for } i < -1 \text{ with}$$

$$h_3 = s_{-i} t_{-1-i}^{-1} t_{-2-i}^{-1} \dots t_1^{-1} k_{j-1} k_{j-2} \dots k_3 k_2.$$

Although Theorem 18 is very long and complicated it has some simple and interesting consequences.

THEOREM 19. For $g > 1$ the mapping class group \mathcal{M}_g of a handlebody B of genus g is generated by five elements: $a_1, s_1, r_{1,2}, t_1$ and $u = t_1 \dots t_{g-1}$.

Proof. Let G be the subgroup of \mathcal{M}_g generated by $a_1, s_1, r_{1,2}, t_1, u$. We have to prove that G contains the generators $a_i, s, d_{1,2}, t_j$ and $r_{i,j}$ of \mathcal{M} . We have $u * t_i = t_{i+1}$ by (P5), so $t_j \in G$ for all j . Now $t_i * a_i = a_{i+1}$ by (P7), so $a_i \in G$ for all i . By (P10) we have $r_{1,2} * a_2 = d_{1,2} \in G$. Now all elements $d_{i,j}, s_j, k_j$ belong to G , by Definition 3. Finally, $r_{i,j} \in G$ by (P12) and induction. ■

THEOREM 20. The abelianized mapping class group $A = \mathcal{M}_g / [\mathcal{M}_g, \mathcal{M}_g]$ is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ for $g = 2$ and to \mathbb{Z}_2 for $g > 2$.

Proof. Elements conjugate in \mathcal{M}_g become equal in A . By (P7) all elements a_i become equal in A to an element which we denote by a . We have $r_{1,2} * a_2 = d_{1,2}$ by (P10), and by Definition 3 all $d_{i,j}$ with $i \neq -j$ become equal to a in A . By (P5) all elements t_i become equal in A to an element which we denote by t . Also all elements $r_{i,j}$ are equal to t in A by (P11). All elements s_i become equal in A to an element which we denote by s . So A is generated by t, s and a . Now $t^2 = s^2$ by (P9), and $t^2 = a^{-2}$

by (P6). Consider the relation in (P12) defining $r_{1,3}$. Reducing modulo the commutator we get $t = s^3 t^{-2} a$ in A . Since $t^2 = s^2$ we get $t = sa$. Squaring both sides we get $t^2 = s^2 a^2$, hence $a^2 = t^2 = s^2 = 1$. So A is generated by a and s and $a^2 = s^2 = 1$.

We now consider separately the cases $g = 2$ and $g > 2$. If $g = 2$ then all edges of X are equivalent to $e_{1,2}$. The relations (P12) become empty. We have very few standard curves and very few corresponding elements d_I : d_{I_0} , d_I , where $I = I_0 - k$ for $k \in I_0$, and $d_{i,j}$ with $i < j$. In A the elements $d_{-1,1}$ and $d_{-2,2}$ become trivial by the definition and by the previous relations. We have seen that $d_{i,j}$ becomes equal to a for $i \neq -j$ so d_{I_0} is equal to 1 in A and d_I is equal to a in A for $I = I_0 - k$. All relations (P1)–(P11) become trivial modulo the commutator and the relations $a^2 = s^2 = 1$. Thus $A = \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Consider now the case $g > 2$. From the definition of $c_{1,3}$ and from the previous relations we get $c_{1,3} = 1$ in A . By (P10)(a) we get $a = 1$, so A is generated by s and $s^2 = 1$. Now all relations (P1)–(P12) become trivial so $A = \mathbb{Z}_2$. ■

2.1. The stabilizer of a vertex. The main task of this subsection is to establish a presentation of the stabilizer H of the vertex v_0 of X .

PROPOSITION 21. *The group H admits a presentation with generators a_i , $i = 1, \dots, g$, $d_{1,2}$, s and t_i , $i = 1, \dots, g - 1$, and with relations (P1)–(P8) of Theorem 18.*

The group H can be defined by two exact sequences:

$$(1) \quad 1 \rightarrow \mathbb{Z}_2^g \rightarrow \pm\Sigma_g \rightarrow \Sigma_g \rightarrow 1,$$

$$(2) \quad 1 \rightarrow H_0 \rightarrow H \rightarrow \pm\Sigma_g \rightarrow 1.$$

Before defining the objects and the homomorphisms in these sequences we explain how from an exact sequence $1 \rightarrow A \rightarrow B \rightarrow C \rightarrow 1$ and from finite presentations of the groups A and C we can construct a finite presentation of B .

- We write generators a_i and relations Q_j in A .
- We write generators c_i and relations W_j in C .
- We lift generators c_i to elements b_i in B .
- We substitute b_i for c_i in W_j getting a word R_j in the letters b_i , which represents an element d_j of A . We write d_j as a product of a_i 's.
- Finally, for every a_i and b_j the conjugate $b_j * a_i$ represents an element $a_{i,j}$ of A , which we write as a product of the generators a_i .
- B has a presentation with generators a_i and b_j and with relations $Q_i = 1$, $R_j = d_j$, and $b_j a_i b_j^{-1} = a_{i,j}$.

We now describe the sequence 1 and the group $\pm\Sigma_g$. This is the group of permutations of the set $I_0 = \{-g, 1-g, \dots, -1, 1, 2, \dots, g\}$ such that $\sigma(-i) = -\sigma(i)$. The homomorphism $\pm\Sigma_g \rightarrow \Sigma_g$ forgets the signs. A generator of the kernel changes the sign of one letter. The sequence splits, Σ_g can be considered as the subgroup of the permutations which take positive numbers to positive numbers. Let $\tau_i = (i, i+1)$ be a transposition in Σ_g for $i = 1, \dots, g-1$. Then

- (S1) $[\tau_i, \tau_j] = 1$ for $|i-j| > 1$,
- (S2) $\tau_i * \tau_{i+1} = \tau_{i+1}^{-1} * \tau_i$ for $i = 1, \dots, g-2$,
- (S3) $\tau_i^2 = 1$ for $i = 1, \dots, g-1$.

This defines a presentation of Σ_g . Further, let σ_i for $i = 1, \dots, g$ denote the change of sign of the i th letter in a signed permutation. Then $\sigma_i^2 = 1$ and $[\sigma_i, \sigma_j] = 1$ for all i, j . Finally, conjugation gives $\tau_i * \sigma_i = \sigma_{i+1}$, $\tau_i * \sigma_{i+1} = \sigma_i$ and $[\sigma_j, \tau_i] = 1$ for $j \neq i$ and $j \neq i+1$. In fact, it suffices to use one generator $\sigma = \sigma_1$ and the relations $\sigma_i = (\tau_{i-1}\tau_{i-2}\dots\tau_1) * \sigma$. We get the relations

- (S4) $\sigma^2 = 1$,
- (S5) $[(\tau_i\tau_{i-1}\dots\tau_1) * \sigma, \tau_j] = 1$ for $j \neq i$ and $j \neq i+1$,
- (S6) $[(\tau_i\tau_{i-1}\dots\tau_1) * \sigma, (\tau_j\tau_{j-1}\dots\tau_1) * \sigma] = 1$ for all i, j .

