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How to recognize a true $\mathbf{\Sigma}_3^0$ set

by

Etienne Matheron (Talence)

Abstract. Let X be a Polish space, and let $(\mathcal{A}_p)_{p\in\omega}$ be a sequence of G_δ hereditary subsets of $\mathcal{K}(X)$ (the space of compact subsets of X). We give a general criterion which allows one to decide whether $\bigcup_{p\in\omega}\mathcal{A}_p$ is a true $\mathbf{\Sigma}_3^0$ subset of $\mathcal{K}(X)$. We apply this criterion to show that several natural families of thin sets from harmonic analysis are true $\mathbf{\Sigma}_3^0$.

1. Introduction. In this paper, we are interested in a particular instance of the following problem: let $\mathcal{X}$ be a separable metric space, and denote by $\mathbf{\Sigma}_\xi^0(\mathcal{X})$ (resp. $\mathbf{\Pi}_\xi^0(\mathcal{X})$) the additive (resp. multiplicative) Borel classes of $\mathcal{X}$. The problem is to find some simple criterion allowing one to decide whether a given $\mathbf{\Sigma}_\xi^0$ set $\mathcal{A}\subseteq\mathcal{X}$ is a "true" $\mathbf{\Sigma}_\xi^0$, that is, a $\mathbf{\Sigma}_\xi^0$ set which is not $\mathbf{\Pi}_\xi^0$.

As a matter of fact, we will limit ourselves to the third level of the Borel hierarchy ($G_{\delta\sigma}$ and $F_{\sigma\delta}$ sets). Moreover, since the examples we have in mind are ideals of compact sets coming from harmonic analysis, we will concentrate on proving criteria of "true-$\mathbf{\Sigma}_3^0$-ness" for ideals of compact subsets of some Polish space X. We denote by $\mathcal{K}(X)$ the space of all compact subsets of X, equipped with its natural (Polish) topology, generated by the sets $\{K\in\mathcal{K}(X) : K\cap V\neq\emptyset\}$ and $\{K\in\mathcal{K}(X) : K\subseteq V\}$, where V is an open subset of X. For any subset M of X, we let $\mathcal{K}(M)=\{K\in\mathcal{K}(X) : K\subseteq M\}$.

In this particular setting, it turns out that the simplest nontriviality condition is enough to ensure true-$\mathbf{\Sigma}_3^0$-ness. To be precise, let $(\mathcal{A}_p)_{p\in\omega}$ be a sequence of dense G_δ hereditary subsets of $\mathcal{K}(X)$, and let $\mathcal{A}=\bigcup_{p\in\omega}\mathcal{A}_p$. Assume that $\mathcal{A}$ is an ideal of $\mathcal{K}(X)$, and that the union is "nontrivial" in the following sense: for each nonempty open set $V\subseteq X$ and for each $p\in\omega$, $\mathcal{A}_p\cap\mathcal{K}(V)$ is a proper subset of $\mathcal{A}\cap\mathcal{K}(V)$. Then one can conclude that $\mathcal{A}$ is a true $\mathbf{\Sigma}_3^0$ set.

We prove this in the first part of the paper together with some related results. We apply these results in the second part to show that quite a lot of

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natural families of thin sets from harmonic analysis happen to be true $\mathbf{\Sigma}^0_3$ (a rather curious descriptive phenomenon). In particular, we show that if $\mathbf{G}$ is a second-countable nondiscrete locally compact abelian group, then the family $\mathcal{H}$ of compact Helson subsets of $\mathbf{G}$ is a true $\mathbf{\Sigma}^0_3$. The same result holds within any M_0 set, and $\mathcal{H}$ is also a true $\mathbf{\Sigma}^0_3$ *inside the countable sets*. In the case of the circle group, we already proved these results in [M]. However, the proofs given there were somewhat obscured by an immoderate use of constructions which are very classical in harmonic analysis, but still rather technical. In the present paper, we actually use very little harmonic analysis.

2. General results. In this section, X is a Polish space, $\mathcal{K}(X)$ is the space of compact subsets of X, and $\mathcal{K}_\omega(X)$ is the family of countable compact subsets of X.

Definition 1. Let $\mathcal{A}$ be a subset of $\mathcal{K}(X)$.

(a) $\mathcal{A}$ is said to be *hereditary* if it is downward closed under inclusion.

(b) $\mathcal{A}$ is an *ideal* of $\mathcal{K}(X)$ if it is hereditary and stable under finite unions.

(c) If $\mathcal{A}$ is hereditary, we say that $\mathcal{A}$ is a *big* subset of $\mathcal{K}(X)$ if it contains a dense G_δ *hereditary* subset of $\mathcal{K}(X)$.

It is quite possible that any comeager hereditary subset of $\mathcal{K}(X)$ is big, but we are unable to prove it.

Definition 2. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two subsets of $\mathcal{K}(X)$. We say that $\mathcal{M}_1$ *is nowhere contained in* $\mathcal{M}_2$ if for each nonempty open set $V \subseteq X$, $\mathcal{M}_1 \cap \mathcal{K}(V)$ is not contained in $\mathcal{M}_2$.

We can now state the main results of this section.

Theorem A. *Let $(\mathcal{A}_p)_{p\in\omega}$ be a sequence of nonempty hereditary subsets of $\mathcal{K}(X)$, and let $\mathcal{A} = \bigcup_{p\in\omega} \mathcal{A}_p$. Assume that $\mathcal{A}$ is a big ideal of $\mathcal{K}(X)$.*

(a) *If $\mathcal{A}$ is nowhere contained in any $\mathcal{A}_p$, then $\mathcal{A}$ is not $\mathbf{\Pi}^0_3$ in $\mathcal{K}(X)$.*

(b) *If the perfect sets in $\mathcal{A}$ are nowhere contained in any $\mathcal{A}_p$, then the family of perfect sets in $\mathcal{A}$ is not $\mathbf{\Pi}^0_3$ in $\mathcal{K}(X)$.*

(c) *If the finite sets in $\mathcal{A}$ are nowhere contained in any $\mathcal{A}_p$, then $\mathcal{A} \cap \mathcal{K}_\omega(X)$ is not relatively $\mathbf{\Pi}^0_3$ in $\mathcal{K}_\omega(X)$.*

Theorem A follows immediately from a more precise and less readable result, Theorem B below, which we will state after a few definitions.

In the sequel, we will use the notations $\mathcal{B}$ and $\mathcal{B}_1$ for the following families of compact sets: either $\mathcal{B} = \mathcal{B}_1 = \mathcal{K}(X)$, or $\mathcal{B} = \mathcal{B}_1 =$ the family of perfect compact subsets of X, or else $\mathcal{B} = \{\emptyset\} \cup \{\{x\} : x \in X\}$ and $\mathcal{B}_1 = \{K \in \mathcal{K}(X) : K$ is of the form $\{x\} \cup \{x_n : n \in \omega\}$, where $(x_n) \subseteq X$ and $x_n \to x\}$. Notice that in each case, $\emptyset \in \mathcal{B}$ and $\mathcal{B}$ is a G_δ subset of $\mathcal{K}(X)$.

If $\mathcal{M}$ is a subset of $\mathcal{K}(X)$, we denote by $\mathcal{M}^{\mathrm{f}}$ the family of compact subsets of X which are finite unions of elements of $\mathcal{M}$.

We denote by $\mathbf{2}^\omega$ the Cantor space of all infinite 0-1 sequences, endowed with its usual (compact, metrizable) topology.

If $\alpha \in \mathbf{2}^\omega$ and $p \in \omega$, we define $\alpha_p \in \mathbf{2}^\omega$ by $\alpha_p(q) = \alpha(\langle p, q\rangle)$, where $(p,q) \mapsto \langle p, q\rangle$ is any fixed bijection from ω^2 onto ω.

Finally, let $\mathbf{Q} = \{\alpha \in \mathbf{2}^\omega : \exists k\ \forall n \geq k\ \alpha(n) = 0\}$, and $\mathbf{W} = \{\alpha \in \mathbf{2}^\omega : \exists p\ \alpha_p \notin \mathbf{Q}\}$. It is well known that $\mathbf{W}$ is a true $\mathbf{\Sigma}_3^0$ subset of $\mathbf{2}^\omega$ (see [Ke2]).

Our slightly more precise version of Theorem A now reads as follows. To deduce Theorem A from it, one just has to take $\mathcal{B} = \mathcal{K}(X)$ in case (a), $\mathcal{B} =$ the perfect subsets of X in case (b), and $\mathcal{B} = \{\emptyset\} \cup \{\{x\} : x \in X\}$ in case (c).

THEOREM B. *Let $(\mathcal{A}_p)_{p\in\omega}$ be a sequence of (nonempty) hereditary subsets of $\mathcal{K}(X)$, and let $\mathcal{A}$ be any big subset of $\mathcal{K}(X)$. Assume that $(\mathcal{A} \cap \mathcal{B})^{\mathrm{f}}$ is nowhere contained in any $\mathcal{A}_p$. Then there exists a continuous map $\alpha \mapsto E(\alpha)$ from $\mathbf{2}^\omega$ into $\mathcal{K}(X)$ such that*:

- *For each $\alpha \in \mathbf{2}^\omega$, $E(\alpha) \in \mathcal{B}_1$.*
- *If $\alpha \in \mathbf{W}$, then $E(\alpha) \in \mathcal{A}^{\mathrm{f}}$.*
- *If $\alpha \notin \mathbf{W}$, then $E(\alpha) \notin \bigcup_{p\in\omega} \mathcal{A}_p$.*

In particular, there is no $\mathbf{\Pi}_3^0$ set $\mathcal{M} \subseteq \mathcal{K}(X)$ such that $\mathcal{A}^{\mathrm{f}} \cap \mathcal{B}_1 \subseteq \mathcal{M} \cap \mathcal{B}_1 \subseteq \bigcup_{p\in\omega} \mathcal{A}_p$.

As an immediate consequence, we get a kind of "Baire category theorem" for big $\mathbf{\Pi}_3^0$ ideals:

COROLLARY. *Let $\mathcal{A} \subseteq \mathcal{K}(X)$ be a big $\mathbf{\Pi}_3^0$ ideal. If $(\mathcal{A}_p)_{p\in\omega}$ is a sequence of hereditary subsets of $\mathcal{K}(X)$ such that $\mathcal{A} \subseteq \bigcup_{p\in\omega} \mathcal{A}_p$, then there exist an integer p and a nonempty open set $V \subseteq X$ such that $\mathcal{A} \cap \mathcal{K}(V) \subseteq \mathcal{A}_p$.*

Some simple remarks may help to justify the hypotheses of Theorem B. Assume that X is perfect.

1) If $\mathcal{A}$ is not big, the result is not true. For example, let $\mathcal{A}$ be the ideal of finite sets and $\mathcal{A}_p = \{K \in \mathcal{K}(X) : \mathrm{card}(K) \leq p\}$ $(p \in \omega)$; then $\mathcal{A}$ is nowhere contained in any $\mathcal{A}_p$, but it is obviously an F_σ set.

2) One cannot drop the hypothesis that $\mathcal{A}$ is an ideal. For example, let $D = \{x_n : n \in \omega\}$ be a countable dense subset of X, and let $G = X \setminus D$. Define $\mathcal{A} = \mathcal{K}(G) \cup \bigcup_{n\in\omega} \mathcal{K}(\{x_n\})$ and $\mathcal{A}_p = \mathcal{K}(G) \cup \bigcup_{n\leq p} \mathcal{K}(\{x_n\})$. Then $\mathcal{A}$ is big, the $\mathcal{A}_p$'s are hereditary and $\mathcal{A}$ is nowhere contained in any $\mathcal{A}_p$; yet $\mathcal{A}$ is $\mathbf{\Pi}_3^0$ (it is the union of a G_δ and a countable set).

3) Finally, one cannot remove the hereditarity assumption on the $\mathcal{A}_p$'s. For example, let $\{K_p : p \in \omega\}$ be any countable dense subset of $\mathcal{K}(X)$ (the K_p's being pairwise distinct), and let $\mathcal{A}_p = \mathcal{K}(X) \setminus \{K_n : n > p\}$.

In the proof of Theorem B, we will make use of the following two lemmas. The first one is easy; the second one is proved by applying the Baire category theorem in $\mathbf{2}^\omega$ (identified with $\mathcal{P}(\omega)$).

Lemma 1. *The map* $\phi : \mathcal{K}(X) \times \mathcal{K}(X) \to \mathcal{K}(X)$ *defined by* $\phi(K, L) = K \cup L$ *is* (*continuous and*) *open.*

Lemma 2 ([Ke1]). *Let* $\mathcal{G} \subseteq \mathcal{K}(X)$ *be* G_δ, $F \in \mathcal{K}(X)$, *and let* $(F_m)_{m\in\omega}$ *be a sequence converging to* F *in* $\mathcal{K}(X)$. *Assume that for each finite set* $I \subseteq \omega$, *the set* $F \cup \bigcup_{m\in I} F_m$ *belongs to* $\mathcal{G}$. *Then the compact set* $F \cup \bigcup_{m\in\omega} F_m$ *is the union of two elements of* $\mathcal{G}$.

Next, we introduce some notation.

Recall that we denote by $\langle p, q\rangle$ the image of a pair (p, q) under some fixed bijection from ω^2 onto ω. The image of an integer n under the inverse map will be denoted by $((n)_0, (n)_1)$.

Let $\mathbf{2}^{<\omega}$ be the set of all finite 0-1 sequences (including the empty sequence). We write $|s|$ for the length of a sequence $s \in \mathbf{2}^{<\omega}$. If $s \in \mathbf{2}^{<\omega}$ (and $s \neq \emptyset$), we denote by s' the immediate predecessor of s in the extension ordering.