The group $\pm\Sigma_g$ has a presentation with generators $\sigma, \tau_1, \dots, \tau_{g-1}$ and with defining relations (S1)–(S6).

We now describe the sequence (2). A homeomorphism in H may permute the curves α_i and may change their orientations. We fix an orientation of each curve α_i and define a homomorphism $\phi_1 : H \rightarrow \pm\Sigma_g$ as follows: a homeomorphism h is mapped to a permutation $i \mapsto \pm j$ if $h(\alpha_i) = \alpha_j$ and the sign is “+” if the orientations of $h(\alpha_i)$ and of α_j agree, and “−” otherwise. If h preserves the isotopy class of α_i and preserves its orientation then it is isotopic to a homeomorphism fixed on α_i . The kernel of ϕ_1 is the subgroup H_0 of the elements of H represented by the homeomorphisms which keep the curves $\alpha_1, \dots, \alpha_g$ pointwise fixed. Such a homeomorphism induces a homeomorphism of the surface S_0 obtained by cutting S open along the curves α_i .

We shall study the mapping class group of S_0 and from it we shall deduce a presentation of H_0 . We want to make the presentation symmetric with respect to all holes so it is relatively complicated but it will simplify the subsequent treatment of the mapping class group \mathcal{M} . We use the notation from Definition 3.

LEMMA 22. *The mapping class group of S_0 has a presentation with generators $d_i, i \in I_0$, and $d_{i,j}, i < j \in I_0$, and with relations*

- (Q1) $[d_i, d_j] = 1$ and $[d_i, d_{j,k}] = 1$ for all $i, j, k \in I_0$,
- (Q2) *pure braid relations*

- (a) $d_{r,s}^{-1} * d_{i,j} = d_{i,j}$ if $r < s < i < j \in I_0$ or $i < r < s < j \in I_0$,
 - (b) $d_{r,i}^{-1} * d_{i,j} = d_{r,j} * d_{i,j}$ if $r < i < j \in I_0$,
 - (c) $d_{i,s}^{-1} * d_{i,j} = (d_{i,j} d_{s,j}) * d_{i,j}$ if $i < s < j \in I_0$,
 - (d) $d_{r,s}^{-1} * d_{i,j} = [d_{r,j}, d_{s,j}] * d_{i,j}$ if $r < i < s < j \in I_0$,
- (Q3) $d_{I_0} = 1$,
- (Q4) $d_k = d_{I_k}$ for every $k \in I_0$ where $I_k = I_0 - \{k\}$.

Proof. We start with a big disk D in S_0 which contains all holes ∂_i , $i \in I_0$. When we glue a disk with a distinguished centre to each curve ∂_i we get a disk with $2g$ distinguished points. Its mapping class group is isomorphic to the pure braid group P_{2g} with generators $d_{i,j}$ and with relations (Q2). In the passage from the mapping class group of D to the mapping class group of the punctured disk we kill exactly the twists d_i , which commute with everything. One can check that the removal of the disks does not affect the relations (Q2) so the mapping class group of D has a presentation with relations (Q1) and (Q2). Now we glue a disk with a distinguished centre P to the boundary ∂ of D . This kills the twist d around the boundary ∂ and we get (Q3), by Lemma 17. Finally, we forget the distinguished point p and get the sphere S_0 . By [2], Theorem 4.3, the kernel of the homomorphism induced by this forgetful map is generated by spin maps with respect to some set of generators of $\pi_1(S_0, P)$.

We can choose a base point P on the x -axis to the right of all the holes and choose generators of $\pi_1(S_0, P)$ to be simple loops λ_k such that λ_k moves parallel to the x -axis, turns around the hole ∂_k and comes back to P parallel to the x -axis (see Figure 10). A spin map with respect to λ_k is isotopic to a quotient of twists with respect to the boundaries of a regular neighbourhood of λ_k . We can homotope these boundaries into D , without passing over P . Then one is isotopic to ∂_k and the other is isotopic to the standard curve δ_I with $I = I_0 - \{k\}$. By Lemma 17 we get (Q4). ■

We now go back to the presentation of H_0 . When we glue back the corresponding pairs of boundary components of S_0 we get the surface S . This glueing map induces a homomorphism from the mapping class group of S_0 into H_0 . In fact, it is onto H_0 —a homeomorphism fixed on $\alpha_1, \dots, \alpha_g$ induces a homeomorphism of S_0 . One can prove that the kernel is generated by the products $d_i d_i^{-1}$ so both twists are identified with a_i in H_0 . Considering the presentation of the mapping class group of S_0 established in Lemma 22 we see that H_0 has a presentation with generators a_1, \dots, a_g and $d_{i,j}$ for $i < j \in I_0$ and with relations (P1)–(P4). In these relations $d_{i,j}$ is represented by a Dehn twist along $\delta_{i,j}$.

We now lift the generators of $\pm\Sigma_g$. Recall that $t_i(\alpha_i) = \alpha_{i+1}$, and $t_i(\alpha_{i+1}) = \alpha_i$, and $t_i(\alpha_k) = \alpha_k$ for $k \neq i, i+1$. $s(\alpha_1)$ is equal to α_1 with the

opposite orientation and s is fixed on the other α_i 's. Thus we can lift τ_i to t_i and σ to s .

When we lift the relations in $\pm\Sigma_g$ and establish the conjugates of the generators of H_0 by the lifted generators of $\pm\Sigma_g$ we get a presentation of H . In order to prove Proposition 21 we consider an abstract group defined by the generators and relations in Proposition 21. We map this group onto H where a_i , $d_{1,2}$, s and t_i are mapped onto corresponding elements in H . By the previous discussion the products $d_{i,j}$ from Definition 3 are mapped onto corresponding twists. Therefore relations (P1)–(P4) are mapped onto true relations in H . Now we check that (P5)–(P8) are also mapped onto true relations in H . We use Lemma 15 and relations (i)–(iv) of Lemma 16.

Relation (P5) follows from the definition of t_i and from (i) and (iv).

We already know that $d_{-1,1} = s^2 a_1^4$. By (i) and (ii), $t_i^2 a_{i+1}^2 a_i^2 = (a_i e_i a_{i+1})^4$ for $i < g$. The boundary of a regular neighbourhood of $\alpha_i \cup \varepsilon_i \cup \alpha_{i+1}$ is equal to $\delta_{i,i+1} \cup \delta_{-i-1,-i}$. Thus (P6) follows from (iii).

The relations in (P7) follow easily from Lemma 15 and (i).

Since $t_1(\alpha_1) = \alpha_2$ and $s(\alpha_2) = \alpha_2$ we have $[t_1 s t_1, a_1] = 1$ by Lemma 15. Also $t_1 s t_1 = e_1 a_1 b_1 a_2 e_1 a_1^2 e_1 a_2 b_1 a_1 b_1$ by (i) and (ii), and the last expression commutes with b_1 by (i) and (ii), hence it commutes with s . $d_{-2,2}$ is supported inside $\delta_{-2,2}$ and s is supported inside $\delta_{-1,1}$, so they commute. We have further $[s, d_{2,3}] = 1$ and $[t_i, d_{1,2}] = 1$ for $i \neq 2$ by (i). Thus (P8) is also satisfied in H .