If $\alpha \in \mathbf{2}^\omega$ and $n \in \omega$, we denote by $\alpha\lceil_n$ the length-n initial segment of α; thus, if $n \geq 1$, then $\alpha\lceil_n = (\alpha(0), \ldots, \alpha(n-1))$.

Next, we define inductively a sequence $(\theta_p)_{p\in\omega}$ of functions from $\mathbf{2}^{<\omega}$ into $\omega \cup \{+\infty\}$ in the following way:

(0) $\theta_p(\emptyset) = +\infty$ for all $p \in \omega$.

(i) If $|s| = n + 1$ and $(n)_0 > p$, then $\theta_p(s) = \theta_p(s')$.

(ii) If $|s| = n + 1$, $(n)_0 \leq p$ and $s(n) = 0$, then

$$\theta_p(s) = \begin{cases} \theta_p(s') & \text{if } \theta_p(s') < +\infty, \\ n & \text{if } \theta_p(s') = +\infty. \end{cases}$$

(iii) If $|s| = n + 1$, $(n)_0 \leq p$ and $s(n) = 1$, then $\theta_p(s) = +\infty$.

In other words, if we define $A_p(s) = \{m < |s| : (m)_0 \leq p\}$, then $\theta_p(s) = \min\{m \in A_p(s) : \forall m' \in A_p(s),\ m' \geq m,\ s(m') = 0\}$ (with the convention that $\min(\emptyset) = +\infty$). Thus, if we denote by $s_p : A_p(s) \to \{0, 1\}$ the restriction of s to $A_p(s)$, then $\theta_p(s)$ indicates the beginning of the longest "cofinal" 0-segment in s_p.

Finally, we define another sequence of functions from $\mathbf{2}^{<\omega}$ into $\omega \cup \{+\infty\}$ by

$$m_p(s) = \max(\langle p, 0\rangle, \theta_p(s)).$$

The following facts will be useful later.

Claim 1. *Let* $p \in \omega$ *and* $\alpha \in \mathbf{2}^\omega$. *Assume that* $\alpha_l \in \mathbf{Q}$ *for all* $l \leq p$.

(a) *The sequence* $(m_p(\alpha\lceil_n))_{n\in\omega}$ *is eventually finite-valued and constant.*

(b) *If we let* $M_p = \lim_{n\to\infty} m_p(\alpha\lceil n)$, *then* $\alpha(M_p) = 0$ *and*

$$\forall n \geq M_p \quad m_p(\alpha\lceil n+1) = M_p \text{ and } ((n)_0 > p \text{ or } \alpha(n) = 0).$$

Proof. Since for each $s \in \mathbf{2}^{<\omega}$, $m_p(S) \geq \theta_p(s)$ and the first coordinate of $m_p(s)$ is $\leq p$, we may content ourselves with proving (a) and (b) with θ_p in place of m_p.

By definition of $\mathbf{Q}$, there is a smallest integer N with the following properties:

$$(N)_0 \leq p, \quad \forall n \geq N \quad (n)_0 \leq p \Rightarrow \alpha(n) = 0.$$

The claim will be proved if we can show that $\theta_p(\alpha\lceil n+1) = N$ for all $n \geq N$.

(i) First, $\theta_p(\alpha\lceil n+1) = \theta_p(\alpha\lceil N+1)$ for every integer $n \geq N$. This follows by induction from the definition of the function θ_p: by the choice of N, we have $\alpha(N) = 0$ and $(N)_0 \leq p$, hence $\theta_p(\alpha\lceil N+1) < +\infty$; and if $n > N$, then either $(n)_0 > p$ or $\alpha(n) = 0$, so $\theta_p(\alpha\lceil n+1) = \theta_p(\alpha\lceil n)$ for all $n > N$.

(ii) By (i), it is now enough to check that $\theta_p(\alpha\lceil N+1) = N$. Let

$$N_1 = \begin{cases} \text{the greatest } n < N \text{ such that } (n)_0 \leq p & \text{if there is any,} \\ -1 & \text{if there is no such } n. \end{cases}$$

By the choice of N, we have $\alpha(N_1) = 1$ if $N_1 \geq 0$; thus, in both cases $\theta_p(\alpha\lceil N_1+1) = +\infty$. Now, by the choice of N_1, we also have $(n)_0 > p$ for all n such $N_1 < n < N$. This implies that $\theta_p(\alpha\lceil N) = \theta_p(\alpha\lceil N_1+1) = +\infty$. Thus $\theta_p(\alpha\lceil N+1) = N$.

Proof of Theorem B. The result is trivial if X is not perfect (one just has to let $E(\alpha) \equiv \{x_0\}$, where x_0 is an isolated point of X). Hence, from now on, X will be perfect.

For simplicity, we assume first that X is compact. We fix some metric compatible with the topology of X and we choose a complete metric δ for $\mathcal{B}$, which is possible since $\mathcal{B}$ is a G_δ subset of $\mathcal{K}(X)$.

Since each $\mathcal{A}_p$ is hereditary, the hypotheses of Theorem B remain unchanged if we replace $\mathcal{A}_p$ by $\bigcup_{l\leq p} \mathcal{A}_l$. Thus, we may assume that the sequence $(\mathcal{A}_p)_{p\in\omega}$ is nondecreasing.

Finally, let $\mathcal{G}$ be a dense G_δ hereditary subset of $\mathcal{K}(X)$ contained in $\mathcal{A}$. We can write $\mathcal{G} = \bigcap_{n\in\omega} \mathcal{U}^n$, where $(\mathcal{U}^n)_{n\in\omega}$ is a nonincreasing sequence of *hereditary* open subsets of $\mathcal{K}(X)$ (if $(\mathcal{W}^n)_{n\in\omega}$ is any nonincreasing sequence of open sets such that $\mathcal{G} = \bigcap_{n\in\omega} \mathcal{W}^n$, let $\mathcal{U}^n = \{K \in \mathcal{K}(X) : \forall L \subseteq K \; L \in \mathcal{W}^n\}$).

Claim 2. *For each positive integer* N, *the set* $\{(x, K_1, \ldots, K_N) \in X \times \mathcal{B}^N : \{x\} \cup \bigcup_{i=1}^N K_i \in \mathcal{G}\}$ *is dense in* $X \times \mathcal{B}^N$.