It remains to lift the relations of $\pm\Sigma_g$ to H , compute the conjugation relations of the generators of H_0 by the lifted generators of $\pm\Sigma_g$, and prove that all these relations follow from (P1)–(P8).

Relations (S1) and (S2) lift to (P5).

Relations (S3) and (S4) lift to (P6).

Relation (S5) lifts to $[(t_i t_{i-1} \dots t_1) * s, t_j] = 1$ for $j \neq i$ and $j \neq i+1$ and this follows from (P5) and (P7).

We shall deal with (S6) a little later. We now pass to the conjugation by s and t_k . Since $s^2 \in H_0$ and $t_k^2 \in H_0$ by (P6), it suffices to know the result of the conjugation of each generator of H_0 by either s or s^{-1} , and by either t_k or t_k^{-1} (the other follows). We have $s * a_i = a_i$ by (P7). The result of the conjugation $t_j * a_i$ is determined by (P7).

We now prove that the result of the conjugation of $d_{i,j}$, by either t_k or t_k^{-1} , and by either s or s^{-1} , is determined by (P5), (P7), (P8) and the definitions of $d_{i,j}$. Consider first the conjugation by s .

If $i > 1$ then $d_{i,j} = (t_{i-1} \dots t_2 t_{j-1} \dots t_3) * d_{2,3}$ by (P5), so $s * d_{i,j} = d_{i,j}$ by (P7) and (P8).

If $i = 1$ then $s^{-1} * d_{1,j} = d_{-1,j}$ by (P7) and the definitions.

If $i < -1$ and $j + i > 0$ then $d_{i,j} = (t_{-i-1}^{-1} \dots t_2^{-1} t_{j-1} \dots t_3 t_1^{-1} s^{-1} t_2) * d_{1,2}$ by (P5) and (P7). Now

$$\begin{aligned} (s^{-1} t_1^{-1} s^{-1} t_2) * d_{1,2} &= s^{-1} t_1^{-1} s^{-1} t_1^{-1} * d_{2,3} \\ &= (t_1^{-1} s^{-1} t_1^{-1}) * d_{2,3} \quad (\text{by (P7) and (P8)}) \\ &= (t_1^{-1} s^{-1} t_2) * d_{1,2}. \end{aligned}$$

Thus $s^{-1} * d_{i,j} = d_{i,j}$.

The case of $i < -1$ and $j > 1$ and $i + j < 0$ is similar.

If $j = -1$ then $d_{i,j} = (t_{-i-1}^{-1} \dots t_1^{-1} s^{-1} t_1^{-1} s^{-1}) * d_{1,2}$. Now

$$(s t_1^{-1} s^{-1} t_1^{-1} s^{-1}) * d_{1,2} = (t_1^{-1} s^{-1}) * d_{1,2} \quad (\text{by (P7)})$$

hence $s * d_{i,-1} = d_{i,1}$.

If $j < -1$ then $d_{i,j} = (t_{-j-1}^{-1} \dots t_2^{-1} t_{-i-1}^{-1} \dots t_3^{-1} t_1^{-1} t_2^{-1} s^{-1} t_1^{-1} s^{-1}) * d_{1,2}$.

Now

$$\begin{aligned} s^{-1} * ((t_1^{-1} t_2^{-1} s^{-1} t_1^{-1} s^{-1}) * d_{1,2}) &= (s^{-1} t_1^{-1} s^{-1} t_1^{-1} t_1 t_2^{-1} t_1^{-1} s^{-1}) * d_{1,2} \\ &= (t_1^{-1} s^{-1} t_1^{-1} s^{-1} t_2^{-1} t_1^{-1} t_2 s^{-1}) * d_{1,2} \\ &= (t_1^{-1} s^{-1} t_1^{-1} t_2^{-1}) * (s^{-1} t_1^{-1} s^{-1} t_2) * d_{1,2} \\ &= (t_1^{-1} s^{-1} t_1^{-1} t_2^{-1} t_1^{-1} s^{-1} t_2) * d_{1,2} \quad (\text{as before}) \\ &= (t_1^{-1} s^{-1} t_2^{-1} t_1^{-1} s^{-1}) * d_{1,2} \quad (\text{by (P7) and (P5)}). \end{aligned}$$

Hence $s^{-1} * d_{i,j} = d_{i,j}$.

If $i = -j$ then $s * d_{k,k+1} = d_{k,k+1}$ for $k > 1$, as above, hence $s * d_{i,j} = (t_{j-1}^{-1} d_{j-1,j} \dots t_2^{-1} d_{2,3} s) * d_{-2,2} = d_{i,j}$ by (P7).

We now check conjugation by t_k . If $i \neq -j$ then it follows easily from (P5), (P7) and (P8) that $d_{i,j}$ either remains invariant under the conjugation, or is transformed into a ‘‘neighbouring’’ $d_{p,q}$. In particular, t_k commutes with $d_{i,i+1}$ if $k < i - 1$ or $k = i$ or $k > i + 1$. Now, by (P5) and the definitions, $t_k * d_{-j,j} = d_{-j,j}$ if $k \neq j - 1, j$, $t_j^{-1} * d_{-j,j} = d_{j,j+1}^{-1} * d_{-j,j}$ and $t_{j-1} * d_{-j,j} = d_{j-1,j} * d_{-j,j}$.

Finally, we lift (S6). We start with the case $[\tau_1 * \xi_1, \xi_1]$. It lifts to

$$\begin{aligned} t_1 s t_1^{-1} s t_1 s^{-1} t_1^{-1} s^{-1} &= t_1 s t_1^{-1} s s^{-1} t_1^{-1} s^{-1} t_1 \quad (\text{by (P7)}) \\ &= t_1 s t_1^{-2} s^{-1} t_1^{-1} t_1^2. \end{aligned}$$

We have $t_1^2 \in H_0$ by (P6), and the conjugation of an element of H_0 by s and t_i is already determined by (P5)–(P8). So we know how to lift $[\tau_1 * \xi_1, \xi_1] = 1$. In the general case we have a commutator $[(t_i t_{i-1} \dots t_1) * s, (t_j t_{j-1} \dots t_1) * s]$. If $i > j$ then, by (P5) and (P7), this commutator is equal to the conjugate

of $[t_1 * s, s]$ by $t_j t_{j-1} \dots t_1 t_i t_{i-1} \dots t_2$. This is a conjugation of an element of H_0 by t_k 's so the result is determined by (P5)–(P8).