Proof. By Lemma 1, the set $\{(K_0, K_1, \ldots, K_N) \in \mathcal{K}(X)^{N+1} : \bigcup_{i=0}^N K_i \in \mathcal{G}\}$ is a dense G_δ subset of $\mathcal{K}(X)^{N+1}$; and since $\mathcal{G}$ is hereditary, this implies that the set $\{(x, K_1, \ldots, K_N) \in X \times \mathcal{K}(X)^N : \{x\} \cup \bigcup_{i=1}^N K_i \in \mathcal{G}\}$

is a dense G_δ subset of $X \times \mathcal{K}(X)^N$. Thus, the claim is true if $\mathcal{B} = \mathcal{K}(X)$. If $\mathcal{B}$ is the family of perfect sets, which is comeager in $\mathcal{K}(X)$ because X is perfect, the claim follows from the Baire category theorem. Finally, if $\mathcal{B} = \{\emptyset\} \cup \{\{x\} : x \in X\}$, we use again the fact that $\mathcal{G}$ is hereditary.

Now, we shall construct inductively a sequence $(j_m)_{m\in\omega}$ of positive integers and, for each $s \in \mathbf{2}^{<\omega}$, a compact set $E(s) \subseteq X$ and a nonempty open set $V(s) \subseteq X$.

For $s \neq \emptyset$, $E(s)$ will be written as $E(s) = \bigcup_{m<|s|} E^m(s)$, where each $E^m(s)$ is compact and of the form $E^m(s) = \bigcup_{j=1}^{j_m} E_j^m(s)$ ($E_j^m(s)$ compact).

We also construct (for $s \in \mathbf{2}^{<\omega} \setminus \{\emptyset\}$, $0 \le m < |s|$, and $1 \le j \le j_m$) nonempty open sets $V_j^m(s) \subseteq X$, and we let $V^m(s) = \bigcup_{j=1}^{j_m} V_j^m(s)$.

The closure of any set A involved in the construction will be denoted by $\overline{A}$.

The following requirements have to be fulfilled (to avoid typographic heaviness, we have omitted more often than not obvious information like "if $|s| \ge 1$", "$m < |s|$" or "$j \le j_m$").

(1) $E_j^m(s) \in \mathcal{A} \cap \mathcal{B}$ and $E^m(s) \neq \emptyset$.

(2) $$\begin{cases} E_j^m(s) \subseteq V_j^m(s) \subseteq \overline{V_j^m(s)} \subseteq V_j^m(s'), \\ \text{the } \overline{V^m(s)} \text{ are pairwise disjoint and disjoint from } \overline{V(s)}, \\ \overline{V(s)} \subseteq V(s'), \quad \overline{V^n(s)} \subseteq V(s') \quad \text{if } |s| = n+1. \end{cases}$$

(3) $\delta(E_j^m(s), E_j^m(s')) < 2^{-|s|}$, $\quad \operatorname{diam} V(s) < 2^{-|s|}$.

(4) If $|s| = n+1$, then $E_j^{m_p(s)}(s) = E_j^{m_p(s)}(s')$ for each $p < (n)_0$ (notice that $m_p(s) = m_p(s')$ here, because $p < (n)_0$).

(5) If $|s| = n+1$ and $s(n) = 0$, then $E_j^m(s) = E_j^m(s')$ for all $m < |s'|$.

(6) If $|s| = n+1$ and $s(n) = 0$, then $E^n(s) \notin \mathcal{A}_n$.

(7) If $|s| = n+1$ and $s(n) = 1$, then
$$\overline{V(s)} \cup \bigcup \{\overline{V^m(s)} : m < |s|,\ \forall p < (n)_0\ m \neq m_p(s)\} \in \mathcal{U}^{|s|}.$$

To begin the construction, we choose a nonempty open set $V(\emptyset) \subseteq X$ of diameter < 1, and we let $E(\emptyset) = \emptyset$.

Assume that the sets $E(t)$ and $V(t)$ have been constructed for all sequences t of length $\le n$. We have to define the positive integer j_n and the sets $V(s)$, $E_j^m(s)$, $V_j^m(s)$ ($0 \le m \le n$, $1 \le j \le j_m$) for every sequence s of length $n+1$.

(a) First, for each sequence t of length n, we choose two nonempty open sets $W_1(t), W_2(t) \subseteq V(t)$ with $W_1(t) \cap W_2(t) = \emptyset$. This is possible because X is perfect.

(b) Next, we define j_n and the sets $E_j^m(s)$ and $V_j^m(s)$ for all sequences s of length $n+1$ such that $s(n) = 0$.

(i) By (5), there is nothing to do for an integer $m < n$.

(ii) Let $S_0 = \{s \in \mathbf{2}^{<\omega} : |s| = n+1,\ s(n) = 0\}$. Since $(\mathcal{A}\cap\mathcal{B})^{\mathrm{f}}$ is nowhere contained in $\mathcal{A}_n$, we can find a positive integer j_n and, for all $s \in S_0$, compact sets $E_1^n(s), \ldots, E_{j_n}^n(s) \subseteq W_1(s')$ such that each $E_j^n(s)$ belongs to $\mathcal{A}\cap\mathcal{B}$, but $\bigcup_{j=1}^{j_n} E_j^n(s) \notin \mathcal{A}_n$. Notice that we can choose the same integer j_n for all sequences s because, since $\emptyset \in \mathcal{A}\cap\mathcal{B}$, we may always add the empty set (as many times as necessary) to the sets $E_j^n(s)$.

At this point, (4), (5) and (6) are satisfied, as well as (1) for $s \in S_0$ ($E^n(s)$ is nonempty because $\emptyset \in \mathcal{A}_n$). Then we can choose for each $s \in S_0$ nonempty open sets $V(s) \subseteq W_2(s')$ and $V_j^m(s) \supseteq E_j^m(s)$ in order to get (2) and (3).

(c) Now, let s be a sequence of length $n+1$ such that $s(n) = 1$.

(i) By (4), we must let $E_j^{m_p(s)}(s) = E_j^{m_p(s)}(s')$ for all the integers $p < (n)_0$ such that $m_p(s) = m_p(s') < |s'|$.

(ii) Let $I(s) = \{m < |s| : \forall p < (n)_0\ m \neq m_p(s)\}$ and $N = \sum_{m\in I(s)} j_m$. By Claim 2, the set $\{(x, K_1, \ldots, K_N) \in X \times \mathcal{B}^N : \{x\} \cup \bigcup_{i=1}^N K_i \in \mathcal{G}\}$ is dense in $X \times \mathcal{B}^N$. Therefore, we can find a point $x(s) \in X$ and compact sets $E_j^m(s) \in \mathcal{B}$ ($m \in I(s),\ 1 \le j \le j_m$) such that

$$(*)\qquad \begin{cases} \delta(E_j^m(s), E_j^m(s')) < 2^{-n-1} \text{ and } E_j^m(s) \subseteq V_j^m(s') \text{ if } m < n, \\ E_j^n(s) \subseteq W_1(s') \text{ and } E_j^n(s) \neq \emptyset, \\ x(s) \in W_2(s')\,, \\ \{x(s)\} \cup \bigcup\{E_j^m(s) : m \in I(s),\ 1 \le j \le j_m\} \in \mathcal{G}. \end{cases}$$

We can also ensure that $E_j^m(s) \neq \emptyset$ whenever $E_j^m(s') \neq \emptyset$, because $\emptyset$ is an isolated point in $\mathcal{K}(X)$. Moreover, since $\mathcal{G}$ is hereditary (and contained in $\mathcal{A}$), the last condition implies that each $E_j^m(s)$ belongs to $\mathcal{A}$; hence (1) is true for $m \in I(s)$ (of course, it was also true for $m \notin I(s)$).