This concludes the proof of Proposition 21. ■

2.2. Proof of the main theorem. We now return to the action of \mathcal{M} on X . A homeomorphism h of B takes meridian curves to meridian curves and cut-systems to cut-systems. If we let $h(\langle \gamma_1, \dots, \gamma_g \rangle) = \langle h(\gamma_1), \dots, h(\gamma_g) \rangle$ we get an action of \mathcal{M} on the vertices of X which extends to a simplicial action on X . Conversely, if h is a homeomorphism of S which takes a cut-system $v = \langle \gamma_1, \dots, \gamma_g \rangle$ onto a cut-system w then h belongs to \mathcal{M} (h can be extended to the interior of B).

We now sketch the rest of the proof of Theorem 18. We prove that \mathcal{M} acts transitively on the vertices of X and that there are only a finite number of edge-orbits of the action. We choose representatives $\mathbf{e}_{i,j}$ of the edge-orbits and prove that the chosen element $r_{i,j}$ of \mathcal{M} switches the vertices of $\mathbf{e}_{i,j}$. We prove that \mathcal{M} is generated by $r_{i,j}$'s together with the generators of H .

Now to every word in the generators we assign an edge-path in X . To a word representing the identity in \mathcal{M} corresponds a closed edge-path. We establish relations between words which induce the same edge-path or edge-paths which differ by cancelling trivial “tails” and we prove that the relations follow from (P9) and (P10). We show that there are only a finite number of face-orbits of the action of \mathcal{M} on X . We choose representatives of face-orbits and for each of them we choose a word which represents the identity in \mathcal{M} and to which corresponds the boundary edge-path of the face. These words correspond to (P11) and (P12). Now if a word w in the generators represents the identity in \mathcal{M} then it induces a closed edge-path. By Theorem 1 this path is a sum of triangular paths (with tails). It follows easily that w is a product of the relators (P1)–(P12).

We now prove that \mathcal{M} acts transitively on the vertices of X . If v and w are cut-systems and we cut S open along the curves of v (respectively w) we get a sphere with $2g$ holes. We can construct a homeomorphism of one sphere onto the other which respects the identification of the boundary components corresponding to the same curve of v (respectively w). It induces a homeomorphism of S which takes v onto w and thus belongs to \mathcal{M} .

We want to classify the edges of X modulo the action of \mathcal{M} . Let $\mathbf{e} = (v, w)$ be an edge of X connecting the vertices v and w . When we cut S open along the curves of v we get a sphere S_1 with $2g$ holes. The vertex w has one new curve γ disjoint from the curves of v . When we cut S_1 along γ we get two components D_1 and D_2 homeomorphic to spheres with holes. We may assume that D_1 has at most $g + 1$ holes. Some of its holes come in pairs corresponding to the same curve of v and some are single holes, with the corresponding holes in D_2 . Let now $\mathbf{e}' = (v', w')$ be another edge

and let D'_1 and D'_2 be the corresponding spheres with holes. There exists a homeomorphism h of S which takes v onto v' and w onto w' (and thus induces a homeomorphism of D_1 onto D'_1) if and only if the numbers of pairs of holes and of single holes in D_1 are the same as in D'_1 .

Consider the vertex $v_0 = \langle \alpha_1, \dots, \alpha_g \rangle$ which was fixed in the previous section. For any pair of natural numbers p, q such that $2p+q \leq g$ there exists a unique curve $\gamma_{i,j}$ (see Definition 2) such that $\gamma_{i,j}$ contains inside exactly p pairs of holes of S_0 and q single holes of S_0 and $i < j \in I_0$ and $i = 1$ or $(i < 0, i+j > 0, j-i \leq g)$. Figure 12 shows a curve $\gamma_{i,j-1}$ for $i < 0$. If we replace a curve α_j of the cut-system $v_0 = \langle \alpha_1, \dots, \alpha_g \rangle$ by the curve $\gamma_{i,j}$ we get another cut-system $v_{i,j}$ which is connected to v_0 by an edge $\mathbf{e}_{i,j} = (v_0, v_{i,j})$ in X . By the above argument every edge \mathbf{e} of X can be mapped by an element of \mathcal{M} onto the unique edge $\mathbf{e}_{i,j}$. We say that \mathbf{e} has *type* (i, j) .

For each edge type (i, j) with $i, j \in I_0, i \leq 1, i+j > 0, j-i \leq g$, we have chosen an element $r_{i,j} = b_j a_j c_{i,j} b_j$ of \mathcal{M} . By Lemma 16(ii), $r_{i,j}(\alpha_j) = \gamma_{i,j}$, $r_{i,j}(\gamma_{i,j}) = \alpha_j$ and clearly $r_{i,j}(\alpha_k) = \alpha_k$ for $k \neq j$. Thus $r_{i,j}(v_0) = v_{i,j}$, $r_{i,j}(v_{i,j}) = v_0$ and $r_{i,j}^2 \in H$.

We now describe precisely a construction from [11] and [7] which will let us determine a presentation of \mathcal{M} .

To every edge-path $\mathbf{p} = (v_0, v_1, \dots, v_k)$ which begins at v_0 we assign a product of elements of \mathcal{M} of the form $g = h_1 r_1 h_2 r_2 \dots h_k r_k h_{k+1}$ such that for $m = 1, \dots, k$ we have $h_1 r_1 \dots h_m r_m(v_0) = v_m$, where $h_m \in H$ and $r_m = r_{i,j}$ if the edge (v_{m-1}, v_m) has type (i, j) . We call such a product an *h-product* corresponding to \mathbf{p} . We construct it as follows. If the edge (v_0, v_1) has type (i, j) then there exists $h_1 \in \mathcal{M}$ such that $h_1(v_0) = v_0$ and $h_1(v_{i,j}) = v_1$. Then for $r_1 = r_{i,j}$ we have $h_1 r_1(v_0) = v_1$. Next we transport the second edge to v_0 . We have $(h_1 r_1)^{-1}(v_1) = v_0$ and $(h_1 r_1)^{-1}(v_2) = v'_1$. If the second edge has type (p, q) and we let $r_2 = r_{p,q}$ then there exists $h_2 \in H$ such that $h_2 r_2(v_0) = v'_1$ and $h_1 r_1 h_2 r_2(v_0) = v_2$, and so on. Observe that the elements h_i in the *h-product* corresponding to an edge-path \mathbf{p} are not uniquely determined. The construction implies

LEMMA 23. *The generators of H together with the elements $r_{i,j}$ generate the group \mathcal{M} .*

PROOF. Let $g \in \mathcal{M}$. Then $g(v_0)$ is a vertex of X and it can be connected to v_0 by an edge-path $\mathbf{p} = (v_0, v_1, \dots, v_k = g(v_0))$. Let $g_1 = h_1 r_1 h_2 r_2 \dots h_k r_k$ be an *h-product* corresponding to \mathbf{p} . Then $g_1(v_0) = v_k = g(v_0)$, therefore $g^{-1} g_1$ leaves v_0 fixed and belongs to the stabilizer H of v_0 . It follows that $g = h_1 r_1 h_2 r_2 \dots h_k r_k h_{k+1}$. ■

By the inverse process we define an edge-path induced by the *h-product* $g = h_1 r_1 h_2 r_2 \dots h_k r_k h_{k+1}$. The edge-path starts at v_0 , then $v_1 = h_1 r_1(v_0)$, $v_2 = h_1 r_1 h_2 r_2(v_0)$ and so on. The last vertex of the path is $v_k = g(v_0)$.