(iii) It is now easy to choose open sets $V(s) \ni x(s)$ and $V_j^m(s) \supseteq E_j^m(s)$ in order to get (2), (3) and (7).

This concludes the inductive step.

It follows from (1) and (3) that if $m \in \omega$ and $j \le j_m$ are fixed, then for any $\alpha \in \mathbf{2}^\omega$, the sequence $(E_j^m(\alpha\lceil n))_{n>m}$ converges to a compact set $E_j^m(\alpha) \in \mathcal{B}$.

For each $\alpha \in \mathbf{2}^\omega$ and each $m \in \omega$, let $E^m(\alpha) = \bigcup_{j\le j_m} E_j^m(\alpha)$. By (1), all the $E^m(\alpha)$'s are nonempty (because $\emptyset$ is isolated in $\mathcal{K}(X)$). Moreover, conditions (2) and (3) imply that the sequence $(E^m(\alpha))_{m\in\omega}$ converges in $\mathcal{K}(X)$ to the singleton $\{x_\alpha\} = \bigcap_{n\in\omega} V(\alpha\lceil n)$.

Thus, the set $E(\alpha) = \{x_\alpha\} \cup \bigcup_{m\in\omega} E^m(\alpha)$ is compact, and in fact it belongs to $\mathcal{B}_1$. Furthermore, it follows from (2) and (3) that the map $\alpha \mapsto E(\alpha)$ is continuous.

Claim 3. *Let $\alpha \in \mathbf{2}^\omega$ and $p \in \omega$. Assume that $\alpha_l \in \mathbf{Q}$ for all $l \leq p$, and let $M_p = \lim_{n\to\infty} m_p(\alpha\lceil n)$. Then $E^{M_p}(\alpha) = E^{M_p}(\alpha\lceil M_p+1)$.*

Proof. By Claim 1, the integer M_p is well defined. Moreover, we know that for each $n \geq M_p$, $m_p(\alpha\lceil n+1) = M_p$, and either $(n)_0 > p$ or $\alpha(n) = 0$.

This implies that $E^{M_p}(\alpha\lceil n+1) = E^{M_p}(\alpha\lceil n)$ for each $n > M_p$. Indeed, we can use (4) if $(n)_0 > p$ and (5) if $\alpha(n) = 0$. Thus $E^{M_p}(\alpha) = E^{M_p}(\alpha\lceil M_p+1)$.

Let us now fix $\alpha \in \mathbf{2}^\omega$.

Case 1. Assume $\alpha \notin \mathbf{W}$. By Claims 1 and 3, all sequences $(m_p(\alpha\lceil n))_{n\in\omega}$ are eventually constant, and if we let $M_p = \lim_{n\to\infty} m_p(\alpha\lceil n)$ ($p \in \omega$), then $\alpha(M_p) = 0$ and $E^{M_p}(\alpha) = E^{M_p}(\alpha\lceil M_p+1)$. Hence, by (6), $E^{M_p}(\alpha) \notin \mathcal{A}_{M_p}$. Since each $\mathcal{A}_{M_p}$ is hereditary, this implies that $E(\alpha) \notin \bigcup_{p\in\omega} \mathcal{A}_{M_p}$. Now $M_p \geq \langle p, 0\rangle$ for all p, hence $\lim_{p\to\infty} M_p = +\infty$ (this was the reason for using the functions m_p rather than the θ_p's), and consequently $E(\alpha) \notin \bigcup_{p\in\omega} \mathcal{A}_p$.

Case 2. Assume $\alpha \in \mathbf{W}$. We have to show that $E(\alpha) \in \mathcal{A}^{\mathrm{f}}$.

Let p_0 be the smallest integer such that $\alpha_p \notin \mathbf{Q}$. For each $p < p_0$, let as usual $M_p = \lim_{n\to\infty} m_p(\alpha\lceil n)$ (which is well defined by Claim 1) and let

$$E_1(\alpha) = \bigcup_{p<p_0} E^{M_p}(\alpha),$$
$$E_2(\alpha) = \{x_\alpha\} \cup \bigcup \{E^m(\alpha) : m \neq M_p \text{ for all } p < p_0\}.$$

Since $E(\alpha) = E_1(\alpha) \cup E_2(\alpha)$, it is enough to check that $E_1(\alpha) \in \mathcal{A}^{\mathrm{f}}$ and $E_2(\alpha) \in \mathcal{G}^{\mathrm{f}}$.

(i) By the choice of p_0, $\alpha_p \in \mathbf{Q}$ for each $p < p_0$. Hence, by Claim 3 and condition (1), $E_1(\alpha) = \bigcup_{p<p_0} E^{M_p}(\alpha\lceil M_p+1) \in \mathcal{A}^{\mathrm{f}}$.

(ii) Let $I(\alpha) = \{m \in \omega : \forall p < p_0 \quad m \neq M_p\}$. It follows from (7) that $\overline{V(\alpha\lceil n+1)} \cup \bigcup \{\overline{V^m(\alpha\lceil n+1)} : m \in I(\alpha),\ m < n+1\} \in \mathcal{U}^n$ for each integer $n > \max\{M_p : p < p_0\}$ such that $(n)_0 = p_0$ and $\alpha(n) = 1$. Since there are infinitely many such n's (because $\alpha_{p_0} \notin \mathbf{Q}$) and each open set $\mathcal{U}^n$ is hereditary, this implies (by (2)) that for any finite set $I \subseteq I(\alpha)$, $\{x_\alpha\} \cup \bigcup_{m\in I} E^m(\alpha) \in \mathcal{G}$. Thus, from Lemma 2 we get $E_2(\alpha) \in \mathcal{G}^{\mathrm{f}}$.

The proof of Theorem B is now complete when X is assumed to be compact.

When X is not compact, we may always view it as a dense G_δ subset of some compact metric space $\widetilde{X}$. Write $X = \bigcap_{n\in\omega} W_n$, where the W_n's are open subsets of $\widetilde{X}$. Then we can perform our construction in $\widetilde{X}$ and moreover, we can easily ensure at each step that the open sets $V_j^m(s)$ and $V(s)$ are all contained in $W_{|s|}$. Hence, in the end, $E(\alpha) \subseteq X$ for each $\alpha \in \mathbf{2}^\omega$. This concludes the whole proof.