REMARK 2. An h -product g represents an element in H if and only if $v_k = v_0$. This happens if and only if the corresponding edge-path is closed. We can multiply such a g by a suitable element of H on the right and get an h -product which represents the identity in \mathcal{M} and induces the same edge-path as g .

We now prove that relations (P9) are satisfied in \mathcal{M} .

PROOF. By Lemma 16(i), (ii),

$$\begin{aligned} r_{i,j} &= b_j a_j c_{i,j} b_j^2 a_j c_{i,j} b_j = s_j b_j^{-1} a_j^{-1} c_{i,j} b_j^2 a_j c_{i,j} b_j \\ &= s_j b_j^{-1} c_{i,j} a_j^{-1} b_j^2 a_j c_{i,j} b_j = s_j b_j^{-1} c_{i,j} b_j a_j^2 b_j^{-1} c_{i,j} b_j \\ &= s_j c_{i,j} b_j c_{i,j}^{-1} a_j^2 c_{i,j} b_j c_{i,j}^{-1} = s_j c_{i,j} s_j c_{i,j}^{-1}. \blacksquare \end{aligned}$$

We next prove that relations (P10) are satisfied in \mathcal{M} . For a fixed edge type (i, j) they are related to the stabilizer $H_{i,j}$ of the edge $\mathbf{e}_{i,j}$ in H . Every element $h \in H_{i,j}$ takes α_j onto itself and takes $\gamma_{i,j}$ onto itself, and permutes the other α_i 's. Since $r_{i,j}$ switches α_j and $\gamma_{i,j}$ and is fixed on the other α_k 's we have $r_{i,j} h r_{i,j}^{-1} \in H_{i,j}$. To make this statement into a finite set of relations we must find a finite set of generators for $H_{i,j}$ and find the corresponding conjugate of each generator.

LEMMA 24. *The group $H_{i,j}$ which stabilizes the edge $\mathbf{e}_{i,j}$ is generated by the following elements:*

- a_1, \dots, a_g ,
- t_k where $k < |i|$, or $k > j$, or $|i| < k < j - 1$, or $k = i = 1 < j - 1$,
- s_k where $k < |i|$, or $k > j$, or $k = -i > 0$,
- $d_{k,m}$ where $k, m \in \{p | i \leq p \leq j\}$ or $k, m \in \{p | p < i \text{ or } j < p\}$,
- z_j if $i = 1$ and $j = g$, or $i < 0$ and $j - i = g$.

PROOF. A homeomorphism h in $H_{i,j}$ takes $\gamma_{i,j}$ onto itself. If $\gamma_{i,j}$ contains exactly g holes then h may switch the two components of $S_0 - \gamma_{i,j}$. This can be done by the homeomorphism z_j , which preserves $\gamma_{i,j}$ and α_j . Modulo z_j we may assume that h leaves each component of $S_0 - \gamma_{i,j}$ invariant. It must leave ∂_j fixed, and may switch ∂_k and ∂_{-k} if they are in the same component of $S_0 - \gamma_{i,j}$. This corresponds to the reversal of the orientation of α_k and can be done by s_k . Also, h may permute single holes in a component of $S_0 - \gamma_{i,j}$ (and permute in the same way the corresponding holes in the other component). It may also permute pairs of holes $\partial_k, \partial_{-k}$ lying in the same component of $S_0 - \gamma_{i,j}$. All these permutations are induced by the suitable t_k 's (see Definition 4). Modulo these homeomorphisms we may assume that h is fixed on all curves ∂_k . The restriction of h to each component of $S_0 - \gamma_{i,j}$ belongs to the mapping class group of the component. As in Lemma 22 each mapping class group is generated by the twists d_i for ∂_i in the component

and the twists $d_{k,m}$ for $\delta_{k,m}$ in the component. (The curves $\delta_{k,m}$ with $k < i$ and $m > j$ can be pushed over the point at infinity and drawn around the right hand side of Figure 10. Then they look more like the curves in Figure 8 containing two holes in the component outside $\gamma_{i,j}$.) ■

We now have to compute the result of the conjugation of the generators of $H_{i,j}$ by $r_{i,j}$.

LEMMA 25. *Relations (P10) are satisfied in \mathcal{M} and they determine the result (in H) of the conjugation of the generators of $H_{i,j}$ by $r_{i,j}$.*

PROOF. Relations (a)–(d) of (P10) follow immediately from the definitions and from Lemma 16(i), (ii).

z_j takes each of the curves α_j , β_j , and $\gamma_{i,j}$ onto itself so it commutes with $r_{i,j}$ by Lemma 15.

The last four relations follow by Lemma 15. One has to draw several pictures to prove it. Since $r_{i,j}^2 \in H$ (by (P9)) it suffices to know the result of the conjugation of each generator of $H_{i,j}$ by either $r_{i,j}$ or $r_{i,j}^{-1}$. Not all generators $d_{k,m}$ from Lemma 24 appear in the last four relations of (P10) but the other generators are obtained from these by conjugation by suitable products of s and t_k 's which are already in $H_{i,j}$. ■

From (P9) and (P10) we get information about h -products.

CLAIM 1. *If two h -products represent the same element in \mathcal{M} and induce the same edge-path then they are equivalent modulo (P1)–(P10).*

PROOF. If two h -products $g_1 = h_1 r_1 \dots r_k h_{k+1}$ and $g_2 = f_1 r'_1 \dots r'_k f_{k+1}$ induce the same edge-path $\mathbf{p} = (v_0, v_1, \dots, v_k)$ then $h_1 r_1(v_0) = f_1 r'_1(v_0)$. Therefore $h_1^{-1} f_1 r'_1(v_0) = r_1(v_0)$. Since $h_1^{-1} f_1$ preserves the edge type of the edge $(v_0, r'_1(v_0))$, say (i, j) , we have $r_1 = r'_1 = r_{i,j}$ and $h_1^{-1} f_1 \in H_{i,j}$. By (P10) we have $f_1 r'_1 f_2 = h_1 h_1^{-1} f_1 r_1 f_2 = h_1 r_1 h'_1 f_2 = h_1 r_1 f'_2$. Therefore g_2 is equivalent to a new h -product $h_1 r_1 f'_2 r'_2 f_3 r'_3 \dots r'_k f_{k+1}$ which induces the same edge-path \mathbf{p} . If we apply $r_1^{-1} h_1^{-1}$ to the vertices (v_1, \dots, v_k) of \mathbf{p} we get a shorter edge-path which starts at v_0 and is induced by two shorter h -products $h_2 r_2 \dots r_k h_{k+1}$ and $f'_2 r'_2 \dots r'_k f_{k+1}$. Claim 1 follows by induction on k . ■

Two different edge-paths may be homotopic in the 1-skeleton X^1 . This means that there is a backtracking v_i, v_{i+1}, v_i along the edge-path.