3. Applications. In this section, $\mathbf{G}$ is a second-countable nondiscrete locally compact abelian group, with dual group $\mathbf{\Gamma}$. We denote by $C_0(\mathbf{G})$, $M(\mathbf{G})$, $A(\mathbf{G})$ and $PM(\mathbf{G})$ respectively: the space of continuous complex-valued functions on $\mathbf{G}$ vanishing at infinity, the space of finite (complex) measures on $\mathbf{G}$, the Fourier transform of the convolution algebra $L^1(\mathbf{\Gamma})$, and the space of pseudomeasures on $\mathbf{G}$ (the dual space of $A(\mathbf{G})$).

Let $q(\mathbf{G}) = \sup\{n \in \omega : \text{every neighbourhood of } 0_{\mathbf{G}} \text{ contains elements of order} \geq n\}$. We define $\mathbf{G}_q = \{x \in \mathbf{G} : x \text{ is of order} \leq q(\mathbf{G})\}$, and $\mathbb{T}_q = \{z \in \mathbb{T} : z^{q(\mathbf{G})} = 1\}$ (with the convention that $z^\infty = 1$ for any $z \in \mathbb{T}$). Notice that $\mathbf{G}_q$ is a clopen subgroup of $\mathbf{G}$, by definition of $q(\mathbf{G})$.

Definition 1. Let K be a compact subset of $\mathbf{G}$.

1) K is said to be a *Helson set* if every continuous function on K can be extended to a function in $A(\mathbf{G})$.

2) K is said to be *without true pseudomeasure* (for short, WTP) if every pseudomeasure supported by K is actually a measure.

3) K is said to be *independent* if there is no nontrivial relation of the form $\sum_{i=1}^n m_i x_i = 0$, where $m_1, \ldots, m_n \in \mathbb{Z}$ and $x_1, \ldots, x_n \in K$ (that is, $\sum m_i\, x_i = 0 \Rightarrow \forall i\ m_i, x_i = 0$; when $\mathbf{G} = \mathbb{T}$, this is not exactly the usual definition).

4) K is said to be a K_q *set* if it is totally disconnected, all its elements have order $q(\mathbf{G})$, and the restrictions of characters of $\mathbf{G}$ are uniformly dense in $C(K, \mathbb{T}_q)$, the space of continuous functions from K into $\mathbb{T}_q$ (when $q(\mathbf{G}) = +\infty$, K_q sets are usually called *Kronecker sets*).

5) K is said to be a U_0' *set* if there is some constant c such that

$$\forall \mu \in M_+(K) \qquad \|\mu\|_{PM} \leq c \varlimsup_{\gamma \to \infty} |\widehat{\mu}(\gamma)|.$$

It is clear that $\mathcal{H}$ (the family of Helson subsets of $\mathbf{G}$), WTP, K_q and U_0' are hereditary subsets of $\mathcal{K}(\mathbf{G})$ (for K_q sets, this is because they are assumed to be totally disconnected).

There are natural constants associated with a given Helson or U_0' set. Namely, for each $K \in \mathcal{K}(\mathbf{G})$, define

$$\eta_0(K) = \inf\{\varlimsup_{\gamma \to \infty} |\widehat{\mu}(\gamma)|/\|\mu\|_{PM} : \mu \in M_+(K),\ \mu \neq 0\},$$
$$\alpha(K) = \inf\{\|\mu\|_{PM}/\|\mu\|_M : \mu \in M(K),\ \mu \neq 0\}.$$

Then (by definition) $K \in U_0' \Leftrightarrow \eta_0(K) > 0$ and (by standard functional-analytic arguments) $K \in \mathcal{H} \Leftrightarrow \alpha(K) > 0$. The number $\alpha(K)$ is the *Helson constant* of K. There is also a "WTP constant", whose definition should be reasonably clear.

Lemma 1. *K_q is a G_δ subset of $\mathcal{K}(\mathbf{G})$, and $\mathcal{H}$, WTP and U_0' are $\mathbf{\Sigma}_3^0$.*

The proofs are standard complexity calculations. For Helson sets, for example, the main point is that for each positive ε, the set $\mathcal{H}_\varepsilon = \{K \in \mathcal{K}(X) : \alpha(K) \geq \varepsilon\}$ is G_δ.

Definition 2. Let E be a closed subset of $\mathbf{G}$.

(a) E is said to be a U_0 *set* (or a set of *extended uniqueness*) if $\mu(E) = 0$ for every positive measure on $\mathbf{G}$ whose Fourier transform vanishes at infinity.

(b) E is an M_0 *set* if $E \notin U_0$, and an M_0^p *set* if $\overline{E \cap V} \in M_0$ for each open set $V \subseteq \mathbf{G}$ such that $E \cap V \neq \emptyset$.

Definition 3. Let p be a positive integer. By a *net of length* p, we mean any set of cardinality 2^p of the form $\{a + \sum_{i=1}^p \varepsilon_i l_i : \varepsilon_i = 0, 1\}$, where $a, l_1, \ldots, l_p$ are fixed elements of $\mathbf{G}$.

We shall use the following results. Almost all the proofs can be found in [GMG], and a lot of them in [KL] (see also [LP]).

1) $WTP \subseteq \mathcal{H} \subseteq U_0' \subseteq U_0$.

2) $\mathcal{H}$, WTP *and* U_0' *are translation-invariant ideals of* $\mathcal{K}(\mathbf{G})$; U_0 *is a translation-invariant* σ*-ideal of closed sets*.

3) *Finite sets are* WTP. *A finite set is a* K_q *set if and only if it is independent and all its elements have order* $q(\mathbf{G})$.

4) K_q *sets are Helson*; *in fact*, $\alpha_q = \inf\{\alpha(K) : K \in K_q\}$ *is* > 0.

5) *If* $F \subseteq \mathbf{G}$ *is a* p*-net*, *then* $\alpha(F) \leq (\sqrt{2})^{-p}$. *Hence*, *if a compact set* K *contains arbitrarily long nets*, *then* K *is not Helson.*

Before applying the results of Section 2, we prove some general facts about K_q sets.

For each integer m such that $0 < m < q(\mathbf{G})$, we let $\mathbf{N}_m = \{x \in \mathbf{G} : mx = 0\}$. The $\mathbf{N}_m$'s are closed subgroups of $\mathbf{G}$ and, by definition of $q(\mathbf{G})$, they are nowhere dense in $\mathbf{G}$.

We denote by $\mathcal{I}$ the σ-ideal generated by all translates of the $\mathbf{N}_m$'s, that is, the family of all subsets of $\mathbf{G}$ which can be covered by countably many translates of the $\mathbf{N}_m$'s.

Finally, we say that a set $E \subseteq \mathbf{G}$ is $\mathcal{I}$*-perfect* if no nonempty relatively open subset of E belongs to $\mathcal{I}$. By the Baire category theorem, every open subset of $\mathbf{G}$ is $\mathcal{I}$-perfect.

Lemma 2. *Let* $F \subseteq \mathbf{G}$ *be a finite* K_q *set*, *and let* $A = \{x \in \mathbf{G} : x \in \mathbf{G}_q$ *and* $F \cup \{x\} \notin K_q\}$. *Then* $A \in \mathcal{I}$.