CLAIM 2. *If two h -products represent the same element in \mathcal{M} and induce edge-paths which are equal modulo backtracking then the h -products are equivalent modulo (P1)–(P10).*

PROOF. Consider an h -product $g = g_i h_{i+1} r_{i+1} h_{i+2} r_{i+2}$, where g_i is an h -product inducing a shorter edge-path \mathbf{p} and the edge-path induced by g has a backtracking at the end. Then $g_i(v_0) = v_i$, $g_i h_{i+1} r_{i+1}(v_0) =$

v_{i+1} , and $g_i h_{i+1} r_{i+1} h_{i+2} r_{i+2}(v_0) = v_i$. Since r_{i+1} and r_{i+2} correspond to the same edge (v_i, v_{i+1}) of a fixed edge type, they must be equal. Clearly the h -product $g_i h_{i+1} r_{i+1} r_{i+1}$ induces the same edge-path. In particular $g_i h_{i+1} r_{i+1}^2(v_0) = v_i$, hence there exists $h' \in H$ such that $g_i h_{i+1} r_{i+1}^2 h' = g_i h_{i+1} r_{i+1} h_{i+2} r_{i+2}$. Now by Claim 1 the h -products are equivalent modulo (P10). But $g_i h_{i+1} r_{i+1}^2 h'$ is equivalent modulo (P9) to a shorter h -product which induces the edge-path \mathbf{p} . Claim 2 follows by induction on the number of backtrackings. ■

The final set of relations corresponds to the faces of X , the closed triangular edge-paths. To every closed edge-path corresponds an h -product which is equal in \mathcal{M} to an element of H . This gives a relation in \mathcal{M} . We shall see that it suffices to consider the triangles modulo the action of \mathcal{M} .

Up to an action of \mathcal{M} we may assume that the path starts with v_0 and that the second vertex is $v_{i,j}$. The third vertex has only one curve which does not belong to $\langle \alpha_1, \dots, \alpha_g \rangle$, because the next vertex is again v_0 .

CASE 1. Suppose first that $\gamma_{i,j}$ remains fixed in the next simple move. Then some α_k is replaced by α_j . ∂_k must be a single hole inside $\gamma_{i,j}$ with ∂_{-k} outside $\gamma_{i,j}$. Applying the product $t_{j-2} t_{j-3} \dots t_k$ we can map ∂_k onto ∂_{j-1} leaving α_j and $\gamma_{i,j}$ fixed. Thus we may assume that our path, read backwards, has the form

$$\mathbf{p}_1 = \langle \alpha_{j-1}, \alpha_j \rangle \rightarrow \langle \gamma_{i,j}, \alpha_j \rangle \rightarrow \langle \gamma_{i,j}, \alpha_{j-1} \rangle \rightarrow \langle \alpha_j, \alpha_{j-1} \rangle.$$

This path can be represented by the h -product $(t_{j-1} r_{i,j})^3$. Indeed, for $g = t_{j-1} r_{i,j}$ we have $g(\alpha_j) = \gamma_{i,j}$, $g(\gamma_{i,j}) = \alpha_{j-1}$, $g(\alpha_{j-1}) = \alpha_j$ and g leaves other curves α_k fixed.

Since g^3 represents a closed edge-path it represents an element of H in \mathcal{M} . The relation $r_{i,j} * t_{j-1} = t_{j-1}^{-1} * r_{i,j}$ of (P11) is satisfied in \mathcal{M} by Lemma 16(iv). From (P11) we get $g^3 = t_{j-1} r_{i,j}^2 t_{j-1} r_{i,j}^2$. The last expression is equivalent to an element f_1 of H modulo (P9). We let $V_1 = (t_{j-1} r_{i,j})^3 f_1^{-1}$. Then V_1 represents the edge-path \mathbf{p}_1 and is equal to the identity modulo (P1)–(P11).

CASE 2. Suppose now that $\gamma_{i,j}$ changes in the second simple move of a triangular path. Then $g - 1$ curves remain fixed at all three vertices of the triangular path and only one curve changes. We call such a path a *triangular path of the second kind*.

LEMMA 26. *Every triangle of the second kind is a sum of triangles of the second kind with at least one edge of type (1,2).*

PROOF. By the *length* of an edge (u, v) we mean the minimal number of holes (boundary components) in a component of $S - (u \cup v)$, where $(u \cup v)$ denotes the union of the $g + 1$ curves appearing in the cut-systems u and v

together. The length is an invariant of the edge type. An edge of type (1, 2) has length 3 and an edge of any other type is longer.

Consider a triangle of the second kind with all edges of length greater than 3. We start with a cut-system $\langle \alpha_1, \dots, \alpha_g \rangle$. We have a path of simple moves $\langle \alpha_i \rangle \rightarrow \langle \beta \rangle \rightarrow \langle \gamma \rangle \rightarrow \langle \alpha_i \rangle$. We assume that the edge $\langle \alpha_i \rangle \rightarrow \langle \beta \rangle$ is the shortest edge of the path. All curves are disjoint. The curve γ splits S_0 into two components and β lies in one of them. One boundary component corresponding to α_i , say ∂_i , lies inside β and the other lies outside γ , in the other component of $S_0 - \gamma$. We want to split the triangle into smaller, simpler triangles. Every curve disjoint from $\alpha_1, \dots, \alpha_g$ is a meridian curve. We choose β_1 inside β such that β_1 contains ∂_i and exactly one more boundary component ∂_j inside β . Our triangle splits into three triangles $\langle \alpha_i \rangle \rightarrow \langle \beta \rangle \rightarrow \langle \beta_1 \rangle \rightarrow \langle \alpha_i \rangle$, $\langle \alpha_i \rangle \rightarrow \langle \beta_1 \rangle \rightarrow \langle \gamma \rangle \rightarrow \langle \alpha_i \rangle$, and $\langle \beta_1 \rangle \rightarrow \langle \beta \rangle \rightarrow \langle \gamma \rangle \rightarrow \langle \beta_1 \rangle$. The first two have an edge of length 3 and the last one has an edge $\langle \beta_1 \rangle \rightarrow \langle \beta \rangle$ shorter than $\langle \alpha_i \rangle \rightarrow \langle \beta \rangle$. The required result follows by induction. ■

We further investigate the triangles of the second kind with one edge of type (1, 2). Up to the action of \mathcal{M} we may assume that the first edge of the path is an edge $\mathbf{e}_{i,j}$, the second is of type (1, 2), and the third is the longest. The path has the form $\langle \alpha_j \rangle \rightarrow \langle \gamma_{i,j} \rangle \rightarrow \langle \gamma \rangle \rightarrow \langle \alpha_j \rangle$.