Proof. We know that $F \cup \{x\}$ is a K_q set if and only if x has order $q(\mathbf{G})$ and $F \cup \{x\}$ is independent. Moreover, if $x \in \mathbf{G}$ has order $\leq q(\mathbf{G})$, it is easy to check that for each $m \in \mathbb{Z}$, there is an integer m' such that $|m'| < q(\mathbf{G})$ and $mx = m'x$.

Now, let $\mathrm{Gp}(F)$ be the subgroup of $\mathbf{G}$ generated by F. From the two preceding remarks, we easily deduce that A is contained in the set A' defined by

$$x \in A' \Leftrightarrow \exists m \ (0 < |m| < q(\mathbf{G}) \text{ and } mx \in \mathrm{Gp}(F)).$$

If $0 < |m| < q(\mathbf{G})$ and $y \in \mathbf{G}$, the set $E_{m,y} = \{x \in \mathbf{G} : mx = y\}$ belongs to $\mathcal{I}$. Since $\mathrm{Gp}(F)$ is countable, it follows that $A' \in \mathcal{I}$. This concludes the proof.

THEOREM 1. *Let $E \subseteq \mathbf{G}$ be a closed $\mathcal{I}$-perfect set contained in $\mathbf{G}_q$. Then $K_q \cap \mathcal{K}(E)$ is dense in $\mathcal{K}(E)$. In fact, for any finite K_q set $F \subseteq \mathbf{G}$, the set $\mathcal{G}_F = \{K \in \mathcal{K}(E) : K \cup F \in K_q\}$ is a dense G_δ hereditary subset of $\mathcal{K}(E)$.*

Proof. Since K_q is hereditary, the second statement implies the first. So let us fix a finite K_q set $F \subseteq \mathbf{G}$.

It is clear that $\mathcal{G}_F$ is G_δ and hereditary.

Now, let $V_1, \ldots, V_k$ be nonempty open subsets of E. Since each V_i is $\mathcal{I}$-perfect, we can apply Lemma 2 k times to get $x_1, \ldots, x_k \in \mathbf{G}$ such that $x_i \in V_i$ for all i and $F \cup \{x_1, \ldots, x_k\} \in K_q$. This shows that $\mathcal{G}_F$ is dense in $\mathcal{K}(E)$.

In the circle group, Theorem 1 simply says that the Kronecker sets are dense in any perfect subset of $\mathbb{T}$, which is a well known fact. When $q(\mathbf{G}) < \infty$, simple examples show that even if $P \subseteq \mathbf{G}$ is perfect and all its elements have order $q(\mathbf{G})$, the K_q sets contained in P need not be dense in $\mathcal{K}(P)$.

COROLLARY. *Let $E \subseteq \mathbf{G}$ be an $\mathcal{I}$-perfect set. Then $WTP \cap \mathcal{K}(E)$ is a big subset of $\mathcal{K}(E)$.*

Proof. It is easy to check (using the Baire category theorem and the separability of $\mathbf{G}$) that, given any nonempty closed set $F \subseteq \mathbf{G}$, there exist a point $a \in \mathbf{G}$ and an open set V such that $V \cap F \neq \emptyset$ and $a + \overline{F \cap V} \subseteq \mathbf{G}_q$. Thus we may and do assume that E is contained in $\mathbf{G}_q$ (because WTP is translation-invariant and every open subset of E is $\mathcal{I}$-perfect).

Now, let α_q be the constant introduced above, and let $\mathcal{G}$ be the family of all compact sets $K \subseteq E$ with the following property:

$$\forall S \in \mathbf{B}_1(PM(K)) \ \forall f \in A(\mathbf{G}) \quad |\langle S, f\rangle| \le \alpha_q \sup\{|f(x)| : x \in E\}.$$

Since $A(\mathbf{G})$ is dense in $C_0(\mathbf{G})$, $\mathcal{G}$ is contained in WTP. Moreover, using the separability of $A(\mathbf{G})$, one easily checks that $\mathcal{G}$ is a G_δ subset of $\mathcal{K}(E)$, which is obviously hereditary.

Finally, since finite sets are WTP, $\mathcal{G}$ contains every finite K_q subset of E; hence, by Theorem 1 (and the fact that K_q is hereditary), $\mathcal{G}$ is dense in $\mathcal{K}(E)$. This concludes the proof.

Now we turn to applications of the results of Section 2. We show first that WTP, $\mathcal{H}$ and U_0' are true $\mathbf{\Sigma}_3^0$ within any M_0 set; then we prove that $\mathcal{H}$ is true $\mathbf{\Sigma}_3^0$ inside the countable sets.

LEMMA 3. *Every closed set in $\mathcal{I}$ is a U_0 set.*

Proof. For each integer m such that $0 < m < q(\mathbf{G})$, $\mathbf{N}_m$ is a closed but nonopen subgroup of $\mathbf{G}$. By a result of V. Tardivel [T], this implies that $\mathbf{N}_m \in U_0$. Since U_0 is a translation-invariant σ-ideal of closed sets, the lemma follows.

LEMMA 4. *Let $E \subseteq \mathbf{G}$ be a nonempty M_0^p set and for $p \in \omega$, define $\mathcal{A}_p = \{K \in \mathcal{K}(E) : \eta_0(K) \geq 2^{-p}\}$. Then the perfect WTP sets contained in E are nowhere contained in any $\mathcal{A}_p$.*

Proof. By Lemma 3, E is $\mathcal{I}$-perfect. Hence, by Theorem 1 (Corollary), $WTP \cap \mathcal{K}(E)$ is a big subset of $\mathcal{K}(E)$. Now, by a result of Kechris [Ke1], if F is any M_0^p set and $\mathcal{G}$ is any dense G_δ hereditary subset of $\mathcal{K}(F)$, then, for each integer $N \geq 1$, there exist perfect sets $K_1, \ldots, K_N \in \mathcal{G}$ such that $\eta_0(K_1 \cup \ldots \cup K_N) < 4/N$. This proves the lemma.

Let $\mathcal{P}$ be the family of perfect compact subsets of $\mathbf{G}$.

THEOREM 2. *If $E \subseteq \mathbf{G}$ is an M_0 set, then there is no $\mathbf{\Pi}_3^0$ set such that $\mathcal{P} \cap WTP \cap \mathcal{K}(E) \subseteq \mathcal{M} \subseteq U_0'$. In particular, the families of perfect WTP sets, perfect Helson sets and perfect U_0' sets contained in E are true $\mathbf{\Sigma}_3^0$ in $\mathcal{K}(E)$.*

Proof. Since U_0 is a σ-ideal of closed sets, every M_0 set contains a nonempty M_0^p set. Hence, Theorem 2 follows from Lemma 4 and Theorem B.

LEMMA 5. *Let p be a positive integer, and let V be a nonempty open subset of $\mathbf{G}$ contained in $\mathbf{G}_q$. Then there is a finite set $F \subseteq \mathbf{G}$ such that*

- *$F \subseteq V$.*
- *F is a net of length p.*
- *Each element of F has order $q(\mathbf{G})$.*

Proof. By Theorem 1 applied to $\mathbf{G}_q$, the set $\{(x_0, \ldots, x_p) \in \mathbf{G}_q^{p+1} : \{x_0, \ldots, x_p\} \in K_q\}$ is dense in $\mathbf{G}_q^{p+1}$. Now, for each $\overline{x} = (x_0, \ldots, x_p) \in \mathbf{G}_q^{p+1}$, let $F(\overline{x}) = \{x_0 + \sum_{i=1}^p \varepsilon_i x_i : \varepsilon_i = 0, 1\}$. The map F is clearly continuous, so the set $\{\overline{x} \in \mathbf{G}_q^{p+1} : F(\overline{x}) \subseteq V,\ x_i \neq x_j \text{ for all } i \neq j\}$ is a nonempty open subset of $\mathbf{G}_q^{p+1}$. It follows that there exist pairwise distinct $a, l_1, \ldots, l_p \in \mathbf{G}_q$ such that $F(a, l_1, \ldots, l_p) \subseteq V$ and $\{a, l_1, \ldots, l_p\}$ is a K_q set.

It is then easy to see that $F = F(a, l_1, \ldots, l_p)$ has cardinality 2^p and that each element of F has order $q(\mathbf{G})$. This proves the lemma.

THEOREM 3. *There exists a continuous map $\alpha \mapsto E(\alpha)$ from $\mathbf{2}^\omega$ into $\mathcal{K}(\mathbf{G})$ such that*

- *For each $\alpha \in \mathbf{2}^\omega$, $E(\alpha)$ is a convergent sequence.*
- *If $\alpha \in \mathbf{W}$, then $E(\alpha)$ is the union of finitely many K_q sets.*
- *If $\alpha \notin \mathbf{W}$, then $E(\alpha)$ contains arbitrarily long nets.*

In particular, *the countable Helson sets form a true $\mathbf{\Sigma}_3^0$ subset of $\mathcal{K}_\omega(\mathbf{G})$.*

Proof. By Theorem 1 and Lemma 5, we can apply Theorem B with $X = \mathbf{G}_q$, $\mathcal{G} = K_q \cap \mathcal{K}(X)$, $\mathcal{B} = \{\emptyset\} \cup \{\{x\} : x \in X\}$ and $\mathcal{A}_p = \{K \in \mathcal{K}(X) :$ K does not contain any p-net$\}$.

Notice that $\mathcal{H} \cap \mathcal{K}_\omega(\mathbf{G})$ is not $\mathbf{\Sigma}_3^0$ in $\mathcal{K}(\mathbf{G})$. In fact, it is not even Borel. To see this, take an independent M_0 set $E \subseteq \mathbf{G}$ (a *Rudin set*, see [LP]). Then $\mathcal{K}_\omega(E)$ is not Borel in $\mathcal{K}(\mathbf{G})$, since E is uncountable. But every countable independent compact set is Helson (see [KL]). Hence $\mathcal{H} \cap \mathcal{K}_\omega(E) = \mathcal{K}_\omega(E)$ is not Borel in $\mathcal{K}(\mathbf{G})$. The same example shows that we cannot "localize" Theorem 3 within an arbitrary M_0 set.

To conclude this paper, we briefly discuss other examples of natural $\mathbf{\Sigma}_3^0$ in harmonic analysis.

Very close to the U_0' sets are the U' sets (see [KL]) and the U_2' sets of R. Lyons [Ly], which also form $\mathbf{\Sigma}_3^0$ subsets of $\mathcal{K}(\mathbb{T})$. In fact, one has the inclusions $WTP \subseteq U' \subseteq U_2' \subseteq U_0'$, so (by Theorem 2) U' and U_2' are true $\mathbf{\Sigma}_3^0$. For U' sets, there is a more precise "local" result: if E is any closed set of multiplicity, then $U' \cap \mathcal{K}(E)$ is a true $\mathbf{\Sigma}_3^0$ of $\mathcal{K}(E)$. This can be deduced from our criteria, using the family of Dirichlet sets rather than the family of K_q sets.

Other examples in the circle group are the *p-Helson sets* introduced by M. Gregory [G]. A compact set $K \subseteq \mathbb{T}$ is *p-Helson* (say $K \in \mathcal{H}_{(p)}$) if every continuous function on K is the restriction of a continuous function on $\mathbb{T}$ whose Fourier series belongs to $l^p(\mathbb{Z})$. Obviously, $\mathcal{H}_{(1)} = \mathcal{H}$, $\mathcal{H}_{(p)} \subseteq \mathcal{H}_{(p')}$ if $p \le p'$, and $\mathcal{H}_{(2)} = \mathcal{K}(\mathbb{T})$.

It is shown in [G] that for $p > 1$, $\mathcal{H}_{(p)}$ is a σ-ideal of $\mathcal{K}(\mathbb{T})$, and that a compact set K is p-Helson if and only if for all $\mu \in M(K)$, $\|\mu\|_M = \inf\{\|\mu - \lambda\|_M + \|\widehat{\lambda}\|_{l^q} : \lambda \in M_q(\mathbb{T})\}$, where $M_q(\mathbb{T}) = \{\lambda \in M(\mathbb{T}) : \widehat{\lambda} \in l^q(\mathbb{Z})\}$ (and $q = p/(p-1)$). Using this, it is not hard to check that $\mathcal{H}_{(p)}$ is G_δ in $\mathcal{K}(\mathbb{T})$.

Now, let $\widetilde{\mathcal{H}}$ be the family of compact subsets of $\mathbb{T}$ which are p-Helson for some $p \in]1, 2[$. It follows from the preceding remarks that $\widetilde{\mathcal{H}}$ is a big $\mathbf{\Sigma}_3^0$ ideal of $\mathcal{K}(\mathbb{T})$. Moreover, it is also shown in [G] that for any $p < 2$, $\mathcal{H}_{(p)}$ is a proper subset of $\widetilde{\mathcal{H}}$; and it is not difficult to deduce from the proof that this is true in any open subset of $\mathbb{T}$. Thus Theorem A applies, and we conclude that $\widetilde{\mathcal{H}}$ is a true $\mathbf{\Sigma}_3^0$ subset of $\mathcal{K}(\mathbb{T})$.

On the other hand, our criteria do not apply to the family of H-*sets* of the circle group, which is also a true $\mathbf{\Sigma}_3^0$ (see [Li]), because H is not an ideal of $\mathcal{K}(\mathbb{T})$.

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Université Bordeaux 1
351, Cours de la libération
33405 Talence Cedex, France
E-mail: matheron@math.u-bordeaux.fr

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