The curve γ is disjoint from $\gamma_{i,j}$, it is separated from $\gamma_{i,j}$ by a single hole ∂_k , $k \neq j$, and the edge represented by the simple move $\langle \gamma \rangle \rightarrow \langle \alpha_j \rangle$ is the longest edge of the path. It follows that γ must lie outside $\gamma_{i,j}$ and up to the action of $H_{i,j}$ we may assume that $\gamma = \gamma_{i,j+1}$, or $\gamma = \gamma_{i-1,j}$ ($\gamma_{-1,j}$ if $i = 1$). For each of these triangles we have a relation in \mathcal{M} . Some of these relations are listed in (P12) and the other relations follow from these.

Consider first the case $i = 1$ and the path

$$\mathbf{p}_2 = \langle \alpha_j \rangle \rightarrow \langle \gamma_{1,j+1} \rangle \rightarrow \langle \gamma_{1,j} \rangle \rightarrow \langle \alpha_j \rangle.$$

Let

$$h_2 = k_j^{-1} t_{j-1}^{-1} t_{j-2}^{-1} \dots t_1^{-1} k_j k_{j-1} \dots k_2.$$

We want to prove that

$$g = k_j^{-1} r_{1,j+1} k_j h_2 r_{1,2} h_2^{-1} s_j^{-1} r_{1,j}$$

represents the edge-path \mathbf{p}_2 . Recall that $r_{1,j}(\alpha_j) = \gamma_{1,j}$, $k_j(\alpha_j) = t_j(\alpha_j) = \alpha_{j+1}$ and $k_j(\beta_j) = \beta_{j+1}$ for all j . It follows that $h_2(\alpha_2) = \alpha_j$, $h_2(\beta_2) = \beta_j$. Also $k_j^{-1} * r_{1,j+1} = b_j a_j c_{1,j+1} b_j$ by Lemma 15. It is also easy to check that $h_2(\delta_{1,2}) = \delta_{j,j+1}$. Therefore $h_2 * r_{1,2} = b_j a_j d_{j,j+1} b_j$. We get $g = b_j a_j c_{1,j+1} b_j b_j d_{j,j+1} c_{1,j} b_j$. We have to check that

$$k_j^{-1} r_{1,j+1}(v_0) = \langle \alpha_1, \dots, \alpha_{j-1}, \gamma_{1,j+1}, \alpha_{j+1}, \dots, \alpha_g \rangle,$$

$$k_j^{-1} r_{1,j+1} k_j b_j a_j d_{j,j+1} b_j(v_0) = \langle \alpha_1, \dots, \alpha_{j-1}, \gamma_{1,j}, \alpha_{j+1}, \dots, \alpha_g \rangle, \quad g(v_0) = v_0.$$

Everything follows from what we already know with the exception of the following two facts, which have to be checked by drawings:

$$k_j^{-1}r_{1,j+1}k_j b_j a_j d_{j,j+1} b_j (\alpha_j) = \gamma_{1,j} \quad \text{and} \quad g(\alpha_j) = \alpha_j.$$

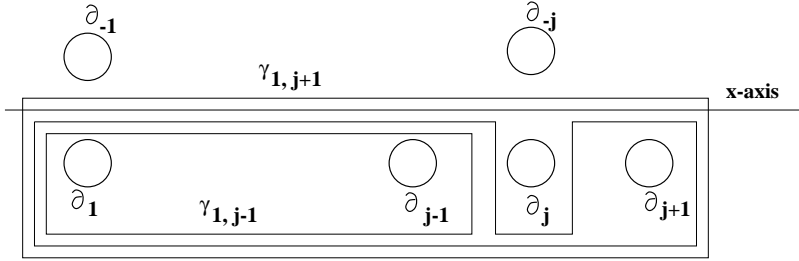


Fig. 11. The first lantern relation for triangles of type 2

We want to find an element of H which is equal to g in \mathcal{M} . Consider the domain in S_0 bounded by the four curves $\gamma_{1,j+1}$, $\gamma_{1,j-1}$, ∂_j and ∂_{j+1} (see Figure 11). By the lantern relation (Lemma 17)

$$c_{1,j+1}c_{1,j-1}a_j a_{j+1} = d_{j,j+1}c_{1,j}(t_j c_{1,j} t_j^{-1})$$

($t_j c_{1,j} t_j^{-1} = d_I$ with $I = \{1, 2, \dots, j-1, j+1\}$). Therefore $d_{j,j+1}c_{1,j} = a_j c_{1,j+1} c_{1,j-1} a_{j+1} t_j c_{1,j}^{-1} t_j^{-1}$. Also $c_{1,j-1}$, a_{j+1} and $t_j c_{1,j}^{-1} t_j^{-1}$ commute with b_j , by Lemma 16. Hence

$$\begin{aligned} g &= b_j a_j c_{1,j+1} b_j b_j d_{j,j+1} c_{1,j} b_j \\ &= (b_j a_j c_{1,j+1} b_j)^2 a_{j+1} c_{1,j-1} t_j c_{1,j}^{-1} t_j^{-1} = k_j^{-1} r_{1,j+1}^2 k_j a_{j+1} c_{1,j-1} t_j c_{1,j}^{-1} t_j^{-1} \\ &= k_j^{-1} s_{j+1} c_{1,j+1} s_{j+1} c_{1,j+1}^{-1} k_j a_{j+1} c_{1,j-1} t_j c_{1,j}^{-1} t_j^{-1} = f_2 \in H. \end{aligned}$$

From this equation we can compute $r_{1,j+1}$ in terms of $r_{1,k}$, $k < j+1$, and the generators of H . We get a formula from (P12). The relation $g = f_2$ becomes trivial modulo (P1)–(P12). If we let $V_2 = g f_2^{-1}$ then V_2 is an h -product which represents the path \mathbf{p}_2 and is equal to the identity modulo (P1)–(P12).

Consider now the edge-path

$$\mathbf{p}_3 = \langle \alpha_j \rangle \rightarrow \langle \gamma_{i,j} \rangle \rightarrow \langle \gamma_{i-1,j} \rangle \rightarrow \langle \alpha_j \rangle$$

(or $\mathbf{p}_3 = \langle \alpha_j \rangle \rightarrow \langle \gamma_{1,j} \rangle \rightarrow \langle \gamma_{-1,j} \rangle \rightarrow \langle \alpha_j \rangle$ if $i = 1$). We let

$$h_3 = s_{1-i} t_{-i}^{-1} t_{-1-i}^{-1} \dots t_2^{-1} t_1^{-1} k_{j-1} k_{j-2} \dots k_2$$

($h_3 = s_1 k_{j-1} k_{j-2} \dots k_2$ if $i = 1$).

We consider in detail the case $i < 0$. The case $i = 1$ requires only a suitable change of indices. We want to prove that

$$g = r_{i,j} s_j^{-1} h_3 r_{1,2} h_3^{-1} r_{i-1,j}$$

is an h -product representing \mathbf{p}_3 . It is easy to check that $h_3(\beta_2) = \beta_j$, $h_3(\alpha_2) = \alpha_j$ and $h_3(\delta_{1,2}) = \gamma$, where γ is the curve in Figure 12. By Lemma 15, $h_3 r_{1,2} h_3^{-1} = b_j a_j c b_j$, where c is the twist around γ . Thus g is equal in \mathcal{M} to $g = b_j c_{i,j} c b_j b_j a_j c_{i-1,j} b_j$. We have to check that

$$\begin{aligned} r_{i,j}(v_0) &= \langle \alpha_1, \dots, \alpha_{j-1}, \gamma_{i,j}, \alpha_{j+1}, \dots, \alpha_g \rangle, \\ b_j c_{i,j} c b_j(v_0) &= \langle \alpha_1, \dots, \alpha_{j-1}, \gamma_{i-1,j}, \alpha_{j+1}, \dots, \alpha_g \rangle, \end{aligned}$$

and $g(v_0) = v_0$. Everything follows from relations which we already know with the exception of the following two facts, which have to be checked by drawings:

$$b_j c_{i,j} c b_j(\alpha_j) = \gamma_{i-1,j} \quad \text{and} \quad g(\alpha_j) = \alpha_j.$$

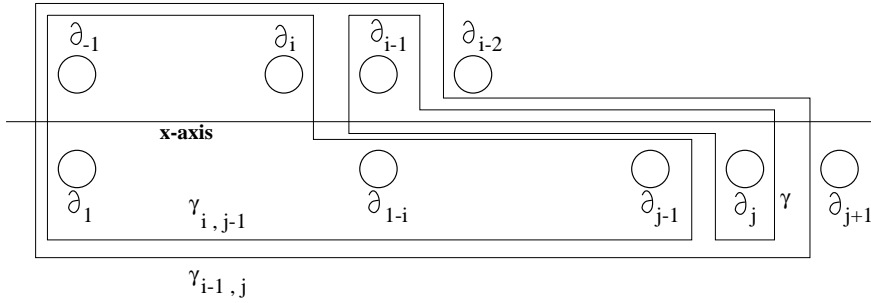


Fig. 12. The second lantern relation for triangles of type 2

We now want to find an element of H which is equal to g in \mathcal{M} . Consider the domain in S_0 bounded by the four curves $\gamma_{i-1,j}$, $\gamma_{i,j-1}$, ∂_{i-1} and ∂_j (see Figure 12). By the lantern relation (Lemma 17)

$$c_{i-1,j} c_{i,j-1} a_{i-1} a_j = c_{i-1,j-1} c_{i,j} c.$$

Therefore $c_{i,j} c = c_{i-1,j-1}^{-1} c_{i,j-1} a_{i-1} c_{i-1,j} a_j$. Moreover $c_{i-1,j-1}$, $c_{i,j-1}$ and a_{i-1} commute with b_j by Lemma 16. Therefore

$$\begin{aligned} g &= b_j c_{i,j} c b_j b_j a_j c_{i-1,j} b_j = c_{i-1,j-1}^{-1} c_{i,j-1} a_{i-1} r_{i-1,j}^2 \\ &= c_{i-1,j-1}^{-1} c_{i,j-1} a_{i-1} s_j c_{i-1,j} s_j c_{i-1,j}^{-1} = f_3 \in H. \end{aligned}$$

From this equation we can compute $r_{i-1,j}$ in terms of $r_{k,j}$, $j - k < j + 1 - i$, and the generators of H . We get a relation from (P12). The relation $g = f_3$ becomes trivial modulo (P1)–(P12). If we let $V_3 = g f_3^{-1}$ then V_3 is an h -product which represents the path \mathbf{p}_3 and is equal to the identity modulo (P1)–(P12).

Consider now the diagram in Figure 13. Every triangle in this diagram has at least one edge of type (1,2). (In the case of $i = -1$ we should replace $i + 1$ by 1.) By the previous discussion every triangle with an edge of type (1,2) is equivalent modulo the action of \mathcal{M} to the outside triangle

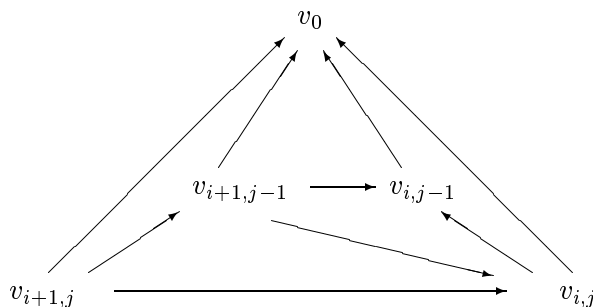


Fig. 13. Reduction for triangular paths

on the diagram or to the triangle on the right hand side of the diagram. The outside triangle and the right hand side triangle have the longest edge of length $j - i + 2$ ($j + 2$ if $i = -1$) and the other triangles have the longest edge of a smaller length.

If $i \neq 1$ then the outer triangle is represented by an h -product V_3 which is trivial modulo (P1)–(P12), by the definition of $r_{i,j}$. The triangle on the right hand side is a sum of the remaining triangles. If $i = 1$ then the triangle on the right hand side is represented by an h -product V_2 which is trivial modulo (P1)–(P12), by the definition of $r_{i,j}$. The outside triangle is a sum of the remaining triangles.

We can now complete the proof of Theorem 18. Let W be a word in the generators of \mathcal{M} which represents the identity in \mathcal{M} . It has the form of an h -product g which represents a closed edge-path \mathbf{p} . We want to prove that g is equivalent to the identity modulo (P1)–(P12). By Proposition 1 and the above discussion, \mathbf{p} is equal modulo backtracking to a sum of paths of the form $\mathbf{q}_1\mathbf{q}_2\mathbf{q}_1^{-1}$, where \mathbf{q}_1 starts at v_0 and \mathbf{q}_2 is equivalent modulo the action of \mathcal{M} to one of the special paths $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ or their inverses. By Claims 1 and 2 it suffices to assume that g itself represents the edge-path $\mathbf{q}_1\mathbf{q}_2\mathbf{q}_1^{-1}$.

Let g_1 be an h -product representing \mathbf{q}_1 and let f be an element of \mathcal{M} such that $\mathbf{q}_2 = f(\mathbf{p}_i)$. Then $g_1(v_0) = f(v_0)$ is the first vertex of \mathbf{q}_2 , hence $g_1^{-1}f = h \in H$. If V_i is the h -product representing \mathbf{p}_i then the h -product g_1hV_i represents $\mathbf{q}_1\mathbf{q}_2$ and there exists an h -product $g_1hV_i g_3$ which represents the identity in \mathcal{M} and represents the edge-path $\mathbf{q}_1\mathbf{q}_2\mathbf{q}_1^{-1}$. Clearly g_1hg_3 represents the edge-path $\mathbf{q}_1\mathbf{q}_1^{-1}$ null-homotopic by backtracking. By Claim 1, g is equivalent to $g_1hV_i g_3$, which is equivalent to g_1hg_3 , which is equivalent to the identity modulo (P1)–(P12).

A path inverse to \mathbf{p}_i is represented by some other h -product V_i' but then V_iV_i' represents a path contractible by backtracking. Thus V_iV_i' is equivalent to the identity modulo (P1)–(P10) and V_i is equivalent to the identity so V_i' is also equivalent to the identity. This concludes the proof of Theorem 18. ■

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